



Lasry-Lions Regularization and a Lemma of Ilmanen

Patrick Bernard

► To cite this version:

Patrick Bernard. Lasry-Lions Regularization and a Lemma of Ilmanen. *Rendiconti del Seminario Matematico della Università di Padova*, 2010, 124, pp.221-229. 10.4171/RSMUP/124-15 . hal-00448255v2

HAL Id: hal-00448255

<https://hal.science/hal-00448255v2>

Submitted on 4 Jan 2016

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Lasry-Lions Regularization and a Lemma of Ilmanen

PATRICK BERNARD (*)

Let H be a Hilbert space. We define the following inf (sup) convolution operators acting on bounded functions $u : H \rightarrow \mathbb{R}$:

$$T_t u(x) := \inf_y \left(u(y) + \frac{1}{t} \|y - x\|^2 \right)$$

and

$$\tilde{T}_t u(x) := \sup_y \left(u(y) - \frac{1}{t} \|y - x\|^2 \right).$$

We have the relation

$$T_t(-u) = -\tilde{T}_t(u).$$

Recall that these operators form semi-groups, in the sense that

$$T_t \circ T_s = T_{t+s} \quad \text{and} \quad \tilde{T}_t \circ \tilde{T}_s = \tilde{T}_{t+s}$$

for all $t \geq 0$ and $s \geq 0$, as can be checked by direct calculation. Note also that

$$\inf u \leq T_t u(x) \leq u(x) \leq \tilde{T}_t u(x) \leq \sup u$$

for each $t \geq 0$ and each $x \in H$. A function $u : H \rightarrow \mathbb{R}$ is called k -semi-concave, $k > 0$, if the function $x \rightarrow u(x) - \|x\|^2/k$ is concave. We will occasionally consider semi-concave functions which take values in $[-\infty, +\infty)$. The function u is called k -semi-convex if $-u$ is k -semi-concave. A function u is t -semi-concave and upper semi-continuous if and only if it belongs to the image of the operator T_t , this follows from Lemma 1 and Lemma 3 below. A function is called semi-concave if it is k -semi-concave for some $k > 0$. A function u is said $C^{1,1}$ if it is Frechet differentiable and if the gradient of u is Lipschitz. Note that a continuous function $u : H \rightarrow \mathbb{R}$ is $C^{1,1}$

(*) Indirizzo dell'A.: CEREMADE, Université Paris-Dauphine, Place du Maréchal de Lattre de Tassigny, 75775, Paris Cedex 16, France.

E-mail: patrick.bernard@ceremade.dauphine.fr

if and only if it is semi-concave and semi-convex, see Lemma 5. Let us recall two important results in that language:

THEOREM 1 (Lasry-Lions, [6]). *Let u be a bounded function. For $0 < s < t$, the function $\tilde{T}_s \circ T_t u$ is $C^{1,1}$ and, if u is uniformly continuous, then it converges uniformly to u when $t \rightarrow 0$.*

THEOREM 2 (Ilmanen, [5]). *Let $u \geq v$ be two bounded functions on H such that u and $-v$ are semi-concave. Then there exists a $C^{1,1}$ function w such that $u \geq w \geq v$.*

Our goal in the present paper is to “generalize” simultaneously both of these results as follows:

THEOREM 3. *The operator $R_t := \tilde{T}_t \circ T_{2t} \circ \tilde{T}_t$ has the following properties:*

- *Regularization : For each function $f : H \rightarrow \mathbb{R}$ and each $t > 0$, the function $R_t(f)$ is $C^{1,1}$ provided it is locally bounded. This holds for all $t > 0$ if f is bounded.*
- *Approximation : If $f : H \rightarrow \mathbb{R}$ is uniformly continuous, then $R_t(f)$ is $C^{1,1}$ and converges uniformly to f as $t \rightarrow 0$.*
- *Pinching: If there exists a k -semi-concave continuous function u and a k -semi-convex continuous function v such that $v \leq f \leq u$, then, for all $t \in]0, k]$, we have $u \geq R_t(f) \geq v$, and $R_t(f)$ is $C^{1,1}$.*

Theorem 3 does not, properly speaking, generalize Theorem 5. However, it offers a new (although similar) answer to the same problem: approximating uniformly continuous functions on Hilbert spaces by $C^{1,1}$ functions with a simple explicit formula.

Because of its symmetric form, the regularizing operator R_t enjoys some nicer properties than the Lasry-Lions operators. For example, if f is $C^{1,1}$, then it follows from the pinching property that $R_t f = f$ for t small enough.

Theorem 2, can be proved using Theorem 3 by taking $w = R_t u$, for t small enough. Note, in view of Lemma 3 below, that $R_t u = \tilde{T}_t \circ T_t u$ when t is small enough.

Theorem 3 can be somehow extended to the case of finite dimensional open sets or manifolds via partition of unity, at the price of loosing the simplicity of explicit expressions. Let M be a paracompact manifold of dimension n , equipped once and for all with an atlas $(\phi_i, i \in \mathfrak{I})$ composed of

charts $\phi_i : B^n \rightarrow M$, where B^n is the open unit ball of radius one centered at the origin in $\mathbb{R}^n$. We assume in addition that the image $\phi_i(B^n)$ is a relatively compact open set, and that the sets $\phi_i(B^n)$, $i \in \mathfrak{I}$ form a locally finite open covering of the manifold M . Let us fix, once and for all, a partition of the unity g_i subordinated to the open covering $(\phi_i(B^n), i \in \mathfrak{I})$. It means that the function g_i is non-negative with support inside $\phi_i(B^n)$, and that $\sum_i g_i = 1$ (note that this sum is finite at each point). Let us define the operator

$$G_t(u) := \sum_i [R_{ta_i}((g_i u) \circ \phi_i)] \circ \phi_i^{-1},$$

where $a_i, i \in \mathfrak{I}$ are positive real numbers. In this expression, we consider each of the terms $[R_{ta_i}((g_i u) \circ \phi_i)] \circ \phi_i^{-1}$ as defined on the whole manifold M with the value 0 outside of $\phi_i(B^n)$. The sum is then locally finite hence well-defined. We say that a function $u : M \rightarrow \mathbb{R}$ is locally semi-concave if, for each $i \in \mathfrak{I}$, there exists a constant b_i such that the function $u \circ \phi_i - \|\cdot\|^2/b_i$ is concave on B^n .

THEOREM 4. *Let $u \geq v$ be two continuous functions on M such that u and $-v$ are locally semi-concave. Then, the real numbers a_i can be chosen such that, for each $t \in]0, 1]$ and each function f satisfying $u \geq f \geq v$, we have:*

- *The function $G_t(f)$ is locally $C^{1,1}$.*
- *If f is continuous, then $G_t(f)$ converges locally uniformly to f as $t \rightarrow 0$.*
- *$u \geq G_t(f) \geq v$.*

We will give some properties, most of which are well-known, of the operators T_t and $\tilde{T}_t$ in Section 1, and derive the proof of the main results in Section 2.

Notes and Acknowledgements. Theorem 2 appears in Ilmanen's paper [5] as Lemma 4G. Several proofs are sketch there but none is detailed. The proof we detail here follows lines similar to one of the sketches of Ilmanen. This statement also has a more geometric counterpart, Lemma 4E in [5]. A detailed proof of this geometric version is given in [2], Appendix. My attention was attracted to these statements and their relations with recent progresses on sub-solutions of the Hamilton-Jacobi equation (see [4, 1, 7]) by Pierre Cardalaguet, Albert Fathi and Maxime Zavidovique. Albert Fathi and Maxime Zavidovique also recently wrote a detailed proof of

Theorem 1, see [3]. This paper also proves how the geometric version follows from Theorem 2. There are many similarities between the tools used in the present paper and those used in [1]. Moreover, Maxime Zavidovique observed in [7] that the existence of $C^{1,1}$ subsolutions of the Hamilton-Jacobi equation in the discrete case can be deduced from Theorem 2. However, it seems that the main result of [1] (the existence of $C^{1,1}$ subsolutions in the continuous case) can't be deduced easily from Theorem 2. Neither can Theorem 2 be deduced from it.

1. The operators T_t and $\tilde{T}_t$ on Hilbert spaces.

The proofs of the theorems follow from standard properties of the operators T_t and $\tilde{T}_t$ that we now recall in details.

LEMMA 1. *For each function $u : H \rightarrow \mathbb{R}$, the function $T_t u$ (which takes values in $[-\infty, +\infty)$), is t -semi-concave and upper semi-continuous. The function $\tilde{T}_t u$ (which takes values in $(-\infty, +\infty]$), is t -semi-convex and lower semi-continuous. Moreover, if u is k -semi-concave, then for each $t < k$ the function $\tilde{T}_t u$ is $(k - t)$ -semi-concave. Similarly, if u is k -semi-convex, then for each $t < k$ the function $T_t u$ is $(k - t)$ -semi-convex.*

PROOF. We shall prove the statements concerning T_t . We have

$$\begin{aligned} T_t u(x) - \|x\|^2/t &= \inf_y (u(y) + \|y - x\|^2/t - \|x\|^2/t) = \\ &= \inf_y (u(y) + \|y\|^2/t - 2x \cdot y/t), \end{aligned}$$

this function is concave and upper semi-continuous as an infimum of continuous linear functions. On the other hand, we have

$$T_t u(x) + \|x\|^2/l = \inf_y (u(y) + \|y - x\|^2/t + \|x\|^2/l).$$

Setting $f(x, y) := u(y) + \|y - x\|^2/t + \|x\|^2/l$, the function $\inf_y f(x, y)$ is a convex function of x if f is a convex function of (x, y) . This is true if u is k -semi-convex, $t < k$, and $l = k - t$ because we have the expression

$$f(x, y) = u(y) + \|y - x\|^2/t + \|x\|^2/l = (u(y) + \|y\|^2/k) + \left\| \sqrt{\frac{l}{kt}} y - \sqrt{\frac{k}{lt}} x \right\|^2.$$

□

Given a uniformly continuous function $u : H \rightarrow \mathbb{R}$, we define its modulus of continuity $\rho(r) : [0, \infty) \rightarrow [0, \infty)$ by the expression $\rho(r) = \sup_{x,e} u(x + re) - u(x)$, where the supremum is taken on all $x \in H$ and all e in the unit ball of H . The function ρ is non-decreasing, it satisfies $\rho(r + r') \leq \rho(r) + \rho(r')$, and it converges to zero in zero (this last fact is equivalent to the uniform continuity of u). We say that a function $\rho : [0, \infty) \rightarrow [0, \infty)$ is a modulus of continuity if it satisfies these properties. Given a modulus of continuity $\rho(r)$, we say that a function u is ρ -continuous if $|u(y) - u(x)| \leq \rho(\|y - x\|)$ for all x and y in H .

LEMMA 2. *If $u : H \rightarrow \mathbb{R}$ is uniformly continuous, then the functions $T_t u$ and $\tilde{T}_t u$ converge uniformly to u when $t \rightarrow 0$. Moreover, given a modulus of continuity ρ , there exists a non-decreasing function $\varepsilon(t) : [0, \infty) \rightarrow [0, \infty)$ satisfying $\lim_{t \rightarrow 0} \varepsilon(t) = 0$ and such that, for each ρ -continuous bounded function u , we have:*

- $T_t u$ and $\tilde{T}_t u$ are ρ -continuous for each $t \geq 0$.
- $u - \varepsilon(t) \leq T_t u(x) \leq u$ and $u \leq \tilde{T}_t u \leq u + \varepsilon(t)$ for each $t \geq 0$.

PROOF. Let us fix $y \in H$, and set $v(x) = u(x + y)$. We have $u(x) - \rho(\|y\|) \leq v(x) \leq u(x) + \rho(\|y\|)$. Applying the operator T_t gives $T_t u(x) - \rho(y) \leq T_t v(x) \leq T_t u(x) + \rho(y)$. On the other hand, we have

$$T_t v(x) = \inf_z (u(z + y) + \|z - x\|^2/t) = \inf_z (u(z) + \|z - (x + y)\|^2/t) = T_t u(x + y),$$

so that

$$T_t u(x) - \rho(\|y\|) \leq T_t u(x + y) \leq T_t u(x) + \rho(\|y\|).$$

We have proved that $T_t u$ is ρ continuous if u is, the proof for $\tilde{T}_t u$ is the same.

In order to study the convergence, let us set $\varepsilon(t) = \sup_{r>0} (\rho(r) - r^2/t)$. We have

$$\varepsilon(t) = \sup_{r>0} (\rho(r\sqrt{t}) - r^2) \leq \sup_{r>0} ((r+1)\rho(\sqrt{t}) - r^2) = \rho(\sqrt{t}) + \rho^2(\sqrt{t})/4.$$

We conclude that $\lim_{t \rightarrow 0} \varepsilon(t) = 0$. We now come back to the operator T_t , and observe that

$$u(y) - \|y - x\|^2/t \geq u(x) - \rho(\|y - x\|) + \|y - x\|^2/t \geq u(x) - \varepsilon(t)$$

for each x and y , so that

$$u - \varepsilon(t) \leq T_t u \leq u.$$

□

LEMMA 3. For each function $u : H \rightarrow (-\infty, +\infty]$, we have $\tilde{T}_t \circ T_t(u) \leq u$ and the equality $\tilde{T}_t \circ T_t(u) = u$ holds if and only if u is t -semi-convex and lower semi-continuous. Similarly, given a function $v : H \rightarrow [-\infty, +\infty)$, we have $T_t \circ \tilde{T}_t(v) \geq v$, with equality if and only if v is t -semi-concave and upper semi-continuous.

PROOF. Let us write explicitly

$$\tilde{T}_t \circ T_t u(x) = \sup_y \inf_z (u(z) + \|z - y\|^2/t - \|y - x\|^2/t).$$

Taking $z = x$, we obtain the estimate $\tilde{T}_t \circ T_t u(x) \leq \sup_y u(x) = u(x)$. Let us now write

$$\tilde{T}_t \circ T_t u(x) + \|x\|^2/t = \sup_y \inf_z (u(z) + \|z\|^2/t + (2y/t) \cdot (x - z))$$

which by an obvious change of variable leads to

$$\tilde{T}_t \circ T_t u(x) + \|x\|^2/t = \sup_y \inf_z (u(z) + \|z\|^2/t + y \cdot (x - z)).$$

We recognize here that the function $\tilde{T}_t \circ T_t u(x) + \|x\|^2/t$ is the Legendre bidual of the function $u(x) + \|x\|^2/t$. It is well-known that a function is equal to its Legendre bidual if and only if it is convex and lower semi-continuous. $\square$

2. Proof of the main results.

PROOF OF THEOREM 3. For each function f and each $t > 0$, the function $\tilde{T}_t \circ T_{2t} \circ \tilde{T}_t f$ is both t -semi-concave and t -semi-convex. It is t -semi-convex by Lemma 1, and it is semi-concave because $T_{2t}(\tilde{T}_t f)$ is $2t$ -semi-concave by Lemma 1, which implies, still by Lemma 1, that $\tilde{T}_t \circ T_{2t} \circ \tilde{T}_t f$ is t -semi-concave. As a consequence, Lemma 5 below implies that the function $R_t f$ is $C^{1,1}$ provided it is locally bounded. The function $R_t(f)$ is bounded if f is bounded, hence it is $C^{1,1}$ in this case.

In the case where f is uniformly continuous, Lemma 2 implies that

$$f - \varepsilon(2t) \leq R_t(f) \leq f + 2\varepsilon(t).$$

As a consequence, $R_t(f)$ is converging uniformly to f , and it is locally bounded hence $C^{1,1}$.

We now consider two continuous functions u and v such that u and $-v$ are k semi-concave, and such that $v \leq u$. We claim that

$$u \geq f \geq v \implies u \geq T_t \circ \tilde{T}_t f \geq v \text{ and } u \geq \tilde{T}_t \circ T_t f \geq v$$

for $t \leq k$. This claim implies that $u \geq \tilde{T}_t \circ T_{2t} \circ \tilde{T}_t f \geq v$ when $u \geq f \geq v$ and $t \leq k$. Let us now prove the claim concerning $\tilde{T}_t \circ T_t$, the other part being similar. Since v is k -semi-convex and continuous, we have $\tilde{T}_t \circ T_t v = v$ for $t \leq k$, by Lemma 3. Then,

$$u \geq f \geq \tilde{T}_t \circ T_t f \geq \tilde{T}_t \circ T_t v = v$$

where the second inequality follows from Lemma 3, and the third from the obvious fact that the operators T_t and $\tilde{T}_t$ are order-preserving.

We have proved that $v \leq R_t(f) \leq u$ if $v \leq f \leq u$ and $t \leq k$. For $t \in]0, k]$, the function $R_t(f)$ is thus locally bounded hence $C^{1,1}$. $\square$

PROOF OF THEOREM 4. Let a_i be chosen such that the functions $(g_i u) \circ \phi_i$ and $-(g_i v) \circ \phi_i$ are a_i -semi-concave on $\mathbb{R}^n$ (when extended by 0 outside of B^n). The existence of real numbers a_i with this property follows from Lemma 4 below. Given $u \geq f \geq v$, we can apply Theorem 3 for each i to the functions

$$(g_i u) \circ \phi_i \geq (g_i f) \circ \phi_i \geq (g_i v) \circ \phi_i$$

extended by zero outside of B^n . We conclude that, for $t \in]0, 1]$, the function $R_{ta_i}((g_i f) \circ \phi_i)$ is $C^{1,1}$ and satisfies

$$(g_i u) \circ \phi_i \geq R_{ta_i}((g_i f) \circ \phi_i) \geq (g_i v) \circ \phi_i.$$

As a consequence, the function

$$[R_{ta_i}((g_i f) \circ \phi_i)] \circ \phi_i^{-1},$$

extended as a function on M equal to 0 outside of $\phi_i(B^n)$, is $C^{1,1}$. The function $G_t(f)$ is thus locally a finite sum of $C^{1,1}$ functions hence it is locally $C^{1,1}$. Moreover, we have

$$u = \sum_i g_i u \geq G_t(f) \geq \sum_i g_i v = v.$$

$\square$

We have used:

LEMMA 4. Let $u : B^n \rightarrow \mathbb{R}$ be a bounded function such that $u - \|\cdot\|^2/a$ is concave, for some $a > 0$. For each compactly supported non-negative C^2 function $g : B^n \rightarrow \mathbb{R}$, the product gu (extended by zero outside of B^n) is semi-concave on $\mathbb{R}^n$.

PROOF. Since u is bounded, we can assume that $u \geq 0$ on B^n . Let $K \subset B^n$ be a compact subset of the open ball B^n which contains the support

of g in its interior. Since the function $u - \|\cdot\|^2/a$ is concave on B_1 it admits super-differentials at each point. As a consequence, for each $x \in B^n$, there exists a linear form l_x such that

$$0 \leq u(y) \leq u(x) + l_x \cdot (y - x) + \|y - x\|^2/a$$

for each $y \in B^1$. Moreover, the linear form l_x is bounded independently of $x \in K$. We also have

$$0 \leq g(y) \leq g(x) + dg_x \cdot (y - x) + C\|y - x\|^2$$

for some $C > 0$, for all x, y in $\mathbb{R}^n$. Taking the product, we get, for $x \in K$ and $y \in B^n$,

$$u(y)g(y) \leq u(x)g(x) + (g(x)l_x + u(x)dg_x) \cdot (y - x) + C\|y - x\|^2 + C\|y - x\|^3 + C\|y - x\|^4$$

where $C > 0$ is a constant independent of $x \in K$ and $y \in B^n$, which may change from line to line. As a consequence, setting $L_x = g(x)l_x + u(x)dg_x$, we obtain the inequality

$$(L) \quad (gu)(y) \leq (gu)(x) + L_x \cdot (y - x) + C\|y - x\|^2$$

for each $x \in K$ and $y \in B^n$. If we set $L_x = 0$ for $x \in \mathbb{R}^n - K$, the relation (L) holds for each $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$. For $x \in K$ and $y \in B^n$, we have already proved it. Since the linear forms L_x , $x \in K$ are uniformly bounded, we can assume that $L_x \cdot (y - x) + C\|y - x\|^2 \geq 0$ for all $x \in K$ and $y \in \mathbb{R}^n - B^n$ by taking C large enough. Then, (L) holds for all $x \in K$ and $y \in \mathbb{R}^n$. For $x \in \mathbb{R}^n - K$ and y outside of the support g , the relation (L) holds in an obvious way, because $gu(x) = gu(y) = 0$, and $L_x = 0$. For $x \in \mathbb{R}^n - K$ and y in the support of g , the relation holds provided that $C \geq \max(gu)/d^2$, where d is the distance between the complement of K and the support of g . This is a positive number since K is a compact set containing the support of g in its interior. We conclude that the function (gu) is semi-concave on $\mathbb{R}^n$. $\square$

For completeness, we also prove, following Fathi:

LEMMA 5. *Let $u : H \rightarrow \mathbb{R}$ be a locally bounded function which is both k -semi-concave and k -semi-convex. Then the function u is $C^{1,1}$, and $6/k$ is a Lipschitz constant for the gradient of u .*

PROOF. It is well known that a locally bounded convex function is continuous. We conclude that u is continuous. Let u be a continuous function which is both k -semi-concave and k -semi-convex. Then, for each $x \in H$,

there exists a unique $l_x \in H$ such that

$$|u(x+y) - u(x) - l_x \cdot y| \leq \|y\|^2/k.$$

We conclude that l_x is the gradient of u at x , and we have to prove that the map $x \mapsto l_x$ is Lipschitz. We have, for each x, y and z in H :

$$\begin{aligned} l_x \cdot (y+z) - \|y+z\|^2/k &\leq u(x+y+z) - u(x) \leq l_x \cdot (y+z) + \|y+z\|^2/k \\ l_{(x+y)} \cdot (-y) - \|y\|^2/k &\leq u(x) - u(x+y) \leq l_{(x+y)} \cdot (-y) + \|y\|^2/k \\ l_{(x+y)} \cdot (-z) - \|z\|^2/k &\leq u(x+y) - u(x+y+z) \leq l_{(x+y)} \cdot (-z) + \|z\|^2/k. \end{aligned}$$

Taking the sum, we obtain

$$|(l_{x+y} - l_x) \cdot (y+z)| \leq \|y+z\|^2/k + \|y\|^2/k + \|z\|^2/k.$$

By a change of variables, we get

$$|(l_{x+y} - l_x) \cdot (z)| \leq \|z\|^2/k + \|y\|^2/k + \|z-y\|^2/k.$$

Taking $\|z\| = \|y\|$, we obtain

$$|(l_{x+y} - l_x) \cdot (z)| \leq 6\|z\|\|y\|/k$$

for each z such that $\|z\| = \|y\|$, we conclude that

$$\|l_{x+y} - l_x\| \leq 6\|y\|/k.$$

□

REFERENCES

- [1] P. BERNARD, Existence of $C^{1,1}$ critical sub-solutions of the Hamilton-Jacobi equation on compact manifolds, Ann. Sci. E. N. S. (4), **40**, no. 3 (2007), pp. 445–452.
- [2] P. CARDALIAGUET, *Front Propagation Problems with Nonlocal Terms 2*, J. Math. An. Appl., **260** (2001), pp. 572–601.
- [3] A. FATHI - M. ZAVIDOVIQUE, *Insertion of $C^{1,1}$ functions and Ilmanen's Lemma*.
- [4] A. FATHI - A. SICONOLFI, *Existence of C^1 critical subsolutions of the Hamilton-Jacobi equation*. Invent. Math., **155**, no. 2 (2004), pp. 363–388.
- [5] T. ILMANEN, *The Level-Set Flow on a Manifold*, Proc. Symposia Pure Math., **54**, no. 1 (1993), pp. 193–204.
- [6] J. M. LASRY - P. L. LIONS, *A Remark on Regularization in Hilbert Spaces*, Israel Math. J., **55**, no. 3 (1996), pp. 257–266.
- [7] M. ZAVIDOVIQUE, *Existence of $C^{1,1}$ critical subsolutions in discrete weak KAM theory*. To appear in Journal of Modern Dynamics.

Manoscritto pervenuto in redazione il 6 luglio 2010.