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## Lasry-Lions Regularization and a Lemma of Ilmanen

PATRICK BERNARD (\*)

Let  $H$  be a Hilbert space. We define the following inf (sup) convolution operators acting on bounded functions  $u : H \rightarrow \mathbb{R}$ :

$$T_t u(x) := \inf_y \left( u(y) + \frac{1}{t} \|y - x\|^2 \right)$$

and

$$\tilde{T}_t u(x) := \sup_y \left( u(y) - \frac{1}{t} \|y - x\|^2 \right).$$

We have the relation

$$T_t(-u) = -\tilde{T}_t(u).$$

Recall that these operators form semi-groups, in the sense that

$$T_t \circ T_s = T_{t+s} \quad \text{and} \quad \tilde{T}_t \circ \tilde{T}_s = \tilde{T}_{t+s}$$

for all  $t \geq 0$  and  $s \geq 0$ , as can be checked by direct calculation. Note also that

$$\inf u \leq T_t u(x) \leq u(x) \leq \tilde{T}_t u(x) \leq \sup u$$

for each  $t \geq 0$  and each  $x \in H$ . A function  $u : H \rightarrow \mathbb{R}$  is called  $k$ -semi-concave,  $k > 0$ , if the function  $x \rightarrow u(x) - \|x\|^2/k$  is concave. We will occasionally consider semi-concave functions which take values in  $[-\infty, +\infty)$ . The function  $u$  is called  $k$ -semi-convex if  $-u$  is  $k$ -semi-concave. A function  $u$  is  $t$ -semi-concave and upper semi-continuous if and only if it belongs to the image of the operator  $T_t$ , this follows from Lemma 1 and Lemma 3 below. A function is called semi-concave if it is  $k$ -semi-concave for some  $k > 0$ . A function  $u$  is said  $C^{1,1}$  if it is Frechet differentiable and if the gradient of  $u$  is Lipschitz. Note that a continuous function  $u : H \rightarrow \mathbb{R}$  is  $C^{1,1}$

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if and only if it is semi-concave and semi-convex, see Lemma 5. Let us recall two important results in that language:

**THEOREM 1** (Lasry-Lions, [6]). *Let  $u$  be a bounded function. For  $0 < s < t$ , the function  $\tilde{T}_s \circ T_t u$  is  $C^{1,1}$  and, if  $u$  is uniformly continuous, then it converges uniformly to  $u$  when  $t \rightarrow 0$ .*

**THEOREM 2** (Ilmanen, [5]). *Let  $u \geq v$  be two bounded functions on  $H$  such that  $u$  and  $-v$  are semi-concave. Then there exists a  $C^{1,1}$  function  $w$  such that  $u \geq w \geq v$ .*

Our goal in the present paper is to “generalize” simultaneously both of these results as follows:

**THEOREM 3.** *The operator  $R_t := \tilde{T}_t \circ T_{2t} \circ \tilde{T}_t$  has the following properties:*

- *Regularization : For each function  $f : H \rightarrow \mathbb{R}$  and each  $t > 0$ , the function  $R_t(f)$  is  $C^{1,1}$  provided it is locally bounded. This holds for all  $t > 0$  if  $f$  is bounded.*
- *Approximation : If  $f : H \rightarrow \mathbb{R}$  is uniformly continuous, then  $R_t(f)$  is  $C^{1,1}$  and converges uniformly to  $f$  as  $t \rightarrow 0$ .*
- *Pinching: If there exists a  $k$ -semi-concave continuous function  $u$  and a  $k$ -semi-convex continuous function  $v$  such that  $v \leq f \leq u$ , then, for all  $t \in ]0, k]$ , we have  $u \geq R_t(f) \geq v$ , and  $R_t(f)$  is  $C^{1,1}$ .*

Theorem 3 does not, properly speaking, generalize Theorem 5. However, it offers a new (although similar) answer to the same problem: approximating uniformly continuous functions on Hilbert spaces by  $C^{1,1}$  functions with a simple explicit formula.

Because of its symmetric form, the regularizing operator  $R_t$  enjoys some nicer properties than the Lasry-Lions operators. For example, if  $f$  is  $C^{1,1}$ , then it follows from the pinching property that  $R_t f = f$  for  $t$  small enough.

Theorem 2, can be proved using Theorem 3 by taking  $w = R_t u$ , for  $t$  small enough. Note, in view of Lemma 3 bellow, that  $R_t u = \tilde{T}_t \circ T_t u$  when  $t$  is small enough.

Theorem 3 can be somehow extended to the case of finite dimensional open sets or manifolds via partition of unity, at the price of loosing the simplicity of explicit expressions. Let  $M$  be a paracompact manifold of dimension  $n$ , equipped once and for all with an atlas  $(\phi_i, i \in \mathfrak{S})$  composed of

charts  $\phi_i : B^n \rightarrow M$ , where  $B^n$  is the open unit ball of radius one centered at the origin in  $\mathbb{R}^n$ . We assume in addition that the image  $\phi_i(B^n)$  is a relatively compact open set, and that the sets  $\phi_i(B^n)$ ,  $i \in \mathfrak{S}$  form a locally finite open covering of the manifold  $M$ . Let us fix, once and for all, a partition of the unity  $g_i$  subordinated to the open covering  $(\phi_i(B^n), i \in \mathfrak{S})$ . It means that the function  $g_i$  is non-negative with support inside  $\phi_i(B^n)$ , and that  $\sum_i g_i = 1$  (note that this sum is finite at each point). Let us define the operator

$$G_t(u) := \sum_i [R_{ta_i}((g_i u) \circ \phi_i)] \circ \phi_i^{-1},$$

where  $a_i, i \in \mathfrak{S}$  are positive real numbers. In this expression, we consider each of the terms  $[R_{ta_i}((g_i u) \circ \phi_i)] \circ \phi_i^{-1}$  as defined on the whole manifold  $M$  with the value 0 outside of  $\phi_i(B^n)$ . The sum is then locally finite hence well-defined. We say that a function  $u : M \rightarrow \mathbb{R}$  is locally semi-concave if, for each  $i \in \mathfrak{S}$ , there exists a constant  $b_i$  such that the function  $u \circ \phi_i - \|\cdot\|^2/b_i$  is concave on  $B^n$ .

**THEOREM 4.** *Let  $u \geq v$  be two continuous functions on  $M$  such that  $u$  and  $-v$  are locally semi-concave. Then, the real numbers  $a_i$  can be chosen such that, for each  $t \in ]0, 1]$  and each function  $f$  satisfying  $u \geq f \geq v$ , we have:*

- *The function  $G_t(f)$  is locally  $C^{1,1}$ .*
- *If  $f$  is continuous, then  $G_t(f)$  converges locally uniformly to  $f$  as  $t \rightarrow 0$ .*
- *$u \geq G_t(f) \geq v$ .*

We will give some properties, most of which are well-known, of the operators  $T_t$  and  $\tilde{T}_t$  in Section 1, and derive the proof of the main results in Section 2.

*Notes and Acknowledgements.* Theorem 2 appears in Ilmanen's paper [5] as Lemma 4G. Several proofs are sketch there but none is detailed. The proof we detail here follows lines similar to one of the sketches of Ilmanen. This statement also has a more geometric counterpart, Lemma 4E in [5]. A detailed proof of this geometric version is given in [2], Appendix. My attention was attracted to these statements and their relations with recent progresses on sub-solutions of the Hamilton-Jacobi equation (see [4, 1, 7]) by Pierre Cardialaguet, Albert Fathi and Maxime Zavidovique. Albert Fathi and Maxime Zavidovique also recently wrote a detailed proof of

Theorem 1, see [3]. This paper also proves how the geometric version follows from Theorem 2. There are many similarities between the tools used in the present paper and those used in [1]. Moreover, Maxime Zavidovique observed in [7] that the existence of  $C^{1,1}$  subsolutions of the Hamilton-Jacobi equation in the discrete case can be deduced from Theorem 2. However, it seems that the main result of [1] (the existence of  $C^{1,1}$  subsolutions in the continuous case) can't be deduced easily from Theorem 2. Neither can Theorem 2 be deduced from it.

### 1. The operators $T_t$ and $\tilde{T}_t$ on Hilbert spaces.

The proofs of the theorems follow from standard properties of the operators  $T_t$  and  $\tilde{T}_t$  that we now recall in details.

**LEMMA 1.** *For each function  $u : H \rightarrow \mathbb{R}$ , the function  $T_t u$  (which takes values in  $[-\infty, +\infty)$ ), is  $t$ -semi-concave and upper semi-continuous. The function  $\tilde{T}_t u$  (which takes values in  $(-\infty, +\infty]$ ), is  $t$ -semi-convex and lower semi-continuous. Moreover, if  $u$  is  $k$ -semi-concave, then for each  $t < k$  the function  $\tilde{T}_t u$  is  $(k - t)$ -semi-concave. Similarly, if  $u$  is  $k$ -semi-convex, then for each  $t < k$  the function  $T_t u$  is  $(k - t)$ -semi-convex.*

**PROOF.** We shall prove the statements concerning  $T_t$ . We have

$$\begin{aligned} T_t u(x) - \|x\|^2/t &= \inf_y (u(y) + \|y - x\|^2/t - \|x\|^2/t) = \\ &= \inf_y (u(y) + \|y\|^2/t - 2x \cdot y/t), \end{aligned}$$

this function is concave and upper semi-continuous as an infimum of continuous linear functions. On the other hand, we have

$$T_t u(x) + \|x\|^2/l = \inf_y (u(y) + \|y - x\|^2/t + \|x\|^2/l).$$

Setting  $f(x, y) := u(y) + \|y - x\|^2/t + \|x\|^2/l$ , the function  $\inf_y f(x, y)$  is a convex function of  $x$  if  $f$  is a convex function of  $(x, y)$ . This is true if  $u$  is  $k$ -semi-convex,  $t < k$ , and  $l = k - t$  because we have the expression

$$f(x, y) = u(y) + \|y - x\|^2/t + \|x\|^2/l = (u(y) + \|y\|^2/k) + \left\| \sqrt{\frac{l}{kt}}y - \sqrt{\frac{k}{lt}}x \right\|^2.$$

□

Given a uniformly continuous function  $u : H \rightarrow \mathbb{R}$ , we define its modulus of continuity  $\rho(r) : [0, \infty) \rightarrow [0, \infty)$  by the expression  $\rho(r) = \sup_{x,e} u(x + re) - u(x)$ , where the supremum is taken on all  $x \in H$  and all  $e$  in the unit ball of  $H$ . The function  $\rho$  is non-decreasing, it satisfies  $\rho(r + r') \leq \rho(r) + \rho(r')$ , and it converges to zero in zero (this last fact is equivalent to the uniform continuity of  $u$ ). We say that a function  $\rho : [0, \infty) \rightarrow [0, \infty)$  is a modulus of continuity if it satisfies these properties. Given a modulus of continuity  $\rho(r)$ , we say that a function  $u$  is  $\rho$ -continuous if  $|u(y) - u(x)| \leq \rho(\|y - x\|)$  for all  $x$  and  $y$  in  $H$ .

LEMMA 2. *If  $u : H \rightarrow \mathbb{R}$  is uniformly continuous, then the functions  $T_t u$  and  $\tilde{T}_t u$  converge uniformly to  $u$  when  $t \rightarrow 0$ . Moreover, given a modulus of continuity  $\rho$ , there exists a non-decreasing function  $\varepsilon(t) : [0, \infty) \rightarrow [0, \infty)$  satisfying  $\lim_{t \rightarrow 0} \varepsilon(t) = 0$  and such that, for each  $\rho$ -continuous bounded function  $u$ , we have:*

- $T_t u$  and  $\tilde{T}_t u$  are  $\rho$ -continuous for each  $t \geq 0$ .
- $u - \varepsilon(t) \leq T_t u(x) \leq u$  and  $u \leq \tilde{T}_t u \leq u + \varepsilon(t)$  for each  $t \geq 0$ .

PROOF. Let us fix  $y \in H$ , and set  $v(x) = u(x + y)$ . We have  $u(x) - \rho(\|y\|) \leq v(x) \leq u(x) + \rho(\|y\|)$ . Applying the operator  $T_t$  gives  $T_t u(x) - \rho(y) \leq T_t v(x) \leq T_t u(x) + \rho(y)$ . On the other hand, we have

$$T_t v(x) = \inf_z (u(z + y) + \|z - x\|^2/t) = \inf_z (u(z) + \|z - (x + y)\|^2/t) = T_t u(x + y),$$

so that

$$T_t u(x) - \rho(\|y\|) \leq T_t u(x + y) \leq T_t u(x) + \rho(\|y\|).$$

We have proved that  $T_t u$  is  $\rho$  continuous if  $u$  is, the proof for  $\tilde{T}_t u$  is the same.

In order to study the convergence, let us set  $\varepsilon(t) = \sup_{r>0} (\rho(r) - r^2/t)$ . We have

$$\varepsilon(t) = \sup_{r>0} (\rho(r\sqrt{t}) - r^2) \leq \sup_{r>0} ((r + 1)\rho(\sqrt{t}) - r^2) = \rho(\sqrt{t}) + \rho^2(\sqrt{t})/4.$$

We conclude that  $\lim_{t \rightarrow 0} \varepsilon(t) = 0$ . We now come back to the operator  $T_t$ , and observe that

$$u(y) - \|y - x\|^2/t \geq u(x) - \rho(\|y - x\|) + \|y - x\|^2/t \geq u(x) - \varepsilon(t)$$

for each  $x$  and  $y$ , so that

$$u - \varepsilon(t) \leq T_t u \leq u.$$

□

LEMMA 3. For each function  $u : H \rightarrow (-\infty, +\infty]$ , we have  $\check{T}_t \circ T_t(u) \leq u$  and the equality  $\check{T}_t \circ T_t(u) = u$  holds if and only if  $u$  is  $t$ -semi-convex and lower semi-continuous. Similarly, given a function  $v : H \rightarrow [-\infty, +\infty)$ , we have  $T_t \circ \check{T}_t(v) \geq v$ , with equality if and only if  $v$  is  $t$ -semi-concave and upper semi-continuous.

PROOF. Let us write explicitly

$$\check{T}_t \circ T_t u(x) = \sup_y \inf_z (u(z) + \|z - y\|^2/t - \|y - x\|^2/t).$$

Taking  $z = x$ , we obtain the estimate  $\check{T}_t \circ T_t u(x) \leq \sup_y u(x) = u(x)$ . Let us now write

$$\check{T}_t \circ T_t u(x) + \|x\|^2/t = \sup_y \inf_z (u(z) + \|z\|^2/t + (2y/t) \cdot (x - z))$$

which by an obvious change of variable leads to

$$\check{T}_t \circ T_t u(x) + \|x\|^2/t = \sup_y \inf_z (u(z) + \|z\|^2/t + y \cdot (x - z)).$$

We recognize here that the function  $\check{T}_t \circ T_t u(x) + \|x\|^2/t$  is the Legendre bidual of the function  $u(x) + \|x\|^2/t$ . It is well-known that a function is equal to its Legendre bidual if and only if it is convex and lower semi-continuous.  $\square$

## 2. Proof of the main results.

PROOF OF THEOREM 3. For each function  $f$  and each  $t > 0$ , the function  $\check{T}_t \circ T_{2t} \circ \check{T}_t f$  is both  $t$ -semi-concave and  $t$ -semi-convex. It is  $t$ -semi-convex by Lemma 1, and it is semi-concave because  $T_{2t}(\check{T}_t f)$  is  $2t$ -semi-concave by Lemma 1, which implies, still by Lemma 1, that  $\check{T}_t \circ T_{2t} \circ \check{T}_t f$  is  $t$ -semi-concave. As a consequence, Lemma 5 below implies that the function  $R_t f$  is  $C^{1,1}$  provided it is locally bounded. The function  $R_t(f)$  is bounded if  $f$  is bounded, hence it is  $C^{1,1}$  in this case.

In the case where  $f$  is uniformly continuous, Lemma 2 implies that

$$f - \varepsilon(2t) \leq R_t(f) \leq f + 2\varepsilon(t).$$

As a consequence,  $R_t(f)$  is converging uniformly to  $f$ , and it is locally bounded hence  $C^{1,1}$ .

We now consider two continuous functions  $u$  and  $v$  such that  $u$  and  $-v$  are  $k$  semi-concave, and such that  $v \leq u$ . We claim that

$$u \geq f \geq v \implies u \geq T_t \circ \check{T}_t f \geq v \text{ and } u \geq \check{T}_t \circ T_t f \geq v$$

for  $t \leq k$ . This claim implies that  $u \geq \check{T}_t \circ T_{2t} \circ \check{T}_t f \geq v$  when  $u \geq f \geq v$  and  $t \leq k$ . Let us now prove the claim concerning  $\check{T}_t \circ T_t$ , the other part being similar. Since  $v$  is  $k$ -semi-convex and continuous, we have  $\check{T}_t \circ T_t v = v$  for  $t \leq k$ , by Lemma 3. Then,

$$u \geq f \geq \check{T}_t \circ T_t f \geq \check{T}_t \circ T_t v = v$$

where the second inequality follows from Lemma 3, and the third from the obvious fact that the operators  $T_t$  and  $\check{T}_t$  are order-preserving.

We have proved that  $v \leq R_t(f) \leq u$  if  $v \leq f \leq u$  and  $t \leq k$ . For  $t \in ]0, k]$ , the function  $R_t(f)$  is thus locally bounded hence  $C^{1,1}$ .  $\square$

PROOF OF THEOREM 4. Let  $a_i$  be chosen such that the functions  $(g_i u) \circ \phi_i$  and  $-(g_i v) \circ \phi_i$  are  $a_i$ -semi-concave on  $\mathbb{R}^n$  (when extended by 0 outside of  $B^n$ ). The existence of real numbers  $a_i$  with this property follows from Lemma 4 below. Given  $u \geq f \geq v$ , we can apply Theorem 3 for each  $i$  to the functions

$$(g_i u) \circ \phi_i \geq (g_i f) \circ \phi_i \geq (g_i v) \circ \phi_i$$

extended by zero outside of  $B^n$ . We conclude that, for  $t \in ]0, 1]$ , the function  $R_{ta_i}((g_i f) \circ \phi_i)$  is  $C^{1,1}$  and satisfies

$$(g_i u) \circ \phi_i \geq R_{ta_i}((g_i f) \circ \phi_i) \geq (g_i v) \circ \phi_i.$$

As a consequence, the function

$$[R_{ta_i}((g_i f) \circ \phi_i)] \circ \phi_i^{-1},$$

extended as a function on  $M$  equal to 0 outside of  $\phi_i(B^n)$ , is  $C^{1,1}$ . The function  $G_t(f)$  is thus locally a finite sum of  $C^{1,1}$  functions hence it is locally  $C^{1,1}$ . Moreover, we have

$$u = \sum_i g_i u \geq G_t(f) \geq \sum_i g_i v = v.$$

$\square$

We have used:

LEMMA 4. Let  $u : B^n \rightarrow \mathbb{R}$  be a bounded function such that  $u - \|\cdot\|^2/a$  is concave, for some  $a > 0$ . For each compactly supported non-negative  $C^2$  function  $g : B^n \rightarrow \mathbb{R}$ , the product  $gu$  (extended by zero outside of  $B^n$ ) is semi-concave on  $\mathbb{R}^n$ .

PROOF. Since  $u$  is bounded, we can assume that  $u \geq 0$  on  $B^n$ . Let  $K \subset B^n$  be a compact subset of the open ball  $B^n$  which contains the support



of  $g$  in its interior. Since the function  $u - \|\cdot\|^2/a$  is concave on  $B_1$  it admits super-differentials at each point. As a consequence, for each  $x \in B^n$ , there exists a linear form  $l_x$  such that

$$0 \leq u(y) \leq u(x) + l_x \cdot (y - x) + \|y - x\|^2/a$$

for each  $y \in B^1$ . Moreover, the linear form  $l_x$  is bounded independently of  $x \in K$ . We also have

$$0 \leq g(y) \leq g(x) + dg_x \cdot (y - x) + C\|y - x\|^2$$

for some  $C > 0$ , for all  $x, y$  in  $\mathbb{R}^n$ . Taking the product, we get, for  $x \in K$  and  $y \in B^n$ ,

$$u(y)g(y) \leq u(x)g(x) + (g(x)l_x + u(x)dg_x) \cdot (y - x) + C\|y - x\|^2 + C\|y - x\|^3 + C\|y - x\|^4$$

where  $C > 0$  is a constant independent of  $x \in K$  and  $y \in B^n$ , which may change from line to line. As a consequence, setting  $L_x = g(x)l_x + u(x)dg_x$ , we obtain the inequality

$$(L) \quad (gu)(y) \leq (gu)(x) + L_x \cdot (y - x) + C\|y - x\|^2$$

for each  $x \in K$  and  $y \in B^n$ . If we set  $L_x = 0$  for  $x \in \mathbb{R}^n - K$ , the relation (L) holds for each  $x \in \mathbb{R}^n$  and  $y \in \mathbb{R}^n$ . For  $x \in K$  and  $y \in B^n$ , we have already proved it. Since the linear forms  $L_x$ ,  $x \in K$  are uniformly bounded, we can assume that  $L_x \cdot (y - x) + C\|y - x\|^2 \geq 0$  for all  $x \in K$  and  $y \in \mathbb{R}^n - B^n$  by taking  $C$  large enough. Then, (L) holds for all  $x \in K$  and  $y \in \mathbb{R}^n$ . For  $x \in \mathbb{R}^n - K$  and  $y$  outside of the support  $g$ , the relation (L) holds in an obvious way, because  $gu(x) = gu(y) = 0$ , and  $L_x = 0$ . For  $x \in \mathbb{R}^n - K$  and  $y$  in the support of  $g$ , the relation holds provided that  $C \geq \max(gu)/d^2$ , where  $d$  is the distance between the complement of  $K$  and the support of  $g$ . This is a positive number since  $K$  is a compact set containing the support of  $g$  in its interior. We conclude that the function  $(gu)$  is semi-concave on  $\mathbb{R}^n$ .  $\square$

For completeness, we also prove, following Fathi:

**LEMMA 5.** *Let  $u : H \rightarrow \mathbb{R}$  be a locally bounded function which is both  $k$ -semi-concave and  $k$ -semi-convex. Then the function  $u$  is  $C^{1,1}$ , and  $6/k$  is a Lipschitz constant for the gradient of  $u$ .*

**PROOF.** It is well known that a locally bounded convex function is continuous. We conclude that  $u$  is continuous. Let  $u$  be a continuous function which is both  $k$ -semi-concave and  $k$ -semi-convex. Then, for each  $x \in H$ ,

there exists a unique  $l_x \in H$  such that

$$|u(x+y) - u(x) - l_x \cdot y| \leq \|y\|^2/k.$$

We conclude that  $l_x$  is the gradient of  $u$  at  $x$ , and we have to prove that the map  $x \mapsto l_x$  is Lipschitz. We have, for each  $x, y$  and  $z$  in  $H$ :

$$\begin{aligned} l_x \cdot (y+z) - \|y+z\|^2/k &\leq u(x+y+z) - u(x) \leq l_x \cdot (y+z) + \|y+z\|^2/k \\ l_{(x+y)} \cdot (-y) - \|y\|^2/k &\leq u(x) - u(x+y) \leq l_{(x+y)} \cdot (-y) + \|y\|^2/k \\ l_{(x+y)} \cdot (-z) - \|z\|^2/k &\leq u(x+y) - u(x+y+z) \leq l_{(x+y)} \cdot (-z) + \|z\|^2/k. \end{aligned}$$

Taking the sum, we obtain

$$|(l_{x+y} - l_x) \cdot (y+z)| \leq \|y+z\|^2/k + \|y\|^2/k + \|z\|^2/k.$$

By a change of variables, we get

$$|(l_{x+y} - l_x) \cdot (z)| \leq \|z\|^2/k + \|y\|^2/k + \|z-y\|^2/k.$$

Taking  $\|z\| = \|y\|$ , we obtain

$$|(l_{x+y} - l_x) \cdot (z)| \leq 6\|z\|\|y\|/k$$

for each  $z$  such that  $\|z\| = \|y\|$ , we conclude that

$$\|l_{x+y} - l_x\| \leq 6\|y\|/k.$$

□

## REFERENCES

- [1] P. BERNARD, Existence of  $C^{1,1}$  critical sub-solutions of the Hamilton-Jacobi equation on compact manifolds, *Ann. Sci. E. N. S.* (4), **40**, no. 3 (2007), pp. 445–452.
- [2] P. CARDALIAGUET, *Front Propagation Problems with Nonlocal Terms 2*, *J. Math. An. Appl.*, **260** (2001), pp. 572–601.
- [3] A. FATHI - M. ZAVIDOVIQUE, *Insertion of  $C^{1,1}$  functions and Ilmanen's Lemma*.
- [4] A. FATHI - A. SICONOLFI, *Existence of  $C^1$  critical subsolutions of the Hamilton-Jacobi equation*. *Invent. Math.*, **155**, no. 2 (2004), pp. 363–388.
- [5] T. ILMANEN, *The Level-Set Flow on a Manifold*, *Proc. Symposia Pure Math.*, **54**, no. 1 (1993), pp. 193–204.
- [6] J. M. LASRY - P. L. LIONS, *A Remark on Regularization in Hilbert Spaces*, *Israel Math. J.*, **55**, no. 3 (1996), pp. 257–266.
- [7] M. ZAVIDOVIQUE, *Existence of  $C^{1,1}$  critical subsolutions in discrete weak KAM theory*. To appear in *Journal of Modern Dynamics*.

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