HAL open science

Spectral triples and manifolds with boundary

Bruno Iochum, Cyril Levy

▶ To cite this version:

Bruno Iochum, Cyril Levy. Spectral triples and manifolds with boundary. Journal of Functional Analysis, 2011, 260 (1), pp.117-134. 10.1016/j.jfa.2010.09.006 . hal-00448475v2

HAL Id: hal-00448475

https://hal.science/hal-00448475v2

Submitted on 10 Sep 2010

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

CENTRE DE PHYSIQUE THÉORIQUE [1]
CNRS–Luminy, Case 907
13288 Marseille Cedex 9
FRANCE

Spectral triples and manifolds with boundary

B. Iochum[1,2], C. Levy[3]

Abstract

We investigate manifolds with boundary in noncommutative geometry. Spectral triples associated to a symmetric differential operator and a local boundary condition are constructed. We show that there is no tadpole for classical Dirac operators with a chiral boundary condition on certain manifolds.

PACS numbers: 11.10.Nx, 02.30.Sa, 11.15.Kc
MSC–2000 classes: 46H35, 46L52, 58B34
CPT-P003-2010

[1] UMR 6207
– Unité Mixte de Recherche du CNRS et des Universités Aix-Marseille I, Aix-Marseille II et de l'Université du Sud Toulon-Var (Aix-Marseille Université)
– Laboratoire affilié à la FRUMAM – FR 2291
[2] Also at Université de Provence, iochum@cpt.univ-mrs.fr
[3] University of Copenhagen, Department of mathematical sciences, Universitetsparken 5, DK-2100 Copenhagen, Denmark, levy@math.ku.dk

1 Introduction

Noncommutative geometry has, in its numerous motivations, the conceptual understanding of different aspects of physics [14,21]. In particular, the spectral approach which is deeply encoded in the notion of a spectral triple, is not only motivated by the algebra of quantum observables $\mathcal{A}$ acting on the Hilbert space $\mathcal{H}$ of physical states, but also by classical physics like general relativity. For instance, a Riemannian compact spin manifold can be reconstructed only via properties of a commutative spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ [19] where the last piece $\mathcal{D}$ is a selfadjoint operator acting on $\mathcal{H}$ playing the role of a Dirac operator which can fluctuates: $\mathcal{D}$ is then replaced by $\mathcal{D}_A := \mathcal{D} + A$ where A is a selfadjoint one-form.
The spectral action $\mathcal{S}$ of Chamseddine–Connes [11] associated to a triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is the trace of $\Phi(\mathcal{D}_A^2/\Lambda^2)$ where Φ is a positive function and Λ plays the role of a cut-off. This can be written (under some conditions on the spectrum) as a series of noncommutative integrals

$$\mathcal{S}(\mathcal{D}_A, \Phi, \Lambda) \;=\; \sum_{k \in Sd^+} \Phi_k \, \Lambda^k \fint |\mathcal{D}_A|^{-k} + \Phi(0)\, \zeta_{\mathcal{D}_A}(0) + \mathcal{O}_{\Lambda\to\infty}(\Lambda^{-1}) \tag{1}$$

where $\Phi_k = \frac{1}{2}\int_0^\infty \Phi(t)\, t^{k/2-1}\, dt$, Sd^+ is the strictly positive part of the dimension spectrum of the spectral triple and the noncommutative integral $\fint X$ for X in the algebra $\Psi(\mathcal{A})$ of pseudodifferential operators, is defined by $\fint X := \operatorname*{Res}_{s=0} \operatorname{Tr}\big(X|\mathcal{D}|^{-s}\big)$.
Since $\fint$ is a trace on $\Psi(\mathcal{A})$ (non necessarily positive), it coincides (up to a scalar) with the Wodzicki residue [58, 59] in the case of a commutative geometry where $\mathcal{A}$ is the algebra of C^∞ functions on a manifold M without boundary: in a chosen coordinate system and local trivialization (x, ξ) of T^*M, this residue is

$$Wres(X) := \int_M \int_{S_x^*M} \operatorname{Tr}\left(\sigma_{-d}^X(x,\xi)\right) b|d\xi|\,|dx|,$$

where σ_{-d}^X is the symbol of the classical pseudodifferential operator X which is homogeneous of degree $-d := -\dim(M)$, $d\xi$ is the normalized restriction of the volume form to the unit sphere $S_x^*M \simeq \mathbb{S}^{d-1}$. The Dixmier's trace Tr_ω [23] concerns compact operators X with singular values $\{\,\mu_k\,\}$ satisfying $\sup_{N\to\infty} a_N < \infty$ where $a_N = \log(N)^{-1}\sum_{k=1}^N \mu_k$ and $\operatorname{Tr}_\omega(X)$ is defined after a choice of an averaging procedure ω such that $\operatorname{Tr}_\omega(X) = \lim_N a_N$ when a_N converges. As shown in [18], $\fint$ coincides (still up to a universal scalar) with Tr_ω when X has order $-d$.
When M has a boundary, the choice of an appropriate differential calculus is delicate. In the noncommutative framework, the links between Boutet de Monvel's algebra, Wodzicki's residue, Dixmier's trace or Kontsevich–Vishik's trace [42] have been clarified [1,25,32,33,48,52] including the case of log-polyhomogeneous symbols [45].
From a physics point of view, applications of noncommutative integrals on manifolds to classical gravity has begun with Connes' remark that $\fint \mathcal{D}^{-2}$ coincides in dimension 4 with Einstein–Hilbert action, a fact recovered in [40,41]. Then, a generalization to manifolds with boundaries was proposed in [53, 55–57]. From the quantum side, a noncommutative approach of the unit disk is proposed in [10].

Nevertheless, a construction of spectral triples in presence of boundary is not an easy task, although a general approach of boundary spectral triples has been announced in [20].
First examples appear with isolated conical singularity in [46], a work related to some extend to [44, 51] when the spectrum dimension is computed. The difficulty is to find an appropriate

boundary condition which preserves not only the selfadjointness of the realization of $\mathcal{D}$ but also the ellipticity. A special case of boundary also appears in non-compact manifolds when one restricts the operator $\mathcal{D}$ to a bounded closed region: for instance, this trick was used in [50, 60]. The choice of a chiral boundary condition, already considered in [8] for mathematical reasons, is preferred in [13] for physical reasons: firstly, it is consistent with a selfadjoint and elliptic realization, and secondly, it is a local boundary condition contrary to the standard APS' one which is global [2]. Thirdly, it gives a similar ratio and signs for the second term of the spectral action (the first one being the cosmological constant), namely the scalar curvature of the manifold and the extrinsic curvature of the boundary, as in the Euclidean action used in gravitation [34]. Since there are a lot of possible choices, this last consideration deserves attention.

Here, we first show a construction for manifolds with boundary that actually produces a spectral triple, and then give conditions on the algebra of functions on that manifold to get a regular triple (remark that the spectral action has only been computed, until now, for spectral triples which are regular). While in field theory, the one-loop calculation divergences, anomalies and different asymptotics of the effective action are directly obtained from the heat kernel method [26, 27, 54], we try to avoid this perturbation approach already used in [39] to prove that there are no tadpoles when a reality operator J exists. Tadpoles are the A-linear terms in (1), like for example $⨍ A\mathcal{D}^{-1}$. In quantum field theory, $\mathcal{D}^{-1}$ is the Feynman propagator and $A\mathcal{D}^{-1}$ is a one-loop graph with fermionic internal line and only one external bosonic line A looking like a tadpole.
In section 2, we derive a technical result on regularity of spectral triples which is sufficient to avoid the use of Sobolev spaces of negative order. Then, we recall few basics on the realization of boundary pseudodifferential operators and their stability by powers using the Grubb's approach [30]. In section 4, we define an algebra $\mathcal{A}_{P_T}$ compatible with the realization of an elliptic pseudodifferential boundary system $\{ P, T \}$. A condition on P is given which guarantees the regularity of the associated spectral triple. The motivating example of a classical Dirac operator is considered in section 4.2. Moreover, the construction of a spectral triple on the boundary is revisited in section 4.3. Section 5 is devoted to a reality operator J on a spectral triple with boundary and some consequences on the tadpoles like $⨍ A\mathcal{D}^{-1}$ which can appear in spectral action.

2 Regularity

Let $\mathbb{N}$ be the non-negative integers and $\mathcal{B}(\mathcal{H})$ be the set of bounded operator on a separable Hilbert space $\mathcal{H}$.
We shall use the following definition of a spectral triple:

Definition 2.1. *A spectral triple of dimension d is a triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ such that $\mathcal{H}$ is a Hilbert space and*

- $\mathcal{A}$ is an involutive unital algebra faithfully represented in $B(\mathcal{H})$,

- $\mathcal{D}$ is a selfadjoint operator on $\mathcal{H}$ with compact resolvent and its singular values $(\mu_n(|\mathcal{D}|))_n$ are $\mathcal{O}(n^{1/d})$,

- for any $a \in \mathcal{A}$, $a \operatorname{Dom} \mathcal{D} \subseteq \operatorname{Dom} \mathcal{D}$ and the commutator $[\mathcal{D}, a]$ (with domain $\operatorname{Dom} \mathcal{D}$) as an extension in $B(\mathcal{H})$ denoted da.

Note that $\operatorname{Dom} |\mathcal{D}| = \operatorname{Dom} \mathcal{D}$. We set $\delta(T) := \overline{[|\mathcal{D}|, T]}$, where $\overline{A}$ is the closure of the operator A, with domain

$$\operatorname{Dom} \delta := \{ T \in \mathcal{B}(\mathcal{H}) \, : T \operatorname{Dom} \mathcal{D} \subseteq \operatorname{Dom} \mathcal{D} \text{ and } [|\mathcal{D}|, T] \text{ has closure in } \mathcal{B}(\mathcal{H}) \}.$$

Definition 2.2. *A spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is said to be regular if $\mathcal{A}$ and $d\mathcal{A}$ are included in $\cap_{n\in\mathbb{N}} \operatorname{Dom} \delta^n$.*

As seen in next lemma, it is quite convenient to introduce the following

Definition 2.3. *Given a selfadjoint operator P on a Hilbert space $\mathcal{H}$, let*

$$\mathcal{H}_P^\infty := \cap_{k\geq 1} \operatorname{Dom} |P|^k = \cap_{k\geq 1} \operatorname{Dom} P^k.$$

A linear map from $\mathcal{H}_P^\infty$ into itself which is continuous for the topologies induced by $\mathcal{H}$ is said to be $\mathcal{H}_P^\infty$-bounded.

Note that $\mathcal{H}_P^\infty$ is a core for any power of P or $|P|$. In particular it is a dense subset of $\mathcal{H}$ and $|P|$ is the closure of the essentially selfadjoint operator $|P|_{|\mathcal{H}_P^\infty}$. Note also that for any $k \in \mathbb{N}$, $(1 + P^2)^{k/2}$ is a bijection from $\operatorname{Dom} P^k$ onto $\mathcal{H}$, and thus, $(1 + P^2)^{-k/2}$ is a bijection from $\mathcal{H}$ onto $\operatorname{Dom} P^k$. As a consequence, for any $p \in \mathbb{Z}$, the operators $(1 + P^2)^{p/2}$ send bijectively $\mathcal{H}_P^\infty$ onto itself, and for any $k \in \mathbb{N}$, $|P|^k$ send $\mathcal{H}_P^\infty$ into itself.
Given a selfadjoint P, let δ', δ_1 be defined on operators by

$$\delta'(T) := [|P|, T], \qquad \delta_1(T) := [P^2, T]\,(1 + P^2)^{-1/2}$$

with domains

$$\operatorname{Dom} \delta' \ (\text{resp.} \operatorname{Dom} \delta_1) := \{\, T \in \mathcal{H}_P^\infty\text{-bounded operators} \,:\, \delta'(T) \big(\text{resp. } \delta_1(T)\big) \text{ is } \mathcal{H}_P^\infty\text{-bounded} \,\}.$$

We record the following lemma, proven by A. Connes:

Lemma 2.4. *[19, Lemma 13.1 and 13.2].*
(i) If $T \in \operatorname{Dom} \delta'$, then the bounded closure $\overline{T}$ of T is in $\operatorname{Dom} \delta$ and $\delta(\overline{T}) = \overline{\delta'(T)}$.
So, by induction, if $T \in \cap_{n\in\mathbb{N}} \operatorname{Dom} {\delta'}^n$ then $\overline{T} \in \cap_{n\in\mathbb{N}} \operatorname{Dom} \delta^n$.
(ii) Let T be a $\mathcal{H}_P^\infty$-bounded operator. If $T \in \cap_{n\in\mathbb{N}} \operatorname{Dom} {\delta_1}^n$, then $T \in \cap_{n\in\mathbb{N}} \operatorname{Dom} {\delta'}^n$.
In particular the bounded closure $\overline{T}$ of T belongs to $\cap_{n\in\mathbb{N}} \operatorname{Dom} \delta^n$.

Definition 2.5. *Given a Hilbert space $\mathcal{H}$ and a selfadjoint (possibly unbounded) operator P, we call Sobolev scale on $\mathbb{N}$, a family $(H^k)_{k\in\mathbb{N}}$ of Hilbert spaces such that*
- *$H^0 = \mathcal{H}$,*
- *H^{k+1} is continuously included in H^k for any $k \in \mathbb{N}$,*
- *for any $k \in \mathbb{N}$, $\operatorname{Dom} P^k$ is a closed subset of H^k.*

By closed graph theorem, the last point implies that P^k is continuous from $\operatorname{Dom} P^k$ endowed with the H^k-topology into $\mathcal{H}$.
Similar abstract Sobolev scales, defined as domains of the powers of an abstract differential operator, have been considered in [36]. A corresponding criterion for regularity has been obtained in [36, 4.26 Theorem]. The scale we consider here will correspond in section 4 to the Sobolev spaces (on the manifold with boundary) and not to the domains of the powers of the realization of a first order pseudodifferential operator. In the case without boundary, these scales coincide.
When T is a $\mathcal{H}_P^\infty$-bounded operator, we shall denote $T^{(k)} := [P^2, \cdot]^k(T)$ for any $k \in \mathbb{N}$.

Lemma 2.6. *Let P be a selfadjoint operator and T be a $\mathcal{H}_P^\infty$-bounded operator. Suppose that there is a Sobolev scale $(H^k)_{k\in\mathbb{N}}$ such that for any $k \in \mathbb{N}$, $T^{(k)}$ is continuous from $\mathcal{H}_P^\infty$ with the H^k-topology into $\mathcal{H}_P^\infty$ with the $\mathcal{H}$-topology. Then $\overline{T} \in \cap_{n\in\mathbb{N}} \operatorname{Dom} \delta^n$.*

Proof. The operator $(P-i)^k = \sum_{j=0}^{k} \binom{k}{j}(-i)^{k-j}P^j$ is continuous from $\operatorname{Dom} P^k$ with the H^k-topology into $\mathcal{H}$. Since P is selfadjoint, $P-i$ is a bijection from $\operatorname{Dom} P$ onto $\mathcal{H}$, and by composition, $(P-i)^k$ is a bijective map from $\operatorname{Dom} P^k$ onto $\mathcal{H}$. The inverse mapping theorem now implies that $(P-i)^{-k}$ is continuous from $\mathcal{H}$ onto $\operatorname{Dom} P^k$ with the H^k-topology. Moreover, $(1+P^2)^{-k/2} = (P-i)^{-k}B$ where $B := (P-i)^k(1+P^2)^{-k/2}$ is, by spectral theory, a bijective operator in $\mathcal{B}(\mathcal{H})$. As a consequence, $(1+P^2)^{-k/2}$ is continuous from $\mathcal{H}$ onto $\operatorname{Dom} P^k$ with the H^k-topology. In particular,$(1+P^2)^{-k/2}$ is continuous from $\mathcal{H}_P^\infty$ with the $\mathcal{H}$-topology, into $\mathcal{H}_P^\infty$ with the H^k-topology.
So the hypothesis gives that $T^{(k)}(1+P^2)^{-k/2} = \left([P^2,\cdot]\,(1+P^2)^{-1/2}\right)^k(T) = \delta_1^k(T)$ is a $\mathcal{H}_P^\infty$-bounded operator. The result follows from Lemma 2.4. □

The previous lemma essentially implies that, in order to prove the regularity of a spectral triple $(\mathcal{A},\mathcal{H},\mathcal{D})$, it is sufficient to construct a Sobolev scale $(H^k)_{k\in\mathbb{N}}$ adapted to $\mathcal{H}$ and $\mathcal{D}$, such that the operators $\mathcal{D}^k$ and $T^{(k)}$ behave respectively as operators of "order" k with respect to the Sobolev scale, when T is any element of $\mathcal{A}\cup d\mathcal{A}$.

Remark 2.7. *By Lemma 2.4, it is possible to obtain regularity without using Sobolev spaces of negative order, implicitly used for instance in [29, Theorem 11.1] which follows the original argument of [15]. This shall considerably simplify the proof of the regularity of the spectral triple on manifolds with boundary, since the continuity of realizations of elliptic boundary differential operators is usually established on Sobolev spaces of positive order [5, 6, 9, 30]. Note however that it may be possible to deal with negative orders by using the technique of transposition described in [47].*
Another approach of regularity can be found in [35, 36, 49].

3 Background on elliptic systems on manifolds with boundary

We review in this section a few definitions and basic properties about Sobolev spaces in manifolds with and without boundary and boundary pseudodifferential operators choosing Boutet de Monvel's calculus. More details and proofs can be found in classical references like [30, 38].
Let $\widetilde{M}$ be a smooth compact manifold without boundary of dimension d and $\widetilde{E}$ be a smooth hermitian vector bundle on $\widetilde{M}$. Let M be an open submanifold of $\widetilde{M}$ of dimension d such that $\overline{M}$ (topological closure) is a compact manifold with nonempty boundary $N := \partial\overline{M} = \overline{M}\backslash M$. As a consequence, N is a smooth compact submanifold of $\widetilde{M}$ without boundary of dimension $d-1$.
The sub-bundle of $\widetilde{E}$ on $\overline{M}$ (resp. N) is denoted E (resp. E_N). We denote $H^s(\widetilde{E})$, $H^s(E)$ the Sobolev spaces of order $s\in\mathbb{R}$ respectively on $\widetilde{M}$ with bundle $\widetilde{E}$ and $\overline{M}$ with bundle E. Recall that by definition

$$H^s(E) := H^s(\overline{M},E) := r^+\big(H^s(\widetilde{E})\big)$$

where r^+ is the restriction to M. We refer to [30, p. 496] for the definition of the topology of $H^s(E)$.
Remark that, for a given manifold $\overline{M}$ with boundary $\partial\overline{M}$, it is always possible to construct $\widetilde{M}$ with previous properties. Moreover, there exist constructions of invertible double for Dirac operators and more general first order elliptic operators on closed double of $\overline{M}$ [4–6].
We denote $\Psi^k(\widetilde{E})$ (resp. $\operatorname{Diff}^k(\widetilde{E})$) the space of pseudodifferential (resp. differential) operators of order k on $(\widetilde{M},\widetilde{E})$. Any element of $\Psi^k(\widetilde{E})$ is a linear continuous operator from $H^s(\widetilde{E})$ into $H^{s-k}(\widetilde{E})$, for any $s\in\mathbb{R}$.

We set

$$C^\infty(\overline{M},E) := r^+\left(C^\infty(\widetilde{M},\widetilde{E})\right), \qquad C^\infty(\overline{M}) := r^+\big(C^\infty(\widetilde{M})\big)\,.$$

Despite notations, note that the spaces $H^s(E)$, $C^\infty(\overline{M},E)$ and $C^\infty(\overline{M})$ are spaces of functions defined on M, and not on $\overline{M}$, as r^+ is the restriction on M. Remark that $C^\infty(\overline{M})$ is identified to $C^\infty(\overline{M})\,\mathrm{Id}_{E_{|M}}$, so that $C^\infty(\overline{M})$ can be seen as an algebra of bounded operators on $H^s(E)$, and in particular on $H^0(E) = L^2(\overline{M},E) = r^+L^2(\widetilde{M},\widetilde{E})$.
A differential operator P on $\overline{M}$ is by definition a differential operator on M with coefficients in $C^\infty(\overline{M},E)$. We denote $\mathrm{Diff}^k(\overline{M},E)$ the space of differential operators of order $k\in\mathbb{N}$ on $\overline{M}$.
Any element of $\mathrm{Diff}^k(\overline{M},E)$ can be extended uniquely as a linear continuous operator from $H^s(E)$ into $H^{s-k}(E)$, for any $s\in\mathbb{R}$.
Finally, note that $C^\infty(\widetilde{M},\widetilde{E})$ is dense in any $H^p(\widetilde{E})$ and $C^\infty(\overline{M},E))$ is also dense in any $H^p(E) := r^+H^p(\widetilde{E})$, and moreover,

$$\bigcap_{p\in\mathbb{N}} H^p(\widetilde{E}) = C^\infty(\widetilde{M},\widetilde{E}), \qquad \bigcap_{p\in\mathbb{N}} H^p(E) = C^\infty(\overline{M},E)\,.$$

The extension by zero operator e^+ is a linear continuous operator from $H^s(E)$ into $H^s(\widetilde{E})$ for any $s\in]-\frac{1}{2},\frac{1}{2}[$ such that $e^+(u)=u$ on M and $e^+(u)(x)=0$ for any $u\in C^\infty(\overline{M},E)$ and $x\in\widetilde{M}\backslash M$.
For any $P\in\Psi^k(\widetilde{E})$, we define its truncation to M by

$$P_+ := r^+\,P\,e^+\,.$$

Recall that $P\in\Psi^k(\widetilde{E})$, $k\in\mathbb{Z}$, is said to satisfy the transmission condition if P_+ maps $C^\infty(\overline{M},E)$ into itself which means that "P_+ preserves C^∞ up to the boundary". In particular, any differential operator satisfies this condition.
It turns out that if P satisfies the transmission condition, P_+ can be seen as linear continuous operator from $H^s(E)$ into $H^{s-k}(E)$ for any $s>-\frac{1}{2}$ ([30, 2.5.8 Theorem, 2.5.12 Corollary]).
We refer to [30, 31] for all definitions of elliptic boundary system, normal trace operator and singular Green operator but to fix notations, we recall the Green formula [30, 1.3.2 Proposition]: for $P\in\Psi^k(\widetilde{E})$, $k\in\mathbb{N}$ and for any $u,v\in C^\infty(\overline{M},E)$,

$$(P_+u,v)_M - (u,(P^*)_+v)_M = (\mathfrak{A}_P\,\rho u,\rho v)_N \tag{2}$$

where $(u,v)_X := \int_X u(x)\overline{v(x)}\,dx$ (if defined) and $\rho=\{\,\gamma_0,\cdots,\gamma_{k-1}\,\}$ is the Cauchy boundary operator given by $\gamma_j u = (-i\partial_d)^j\,u_{|N}$ (∂_d being the interior normal derivative) and $\mathfrak{A}_P$ is the Green matrix associated to P.
Here, $x=(x',x_d)$ is an element of $\overline{M}=N\sqcup M$ with $(x',0)\in N$ and x_d denotes a normal coordinate. By [30, Lemma 1.3.1], $P=A+P'$ with $A=\sum_{l=0}^k S_l\,(-i\partial_d)^l$, where S_l is a tangential differential operator of order $k-l$ supported near $N=\partial M$, and P' is a pseudodifferential operator of order k satisfying $(P'_+u,v)_M-(u,(P'^*)_+v)_M=0$. The Green matrix satisfies

$$\mathfrak{A}_P = (\mathfrak{A}_{j,l})_{j,l=0,\cdots,k-1} \text{ with } \mathfrak{A}_{j,l}(x',D') := iS_{j+l+1}(x',0,D') + \text{ lower-order terms}\,.$$

and $\mathfrak{A}_{jl}$ is zero if $j+l+1>k$.

Remark 3.1. *When P is a pseudodifferential operator of order 1, $\mathfrak{A}_P$ is an endomorphism on the boundary N. For instance, if P is a classical Dirac operator acting on a Dirac bundle, then $\mathfrak{A}_P=-i\gamma_d$ where γ is the (selfadjoint) Clifford action and $\{e_i\}$, $i=1,\dots,d$ is a (local) orthonormal frame of TM such that e_d is the inward pointing unit normal, and $\gamma_i:=\gamma(e_i)$. This $\mathfrak{A}_P$ corresponds to $-J_0$ in [4, 5].*

When $P \in \Psi^n(\widetilde{E})$, $n \in \mathbb{N}$, satisfies the transmission condition, G is a singular Green operator of order n and class $\leq n$ and $T := \{T^0, \cdots, T^{n-1}\}$ is a system of normal trace operators associated with the order n, each T^i going from E to E_N, then the *(H^n-) realization of the system* $\{P_+ + G, T\}$ is the operator $(P+G)_T$ defined as the operator acting like $P_+ + G$ with domain

$$\mathrm{Dom}\,(P+G)_T := \{\psi \in H^n(\overline{M}, E) \ : \ T\psi = 0\}.$$

These realizations are always densely defined, and since T is continuous from $H^n(E)$ into $\prod_{j=0}^{n-1} H^{n-j-\frac{1}{2}}(E_N)$, $\mathrm{Dom}\,(P+G)_T$ is a closed subset of $H^n(E)$. Recall that $P_+ + G$ is continuous from $H^s(E)$ into $H^{s-n}(E)$ for any $s > n - \frac{1}{2}$.
We shall assume that all pseudodifferential operators satisfy the transmission property at the boundary.
We record here the following proposition, which is a direct application of [30, 1.4.6 Theorem, Corollary 2.5.12, 2.7.8 Corollary]:

Proposition 3.2. *Let $\{P_+ + G, T\}$ be an elliptic system of order n (G being with class $\leq n$) with T a system of normal trace operators associated with the order n.*
Then for any $k \in \mathbb{N}$, there exist a singular Green operators G_k of class $\leq nk$, and a system of normal trace operators T_k associated with the order nk such that $\big((P+G)_T\big)^k$ is the realization of the elliptic system $\{(P^k)_+ + G_k, T_k\}$ of order nk.
Moreover, $\mathrm{Dom}\,((P+G)_T)^k$ *is a closed subset of $H^{nk}(E)$, and $(P^k)_+ + G_k$ is continuous from $H^s(E)$ into $H^{s-nk}(E)$ for $s > nk - \frac{1}{2}$.*

When P, P' are in $\Psi^\infty(\widetilde{E})$, the leftover of P and P'

$$L(P, P') := (PP')_+ - P_+ P'_+$$

is a singular Green operator of order $k + k'$ and class $\leq k + k$' when the order of P and P' are k and k'.
The following result is a consequence of [30, (2.6.27)], but we give a short proof.

Lemma 3.3. *Let $P, P' \in \Psi^\infty(\widetilde{E})$.*
(i) If P is differential, then $L(P, P') = 0$.
(ii) If P' is an endomorphism on $(\widetilde{M}, \widetilde{E})$ (differential operator of order 0), then $L(P, P') = 0$.

Proof. (i) From the locality of differential operators, we see that $r^+ P e^+ r^+ = r^+ P$. It follows that $L(P, P') = 0$.
(ii) Since $e^+ r^+ P' e^+ = P' e^+$, the result follows. □

4 Spectral triples on manifolds with boundary

4.1 Spectral triples on $\widetilde{M}$ and $\overline{M}$

Since it is a first step to the main theorem of this section, we record the following known fact in noncommutative geometry [22]: any elliptic pseudodifferential operator of first order on compact manifolds, whose square has a scalar principal symbol, yields a regular spectral triple with the algebra of smooth functions.
Recall that the principal symbol $\sigma_d(P)$ of $P \in \Psi^d(\widetilde{E})$ is said to be scalar when it is of the form $\sigma\,\mathrm{Id}_E$ with $\sigma \in C^\infty(T^*M, \mathbb{C})$.

Proposition 4.1. *Let $P \in \Psi^1(\widetilde{E})$ be an elliptic symmetric pseudodifferential operator of order one on $\widetilde{M}$ such that the principal symbol of P^2 is scalar. Then $\big(C^\infty(\widetilde{M}), L^2(\widetilde{E}), P\big)$ is a regular spectral triple of dimension d.*

Proof. Since $P \in \Psi^1(\widetilde{M})$ is an elliptic symmetric operator on $L^2(\widetilde{E})$, it is selfadjoint with domain $H^1(\widetilde{E})$. Any $a \in C^\infty(\widetilde{M})$ is represented by the left multiplication operator on $L^2(\widetilde{E})$ which is bounded. Since a is scalar, the commutator $[P, a]$ is a pseudodifferential operator of order 0 and thus can be extended as a bounded operator on $L^2(\widetilde{E})$.
Ellipticity implies that $\operatorname{Dom} P^k = H^k(\widetilde{E})$ for any $k \in \mathbb{N}$, where here P^k is the composition of operators $P : H^1(\widetilde{E}) \to L^2(\widetilde{E})$ as unbounded operators in $L^2(\widetilde{E})$. The operator $(P - i)^d$ is thus continuous and bijective from $H^d(\widetilde{E})$ onto $\mathcal{H}$, and by inverse mapping theorem $(P - i)^{-d}$ is continuous from $\mathcal{H}$ onto $H^d(\widetilde{E})$. By a classical result (see for instance [30, A.4 Lemma]), $(P - i)^{-d}$ is compact, and the dimension of the given triple is d.
It remains to check the regularity. By Sobolev lemma, $\mathcal{H}_P^\infty = C^\infty(\widetilde{M}, \widetilde{E})$ so that, for any $a \in C^\infty(\widetilde{M})$, $a_{|\mathcal{H}_P^\infty}$ and $da_{|\mathcal{H}_P^\infty}$, are $\mathcal{H}_P^\infty$-bounded operators. Since the principal symbol of P^2 is scalar, $(a_{|\mathcal{H}_P^\infty})^{(k)}$ and $(da_{|\mathcal{H}_P^\infty})^{(k)}$ are pseudodifferential operators of order k defined on $\mathcal{H}_P^\infty$. Applying now Lemma 2.6 with the Sobolev scale $\big(H^k(\widetilde{E})\big)_{k\in\mathbb{N}}$ yields the result. □

In the case of manifolds with boundary, the full algebra $C^\infty(\overline{M})$ cannot yield, in general, regular spectral triples on $\overline{M}$, because there is a conflict between the necessity of selfadjointness for the realization P_T which is implemented by a boundary condition given by a trace operator T, and the fact that the elements of the algebra must preserve all the domains $\operatorname{Dom} {P_T}^k$. Therefore, we have to consider a subalgebra of $C^\infty(\overline{M})$ that will be adapted to a realization P_T:

Definition 4.2. *Let $\{P_+, T\}$ be an elliptic pseudodifferential boundary system of order one, where $P \in \Psi^1(\widetilde{E})$, $T = S\gamma_0$ is a normal trace operator, with $\gamma_0 : u \mapsto u_{|N}$ and S an endomorphism of E_N. Suppose moreover that P_T is selfadjoint (to apply Definition 2.3).*
We define $\mathcal{A}_{P_T}$ as the $*$-algebra of smooth functions $a \in C^\infty(\overline{M})$ such that

$$a\,\mathcal{H}_{P_T}^\infty \subseteq \mathcal{H}_{P_T}^\infty, \qquad a^*\,\mathcal{H}_{P_T}^\infty \subseteq \mathcal{H}_{P_T}^\infty\,.$$

Remark 4.3. *Note that for any $a \in C^\infty(\overline{M})$, we have $a \operatorname{Dom} P_T \subseteq \operatorname{Dom} P_T$. As a consequence, when $a \in \mathcal{A}_{P_T}$, $[P_T, a]$ is an operator with domain $\operatorname{Dom} P_T$, which sends $\mathcal{H}_{P_T}^\infty$ into itself.*

The following lemma provides some lower bounds to $\mathcal{A}_{P_T}$.

Lemma 4.4. *(i) $\mathcal{A}_{P_T}$ contains the algebra*

$$\mathcal{B} := \{\, a \in C^\infty(\overline{M}) \,:\, T\,d^k a\,,\, T\,d^k a^* \in \Psi^\infty(E_N)\,T, \ \textit{for any } k \in \mathbb{N} \,\}$$

where $d^k := [P_+, \cdot\,]^k$.
(ii) If P is a differential operator, $\mathcal{A}_{P_T}$ contains the smooth functions that are constant near the boundary.

Proof. (i) Suppose that $a \in \mathcal{B}$. Since $Ta = a_{|N}\,T$, we directly check that $a \operatorname{Dom} P_T \subseteq \operatorname{Dom} P_T$. By induction, for any $j \in \mathbb{N}$, $[{P_+}^j, a] = \sum_{i=1}^j c_{ij}\, d^i(a)\, {P_+}^{j-i}$ where c_{ij} are scalar coefficients. Choose $k \in \mathbb{N}$ and $\psi \in \operatorname{Dom} {P_T}^k$. Thus, $\psi \in H^k(E)$ and for any $0 \le j \le k-1$, $T({P_+}^j\psi) = 0$. So, there are $R_i \in \Psi^\infty(E_N)$ such that

$$T({P_+}^j a\psi) = \sum_{i=1}^j c_{ij} T\big(d^i(a){P_+}^{j-i}\psi\big) = \sum_{i=1}^j c_{ij} R_i\, T\, {P_+}^{j-i}\psi = 0$$

which proves that $a\psi \in \operatorname{Dom} P_T^k$ since $a\psi \in H^k(E)$. The same can be obtained for a^*.
(ii) If a is a smooth function constant near the boundary, there is $\lambda \in \mathbb{C}$ and $f \in C_c^\infty(M)$ with compact support such that $a = \lambda 1_M + f$, where 1_M is the function equal to 1 on M. The result follows from inclusions $C_c^\infty(M) \operatorname{Dom} P_T^k \subseteq H_c^k(E) \subseteq \operatorname{Dom} P_T^k$. □

Here is the main result of this section:

Theorem 4.5. *Let $P \in \Psi^1(\widetilde{E})$ be a symmetric pseudodifferential operator of order one on $\widetilde{M}$ satisfying the transmission condition.*
Let $S \in C^\infty\big(N, \operatorname{End}(E_N)\big)$ be an idempotent selfadjoint endomorphism on the boundary such that the system $\{ P_+, T := S\gamma_0 \}$ is an elliptic pseudodifferential boundary operator. Then
(i) P_T is selfadjoint if and only if

$$(1-S)\,\mathfrak{A}_P\,(1-S) = 0 \quad \textit{and} \quad S\,\mathfrak{A}_P^{-1}\,S = 0\,. \tag{3}$$

(ii) When P_T is selfadjoint, $\big(C^\infty(\overline{M}), L^2(E), P_T\big)$ is a spectral triple of dimension d.
(iii) When P is a differential operator such that P^2 has a scalar principal symbol and P_T is selfadjoint, the spectral triple $\big(\mathcal{A}_{P_T}, L^2(E), P_T\big)$ is regular.
(iv) Under the hypothesis of (iii), $\mathcal{A}_{P_T}$ is the largest algebra $\mathcal{A}$ in $C^\infty(\overline{M})$ such that the triple $\big(\mathcal{A}, L^2(E), P_T\big)$ is regular.

Proof. (i) Since $\{ P_+, T \}$ is elliptic and $P^* = P$ (viewed as defined on $H^1(\overline{M})$), we can apply [30, 1.6.11 Theorem] with the same notations, except that S is here not surjective: it is an endomorphism only surjective on $E_N^+ := S(E_N)$ with kernel $E_N^- := (1-S)(E_N)$, so E_N is the direct orthogonal sum of E_N^+ and E_N^- and our S is just replaced by the notation $\widehat{S}$ where $\widehat{R}$ is the surjective morphism associated to the endomorphism R from its domain to its range $R(E)$ to avoid confusion. In the notation of [30], we take $B = P_T$, $0 = G = K = G' = \widetilde{G} = T'$, $\rho = \gamma_0$. Thus, P_T is selfadjoint if there is a homeomorphism Ψ from $H^s(E_N^+)$ onto $H^s(E_N^-)$, such that (since $P = P^*$ yields $\mathfrak{A}_P^* = -\mathfrak{A}_P$)

$$-\,C'^*\,\mathfrak{A}_P\,\gamma_0 = \Psi\,\widehat{S}\,\gamma_0\,, \tag{4}$$

with C' satisfying $(\widehat{1-S})C' = Id_{E_N^-}$ and $C'(\widehat{1-S}) = (1-S)$. In other words, C' is the injection from E_N^- into E_N and $C'^* = (\widehat{1-S})$.
By [30, (1.6.52)], when this is the case, Ψ has the form $\Psi = C'^*\mathfrak{A}_P^* C$ with $\widehat{S}C = Id_{E_N^+}$ and $C\widehat{S} = S$ (remark that the matrix $I^\times$ is the number 1 here). Note that $\mathfrak{A}_P$ is invertible as a consequence of the ellipticity of P.
Now, suppose that $(1-S)\mathfrak{A}_P(1-S) = 0$ and $S\,\mathfrak{A}_P^{-1}\,S = 0$. We define $\Psi = -(\widehat{1-S})\,\mathfrak{A}_P\,C$. This is a homeomorphism from $H^s(E_N^+)$ onto $H^s(E_N^-)$. Indeed, if we set $\Psi^{-1} := -\widehat{S}\,\mathfrak{A}_P^{-1}\,C'$, we get

$$\begin{aligned}\Psi \circ \Psi^{-1} &= (\widehat{1-S})\,\mathfrak{A}_P\,C\,\widehat{S}\,\mathfrak{A}_P^{-1}\,C' = (\widehat{1-S})\,\mathfrak{A}_P\,S\,\mathfrak{A}_P^{-1}\,C' \\ &= (\widehat{1-S})\,\mathfrak{A}_P\,\big(S + (1-S)\big)\,\mathfrak{A}_P^{-1}\,C' = (\widehat{1-S})\,C' = Id_{E_N^-}\end{aligned}$$

and we also have $\Psi^{-1} \circ \Psi = Id_{E_N^+}$ using $S\,\mathfrak{A}_P^{-1}\,S = 0$. Moreover,

$$\Psi\,\widehat{S} = -(\widehat{1-S})\,\mathfrak{A}_P\,S = -C'^*\,\mathfrak{A}_P\,(S + (1-S)) = -C'^*\,\mathfrak{A}_P\,.$$

As a consequence, (4) is satisfied and the if part of the assertion follows.
Conversely, suppose that P_T is selfadjoint. From Green's formula (2), we get $(\mathfrak{A}_P\,\gamma_0 u, \gamma_0 v)_N = 0$ for any $u, v \in \operatorname{Dom} P_T$. Since $\gamma_0 : H^1(E) \to H^{1/2}(E_N)$ is surjective, $(\mathfrak{A}_P\,(1-S)\psi, (1-S)\phi)_N = 0$ for any $\psi, \phi \in H^{1/2}(E_N)$ and thus $(1-S)\,\mathfrak{A}_P\,(1-S) = 0$. Again, from [30, Theorem 1.6.11] we get that $\Psi := C'^*\,\mathfrak{A}_P^*\,C$ is a homeomorphism from $H^s(E_N^+)$ onto $H^s(E_N^-)$ and we check as before that $\Psi^{-1} := -\widehat{S}\,\mathfrak{A}_P^{-1}\,C'$ is a right-inverse of Ψ, and thus, is the inverse of Ψ. The equation $\Psi^{-1} \circ \Psi = \mathrm{Id}_{E_N^+}$ yields $\widehat{S}\,\mathfrak{A}_P^{-1}\,S\,\mathfrak{A}_P\,C = 0$, which gives $\widehat{S}\,\mathfrak{A}_P^{-1}\,S\,\mathfrak{A}_P\,S = 0$. Thus,

$$\widehat{S}\,\mathfrak{A}_P^{-1}\,S\,\mathfrak{A}_P = \widehat{S}\,\mathfrak{A}_P^{-1}\,S\,\mathfrak{A}_P\,(1-S) = \widehat{S}\,\mathfrak{A}_P^{-1}\,\big(S + (1-S)\big)\,\mathfrak{A}_P\,(1-S) = \widehat{S}\,(1-S) = 0$$

so $S\,\mathfrak{A}_P^{-1}\,S = 0$.
(ii) Clearly, $C^\infty(\overline{M})$ is represented as bounded operators on $L^2(E)$ by left multiplication. Since P_T is a selfadjoint unbounded operator on $L^2(E)$, $(P_T - i)^d$ is a bijective operator from $\operatorname{Dom} P_T^d$ onto $L^2(E)$.
The system $\{\,P_+, T\,\}$ being elliptic, it follows from Proposition 3.2 that $\operatorname{Dom} {P_T}^d$ is a closed subset of $H^d(E)$ and ${P_T}^d$ is continuous from $\operatorname{Dom} {P_T}^d$, with the topology of $H^d(E)$, into $L^2(E)$. Using the inverse mapping theorem as in the proof of Lemma 2.6 (ii), we see that $(P_T - i)^{-d}$ is a topological isomorphism from $L^2(E)$ onto $\operatorname{Dom} {P_T}^d$ (with the induced topology of $H^d(E)$). Again, [30, A.4 Lemma] implies that $(P_T - i)^{-d}$ is compact, and the singular values $\mu_n(|P_T|)$ are $\mathcal{O}(n^{1/d})$.
Let $a \in C^\infty(\overline{M})$. In particular $\operatorname{Dom}[P_T, a] = \operatorname{Dom} P_T$ and if $\psi \in \operatorname{Dom} P_T$, $[P_T, a]\psi = [P_+, a]\psi$. By Lemma 3.3, $[P_+, a] = [P, \widetilde{a}]_+$ where $\widetilde{a} \in C^\infty(\widetilde{M})$ is such that $(\widetilde{a})_+ = a$. Thus, since $[P, \widetilde{a}]$ is a pseudodifferential operator of order 0 in $(\widetilde{M}, \widetilde{E})$ satisfying the transmission property, $[P_+, a]$ is continuous from $H^s(E)$ into $H^s(E)$ for any $s > -\frac{1}{2}$. In particular, $[P_T, a]$ extends uniquely as a bounded operator da on $L^2(E)$.
(iii) By (ii) $\big(C^\infty(\overline{M}), L^2(E), P_T\big)$ is a spectral triple. Thus, since $\mathcal{A}_{P_T}$ is a $*$-subalgebra of $C^\infty(\overline{M})$, $\big(\mathcal{A}_{P_T}, L^2(E), P_T\big)$ is also a spectral triple (of the same dimension).
Let $A \in \mathcal{A}_{P_T} \cup d\mathcal{A}_{P_T}$. It is clear that A sends $\mathcal{H}^\infty_{P_T}$ into itself. We denote B the associated $\mathcal{H}^\infty_{P_T}$-bounded operator (so $\overline{B} = A$).
Clearly, $H^k(E)$ is a Sobolev scale associated to $L^2(E)$ and P_T. The result will follow by Lemma 2.6 if we check that for any $k \in \mathbb{N}$, $B^{(k)} := [P_T^2, \cdot]^k(B)$ is continuous from $\mathcal{H}^\infty_{P_T}$ with the $H^k(E)$-topology, into $\mathcal{H}^\infty_{P_T}$ with the $L^2(E)$-topology. Since P is differential, Lemma 3.3 yields for any $\psi \in \mathcal{H}^\infty_{P_T}$

$$B^{(k)}\psi = [P_T^2, \cdot]^k(B)\psi = [(P_+)^2, \cdot]^k(B)(\psi) = \big(\,[P^2, \cdot]^k(\widetilde{B})\,\big)_+(\psi)$$

where $\widetilde{B}$ is a differential operator of order 0 on $(\widetilde{M}, \widetilde{E})$ satisfying $\widetilde{B}_+ = \overline{B} = A$. Since P^2 has scalar principal symbol, $[P^2, \cdot]^k(\widetilde{B})$ is a differential operator of order k. The claim follows.
(iv) Suppose that $\mathcal{A}$ is a subalgebra of $C^\infty(M)$ such that $(\mathcal{A}, L^2(\overline{M}, E), P_T)$ is a regular spectral triple, $\mathcal{A}$ acting on $L^2(\overline{M}, E)$ by left multiplication. By regularity, a direct application of the proof of (3) $\Rightarrow$ (4) in [19, Lemma 2.1] yields $a\mathcal{H}^\infty_{P_T} \subseteq \mathcal{H}^\infty_{P_T}$ for any $a \in \mathcal{A}$. This shows that $\mathcal{A} \subseteq \mathcal{A}_{P_T}$. □

Note that if $S = 1$ or $S = 0$ then by previous theorem (i), P_T is not selfadjoint.

Remark 4.6. *The hypothesis "P is a differential operator" in Theorem 4.5 (iii) is crucial in the sense that if P is a non-differential pseudodifferential operator, some non-vanishing leftovers may appear in $B^{(k)}$ and destroy the continuity of order k.*

This phenomenon does not appear in the boundaryless case since it stems from the singularities generated by the cut-off operator e^+r^+ at the boundary.

We conclude this section with a simple one-dimensional example:

Remark 4.7. *Let $\widetilde{M} = \mathbb{S}^1$ and $\overline{M} = \{ (x,y) \in \mathbb{S}^1 \ : \ x \geq 0 \} \simeq [-\frac{\pi}{2}, \frac{\pi}{2}]$. Define P on $H^1(\widetilde{M}, \mathbb{C}^2)$ by $\begin{pmatrix} 0 & \frac{d}{d\theta} \\ -\frac{d}{d\theta} & 0 \end{pmatrix}$, where θ is the polar coordinate $\theta \mapsto (\cos\theta, \sin\theta)$, and let $T = S\gamma_0$ where $S = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. The system (P_+, T) is elliptic of order 1 and P_T is selfadjoint since $\mathfrak{A}_P = \varepsilon \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\mathfrak{A}_P^{-1}$ and $(1-S)\mathfrak{A}_P = \mathfrak{A}_P S$ where ε is the function on $N = \partial\overline{M}$ such that $\varepsilon(0,1) = 1$ and $\varepsilon(0,-1) = -1$.*
By Theorem 4.5, $(\mathcal{A}_{P_T}, L^2(\overline{M}, \mathbb{C}^2), P_T)$ is a regular spectral triple of dimension 1.
The smooth domain $\mathcal{H}^\infty_{P_T}$ is equal to the set of all $\psi = (\psi_1, \psi_2) \in C^\infty(\overline{M}, \mathbb{C}^2)$ such that for all $k \in \mathbb{N}$, $(\psi_1^{(2k)})_{|N} = 0$ and $(\psi_2^{(2k+1)})_{|N} = 0$, where $\phi^{(p)}$ for all $p \in \mathbb{N}$ denotes $(\frac{d}{d\theta})^p\phi$.
Moreover, the algebra $\mathcal{A}_{P_T}$ is equal to the set of all smooth functions $a \in C^\infty(\overline{M})$ such that for all $k \in \mathbb{N}$, $(a^{(2k+1)})_{|N} = 0$. For example, $\theta \mapsto \sin\theta \in \mathcal{A}_{P_T}$ while $\theta \mapsto \cos\theta \notin \mathcal{A}_{P_T}$. In particular $\mathcal{A}_{P_T}$ is strictly included in $C^\infty(\overline{M})$.
We conjecture that $\mathcal{A}_{P_T}$ is different from $C^\infty(\overline{M})$ for any differential operator of first order P and any normal trace operator $T = S\gamma_0$ such that P_T is selfadjoint and $\{ P_+, T \}$ is elliptic.

4.2 Case of a Dirac operator

We now assume that $\widetilde{M}$ is a riemannian manifold with metric g, $d = \dim \widetilde{M}$ is even and $\widetilde{E}$ has a Clifford module structure. This means that there is smooth map $\gamma : T\widetilde{M} \to \mathrm{End}(\widetilde{E})$ such that for any $x, y \in T\widetilde{M}$, $\gamma(x)\gamma(y) + \gamma(y)\gamma(x) = 2g(x,y)$.
We also fix on $\widetilde{E}$ a Hermitian inner product $(\cdot,\cdot)$ such that $\gamma(x)^* = \gamma(x)$ for any $x \in T\widetilde{M}$. Note that for a given γ, such Hermitian inner product always exists. We fix a connection ∇ on $\widetilde{E}$ such that for any $x, y \in T\widetilde{M}$ and $v, w \in \widetilde{E}$,

$$x(v,w) = (\nabla_x v, w) + (v, \nabla_x w)\,,$$
$$\nabla_x\big(\gamma(y)v\big) = \gamma\big(\nabla_x^{LC}(y)\big)v + \gamma(y)\nabla_x v\,.$$

where ∇^{LC} is the Levi-Civita connection of $(\widetilde{M}, g)$. By [27, Lemma 1.1.7], such connection always exists. The Dirac operator associated to the connection ∇ is locally defined by $\mathcal{D} := i\sum_j \gamma_j\nabla_j$ where our conventions are the following: $(e) := \{ e_1, \ldots, e_d \}$ is a local orthonormal frame of the tangent space where e_d is the inward pointing unit vector field, $\gamma_j := \gamma(e_j)$ and $\nabla_j := \nabla_{e_j}$.
The chiral boundary operator χ investigated in [13, 39] is defined the following way: choose $\chi := (-i)^{d/2+1}\gamma_1 \cdots \gamma_{d-1}$, so $\chi_{\widetilde{M}} := i\chi\gamma_d$ (also sometimes denoted γ_{d+1}) is the natural chirality of $\widetilde{M}$. Then $\{\chi, \gamma_d\} = 0$ while $[\chi, \gamma_n] = 0$, $\forall n \in \{ 1, \cdots, d-1 \}$. Let $\Pi_\pm$ be the projections associated to eigenvalues ± 1 of χ ($\chi = \chi^*$ and $\chi^2 = 1$). This defines a particular case of elliptic boundary condition [27, Lemma 1.4.9, Theorem 1.4.11, Lemma 1.5.3]. The map $S := \Pi_-$ is an idempotent selfadjoint endomorphism on N and $1 - S = \Pi_+$. Since $\mathfrak{A}_{\mathcal{D}} = -i\gamma_d = -\mathfrak{A}_{\mathcal{D}}^{-1}$ by Remark 3.1, we get $(1-S)\,\mathfrak{A}_{\mathcal{D}} = \mathfrak{A}_{\mathcal{D}}\,S$ and $\mathcal{D}_T$, where $T := S_{|N}\gamma_0$, is selfadjoint by (3). This type of boundary condition is chosen of course to obtain a selfadjoint boundary Dirac operator, which is not the case if we choose a Dirichlet or a Neumann–Robin condition.
In this framework, Theorem 4.5 shows the following, a fact not considered in [13]:

Theorem 4.8. *The triple* $\big(\mathcal{A}_{\mathcal{D}_T}, L^2(\overline{M}, E), \mathcal{D}_T\big)$ *is a regular spectral triple of dimension* d.

4.3 A spectral triple on the boundary $N := \partial M$

We intend here to construct a spectral triple on the boundary of the manifold $\overline{M}$. The idea is to define a transversal differential elliptic operator on the boundary from a differential elliptic operator on $\widetilde{M}$ and a Riemannian structure.
Let $\mathcal{D}$ be an elliptic symmetric differential operator of order one on $\widetilde{M}$. Recall that a pair (g, h) where g is a metric on $\widetilde{M}$ and h is an hermitian pairing on $\widetilde{E}$ is said of the product-type near the boundary N if there exists a tubular neighborhood U of N in $\widetilde{M}$ such that $(U, g_{|U})$ is isometric to $]-\varepsilon, \varepsilon[\times N$ for some $\varepsilon > 0$ with metric $dx^2 \otimes g_N(x)$ where $g_N(x)$ is a smooth family of metrics on N, and h is such that $h(x) := \mathcal{F}_* h_{|\{x\}\times N}$ is independent of $x \in]-\varepsilon, \varepsilon[$, where $\mathcal{F}$ is the bundle isomorphism between $\widetilde{E}_{|U}$ and $]-\varepsilon, \varepsilon[\times E_N$.
As observed in [5, Section 2.1], we can always suppose that the pair (g, h) on $\widetilde{M}$ is of the product type near the boundary N. The idea is to let the coefficients of $\mathcal{D}$ to absorb any non-product behavior of (g, h). More precisely, if we start with a general pair (g, h) on $\widetilde{M}$ and if (g_1, h_1) is pair of product type near the boundary, we can define $s \in C^\infty\big(\widetilde{M}, L(\widetilde{E})\big)$ such that for any $p \in \widetilde{M}$, $s(p)$ is the isomorphism between the two equivalent quadratic spaces $\big(\widetilde{E}_p, h(p)\big)$ and $\big(\widetilde{E}_p, h_1(p)\big)$. We then define the following application

$$\Psi : u \in C^\infty(\widetilde{M}, \widetilde{E}) \to \rho^{-1/2} s^{-1} u \in C^\infty(\widetilde{M}, \widetilde{E})$$

where $\rho := d\omega(g_1)/d\omega(g)$ and $d\omega(g)$ (resp. $d\omega(g_1)$) is the volume form associated to the metric g (resp. g_1). This map extends as an isometry between $L^2(\widetilde{M}, \widetilde{E}, g, h)$ and $L^2(\widetilde{M}, \widetilde{E}, g_1, h_1)$. Thus, we can deal with the differential operator $\Psi\,\mathcal{D}\,\Psi^{-1}$ which is unitarily equivalent to $\mathcal{D}$.
From now on, we suppose that (g, h) on $(\widetilde{M}, \widetilde{E})$ is of product-type.
Thus, $\mathcal{D}$, as an operator in $L^2(U, E_{|U})$, is unitarily equivalent (see for instance [2, 5, 9]) to an operator of the form

$$I_x \circ \left(\tfrac{d}{dx} + A_x\right)$$

where $I \in C^\infty\big(]-\varepsilon, \varepsilon[, GL(E_N)\big)$, $A \in C^\infty\big(]-\varepsilon, \varepsilon[, \mathrm{Diff}^1(N, E_N)\big)$, each A_x being elliptic and each I_x being an anti-selfadjoint endomorphism. Note that, when $\mathcal{D}$ is a Dirac type operator (in the sense that its square $\mathcal{D}^2$ has $g_x(\xi, \xi)\,\mathrm{Id}$ for principal symbol), I can be chosen as constant $I_x = I_0$ with $I_0^2 = -1$.
We then define the tangential operator

$$\mathcal{D}_N := I_0 \circ A_0,$$

which is an elliptic symmetric first-order differential operator on (N, E_N).
Note that ${\mathcal{D}_N}^2$ has a scalar principal symbol if $\mathcal{D}^2$ has. By Proposition 4.1, we directly obtain a spectral triple on the boundary:

Proposition 4.9. *Let* $\mathcal{D}$ *be an elliptic symmetric differential operator of order one on* $\widetilde{M}$ *such that* $\mathcal{D}^2$ *has a scalar principal symbol. Then* $\big(C^\infty(N), L^2(E_N), \mathcal{D}_N\big)$ *is a regular spectral triple of dimension* $d-1$.

Remark that if $\mathcal{D}$ is a classical Dirac operator associated to a Clifford module, $\mathcal{D}_N$ corresponds to the hypersurface Witten–Dirac operator and has been intensively studied in [28, 37].

5 Reality and tadpoles

5.1 Conjugation operator and dimension spectrum on a commutative triple

Definition 5.1. *A commutative spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ provided with a chirality operator χ (a $\mathbb{Z}/2$ grading on $\mathcal{H}$ which anticommutes with $\mathcal{D}$) is said to be real if there exists an antilinear isometry J on $\mathcal{H}$ such that $JaJ^{-1} = a^*$ for $a \in \mathcal{A}$, $J\mathcal{D} = \epsilon\mathcal{D}J$, $J\chi = \epsilon'\chi J$, and $J^2 = \epsilon''$ where $\epsilon, \epsilon', \epsilon''$ are signs given by the table quoted in [16, 17, 29].*

As we shall see, it turns out that the existence of an operator J only satisfying $J\mathcal{D} = \pm\mathcal{D}J$ and $JaJ^{-1} = a^*$ is enough to impose vanishing tadpoles at any order. We thus introduce a weak definition of conjugation operator:

Definition 5.2. *A conjugation operator on a commutative spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is an antilinear isometry J on $\mathcal{H}$ such that $JaJ^{-1} = a^*$ for any $a \in \mathcal{A}$ and $J\mathcal{D} = \varepsilon\mathcal{D}J$ with $\varepsilon \in \{-1, 1\}$.*

In order to be able to compute the spectral action and the corresponding tadpoles on the spectral triples $\left(\mathcal{A}_{P_T}, L^2(E), P_T\right)$, we shall need the noncommutative integral and a characterization of the dimension spectrum. We first recall a few definitions of the Chamseddine–Connes pseudodifferential calculus [12]. From now on, $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a regular commutative spectral triple of dimension d endowed with conjugation operator J.
We shall use the following convention: X^{-s} when $\Re(s) > 0$ actually means $(X + P_0)^{-s}$ where P_0 is the orthogonal projection on the kernel of X. Note that in our case, $J\mathcal{A}J^{-1} = \mathcal{A}$, by definition of J.
For any $\alpha \in \mathbb{R}$, we define $OP^\alpha := \{T \ : \ |\mathcal{D}|^{-\alpha}T \in \cap_k \operatorname{Dom}\delta^k\}$, where $\delta = \overline{[|\mathcal{D}|, \cdot]}$ as defined in section 2.

Lemma 5.3. *Suppose that the triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ satisfies the first-order condition $[da, b] = 0$ for any $a, b \in \mathcal{A}$ and let A be a one-form (a finite sum of terms of the form adb, for $a, b \in \mathcal{A}$). Then $A^* = -\varepsilon JAJ^{-1}$. In particular, if A is selfadjoint, $A + \varepsilon JAJ^{-1} = 0$.*

Proof. Direct computation. □

Definition 5.4. *Let $\mathcal{D}(\mathcal{A})$ be the polynomial algebra generated by $\mathcal{A}$, $\mathcal{D}$.*
The operator T is said to be pseudodifferential if there exists $d \in \mathbb{Z}$ such that for any $N \in \mathbb{N}$, there exist $p \in \mathbb{N}$, $P \in \mathcal{D}(\mathcal{A})$ and $R \in OP^{-N}$ (p, P and R may depend on N) such that $P\,\mathcal{D}^{-2p} \in OP^d$ and

$$T = P\,\mathcal{D}^{-2p} + R\,.$$

Define $\Psi(\mathcal{A})$ as the set of pseudodifferential operators and $\Psi^k(\mathcal{A}) := \Psi(\mathcal{A}) \cap OP^k$.

Remark 5.5. *We use here the algebra of pseudodifferential operators defined by Chamseddine–Connes in [12] and denoted $\Psi_1(\mathcal{A})$ in [24,39]. This algebra does not a priori contain the operators of type $|\mathcal{D}|^k$ or $|\mathcal{D}+A|^k$ (A one-form) for k odd, contrarily to the larger algebra considered in [24].*

The dimension spectrum $Sd(\mathcal{A}, \mathcal{H}, \mathcal{D})$ of a spectral triple has initially been defined in [15, 22] and is adapted here to the definition of pseudodifferential operator.

Definition 5.6. *A spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is said to be simple if all generalized zeta functions $\zeta_{\mathcal{D}}^P := s \mapsto \operatorname{Tr}\left(P|\mathcal{D}|^{-s}\right)$, where P is any pseudodifferential operator in OP^0, are meromorphic on the complex plane with only simple poles. The set of these poles is denoted $Sd(\mathcal{A}, \mathcal{H}, \mathcal{D})$, and called the dimension spectrum of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$.*

When $(\mathcal{A},\mathcal{H},\mathcal{D})$ is simple, $⨍ T := \operatorname{Res}_{s=0}\operatorname{Tr}\left(T|\mathcal{D}|^{-s}\right)$ is a trace on the algebra $\Psi(\mathcal{A})$. Note that for $k\in\mathbb{N}$,

$$\operatorname*{Res}_{s=d-k}\operatorname{Tr}|\mathcal{D}+A|^{-s} = ⨍ K(A)|\mathcal{D}|^{-(d-k)} = ⨍ |\mathcal{D}+A|^{-(d-k)}$$

where $K(A)$ is a pseudodifferential operator in the sense of Definition 5.4 (see [24, Lemma 4.6, Proposition 4.8]).
Suppose now that $(\mathcal{A},\mathcal{H},\mathcal{D})$ is simple and that $Sd(\mathcal{A},\mathcal{H},\mathcal{D})\subseteq d-\mathbb{N}$. Moreover, we suppose, as in [21, p. 197] that for all selfadjoint one-forms A, $\operatorname{Tr} e^{-t(\mathcal{D}+A)^2}$ has a complete asymptotic expansion in real powers of t when $t\downarrow 0$. Thus, $S(\mathcal{D}+A,\Phi,\Lambda)=\operatorname{Tr}\Phi((\mathcal{D}+A)^2/\Lambda^2)$ satisfies (1) where $\zeta_{\mathcal{D}_A}(0)=\zeta_{\mathcal{D}}(0)+\sum_{q=1}^{d}\frac{(-1)^q}{q}⨍(A\mathcal{D}^{-1})^q$ (see [12]).
We now obtain from [39]:

Proposition 5.7. *Let A be a selfadjoint one-form. The linear term in A in the Λ^{d-k} coefficient of* (1)*, denoted* $\operatorname{Tad}_{\mathcal{D}+A}(d-k)$ *and called tadpole of order $d-k$, satisfies:*

$$\operatorname{Tad}_{\mathcal{D}+A}(d-k) = -(d-k)⨍ A\mathcal{D}|\mathcal{D}|^{-(d-k)-2},\quad \forall k\neq d,$$
$$\operatorname{Tad}_{\mathcal{D}+A}(0) = -⨍ A\mathcal{D}^{-1}.$$

If the triple satisfies the first-order condition $[da,b]=0$ for any $a,b\in\mathcal{A}$, then (see [39, Corollary 3.7] and Lemma 5.3) $\operatorname{Tad}_{\mathcal{D}+A}(d-k)=0$ *for any $k\in\mathbb{N}$.*

5.2 Tadpoles on $\left(\mathcal{A}_{\mathcal{D}_T}, L^2(\overline{M},E),\mathcal{D}_T\right)$

It is known that real the commutative spectral triple based on the classical Dirac operator on a compact spin manifold of even dimension d has no tadpoles [39]. In fact, as we shall see, the same result applies in presence of a boundary.
We now work in the setting of section 4.2. We suppose moreover that $\widetilde{M}$ is a spin manifold, $\widetilde{E}$ is the spin bundle and $\mathcal{D}$ is the classical Dirac operator in the sense of Atiyah–Singer. The spin structure brings us an antilinear isometry J (the ordinary conjugation operator) satisfying $\mathcal{D}J=J\mathcal{D}$, $JbJ^{-1}=b^*$ for all $b\in C^\infty(\widetilde{M})$, and $J\gamma_i=-\gamma_i J$ for any $i\in\{1,\cdots,d\}$ [29, Theorem 9.20]. Moreover, $J\chi_{\widetilde{M}}=\varepsilon'\chi_{\widetilde{M}}J$, and thus $J\chi=\varepsilon'\chi J$, where $\varepsilon'=-1$ if $d/2$ is odd and $\varepsilon'=1$ if $d/2$ is even.
In particular, if $d/2$ is even then $JS=SJ$.
If we set

$$\widetilde{J} := J\chi_{\widetilde{M}}$$

then $\widetilde{J}$ is an antilinear isometry satisfying $\mathcal{D}\widetilde{J}=-\widetilde{J}\mathcal{D}$, $\widetilde{J}b\widetilde{J}^{-1}=b^*$ for all $b\in C^\infty(\widetilde{M})$, and $\widetilde{J}\chi=-\varepsilon'\chi\widetilde{J}$. As a consequence, if $d/2$ is odd, then $\widetilde{J}S=S\widetilde{J}$.

Theorem 5.8. *The spectral triple $\left(\mathcal{A}_{\mathcal{D}_T}, L^2(\overline{M},E),\mathcal{D}_T\right)$ of Theorem 4.8 has no tadpoles.*

Proof. By Theorem 4.8, we know that $\left(\mathcal{A}_{\mathcal{D}_T}, L^2(\overline{M},E),\mathcal{D}_T\right)$ is a regular spectral triple of dimension d. In order to prove it is a simple spectral triple with dimension spectrum included $\{d-k\ :\ k\in\mathbb{N}\}$, it is enough to check that for any $B\in\mathcal{D}(\mathcal{A}_{\mathcal{D}_T})$, the function $\zeta^B_{\mathcal{D}_T}(s):=\operatorname{Tr}\left(B|\mathcal{D}_T|^{-s}\right)$ has only simple poles in $\mathbb{Z}$. Since $\mathcal{D}$ is differential, by Lemma 3.3, B is a differential operator on $(\overline{M},E)$. Since $s\mapsto\operatorname{Tr}\left(A|\mathcal{D}_T|^{-s}\right)$ has a meromorphic extension on $\mathbb{C}$ with only simple poles in $\{d+n-k\ :k\in\mathbb{N}\}$, when A is a differential operator of order n

on $(\overline{M}, E)$ (see for instance [26, Theorem 1.12.2]), we can conclude that $\big(\mathcal{A}_{\mathcal{D}_T}, L^2(\overline{M}, E), \mathcal{D}_T\big)$ is simple with dimension spectrum included in $d - \mathbb{N}$. Moreover, if A is a selfadjoint one-form, $\operatorname{Tr} e^{-t(\mathcal{D}_T+A)^2}$ has a complete asymptotic expansion in real powers of t when $t \to 0$ [27, Theorems 1.4.5 and 1.4.11].
Note that the first order condition $[da, b] = 0$ for any $a, b \in \mathcal{A}_{\mathcal{D}_T}$ is clearly satisfied since da is a differential operator of order 0 on $(\overline{M}, E)$.
It only remains to prove that there is conjugation operator (Definition 5.2) on the simple spectral triple $\big(\mathcal{A}_{\mathcal{D}_T}, L^2(\overline{M}, E), \mathcal{D}_T\big)$. We define $J' := J$ if $d/2$ is even, and $J' := \widetilde{J}$ if $d/2$ is odd.
The operator $J'_+ := r^+ J' e^+$, is an endomorphism on $L^2(\overline{M}, E)$. Clearly, J'_+ is an antilinear isometry satisfying $J'_+ a {J'_+}^{-1} = a^*$ for any $a \in \mathcal{A}_{\mathcal{D}_T}$. By Lemma 3.3, $L(J', \mathcal{D}) = L(\mathcal{D}, J') = 0$ and thus, $J'_+ \mathcal{D}_+ = (J'\mathcal{D})_+ = (-1)^{d/2}(\mathcal{D}J')_+ = (-1)^{d/2}\mathcal{D}_+ J'_+$.
Moreover $J'_{|N}\, S_{|N}\, {J'}^{-1}{}_{|N} = S_{|N}$ and thus

$$T J'_+ = S_{|N}\gamma_0 J'_+ = S_{|N} J'_{|N}\gamma_0 = J'_{|N} S_{|N}\gamma_0 = J'_{|N} T.$$

In particular, J'_+ preserves Dom $\mathcal{D}_T$, and thus $J'_+ \mathcal{D}_T = (-1)^{d/2}\mathcal{D}_T J'_+$. As a consequence, J'_+ is a conjugation operator on $\big(\mathcal{A}_{\mathcal{D}_T}, L^2(\overline{M}, E), \mathcal{D}_T\big)$. Proposition 5.7 now yields the result. □

Acknowledgments

We thank Alain Connes, Ali Chamseddine, Gerd Grubb, Ryszard Nest and Uuye Otgonbayar for helpful discussions.

References

[1] J. Aastrup, R. Nest and E. Schrohe, "Index theory for boundary value problems via continuous fields of C*-algebras", Journal of Functional Analysis **257** (2009), 2645–2692.

[2] M. Atiyah, V. Patodi and I. M. Singer, "Spectral asymmetry and Riemannian geometry,
I. Math. Proc. Cambr. Phil. Soc **77** (1975), 43–69,
II. Math. Proc. Cambr. Phil. Soc **78** (1975), 405–432,
III. Math. Proc. Cambr. Phil. Soc **79** (1976), 71-99.

[3] B. Booß-Bavnbek, "The determinant of elliptic boundary problems for Dirac operators", http://mmf.ruc.dk/ booss/ell/determinant.pdf

[4] B. Booß-Bavnbek and M. Lesch, "The invertible double of elliptic operators", Lett. Math. Phys. **87** (2009), 19–46.

[5] B. Booß-Bavnbek, M. Lesch and C. Zhu, "The Calderón projection: new definitions and applications", Journal of Geometry and Physics **59** (2009), 784–826.

[6] B. Booß-Bavnbek and K. Wojciechowski, *Elliptic Boundary Problems for Dirac Operators*, Birkhäuser, Boston, 1993.

[7] T. Branson and P. Gilkey, "The asymptotics of the Laplacian on a manifold with boundary", Partial Differential Equations **15** (1990), 245–272.

[8] T. Branson and P. Gilkey, "Residues of the eta function for an operator of Dirac type with local boundary conditions", Differential Geometry and its Applications **2** (1992), 249–267.

[9] J. Brünig and M. Lesch, "On boundary value problems for Dirac type operators", Journal of Functional Analysis **185** (2001), 1–62.

[10] A. L. Carey, S. Klimek and K. P. Wojciechowski, "A Dirac type operator on the noncommutative disk", Lett. Math. Phys. **93** (2010), 107–125.

[11] A. Chamseddine and A. Connes, "The spectral action principle", Commun. Math. Phys. **186** (1997), 731–750.

[12] A. Chamseddine and A. Connes, "Inner fluctuations of the spectral action", J. Geom. Phys. **57** (2006), 1–21.

[13] A. Chamseddine and A. Connes, "Quantum gravity boundary terms from the spectral action on noncommutative space", PRL **99** (2007), 071302.

[14] A. Connes, *Noncommutative Geometry*, Academic Press, London and San Diego, 1994.

[15] A. Connes, "Geometry from the spectral point of view", Lett. Math. Phys. **34** (1995), 203–238.

[16] A. Connes, "Noncommutative geometry and reality", J. Math. Phys. **36** (1995), 6194–6231.

[17] A. Connes, "Gravity coupled with matter and the foundation of non commutative geometry", Commun. Math. Phys. **182** (1996), 155–177.

[18] A. Connes, "The action functional in noncommutative geometry", Comm. Math. Phys. **117** (1998), 673–683.

[19] A. Connes, "On the spectral characterization of manifolds", arXiv:0810.2088v1.

[20] A. Connes, "Variation sur le thème spectral", Résumé des cours 2006–2007, http://www.college-de-france.fr/media/ana_geo/UPL53971_2.pdf, 2007.

[21] A. Connes and M. Marcolli, *Noncommutative Geometry, Quantum Fields and Motives*, Colloquium Publications, Vol. 55, American Mathematical Society, 2008.

[22] A. Connes and H. Moscovici, "The local index formula in noncommutative geometry", Geom. Funct. Anal. **5** (1995), 174–243.

[23] J. Dixmier, "Existence de traces non normales", C. R. Acad. Sci. Paris, Sér. A **262** (1966), 1107–1108.

[24] D. Essouabri, B. Iochum, C. Levy and A. Sitarz, "Spectral action on noncommutative torus", J. Noncommut. Geom. **2** (2008), 53–123.

[25] B. Fedosov, F. Golse, E. Leichtnam and E. Schrohe, "The noncommutative residue for manifolds with boundary", J. Funct. Anal. **142** (1996), 1–31.

[26] P. B. Gilkey, *Invariance Theory, the Heat equation, and the Atiyah–Singer Index Theory*, CRC Press, Boca Raton, 1995.

[27] P. B. Gilkey, *Asymptotic Formulae in Spectral Geometry*, Chapman & Hall, 2003.

[28] N. Ginoux, *The Dirac spectrum*, Lecture Notes in Mathematics, Vol. 1976, 2009.

[29] J. M. Gracia-Bondía, J. C. Várilly and H. Figueroa, *Elements of Noncommutative Geometry*, Birkhäuser Advanced Texts, Birkhäuser, Boston, 2001.

[30] G. Grubb, *Functional calculus of pseudodifferential boundary problems*, Second edition, Progress in Mathematics **65**, Birkhäuser, Boston, 1996.

[31] G. Grubb, *Distributions and Operators*, Graduate Texts in Math. **252**, Springer, 2009.

[32] G. Grubb and E. Schrohe, "Trace expansions and the noncommutative residue for manifolds with boundary, J. Reine Angew. Math. (Crelle's Journal) **536** (2001), 167–207.

[33] G. Grubb and E. Schrohe, "Traces and quasi-traces on the Boutet de Monvel algebra", Ann. Inst. Fourier **54** (2004), 1641–1696.

[34] S. Hawking and G. Horowitz, "The gravitational Hamiltonian, action, entropy and surface terms", Class. Quantum Grav. **13** (1996), 1487–1498.

[35] N. Higson, "The local index formula in noncommutative geometry", Lectures given at the School and Conference on Algebraic K-theory and its applications, Trieste, 2002.

[36] N. Higson, "The residue index theorem of Connes and Moscovici", Surveys in noncommutative geometry, Clay Math. Proc., vol. 6, Amer. Math. Soc., Providence, RI, 2006, 71–126.

[37] O. Hijazi, S. Montiel and A. Roldán, "Eigenvalue boundary problems for the Dirac operator", Commun. Math. Phys. **231** (2002), 375–390.

[38] L. Hörmander, *The Analysis of Linear Partial Differential Operators I, II, III*, Springer-Verlag, Berlin, 1989, 2005, 2007.

[39] B. Iochum and C. Levy, "Tadpoles and commutative spectral triples", arXiv:0904.0222 [math-ph], to appear in J. Noncommut. Geom.

[40] W. Kalau and M. Walze, "Gravity, non-commutative geometry, and the Wodzicki residue", J. Geom. Phys. **16** (1995), 327–344.

[41] D. Kastler, " The Dirac operator and gravitation", Comm. Math. Phys. **166** (1995), 633–643.

[42] M. Kontsevich and S. Vishik, "Geometry of determinants of elliptic operators", Functional Analysis on the Eve of the 21'st century, Vol. I (S. Gindikin et al. eds.), Prog. Math. 131, Birkhäuser, Boston, 1995, 173–197.

[43] H. B. Lawson and M.-L. Michelsohn, *Spin Geometry*, Princeton Univ. Press, Princeton, 1989.

[44] M. Lesch, *Operators of Fuchs type, conical singularities and asymptotic methods*, Teubner, Leipzig, 1997.

[45] M. Lesch, "On the noncommutative residue for pseudodifferential operators with log-polyhomogeneous symbols", Ann. Global Anal. Geom. **17** (1999), 151–187.

[46] J.-M. Lescure, "Triplets spectraux pour les variétés à singularité conique isolée", Bull. Soc. Math. France **129** (2001), 593–623.

[47] J.L. Lions and E. Magenes, *Problèmes aux limites non homogènes et applications, Volume 1*, Dunod, Paris, 1968.

[48] R. Nest and E. Schrohe, "Dixmier's trace for boundary value problems", manuscripta math. **96** (1998), 203–218.

[49] U. Otgonbayar, "Pseudo-differential operators and regularity of spectral triples", arXiv:0911.0816 [math.OA].

[50] A. Rennie, "Smoothness and locality for nonunital spectral triples", K-Theory **28** (2003), 127–165.

[51] E. Schrohe, "Noncommutative residues and manifolds with conical singularities", J. Funct. Anal. **150** (1997), 146–174.

[52] E. Schrohe, "Noncommutative residues, Dixmier's trace, and heat trace expansions on manifolds with boundary", In: B. Booss-Bavnbek and K. Wojciechowski (eds), Geometric Aspects of Partial Differential Equations. Contemporary Mathematics, **242** Amer. Math. Soc. Providence, R.I., (1999), 161–186.

[53] W. J. Ugalde, "Some conformal invariants from the noncommutative residue for manifolds with boundary", SIGMA, **3** (2007), 104, 18 pages.

[54] D. V. Vassilevich, "Heat kernel expansion: user's manual, Phys. Rep. **388** (2003), 279–360.

[55] Y. Wang, "Differential forms and the Wodzicki residue for manifold with boundary", J. Geom. Physics, **56** (2006), 731–753.

[56] Y. Wang, "Differential forms and the noncommutative residue for manifolds with boundary in the non-product case" Lett. Math. Phys. (2006), 41–51.

[57] Y. Wang, "Gravity and the noncommutative residue for manifolds with boundary, Lett. Math. Phys. **80** (2007), 37–56.

[58] M. Wodzicki, "Local invariants of spectral asymmetry", Invent. Math. **75** (1984), 143–177.

[59] M. Wodzicki, "Noncommutative residue. Chapter I: Fundamentals", 320–399, in *K-theory, Arithmetic and Geometry*, Yu. I. Manin, ed., Lecture Notes in Mathematics **1289**, Springer, Berlin 1987.

[60] C. Yang, "Isospectral deformations of Eguchi–Hanson spaces as non unital spectral triples", Commun. Math. Phys. **288** (2009), 615–652.