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Lasry-Lions Regularization and a Lemma of Ilmanen

Patrick Bernard (*)

Let $H$ be a Hilbert space. We define the following inf (sup) convolution operators acting on bounded functions $u : H \to \mathbb{R}$:

$$T_t u(x) := \inf_y \left( u(y) + \frac{1}{t} \| y - x \|^2 \right)$$

and

$$T_t u(x) := \sup_y \left( u(y) - \frac{1}{t} \| y - x \|^2 \right).$$

We have the relation

$$T_t(-u) = -T_t(u).$$

Recall that these operators form semi-groups, in the sense that

$$T_t \circ T_s = T_{t+s} \quad \text{and} \quad \tilde{T}_t \circ \tilde{T}_s = \tilde{T}_{t+s}$$

for all $t \geq 0$ and $s \geq 0$, as can be checked by direct calculation. Note also that

$$\inf u \leq T_t u(x) \leq u(x) \leq T_t u(x) \leq \sup u$$

for each $t \geq 0$ and each $x \in H$. A function $u : H \to \mathbb{R}$ is called $k$-semi-concave, $k > 0$, if the function $x \to u(x) - \|x\|^2/k$ is concave. We will occasionally consider semi-concave functions which take values in $[-\infty, +\infty)$. The function $u$ is called $k$-semi-convex if $-u$ is $k$-semi-concave.

A function $u$ is $t$-semi-concave and upper semi-continuous if and only if it belongs to the image of the operator $T_t$, this follows from Lemma 1 and Lemma 3 below. A function is called semi-concave if it is $k$-semi-concave for some $k > 0$. A function $u$ is said $C^{1,1}$ if it is Frechet differentiable and if the gradient of $u$ is Lipschitz. Note that a continuous function $u : H \to \mathbb{R}$ is $C^{1,1}$.

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if and only if it is semi-concave and semi-convex, see Lemma 5. Let us recall two important results in that language:

**Theorem 1** (Lasry-Lions, [6]). Let \( u \) be a bounded function. For \( 0 < s < t \), the function \( T_s \circ T_t u \) is \( C^{1,1} \) and, if \( u \) is uniformly continuous, then it converges uniformly to \( u \) when \( t \to 0 \).

**Theorem 2** (Imanen, [5]). Let \( u \geq v \) be two bounded functions on \( H \) such that \( u \) and \( -v \) are semi-concave. Then there exists a \( C^{1,1} \) function \( w \) such that \( u \geq w \geq v \).

Our goal in the present paper is to “generalize” simultaneously both of these results as follows:

**Theorem 3.** The operator \( R_t := T_t \circ T_2 \circ T_t \) has the following properties:

- **Regularization**: For each function \( f : H \to \mathbb{R} \) and each \( t > 0 \), the function \( R_t(f) \) is \( C^{1,1} \) provided it is locally bounded. This holds for all \( t > 0 \) if \( f \) is bounded.

- **Approximation**: If \( f : H \to \mathbb{R} \) is uniformly continuous, then \( R_t(f) \) is \( C^{1,1} \) and converges uniformly to \( f \) as \( t \to 0 \).

- **Pinching**: If there exists a \( k \)-semi-concave continuous function \( u \) and a \( k \)-semi-convex continuous function \( v \) such that \( v \leq f \leq u \), then, for all \( t \in [0,k] \), we have \( u \geq R_t(f) \geq v \), and \( R_t(f) \) is \( C^{1,1} \).

Theorem 3 does not, properly speaking, generalize Theorem 5. However, it offers a new (although similar) answer to the same problem: approximating uniformly continuous functions on Hilbert spaces by \( C^{1,1} \) functions with a simple explicit formula.

Because of its symmetric form, the regularizing operator \( R_t \) enjoys some nicer properties than the Lasry-Lions operators. For example, if \( f \) is \( C^{1,1} \), then it follows from the pinching property that \( R_t f = f \) for \( t \) small enough.

Theorem 2 can be proved using Theorem 3 by taking \( w = R_t u \) for \( t \) small enough. Note, in view of Lemma 3 bellow, that \( R_t u = T_t \circ T_t u \) when \( t \) is small enough.

Theorem 3 can be somehow extended to the case of finite dimensional open sets or manifolds via partition of unity, at the price of loosing the simplicity of explicit expressions. Let \( M \) be a paracompact manifold of dimension \( n \), equipped once and for all with an atlas \( (\phi_i, i \in \mathcal{A}) \) composed of
charts $\phi_i : B^n \rightarrow M$, where $B^n$ is the open unit ball of radius one centered at the origin in $\mathbb{R}^n$. We assume in addition that the image $\phi_i(B^n)$ is a relatively compact open set, and that the sets $\phi_i(B^n)$, $i \in I$ form a locally finite open covering of the manifold $M$. Let us fix, once and for all, a partition of the unity $g_i$, subordinated to the open covering $(\phi_i(B^n), i \in I)$. It means that the function $g_i$ is non-negative with support inside $\phi_i(B^n)$, and that $\sum g_i = 1$ (note that this sum is finite at each point). Let us define the operator

$$G_t(u) := \sum_i \left[ R_{\text{int}}(g_i u \circ \phi_i) \right] \circ \phi_i^{-1},$$

where $a_i$, $i \in I$ are positive real numbers. In this expression, we consider each of the terms $\left[ R_{\text{int}}((g_i u) \circ \phi_i) \right] \circ \phi_i^{-1}$ as defined on the whole manifold $M$ with the value 0 outside of $\phi_i(B^n)$. The sum is then locally finite hence well-defined. We say that a function $u : M \rightarrow \mathbb{R}$ is locally semi-concave if, for each $i \in I$, there exists a constant $b_i$ such that the function $u \circ \phi_i - \| \cdot \|^2 / b_i$ is concave on $B^n$.

**Theorem 4.** Let $u \geq v$ be two continuous functions on $M$ such that $u$ and $-v$ are locally semi-concave. Then, the real numbers $a_i$ can be chosen such that, for each $t \in [0, 1]$ and each function $f$ satisfying $u \geq f \geq v$, we have:

- The function $G_t(f)$ is locally $C^{1,1}$.
- If $f$ is continuous, then $G_t(f)$ converges locally uniformly to $f$ as $t \rightarrow 0$.
- $u \geq G_t(f) \geq v$.

We will give some properties, most of which are well-known, of the operators $T_t$ and $T_t$ in Section 1, and derive the proof of the main results in Section 2.

**Notes and Acknowledgements.** Theorem 2 appears in Ilmanen’s paper [5] as Lemma 4G. Several proofs are sketch there but none is detailed. The proof we detail here follows lines similar to one of the sketches of Ilmanen. This statement also has a more geometric counterpart, Lemma 4E in [5]. A detailed proof of this geometric version is given in [2], Appendix. My attention was attracted to these statements and their relations with recent progresses on sub-solutions of the Hamilton-Jacobi equation (see [4, 1, 7]) by Pierre Card$\text{e}$alaguet, Albert Fathi and Maxime Zavidovique. Albert Fathi and Maxime Zavidovique also recently wrote a detailed proof of
Theorem 1, see [3]. This paper also proves how the geometric version follows from Theorem 2. There are many similarities between the tools used in the present paper and those used in [1]. Moreover, Maxime Zavidovique observed in [7] that the existence of $C^{1,1}$ subsolutions of the Hamilton-Jacobi equation in the discrete case can be deduced from Theorem 2. However, is seems that the main result of [1] (the existence of $C^{1,1}$ subsolutions in the continuous case) can’t be deduced easily from Theorem 2. Neither can Theorem 2 be deduced from it.

1. The operators $T_t$ and $\bar{T}_t$ on Hilbert spaces.

The proofs of the theorems follow from standard properties of the operators $T_t$ and $\bar{T}_t$ that we now recall in details.

**Lemma 1.** For each function $u : H \to \mathbb{R}$, the function $T_t u$ (which takes values in $[-\infty, +\infty)$), is $t$-semi-concave and upper semi-continuous. The function $\bar{T}_t u$ (which takes values in $(-\infty, +\infty]$), is $t$-semi-convex and lower semi-continuous. Moreover, if $u$ is $k$-semi-concave, then for each $t < k$ the function $T_t u$ is $(k - t)$-semi-concave. Similarly, if $u$ is $k$-semi-convex, then for each $t < k$ the function $T_t u$ is $(k - t)$-semi-convex.

**Proof.** We shall prove the statements concerning $T_t$. We have

\[ T_t u(x) - \|x\|^2/t = \inf_y \left( u(y) + \|y - x\|^2/t - \|x\|^2/t \right) = \inf_y \left( u(y) + \|y\|^2/t - 2x \cdot y/t \right). \]

this function is concave and upper semi-continuous as an infimum of continuous linear functions. On the other hand, we have

\[ T_t u(x) + \|x\|^2/l = \inf_y \left( u(y) + \|y - x\|^2/t + \|x\|^2/l \right). \]

Setting $f(x, y) := u(y) + \|y - x\|^2/t + \|x\|^2/l$, the function $\inf_y f(x, y)$ is a convex function of $x$ if $f$ is a convex function of $(x, y)$. This is true if $u$ is $k$-semi-convex, $t < k$, and $l = k - t$ because we have the expression

\[ f(x, y) = u(y) + \|y - x\|^2/t + \|x\|^2/l = (u(y) + \|y\|^2/k) + \left( \frac{l}{k} \sqrt{t} - \sqrt{l} \right)^2. \]

\[ \square \]
Given a uniformly continuous function \( u : H \to \mathbb{R} \), we define its modulus of continuity \( \rho(r) : [0, \infty) \to [0, \infty) \) by the expression \( \rho(r) = \sup_{x \in B} u(x + r e) - u(x) \), where the supremum is taken on all \( x \in H \) and all \( e \) in the unit ball of \( H \). The function \( \rho \) is non-decreasing, it satisfies \( \rho(r + r') \leq \rho(r) + \rho(r') \), and it converges to zero in zero (this last fact is equivalent to the uniform continuity of \( u \)). We say that a function \( \rho : [0, \infty) \to [0, \infty) \) is a modulus of continuity if it satisfies these properties. Given a modulus of continuity \( \rho(t) \), we say that a function \( u \) is \( \rho \)-continuous if \( |u(y) - u(x)| \leq \rho(\|y - x\|) \) for all \( x \) and \( y \) in \( H \).

**Lemma 2.** If \( u : H \to \mathbb{R} \) is uniformly continuous, then the functions \( T_t u \) and \( T_t \bar{u} \) converge uniformly to \( u \) when \( t \to 0 \). Moreover, given a modulus of continuity \( \rho \), there exists a non-decreasing function \( \varepsilon(t) : [0, \infty) \to [0, \infty) \) satisfying \( \lim_{t \to 0} \varepsilon(t) = 0 \) and such that, for each \( \rho \)-continuous bounded function \( u \), we have:

- \( T_t u \) and \( T_t \bar{u} \) are \( \rho \)-continuous for each \( t \geq 0 \).
- \( u - \varepsilon(t) \leq T_t u(x) \leq u \) and \( u \leq T_t \bar{u} \leq u + \varepsilon(t) \) for each \( t \geq 0 \).

**Proof.** Let us fix \( y \in H \), and set \( v(x) = u(x + y) \). We have \( u(x) - \rho(\|y\|) \leq v(x) \leq u(x) + \rho(\|y\|) \). Applying the operator \( T_t \) gives \( T_t u(x) - \rho(y) \leq T_t v(x) \leq T_t u(x) + \rho(y) \). On the other hand, we have

\[
T_t v(x) = \inf_z (u(z + y) + \|z - x\|^2 / t) = \inf_z (u(z) + \|z - (x + y)\|^2 / t) = T_t u(x + y),
\]

so that

\[
T_t u(x) - \rho(\|y\|) \leq T_t u(x + y) \leq T_t u(x) + \rho(\|y\|).
\]

We have proved that \( T_t \bar{u} \) is \( \rho \)-continuous if \( u \) is, the proof for \( T_t \bar{u} \) being the same.

In order to study the convergence, let us set \( \varepsilon(t) = \sup_{r > 0} \rho(r) - r^2 / t \). We have

\[
\varepsilon(t) = \sup_{r > 0} (\rho(r \sqrt{t}) - r^2) \leq \sup_{r > 0} (r + 1)\rho(\sqrt{t}) - r^2 = \rho(\sqrt{t}) + \rho^2(\sqrt{t}) / 4.
\]

We conclude that \( \lim_{t \to 0} \varepsilon(t) = 0 \). We now come back to the operator \( T_t \), and observe that

\[
u(y) - \|y - x\|^2 / t \geq u(x) - \rho(\|y - x\|) + \|y - x\|^2 / t \geq u(x) - \varepsilon(t)
\]

for each \( x \) and \( y \), so that

\[
u - \varepsilon(t) \leq T_t u \leq u.
\]
Lemma 3. For each function $u : H \rightarrow (-\infty, +\infty)$, we have $\tilde{T}_t \circ T_t(u) \leq u$ and the equality $\tilde{T}_t \circ T_t(u) = u$ holds if and only if $u$ is t-semi-convex and lower semi-continuous. Similarly, given a function $v : H \rightarrow [-\infty, +\infty)$, we have $\tilde{T}_t \circ T_t(v) \geq v$, with equality if and only if $v$ is t-semi-concave and upper semi-continuous.

Proof. Let us write explicitly

$$\tilde{T}_t \circ T_t u(x) = \sup_{y,z} \left( u(z) + \|z - y\|^2 / t - \|y - x\|^2 / t \right).$$

Taking $z = x$, we obtain the estimate $\tilde{T}_t \circ T_t u(x) \leq \sup_y u(z) = u(z).$ Let us now write

$$\tilde{T}_t \circ T_t u(x) + \|x\|^2 / t = \sup_{y,z} \left( u(z) + \|z\|^2 / t + (2y/t) \cdot (x - z) \right)$$

which by an obvious change of variable leads to

$$\tilde{T}_t \circ T_t u(x) + \|x\|^2 / t = \sup_{y,z} \left( u(z) + \|z\|^2 / t + y \cdot (x - z) \right).$$

We recognize here that the function $\tilde{T}_t \circ T_t u(x) + \|x\|^2 / t$ is the Legendre bidual of the function $u(x) + \|x\|^2 / t$. It is well-known that a function is equal to its Legendre bidual if and only if it is convex and lower semi-continuous.

2. Proof of the main results.

Proof of Theorem 3. For each function $f$ and each $t > 0$, the function $\tilde{T}_t \circ T_t f$ is both convex and t-semi-convex. It is t-semi-convex by Lemma 1, and it is semi-convex because $T_{2t}(\tilde{T}_t f)$ is 2t-semi-convex by Lemma 1, which implies, still by Lemma 1, that $\tilde{T}_t \circ T_{2t} \circ T_t f$ is t-semi-convex. As a consequence, Lemma 5 below implies that the function $R_t(f)$ is $C^{1,1}$ provided it is locally bounded. The function $R_t(f)$ is bounded if $f$ is bounded, hence its is $C^{1,1}$ in this case.

In the case where $f$ is uniformly continuous, Lemma 2 implies that

$$f - \varphi(2t) \leq R_t(f) \leq f + 2\varphi(t).$$

As a consequence, $R_t(f)$ is converging uniformly to $f$, and it is locally bounded hence $C^{1,1}$.

We now consider two continuous functions $u$ and $v$ such that $u$ and $-v$ are $k$ semi-concave, and such that $v \leq u$. We claim that

$$u \geq f \geq v \implies u \geq \tilde{T}_t \circ T_t f \geq v \text{ and } u \geq \tilde{T}_t \circ T_t f \geq v$$
for $t \leq k$. This claim implies that $u \geq T_0(T_1 \circ T_2) \circ T_3 f \geq v$ when $u \geq f \geq v$ and $t \leq k$. Let us now prove the claim concerning $T_0 \circ T_1$, the other part being similar. Since $v$ is $k$-semi-convex and continuous, we have $T_0(T_1 v) = v$ for $t \leq k$, by Lemma 3. Then,

$$u \geq f \geq T_0(T_1 f) \geq T_0 T_1 v = v$$

where the second inequality follows from Lemma 3, and the third from the obvious fact that the operators $T_0$ and $T_1$ are order-preserving.

We have proved that $v \leq R_t(f) \leq u$ if $v \leq f \leq u$ and $t \leq k$. For $t \in [0, k]$, the function $R_t(f)$ is thus locally bounded hence $C^{1,1}$.

**Proof of Theorem 4.** Let $a_i$ be chosen such that the functions $(g_i u) \circ \hat{\phi}_i$ and $-(g_i v) \circ \hat{\phi}_i$ are $a_i$-semi-concave on $\mathbb{R}^n$ (when extended by 0 outside of $B^n$). The existence of real numbers $a_i$, with this property follows from Lemma 4 below. Given $u \geq f \geq v$, we can apply Theorem 3 for each $i$ to the functions

$$(g_i u) \circ \hat{\phi}_i \geq (g_i f) \circ \hat{\phi}_i \geq (g_i v) \circ \hat{\phi}_i$$

extended by zero outside of $B^n$. We conclude that, for $t \in [0, 1]$, the function $R_{tn}(g_i f \circ \hat{\phi}_i)$ is $C^{1,1}$ and satisfies

$$(g_i u) \circ \hat{\phi}_i \geq R_{tn}((g_i f) \circ \hat{\phi}_i) \geq (g_i v) \circ \hat{\phi}_i.$$

As a consequence, the function

$$[R_{tn}(g_i f \circ \hat{\phi}_i)] \circ \hat{\phi}_i^{-1},$$

extended as a function on $M$ equal to 0 outside of $\hat{\phi}_i(B^n)$, is $C^{1,1}$. The function $G_t(f)$ is thus locally a finite sum of $C^{1,1}$ functions hence it is locally $C^{1,1}$. Moreover, we have

$$u = \sum_i g_i u \geq G_t(f) \geq \sum_i g_i v = v.$$

We have used:

**Lemma 4.** Let $u : B^n \to \mathbb{R}$ be a bounded function such that $u - \|\cdot\|^2/a$ is concave, for some $a > 0$. For each compactly supported non-negative $C^2$ function $g : B^n \to \mathbb{R}$, the product $gu$ (extended by zero outside of $B^n$) is semi-concave on $\mathbb{R}^n$.

**Proof.** Since $u$ is bounded, we can assume that $u \geq 0$ on $B^n$. Let $K \subset B^n$ be a compact subset of the open ball $B^n$ which contains the support
of \( g \) in its interior. Since the function \( u - \| \cdot \|^2 / a \) is concave on \( B_1 \) it admits super-differentials at each point. As a consequence, for each \( x \in B^n \), there exists a linear form \( L_x \) such that
\[
0 \leq u(y) \leq u(x) + L_x \cdot (y - x) + \| y - x \|^2 / a
\]
for each \( y \in B^1 \). Moreover, the linear form \( L_x \) is bounded independently of \( x \in K \). We also have
\[
0 \leq g(y) \leq g(x) + d g_x \cdot (y - x) + C \| y - x \|^2
\]
for some \( C > 0 \), for all \( x, y \in \mathbb{R}^n \). Taking the product, we get, for \( x \in K \) and \( y \in B^n \),
\[
u(y) g(y) \leq u(x) g(x) + (g(x) L_x + u(x) d g_x) \cdot (y - x) + C \| y - x \|^2 + C \| y - x \|^3 + C \| y - x \|^4
\]
where \( C > 0 \) is a constant independent of \( x \in K \) and \( y \in B^n \), which may change from line to line. As a consequence, setting \( L_x = g(x) L_x + u(x) d g_x \), we obtain the inequality
\[
(L) \quad \quad \quad \quad \quad (gu)(y) \leq (gu)(x) + L_x \cdot (y - x) + C \| y - x \|^2
\]
for each \( x \in K \) and \( y \in B^n \). If we set \( L_x = 0 \) for \( x \in \mathbb{R}^n - K \), the relation \( (L) \) holds for each \( x \in \mathbb{R}^n \) and \( y \in \mathbb{R}^n \). For \( x \in K \) and \( y \in B^n \), we have already proved it. Since the linear forms \( L_x, x \in K \) are uniformly bounded, we can assume that \( L_x \cdot (y - x) + C \| y - x \|^2 \geq 0 \) for all \( x \in K \) and \( y \in \mathbb{R}^n - B^n \) by taking \( C \) large enough. Then, \( (L) \) holds for all \( x \in K \) and \( y \in \mathbb{R}^n \). For \( x \in \mathbb{R}^n - K \) and \( y \) outside of the support \( g \), the relation \( (L) \) holds in an obvious way, because \( gu(x) = gu(y) = 0 \), and \( L_x = 0 \). For \( x \in \mathbb{R}^n - K \) and \( y \) in the support of \( g \), the relation holds provided that \( C \geq \max (gu)/d^2 \), where \( d \) is the distance between the complement of \( K \) and the support of \( g \). This is a positive number since \( K \) is a compact set containing the support of \( g \) in its interior. We conclude that the function \( (gu) \) is semi-concave on \( \mathbb{R}^n \). \( \square \)

For completeness, we also prove, following Fathi:

**Lemma 5.** Let \( u : H \rightarrow \mathbb{R} \) be a locally bounded function which is both \( k \)-semi-concave and \( k \)-semi-convex. Then the function \( u \) is \( C^{1,1} \), and \( 6/k \) is a Lipschitz constant for the gradient of \( u \).

**Proof.** It is well known that a locally bounded convex function is continuous. We conclude that \( u \) is continuous. Let \( u \) be a continuous function which is both \( k \)-semi-concave and \( k \)-semi-convex. Then, for each \( x \in H \),
there exists a unique $l_x \in H$ such that
\[ |u(x + y) - u(x) - l_x \cdot y| \leq \|y\|^2/k. \]

We conclude that $l_x$ is the gradient of $u$ at $x$, and we have to prove that the map $x \mapsto l_x$ is Lipschitz. We have, for each $x$, $y$ and $z$ in $H$:
\[ l_x \cdot (y + z) - \|y + z\|^2/k \leq u(x + y + z) - u(x) \leq l_x \cdot (y + z) + \|y + z\|^2/k \]
\[ l_{x+y} \cdot (-y) - \|y\|^2/k \leq u(x) - u(x + y) \leq l_{x+y} \cdot (-y) + \|y\|^2/k \]
\[ l_{x+y} \cdot (-z) - \|z\|^2/k \leq u(x + y) - u(x + y + z) \leq l_{x+y} \cdot (-z) + \|z\|^2/k. \]

Taking the sum, we obtain
\[ |(l_{x+y} - l_x) \cdot (y + z)| \leq \|y + z\|^2/k + \|y\|^2/k + \|z\|^2/k. \]

By a change of variables, we get
\[ |(l_{x+y} - l_x) \cdot (z)| \leq \|z\|^2/k + \|y\|^2/k + \|z - y\|^2/k. \]

Taking $\|z\| = \|y\|$, we obtain
\[ |(l_{x+y} - l_x) \cdot (z)| \leq 6\|z\|\|y\|/k \]

for each $z$ such that $\|z\| = \|y\|$, we conclude that
\[ \|l_{x+y} - l_x\| \leq 6\|y\|/k. \]

\[\Box\]

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