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HAUSDORFF MEASURE OF VORTICITY NODAL SETS FOR THE 3D HYPERVISCOUS NAVIER STOKES EQUATIONS WITH GENERAL FORCES

ABDELHAFID YOUNSI

ABSTRACT. In this paper, we modified the three dimensional Navier-Stokes equations by adding a l -Laplacian. We provide upper bounds on the two-dimensional Hausdorff measure \mathcal{H}_l^2 of $N_\omega^0 = \{x \in \Omega \subset \mathbb{R}^3 / \omega(x, t) = 0\}$ the level sets of the vorticity ω of solutions. We express them in terms of the Kolmogorov length-scale and the Landau–Lifschitz estimates of the number of degrees of freedom in turbulent flow. We also, under certain hypothesis recover the two-dimensional Hausdorff measure estimates for the usual 3D Navier–Stokes equations with potential force. Moreover, we show that the estimates depend on l , this result suggests that the modified Navier Stokes system is successful model of turbulence and the size of the nodal set $\mathcal{H}_l^2(N_\omega^0)$ leads the way for developing the turbulence theory.

1. INTRODUCTION

In this paper, we provide upper bounds on the two-dimensional Hausdorff measure \mathcal{H}_l^2 of N_ω^0 the level sets associated with the vorticity of modified three dimensional Navier-Stokes equations. We modified the 3D Navier-Stokes system by adding a higher-order viscosity term to the conventional system

$$\begin{aligned} \frac{du}{dt} + \varepsilon(-\Delta)^l u - \nu \Delta u + (u \cdot \nabla) u + \nabla p &= f(x), \text{ in } \Omega \times (0, \infty) \\ \operatorname{div} u &= 0, \text{ in } \Omega \times (0, \infty), \\ p(x + Le_i, t) &= p(x, t), \quad u(x + Le_i, t) = u(x, t) \quad i = 1, \dots, d \quad t \in (0, \infty) \\ u(x, 0) &= u_0(x), \text{ in } \Omega, \end{aligned} \tag{1.1}$$

on $\Omega = (0, L)^d$ with periodic boundary conditions and (e_1, \dots, e_d) is the natural basis of \mathbb{R}^d . Here $\varepsilon > 0$ is the artificial dissipation parameter and $\nu > 0$ is the kinematic viscosity of the fluid, $l > 1$. The functions u is the velocity vector field, p is the pressure, and f is a given force field. For $\varepsilon = 0$, the model is reduced to the Navier-Stokes system.

In the work [31], the strong convergence of the solution of this problem to the solution of the conventional system as the regularization parameter goes to zero, was established for each dimension $d \leq 4$.

Mathematical model for such fluid motion has been used extensively in turbulence simulations (see e.g. [9]) also see Borue and Orsag [3, 4]. For further discussion of theoretical results concerning (1.1), see [1, 2, 23, 31].

For the 3D Navier–Stokes system weak solutions of problem are known to exist by a basic result by J. Leray from 1934 [21], only the uniqueness of weak solutions

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remains as an open problem. Then the known theory of global attractors of infinite dimensional dynamical systems is not applicable to the 3D Navier–Stokes system.

In particular, in case one accepts the point of view that the dimension of a global attractor for the Navier–Stokes equations is associated with the number of degrees of freedom in turbulent flows, then the two-dimensional Hausdorff measure $\mathcal{H}_l^2(N_\omega^0)$ is an important way to the understanding of turbulence theory [31].

We are interested in the three dimensional case. Let P_m be the projection onto the first m eigenspaces of the Stokes operator $A = -\Delta$ and let $N_\omega^0 = \{x \in \Omega \subset \mathbb{R}^3 / \omega(x, t) = 0\}$ the nodal sets of the vorticity ω for solutions of the equation (1.1). We provide an upper bound on the size of the nodal sets $\mathcal{H}_l^2(N_\omega^0)$ and we show that, the bounds necessarily depend on m and l this dependence is a fractional power of l . Thus answering a question raised by J. Avrin [1]. We also obtain here scale-invariant estimates on the two-dimensional Hausdorff measure $\mathcal{H}_l^2(N_\omega^0)$ in terms of the Landau–Lifschitz theory of the number of degrees of freedom in turbulent flow. Since expressing the above estimates in terms of the (dimensionless) Grashoff number G . In order to obtain an upper bound on the Hausdorff measure of level sets associated, we use the method from [19] (see also [6], [7]).

The main purpose of the present article is to study the dependence of the two-dimensional Hausdorff measure $\mathcal{H}_l^2(N_\omega^0)$ on the parameter l . Using a family of Kolmogorov flows as base flows we can deduce also upper bounds on the Hausdorff measure $\mathcal{H}_l^2(N_\omega^0)$. We also find here that the upper bounds on the two-dimensional Hausdorff measure of N_ω^0 converges to the corresponding upper bounds on $\mathcal{H}_1^2(N_\omega^0)$ the two-dimensional Hausdorff measure of the nodal sets of the usual 3D Navier–Stokes as $l = 1$. Under certain hypothesis we recover the two-dimensional Hausdorff measure $\mathcal{H}_1^2(N_\omega^0)$ estimates for the usual 3D Navier–Stokes equations with potential force. We extend the method from [17] to a 3D Navier–Stokes with general forcing modified by l -Laplacian. These estimates are obtained without using the Dirichlet quotients [17].

We note, however, that for the incompressible 3D Navier–Stokes equations with general force, it seems not so easy to get some better estimates on the Hausdorff measure of the level sets associated with the vorticity as in the case of potential force studied in [17, 18] for periodic solutions of the 2D. Related results for the 3D Navier–Stokes equations (with general forcing) can be found in [5]. The upper bounds on the Hausdorff measures of the level sets associated with solutions of some other partial differential equations were obtained in [7],[11], [12], [16], [19], [20], and [22].

The paper is organized as follows. In Section 2, we present the relevant mathematical framework for the paper. In Section 3, we provide upper bounds for the two-dimensional Hausdorff measure \mathcal{H}_l^2 of the level sets associated with the vorticity of the Navier–Stokes system with hyperdissipation.

2. NOTATIONS AND PRELIMINARIES

In this section we introduce notations and the definitions of standard functional spaces that will be used throughout the paper. We denote by $H^m(\Omega)$, the Sobolev space of L_{per} periodic functions. These spaces are endowed with the inner product

$$(u, v) = \sum_{|\beta| \leq m} (D^\beta u, D^\beta v)_{L^2(\Omega)} \text{ and the norm } \|u\|_m = \left(\sum_{|\beta| \leq m} \|D^\beta u\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}.$$

Each $u \in L_{per}$ can be identified with its Fourier expansion

$$u(x) = \sum_{k \in \mathbb{Z}^3} u_k \exp(2i\pi k \cdot \frac{x}{L})$$

where $u_k \in \mathbb{C}^3$ satisfy $\bar{u}_k = u_{-k}$. Then u is in L^2 if and only if

$$\|u\|_{L^2}^2 = |\Omega| \sum_{k \in \mathbb{Z}^3} |u_k|^2 < \infty, \quad |\Omega| = L^3,$$

then the Sobolev space $u \in H^m(\Omega)$, $m \in \mathbb{R}^+$ can be characterized by

$$H^m(\Omega) = \{u, \bar{u}_k = u_{-k}, \sum_{k \in \mathbb{Z}^3} k^{2m} |u_k|^2 < \infty.\}$$

$H^{-m}(\Omega)$ denote the dual space of $H^m(\Omega)$.

We denote by $\dot{H}^m(\Omega)$ the subspace of $H^m(\Omega)$ with, zero average

$$\dot{H}^m(\Omega) = \{u \in H^m(\Omega); \int_{\Omega} u(x) dx = 0\}.$$

For $m = 0$, we have $\dot{H}^m(\Omega) = \dot{L}^2(\Omega)$.

- We introduce the following solenoidal subspaces V_s , $s \in \mathbb{R}^+$ which are important to our analysis

$$V_0(\Omega) = \{u \in \dot{L}^2(\Omega), \operatorname{div} u = 0, u \cdot n|_{\Sigma_i} = -u \cdot n|_{\Sigma_{i+3}}, i = 1, 2, 3\};$$

$$V_1(\Omega) = \{u \in \dot{H}^1(\Omega), \operatorname{div} u = 0, \gamma_0 u|_{\Sigma_i} = \gamma_0 u|_{\Sigma_{i+3}}, i = 1, 2, 3\}.$$

$$V_2(\Omega) = \{u \in \dot{H}^2(\Omega), \operatorname{div} u = 0, \gamma_0 u|_{\Sigma_i} = \gamma_0 u|_{\Sigma_{i+3}}, \gamma_1 u|_{\Sigma_i} = -\gamma_1 u|_{\Sigma_{i+3}}, i = 1, 2, 3\},$$

see [29, Chapter III, Section 2]. We refer the reader to R.Temam [30] for details on these spaces. Here the faces of Ω are numbered as

$$\Sigma_i = \partial\Omega \cap \{x_i = 0\} \text{ and } \Sigma_{i+3} = \partial\Omega \cap \{x_i = L\}, \quad i = 1, 2, 3.$$

Here γ_0, γ_1 are the trace operators and n is the unit outward normal on $\partial\Omega$.

- The space V_0 is endowed with the inner product $(u, v)_{L^2(\Omega)}$ and norm $\|u\| = (u, u)_{L^2(\Omega)}^{1/2}$.
- V_1 is the Hilbert space with the norm $\|u\|_1 = \|u\|_{V_1}$. The norm induced by $\dot{H}^1(\Omega)$ and the norm $\|\nabla u\|$ are equivalent in V_1 .
- V_2 is the Hilbert space with the norm $\|u\|_2 = \|u\|_{V_2}$. In V_2 the norm induced by $\dot{H}^2(\Omega)$ is equivalent to the norm $\|\Delta u\|$.

V'_s denote the dual space of V_s .

Let P be the orthogonal projection in $L^2_{per}(\mathbb{R}^3)^3$ with the range H .

Let $A = -P\Delta$ the Stokes operator. It is easy to check that $Au = -\Delta u$ for every $u \in D(A)$. We recall that the operator A is a closed positive self-adjoint unbounded operator, with $D(A) = \{u \in V_0, Au \in V_0\}$. We have in fact,

$$D(A) = \dot{H}^2(\Omega) \cap V_0 = V_2.$$

The eigenvalues of A are $\{\lambda_j\}_{j=1}^{j=\infty}$, $0 < \lambda_1 \leq \lambda_2 \leq \dots$ and the corresponding orthonormal set of eigenfunctions $\{w_j\}_{j=1}^{j=\infty}$ is complete in V_0

$$Aw_j = \lambda_j w_j, \quad w_j \in D(A), \forall j.$$

The spectral theory of A allows us to define the powers A^l of A for $l \geq 1$, A^l is an unbounded self-adjoint operator in V_0 with a domain $D(A^l)$ dense in $V_2 \subset V_0$. We set here

$$A^l u = (-\Delta)^l u \text{ for } u \in D(A^l) = V_{2l} \cap V_0.$$

The space $D(A^l)$ is endowed with the scalar product and the norm

$$(u, v)_{D(A^l)} = (A^l u, A^l v), \quad \|u\|_{D(A^l)} = \{(u, v)_{D(A^l)}\}^{\frac{1}{2}}. \quad (2.1)$$

In the case for $l > 0$, we have $D(A^l) = \{u \in H, \sum_{j=1}^{\infty} \lambda_j^{2l} (u, w_j)^2 < \infty\}$. For $l \in \mathbb{R}$ the scalar product and the norm in (2.1) can be written alternatively as

$$(u, v)_{D(A^l)} = \sum_{j=1}^{\infty} \lambda_j^{2l} (u, w_j)(v, w_j), \quad \|u\|_{D(A^l)} = \left\{ \sum_{j=1}^{\infty} \lambda_j^{2l} (u, w_j)^2 \right\}^{\frac{1}{2}} \quad (2.2)$$

and for $u \in D(A^l)$ we can write

$$A^l u = \sum_{j=1}^{\infty} \lambda_j^l (u, w_j) w_j.$$

Let us now define the trilinear form $b(\cdot, \cdot, \cdot)$ associated with the inertia terms

$$b(u, v, w) = \sum_{i,j=1}^3 \int_{\Omega} u_i \frac{\partial v_j}{\partial x_i} w_j dx.$$

The continuity property of the trilinear form enables us to define (using Riesz representation Theorem) a bilinear continuous operator $B(u, v); V_2 \times V_2 \rightarrow V_2'$ will be defined by

$$\langle B(u, v), w \rangle = b(u, v, w), \quad \forall w \in V_2. \quad (2.3)$$

Recall that for u satisfying $\nabla \cdot u = 0$ we have

$$b(u, u, u) = 0 \text{ and } b(u, v, w) = -b(u, w, v). \quad (2.4)$$

We recall some well known inequalities that we will be using in what follows. Young's inequality

$$ab \leq \frac{\sigma}{p} a^p + \frac{1}{q\sigma^{\frac{q}{p}}} b^q, \quad a, b, \sigma > 0, \quad p > 1, \quad q = \frac{p}{p-1}. \quad (2.5)$$

Poincaré's inequality

$$\lambda_1 \|u\|^2 \leq \|A^{\frac{1}{2}} u\|^2 \text{ for all } u \in V_0. \quad (2.6)$$

Denoting

$$\|u\|_{G(t)}^2 = \|e^{tA^{\frac{1}{2}}} u\|^2 \text{ and } (u, v)_{G(t)} = (e^{tA^{\frac{1}{2}}} u, e^{tA^{\frac{1}{2}}} v).$$

The set $D(e^{\alpha A})$ is called the Gevrey class of operator of order $\alpha \geq 0$ [10]. Our use of Gevrey classes shall be based on the following consideration.

Denote with $N_h^0 = \{x \in \Omega : h(x) = 0\}$ the zero (nodal) set of a function h in a set Ω , and let \mathcal{H}^2 be the two-dimensional Hausdorff measure operating on subsets of \mathbb{R}^3 (area in this case).

3. LEVEL SETS OF THE VORTICITY FUNCTION

Using the operators defined above, we can write the modified system (1.1) in the evolution form

$$\begin{aligned} \frac{du}{dt} + \varepsilon A^l u + \nu Au + B(u, u) &= f(x), \text{ in } \Omega \times (0, \infty) \\ u_0(x) &= u_0, \text{ in } \Omega. \end{aligned} \quad (3.1)$$

The existence and uniqueness results for initial value problem (1.1) can be found in [23]. The following theorem collects the main result in this work

Theorem 3.1. *For $l \geq \frac{d+2}{4}$, d is the space dimension, for $\varepsilon > 0$ fixed, $f \in L^2(0, T; V_0')$ and $u_0 \in V_0$ be given. There exists a unique weak solution of (1.1) which satisfies $u \in L^2(0, T; V_l) \cap L^\infty(0, T; V_0)$, $\forall T > 0$.*

The modern understanding of turbulence is that it is a collection of weakly correlated vortical motions, which, despite their intermittent and chaotic distribution over a wide range of space and time scales, actually consist of local characteristic 'eddy' patterns that persist as they move around under the influences of their own and other eddies' vorticity fields [15].

In fluid mechanics, the Reynolds number is important in analyzing any type of flow when there is substantial velocity gradient (i.e. shear.) It indicates the relative significance of the viscous effect compared to the inertia effect. The Reynolds number is proportional to inertial force divided by viscous force (see [9])

$$Re = \frac{Ul}{\nu} \quad U^2 = L^{-2} \langle \|u\|_2^2 \rangle \quad (3.2)$$

where l the characteristic scale of the forcing and $\langle \cdot \rangle$ is the long-time-average

$$\langle g(\cdot) \rangle = \lim_{T \rightarrow \infty} \sup \left(\frac{1}{T} \int_0^T g(t) dt \right). \quad (3.3)$$

With Reynolds number calculator we can analyze what makes fluid flow regime laminar and what is needed to force the fluid to flow in turbulent regime. Experimental observations show that for 'fully developed' flow, laminar flow occurs when $Re < R_{te}$ and turbulent flow occurs when $Re > R_{te}$. In the interval between R_{te} and R_{te} , laminar and turbulent flows are possible ('transition' flows) [9] and references therein. The nature of the vortex formed in the fluid flow depends strongly on the Reynolds number ([9]; and references therein). These transition Reynolds numbers are also called critical Reynolds numbers, and were studied by Osborne Reynolds around 1895 [26]. The transition to turbulence and the construction of vortex are delayed by increasing the critical Reynolds number. If we assume that the critical Reynolds number R_{ce} for the onset of vortex shedding is, attempt for

$$\|u\| = \frac{\nu R_{ce}}{l} L, \quad (3.4)$$

then the associated velocity u for each

$$Re \geq R_{ce} \quad (3.5)$$

satisfies the inequality

$$\|u\| \geq \frac{\nu R_{ce}}{l} L = \mu, \quad (3.6)$$

μ is a positive constant.

Another nondimensional quantity that we use often is the so-called Grashof number, which is proportional to the forcing term f . Hence, we define the Grashof numbers in the 3-dimensional case, as in Foias, Manley, Rosa and Temam [9] by

$$Gr(f) = \frac{1}{\nu^2 \lambda_1^{3/4}} \|f\| \quad (3.7)$$

The effects of variation in Grashof number on vortex have been shown in the work of Olson and Titi [24], they keep the spatial structure of the forcing function fixed and vary the Grashof number by varying the amplitude of the forcing function. Namely, they vary the Grashof number by rescaling the forcing function by a multiplicative factor. This is equivalent to changing the viscosity or the size of the domain. As increases, or equivalently as the viscosity decreases, the turbulent flow becomes more energetic and one would expect the number of numerically determining modes to increase as well. There are many reasons to suppose that the existence and intensity of vortex in our work should increase as the Grashof number increases [14, 24, 25]. In [14] zero forcing implies that the attractor has been reduced to zero. Since all solutions decay eventually to zero in the unforced case.

This intuition is supported by existing theoretical critical number $G_{cr}(f)$ for the existence of level curves of representative vorticity fields.

Note the flow for $Gr(f) \geq G_{cr}(f)$ has noticeably more large scale structure compared to the flow for $Gr(f) \leq G_{cr}(f)$. This is consistent with the energy spectra, where most of the energy is in the lowest modes, that is, in the large spatial scales and eddies when the Grashof number is large [24].

The effect of a body force on vorticity production and turbulence generation in a fluid flow is described by the Grashof number.

In addition, we assume without loss of generality that $\|f\|$ is bounded. Then, there exist a maximum Grashof number $G_{\max}r(f)$ and a positive constant ρ such that the body force f satisfies the following inequality

$$\|f\| \leq \nu^2 \lambda_1^{3/4} G_{\max}r(f) = \rho. \quad (3.8)$$

Since $\|f\|$ is strictly positive we get

$$\frac{\|u\|}{\|f\|} \geq \frac{\mu}{\rho} = \frac{LR_{ce}}{\nu \lambda_1^{3/4} l G_{\max}r(f)} = \beta \quad (3.9)$$

this gives a relation between $\|u\|$ and $\|f\|$

$$\|u\| \geq \beta \|f\|. \quad (3.10)$$

Moreover, according to the definition of the Gevrey norm and the relation (3.10) we get

$$\|u\|_G \geq \beta \|f\|_G.$$

The vorticity, $\omega = \nabla \times u$ satisfies the equation

$$\left(\frac{d}{dt} + u \cdot \nabla + \nu \Delta + \varepsilon (-\Delta)^l\right) \omega = \omega \cdot \nabla u + F \quad (3.11)$$

where $F = \nabla \times f$.

Theorem 3.2. [19] *Suppose that a nonzero function $h \in V_1$ satisfies*

$$\|e^{\alpha A} h\|_1 \leq M \|h\|_1$$

Then

$$\mathcal{H}^2(N_h^0) \leq C_1 L^2 (1 + \log M) e^{C_2 L/\alpha}.$$

Hereafter, C_i for $i \in \mathbb{N}$, stand for universal constants. The above statement is a special case of [19, Theorem 2.1]. It will be used in conjunction with the following statement:

Lemma 3.3. [17] *Let $u \in V_0$, and let ω and be its vorticity. If*

$$\| A^{\frac{1}{2}} e^{\alpha A^{\frac{1}{2}}} u \| \leq M \| A^{\frac{1}{2}} u \| \quad (3.12)$$

for some $M > 0$, then, for every $c \in \mathbb{R}$

$$\| e^{\alpha A^{\frac{1}{2}}} (\omega - c) \| \leq M \| \omega - c \| . \quad (3.13)$$

For the rest of the paper, let $u(t)$ be an arbitrary solution of the the modified Navier Stokes system (1.1) with $u(0) = u_0$.

Theorem 3.4. *Let $\|u\| \geq \beta \|f\|$ for any $\alpha \leq \frac{\nu \lambda_1^{\frac{1}{2}}}{4}$ and $\beta \leq \frac{4\sqrt{2}}{\nu}$. Then there exists a universal constant C_3 such that if $\| A^{\frac{1}{2}} u \| \leq C_3$, then*

$$\| A^{\frac{1}{2}} e^{\alpha t A^{\frac{1}{2}}} u \| \leq 2 \| A^{\frac{1}{2}} u_0 \| , \quad t \geq 0. \quad (3.14)$$

Proof. For any $\alpha, t \geq 0$, We take the inner product of (3.1) with u , to obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \| A^{\frac{1}{2}} u \|_{G(t)}^2 &= \alpha \| A^{\frac{3}{4}} u \|_{G(t)}^2 + (A\dot{u}, u)_{G(t)} \\ &= \alpha \| A^{\frac{3}{4}} u \|_{G(t)}^2 - \varepsilon \| A^{\frac{l+1}{2}} u \|_{G(t)}^2 - \nu \| Au \|_{G(t)}^2 - b(u, u, Au)_{G(t)} + (f, Au)_{G(t)}. \end{aligned} \quad (3.15)$$

then using the Young's inequality (2.5) we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \| A^{\frac{1}{2}} u \|_{G(t)}^2 &\leq -\varepsilon \| A^{\frac{l+1}{2}} u \|_{G(t)}^2 + \frac{\nu}{4} \| Au \|_{G(t)}^2 + \frac{\alpha^2}{\nu} \| A^{\frac{1}{2}} u \|_{G(t)}^2 - \nu \| Au \|_{G(t)}^2 \\ &\quad + \left(\frac{\nu}{2} \| Au \|_{G(t)}^2 + \frac{1}{2\nu} \| f \|_{G(t)}^2 \right) + b(u, u, Au)_{G(t)} \end{aligned}$$

From we get

$$\| Au \|_{G(t)}^2 \geq \| f \|_{G(t)}^2$$

this give

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \| A^{\frac{1}{2}} u \|_{G(t)}^2 &\leq -\varepsilon \| A^{\frac{l+1}{2}} u \|_{G(t)}^2 + \frac{\nu}{4} \| Au \|_{G(t)}^2 + \frac{\alpha^2}{\nu} \| A^{\frac{1}{2}} u \|_{G(t)}^2 - \nu \| Au \|_{G(t)}^2 \\ &\quad + \left(\frac{\nu}{2} \| Au \|_{G(t)}^2 + \frac{\beta^2}{2\lambda_1\nu} \| A^{\frac{1}{2}} u \|_{G(t)}^2 \right) + b(u, u, Au)_{G(t)} \\ &\leq -\varepsilon \| A^{\frac{l+1}{2}} u \|_{G(t)}^2 + \frac{-\lambda_1\nu}{4} \| A^{\frac{1}{2}} u \|_{G(t)}^2 + \frac{\alpha^2}{\nu} \| A^{\frac{1}{2}} u \|_{G(t)}^2 \\ &\quad + \frac{\beta^2}{2\lambda_1\nu} \| A^{\frac{1}{2}} u \|_{G(t)}^2 + b(u, u, Au)_{G(t)}. \end{aligned}$$

We get for $\beta^2 = \frac{1}{2\lambda_1\alpha^2}$

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \| A^{\frac{1}{2}} u \|_{G(t)}^2 &\leq -\varepsilon \| A^{\frac{l+1}{2}} u \|_{G(t)}^2 - \frac{\lambda_1\nu}{4} \| A^{\frac{1}{2}} u \|_{G(t)}^2 \\ &\quad + \frac{2\alpha^2}{\nu} \| A^{\frac{1}{2}} u \|_{G(t)}^2 + b(u, u, Au)_{G(t)}. \end{aligned}$$

We use the following inequality from [10] and [17, Section 4]

$$b(u, u, Au)_{G(t)} \leq C_4 \| A^{\frac{1}{2}} u \|_{G(t)}^2 \| Au \|_{G(t)} (1 + \log \frac{\| Au \|_{G(t)}^2}{\lambda_1 \| A^{\frac{1}{2}} u \|_{G(t)}^2})^{\frac{1}{2}} \quad (3.16)$$

to obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|A^{\frac{1}{2}} u\|_{G(t)}^2 &\leq -\varepsilon \|A^{\frac{l+1}{2}} u\|_{G(t)}^2 - \frac{\lambda_1 \nu}{4} \|A^{\frac{1}{2}} u\|_{G(t)}^2 + \frac{2\alpha^2}{\nu} \|A^{\frac{1}{2}} u\|_{G(t)}^2 \\ &\quad + C_4 \|A^{\frac{1}{2}} u\|_{G(t)}^2 \|Au\|_{G(t)} (1 + \log \frac{\|Au\|_{G(t)}^2}{\lambda_1 \|A^{\frac{1}{2}} u\|_{G(t)}^2})^{\frac{1}{2}}. \end{aligned}$$

To establish (3.14) we use the estimate [17]

$$a \mu (1 + \log \frac{\mu^2}{b^2})^{\frac{1}{2}} \leq d \mu^2 + \frac{a^2}{d^2} \log \frac{2a}{bd} \quad a, d > 0, \mu \geq b > 0. \quad (3.17)$$

By applying the Poincaré's inequality (2.6), we have that for $\mu = \|Au\|_{G(t)}$ and $d = \frac{\nu}{8}$

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|A^{\frac{1}{2}} u\|_{G(t)}^2 + \varepsilon \|A^{\frac{l+1}{2}} u\|_{G(t)}^2 &\leq -\frac{\lambda_1 \nu}{8} \|A^{\frac{1}{2}} u\|_{G(t)}^2 + \frac{2\alpha^2}{\nu} \|A^{\frac{1}{2}} u\|_{G(t)}^2 \\ &\quad + C_5 \|A^{\frac{1}{2}} u\|_{G(t)}^4 \log \frac{C_6 \|A^{\frac{1}{2}} u\|_{G(t)}}{\lambda_1^{\frac{1}{2}}}. \end{aligned}$$

Letting $\alpha \leq \frac{\nu \lambda_1^{\frac{1}{2}}}{4}$ we have for $\beta \leq \frac{4\sqrt{2}}{\nu}$ that

$$\frac{1}{2} \frac{d}{dt} \|A^{\frac{1}{2}} u\|_{G(t)}^2 \leq C_5 \|A^{\frac{1}{2}} u\|_{G(t)}^4 \log \frac{C_6 \|A^{\frac{1}{2}} u\|_{G(t)}}{\lambda_1^{\frac{1}{2}}}. \quad (3.18)$$

If $\|A^{\frac{1}{2}} u_0\| < \frac{\lambda_1^{\frac{1}{2}}}{C_6} = C_3$, the term with a logarithm in (3.18) is negative at $t = 0$, and thus (3.18) implies that $\|A^{\frac{1}{2}} u\|_{G(t)}$ is a decreasing function of t . \square

Theorem 3.4. implies that, for any solution $u(t)$ of (1.1), the space analyticity radius of $u(t)$ goes to infinity as $t \rightarrow \infty$.

Let Ω be a periodic box, for simplicity assume $\Omega = (0, L)^3$, A has eigenvalues $0 < \lambda_1 < \lambda_2 < \dots$ with corresponding eigenspaces E_1, E_2, \dots . Let P_m be the projection on the eigenspaces $E_1 \oplus E_2 \oplus \dots \oplus E_m$ and let $Q_m = I - P_m$ we have $\|u\|^2 = \|P_m u\|^2 + \|Q_m u\|^2$ and we also have from (2.2) that

$$\|A^l u\| \leq \lambda_m^l \|u\| \text{ for every } l \geq 0 \text{ and } u \in D(A^l). \quad (3.19)$$

For any $t \geq 0$, let $\omega(t)$ be the vorticity of $u(t)$. We shall, for any fixed $t > 0$, estimate the quantity

$$l(\omega(t)) = \sup_{c \in \mathbb{R}} \mathcal{H}_l^2(N_\omega^c). \quad (3.20)$$

Recall that for a function $h : \Omega \rightarrow \mathbb{R}$, $N_h^0 = \{x \in \Omega : h(x) = 0\}$. We need the following fact

Lemma 3.5. *Let $\|u\| \geq \beta \|f\|$ for any $\beta \geq 0$. Then*

$$\|u(t)\| \geq \|u(0)\| \exp(\eta t) \text{ for every } t \geq 0. \quad (3.21)$$

With $\eta = -(\varepsilon \lambda_m^{\frac{1}{2}} + \frac{1+\beta^2}{2\beta^2})$.

Proof. Taking the scalar product of both sides of (1.1) by $u(t)$ and using (2.4), we have that

$$\frac{1}{2} \frac{d}{dt} \|u\|^2 + \nu \|A^{\frac{1}{2}} u\|^2 + \varepsilon \|A^{\frac{l}{2}} u\|^2 = (f, u) \text{ for } t \geq 0. \quad (3.22)$$

Using (3.19) and the following inequality

$$(f, u) \geq -\frac{1}{2}\|f\|^2 - \frac{1}{2}\|u\|^2 \quad (3.23)$$

Because the increasing sequence $0 \leq \lambda_1 \leq \lambda \leq \lambda_m$ we have

$$\frac{1}{2} \frac{d}{dt} \|u\|^2 \geq (-\nu\lambda \|A^{\frac{1}{2}}u\|^2 - (\varepsilon\lambda_m^{\frac{1}{2}} + \frac{1+\beta^2}{2\beta^2}))\|u\|^2 \quad (3.24)$$

note that since

$$\|A^{\frac{1}{2}}u\| \leq C_3$$

we have that

$$\frac{1}{2} \frac{d}{dt} \|u\|^2 \geq -C_3 - (\varepsilon\lambda_m^{\frac{1}{2}} + \frac{1+\beta^2}{2\beta^2})\|u\| \quad (3.25)$$

if we set $\eta = -(\varepsilon\lambda_m^{\frac{1}{2}} + \frac{1+\beta^2}{2\beta^2})$ then we have from (3.25) that

$$\frac{1}{2} \frac{d}{dt} \|u\|^2 \geq -C_3 + \eta\|u\|.$$

Integrating the above inequality from 0 to t , we get

$$\|u\|^2 \geq \frac{-C_3}{\eta} (1 - \exp(\eta t)) + \|u(0)\|^2 \exp(\eta t) \quad (3.26)$$

or, since

$$\frac{-C_3}{\eta} (1 - \exp(\eta t)) \geq 0.$$

Thus, we have the inequality (3.21). \square

Proposition 3.6. *Let $\|u\| \geq \beta \|f\|$ and $u_0 \neq 0$, and suppose that $\|u_0\| \leq C_3\nu\lambda_1^{\frac{1}{2}}$. Then we have that*

$$l(\omega(t)) \leq C_1 L(1 + \frac{1}{2} \text{Log} \frac{\lambda_m}{\lambda_1} + (\varepsilon\lambda_m^{\frac{1}{2}} + \frac{1+\beta^2}{2\beta^2})t) e^{\frac{C_2 L}{\alpha t}} \text{ for } t \geq 0, \quad (3.27)$$

for any $\alpha \leq \frac{\nu\lambda_1^{\frac{1}{2}}}{4}$ and $\beta \leq \frac{4\sqrt{2}}{\nu}$.

Proof. By Theorem 3.4 and Lemma 3.5, we get for $t \geq 0$ the following

$$\begin{aligned} \|A^{\frac{1}{2}}e^{\alpha t A^{\frac{1}{2}}}u\| &\leq 2 \|A^{\frac{1}{2}}u_0\| \\ &\leq 2\lambda^{\frac{1}{2}} \|u_0\| \end{aligned} \quad (3.28)$$

and use the inequality (3.21) to get

$$\begin{aligned} \|A^{\frac{1}{2}}e^{\alpha t A^{\frac{1}{2}}}u\| &\leq 2\lambda^{\frac{1}{2}} \|u(t)\| \exp(\varepsilon\lambda_m^{\frac{1}{2}} + \frac{1+\beta^2}{2\beta^2})t \\ &\leq 2(\frac{\lambda}{\lambda_1})^{\frac{1}{2}} \|A^{\frac{1}{2}}u(t)\| \exp(\varepsilon\lambda_m^{\frac{1}{2}} + \frac{1+\beta^2}{2\beta^2})t. \end{aligned} \quad (3.29)$$

The rest follows by combining (3.29) with Lemma 3.3 and Theorem 3.2. \square

The foundational result for our two-dimensional Hausdorff measure estimates of $N_\omega^0 = \{x \in \Omega \subset \mathbb{R}^3 / \omega(x, t) = 0\}$ the level sets of the vorticity ω of solutions is

Theorem 3.7. *Let $\|u\| \geq \beta \|f\|$ and $u_0 \neq 0$, and suppose that $\|u_0\| \leq C_3 \nu \lambda_1^{\frac{1}{2}}$. Then*

$$l(\omega(t)) \leq C_7 \lambda_m^{\frac{1}{2}} \text{ for } t \geq t_0, \quad (3.30)$$

with $t_0 = \frac{2C_2 L}{\nu \lambda_1^{\frac{1}{2}}}$ for any $\alpha \leq \frac{\nu \lambda_1^{\frac{1}{2}}}{4}$ and $\beta \leq \frac{4\sqrt{2}}{\nu}$.

Proof. With $t \geq \frac{2C_2 L}{\nu \lambda_1^{\frac{1}{2}}}$ the inequality (3.27) implies

$$l(\omega(t)) \leq C_1 e(2 + \frac{1}{2} \text{Log} \frac{\lambda_m}{\lambda_1} + (\varepsilon \lambda_m^{\frac{1}{2}} + \frac{1 + \beta^2}{2\beta^2}) \frac{C_2 L}{\alpha}) \text{ for } t \geq 0. \quad (3.31)$$

Since $\lambda_m \geq \lambda_1$ (3.31) follows directly from the above inequality. \square

The estimate of the Hausdorff measure \mathcal{H}_t^2 grows in m due to the term $\frac{\lambda_m}{\lambda_1}$ but at a rate lower than $\frac{1}{3}$.

Proposition 3.8. *Let $\|u\| \geq \beta \|f\|$ and $u_0 \neq 0$, and suppose that $\|u_0\| \leq C_3 \nu \lambda_1^{\frac{1}{2}}$. Then*

$$\sup_{t \rightarrow \infty} l(\omega(t)) \leq C_8 m^{\frac{1}{3}} \text{ for } t \geq t_0. \quad (3.32)$$

for any $\alpha \leq \frac{\nu \lambda_1^{\frac{1}{2}}}{4}$ and $\beta \leq \frac{4\sqrt{2}}{\nu}$.

Proof. Note that in the 3D case we have $\lambda_j \geq C_9 L^{-2} j^{\frac{2}{3}}$ for some positive universal constant (see, for example [29, Lemma VI 2.1]). Therefore, Since $\lambda_m \sim \lambda_1 m^{\frac{2}{3}}$ the growth in m of the Hausdorff measure (3.32) is less than $m^{\frac{1}{3}}$. \square

If we impose the condition $\lambda_m \leq (\frac{l_0}{l_\epsilon})^2$ or $\frac{\lambda_m}{\lambda_1} \leq (\frac{l_0}{l_\epsilon})^2$ where $l_0 = \lambda_1^{-\frac{1}{2}}$ represents characteristic macroscopic length, and l_ϵ is the Kolmogorov length scale, i.e. $l_\epsilon = \frac{\nu^{\frac{3}{4}}}{\epsilon^{\frac{1}{4}}}$ where ϵ is Kolmogorov's mean rate of dissipation of energy in turbulent flow (see e.g. [1, 9, 13, 29], and the references contained therein) is defined as

$$\epsilon = \lambda_1^{\frac{3}{2}} \nu \lim_{T \rightarrow \infty} \sup \int_0^T \|A^{\frac{1}{2}}\|_2^2 ds.$$

Substituting this in (3.30) gives

$$\sup_{t \rightarrow \infty} l(\omega(t)) \leq C_{10} (\frac{l_0}{l_\epsilon})^{\frac{1}{3}}. \quad (3.33)$$

Since the (dimensionless) Grashoff number $G = \frac{\sup_{t>0} \|f\|_2^2}{\nu^3 \lambda_1^{\frac{3}{2}}}$ in 3D (see e.g. [1, 9, 29])

is an upper bound for $(\frac{l_0}{l_\epsilon})^2$. Hence, we obtain for the Hausdorff measure of the equation (1.1) the following estimate in terms of the Grashoff number G .

Proposition 3.9. *Let $\|u\| \geq \beta \|f\|$ and $u_0 \neq 0$, and suppose that $\|u_0\| \leq C_3 \nu \lambda_1^{\frac{1}{2}}$.*

Then for any $\alpha \leq \frac{\nu \lambda_1^{\frac{1}{2}}}{4}$ and $\beta \leq \frac{\nu \lambda_1}{2\sqrt{2}}$ we have

$$\sup_{t \rightarrow \infty} l(\omega(t)) \leq C_{11} G^{\frac{1}{6}} \text{ for } t \geq t_0. \quad (3.34)$$

This result holds independently of m , with C_{11} independent of m . The estimate grows in l at a rate lower than $\frac{l}{6}$. If we impose the condition $l = 1$, the estimates become $\sup_{t \rightarrow \infty} l(\omega(t)) \leq C_{11} G^{\frac{1}{6}}$. This result recovers the usual 3D Navier–Stokes equations estimates, for the two-dimensional Hausdorff measure $\mathcal{H}_1^2(N_\omega^0)$ estimates of the level sets associated with the vorticity. Here again our results indicate that under certain conditions the upper bounds for $\mathcal{H}_1^2(N_\omega^0)$ converge to the associated upper bounds of the two-dimensional Hausdorff measure $\mathcal{H}_1^2(N_\omega^0)$ estimates for the usual 3D Navier–Stokes equations with potential force.

4. Conclusion

Proving global regularity for the 3D Navier–Stokes equations is one of the most challenging outstanding problems in nonlinear analysis. The main difficulty in establishing this result lies in controlling certain norms of vorticity. More specifically, the vorticity stretching term in the 3D vorticity equation forms the main obstacle to achieving this control, C. Foias [8] and estimates on the number of degrees of freedom for the Navier–Stokes equations and its closure models are a measure of the complexity of the system J. Avrin [1]. This paper proposed another interesting way to estimate this complexity through bounding the size of the nodal set for the vorticity and expressing this estimate in terms of G .

We provide upper bounds for the two-dimensional Hausdorff measure \mathcal{H}_1^2 of the level sets associated with the vorticity of modified three dimensional Navier–Stokes equations this bounds depend on m and l , this dependence is a fractional power of l . Thus answering a question raised by J. Avrin [1].

Another interesting way to study decaying turbulence in the three-dimensional incompressible Navier–Stokes equations is to provide a numerical investigation of our theoretical results on the size of the nodal set for the vorticity in the dependence of turbulence structure and vortex dynamics, as was done in [24] for the number of numerically determining modes in the 2D Navier–Stokes equations. It would be interesting to see how the turbulence structure depend on l .

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