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HAUSDORFF MEASURE OF VORTICITY NODAL SETS FOR THE 3D HYPERVISCOUS NAVIER STOKES EQUATIONS WITH GENERAL FORCES

ABDELHAFID YOUNSI

ABSTRACT. In this paper, we modified the three dimensional Navier-Stokes equations by adding a l -Laplacian. We provide upper bounds on the two-dimensional Hausdorff measure \mathcal{H}_l^2 of $N_\omega^0 = \{x \in \Omega \subset \mathbb{R}^3 / \omega(x, t) = 0\}$ the level sets of the vorticity ω of solutions. We express them in terms of the Kolmogorov length-scale and the Landau–Lifschitz estimates of the number of degrees of freedom in turbulent flow. We also, under certain hypothesis recover the two-dimensional Hausdorff measure estimates for the usual 3D Navier–Stokes equations with potential force. Moreover, we show that the estimates depend on l , this result suggests that the modified Navier Stokes system is successful model of turbulence and the size of the nodal set $\mathcal{H}_l^2(N_\omega^0)$ leads the way for developing the turbulence theory.

1. INTRODUCTION

In this paper, we provide upper bounds on the two-dimensional Hausdorff measure \mathcal{H}_l^2 of N_ω^0 the level sets associated with the vorticity of modified three dimensional Navier-Stokes equations. We modified the 3D Navier-Stokes system by adding a higher-order viscosity term to the conventional system

$$\begin{aligned} \frac{du}{dt} + \varepsilon(-\Delta)^l u - \nu \Delta u + (u \cdot \nabla) u + \nabla p &= f(x), \text{ in } \Omega \times (0, \infty) \\ \operatorname{div} u &= 0, \text{ in } \Omega \times (0, \infty), \\ p(x + Le_i, t) &= p(x, t), \quad u(x + Le_i, t) = u(x, t) \quad i = 1, \dots, d \quad t \in (0, \infty) \\ u(x, 0) &= u_0(x), \text{ in } \Omega, \end{aligned} \tag{1.1}$$

on $\Omega = (0, L)^d$ with periodic boundary conditions and (e_1, \dots, e_d) is the natural basis of \mathbb{R}^d . Here $\varepsilon > 0$ is the artificial dissipation parameter and $\nu > 0$ is the kinematic viscosity of the fluid, $l > 1$. The functions u is the velocity vector field, p is the pressure, and f is a given force field. For $\varepsilon = 0$, the model is reduced to the Navier-Stokes system.

Mathematical model for such fluid motion has been used extensively in turbulence simulations (see e.g. [9]) also see Borue and Orsag [3, 4]. For further discussion of theoretical results concerning (1.1), see [2, 19, 22].

For the 3D Navier–Stokes system weak solutions of problem are known to exist by a basic result by J. Leray from 1934 [20], only the uniqueness of weak solutions remains as an open problem. Then the known theory of global attractors of infinite dimensional dynamical systems is not applicable to the 3D Navier–Stokes system.

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In particular, in case one accepts the point of view that the dimension of a global attractor for the Navier–Stokes equations is associated with the number of degrees of freedom in turbulent flows, then the two-dimensional Hausdorff measure $\mathcal{H}_l^2(N_\omega^0)$ is an important way to the understanding of turbulence theory.

We are interested in the three dimensional case. Let P_m be the projection onto the first m eigenspaces of the Stokes operator $A = -\Delta$ and let $N_\omega^0 = \{x \in \Omega \subset \mathbb{R}^3 / \omega(x, t) = 0\}$ the nodal sets of the vorticity ω for solutions of the equation (1.1). We provide an upper bound on the size of the nodal sets $\mathcal{H}_l^2(N_\omega^0)$ and we show that, the bounds necessarily depend on m and l this dependence is a fractional power of l . Thus answering a question raised by J. Avrin [1]. We also obtain here scale-invariant estimates on the two-dimensional Hausdorff measure $\mathcal{H}_l^2(N_\omega^0)$ in terms of the Landau–Lifschitz theory of the number of degrees of freedom in turbulent flow. Since expressing the above estimates in terms of the (dimensionless) Grashoff number G . In order to obtain an upper bound on the Hausdorff measure of level sets associated, we use the method from [17] (see also [6], [7]).

The main purpose of the present article is to study the dependence of the two-dimensional Hausdorff measure $\mathcal{H}_l^2(N_\omega^0)$ on the parameter l . Using a family of Kolmogorov flows as base flows we can deduce also upper bounds on the Hausdorff measure $\mathcal{H}_l^2(N_\omega^0)$. We also find here that the upper bounds on the two-dimensional Hausdorff measure of N_ω^0 converges to the corresponding upper bounds on $\mathcal{H}_1^2(N_\omega^0)$ the two-dimensional Hausdorff measure of the nodal sets of the usual 3D Navier–Stokes as $l = 1$. Under certain hypothesis we recover the two-dimensional Hausdorff measure $\mathcal{H}_1^2(N_\omega^0)$ estimates for the usual 3D Navier–Stokes equations with potential force. We extend the method from [15] to a 3D Navier–Stokes with general forcing modified by l -Laplacian. These estimates are obtained without using the Dirichlet quotients [15].

We note, however, that for the incompressible 3D Navier–Stokes equations with general force, it seems not so easy to get some better estimates on the Hausdorff measure of the level sets associated with the vorticity as in the case of potential force studied in [15, 16] for periodic solutions of the 2D. Related results for the 3D Navier–Stokes equations (with general forcing) can be found in [5]. The upper bounds on the Hausdorff measures of the level sets associated with solutions of some other partial differential equations were obtained in [7],[11], [12], [14], [17], [18], and [21].

The paper is organized as follows. In Section 2, we present the relevant mathematical framework for the paper. In Section 3, we provide upper bounds for the two-dimensional Hausdorff measure \mathcal{H}_l^2 of the level sets associated with the vorticity of the Navier–Stokes system with hyperdissipation.

2. NOTATIONS AND PRELIMINARIES

In this section we introduce notations and the definitions of standard functional spaces that will be used throughout the paper. We denote by $H^m(\Omega)$, the Sobolev space of L_{per} periodic functions. These spaces are endowed with the inner product

$$(u, v) = \sum_{|\beta| \leq m} (D^\beta u, D^\beta v)_{L^2(\Omega)} \text{ and the norm } \|u\|_m = \left(\sum_{|\beta| \leq m} (\|D^\beta u\|_{L^2(\Omega)}^2) \right)^{\frac{1}{2}}.$$

Each $u \in L_{per}$ can be identified with its Fourier expansion

$$u(x) = \sum_{k \in \mathbb{Z}^3} u_k \exp(2i\pi k \cdot \frac{x}{L})$$

where $u_k \in \mathbb{C}^3$ satisfy $\bar{u}_k = u_{-k}$. Then u is in L^2 if and only if

$$\|u\|_{L^2}^2 = |\Omega| \sum_{k \in \mathbb{Z}^3} |u_k|^2 < \infty, \quad |\Omega| = L^3,$$

then the Sobolev space $u \in H^m(\Omega)$, $m \in \mathbb{R}^+$ can be characterized by

$$H^m(\Omega) = \{u, \bar{u}_k = u_{-k}, \sum_{k \in \mathbb{Z}^3} k^{2m} |u_k|^2 < \infty.\}$$

$H^{-m}(\Omega)$ denote the dual space of $H^m(\Omega)$.

We denote by $\dot{H}^m(\Omega)$ the subspace of $H^m(\Omega)$ with, zero average

$$\dot{H}^m(\Omega) = \{u \in H^m(\Omega); \int_{\Omega} u(x) dx = 0\}.$$

For $m = 0$, we have $\dot{H}^m(\Omega) = \dot{L}^2(\Omega)$.

- We introduce the following solenoidal subspaces V_s , $s \in \mathbb{R}^+$ which are important to our analysis

$$V_0(\Omega) = \{u \in \dot{L}^2(\Omega), \operatorname{div} u = 0, u \cdot n|_{\Sigma_i} = -u \cdot n|_{\Sigma_{i+3}}, i = 1, 2, 3\};$$

$$V_1(\Omega) = \{u \in \dot{H}^1(\Omega), \operatorname{div} u = 0, \gamma_0 u|_{\Sigma_i} = \gamma_0 u|_{\Sigma_{i+3}}, i = 1, 2, 3\}.$$

$$V_2(\Omega) = \{u \in \dot{H}^2(\Omega), \operatorname{div} u = 0, \gamma_0 u|_{\Sigma_i} = \gamma_0 u|_{\Sigma_{i+3}}, \gamma_1 u|_{\Sigma_i} = -\gamma_1 u|_{\Sigma_{i+3}}, i = 1, 2, 3\},$$

see [24, Chapter III, Section 2]. We refer the reader to R.Temam [25] for details on these spaces. Here the faces of Ω are numbered as

$$\Sigma_i = \partial\Omega \cap \{x_i = 0\} \text{ and } \Sigma_{i+3} = \partial\Omega \cap \{x_i = L\}, \quad i = 1, 2, 3.$$

Here γ_0, γ_1 are the trace operators and n is the unit outward normal on $\partial\Omega$.

- The space V_0 is endowed with the inner product $(u, v)_{L^2(\Omega)}$ and norm $\|u\| = (u, u)_{L^2(\Omega)}^{1/2}$.
- V_1 is the Hilbert space with the norm $\|u\|_1 = \|u\|_{V_1}$. The norm induced by $\dot{H}^1(\Omega)$ and the norm $\|\nabla u\|$ are equivalent in V_1 .
- V_2 is the Hilbert space with the norm $\|u\|_2 = \|u\|_{V_2}$. In V_2 the norm induced by $\dot{H}^2(\Omega)$ is equivalent to the norm $\|\Delta u\|$.

V'_s denote the dual space of V_s .

Let P be the orthogonal projection in $L^2_{per}(\mathbb{R}^3)^3$ with the range H .

Let $A = -P\Delta$ the Stokes operator. It is easy to check that $Au = -\Delta u$ for every $u \in D(A)$. We recall that the operator A is a closed positive self-adjoint unbounded operator, with $D(A) = \{u \in V_0, Au \in V_0\}$. We have in fact,

$$D(A) = \dot{H}^2(\Omega) \cap V_0 = V_2.$$

The eigenvalues of A are $\{\lambda_j\}_{j=1}^{j=\infty}$, $0 < \lambda_1 \leq \lambda_2 \leq \dots$ and the corresponding orthonormal set of eigenfunctions $\{w_j\}_{j=1}^{j=\infty}$ is complete in V_0

$$Aw_j = \lambda_j w_j, \quad w_j \in D(A), \forall j.$$

The spectral theory of A allows us to define the powers A^l of A for $l \geq 1$, A^l is an unbounded self-adjoint operator in V_0 with a domain $D(A^l)$ dense in $V_2 \subset V_0$. We set here

$$A^l u = (-\Delta)^l u \text{ for } u \in D(A^l) = V_{2l} \cap V_0.$$

The space $D(A^l)$ is endowed with the scalar product and the norm

$$(u, v)_{D(A^l)} = (A^l u, A^l v), \quad \|u\|_{D(A^l)} = \{(u, v)_{D(A^l)}\}^{\frac{1}{2}}. \quad (2.1)$$

In the case for $l > 0$, we have $D(A^l) = \{u \in H, \sum_{j=1}^{\infty} \lambda_j^{2l} (u, w_j)^2 < \infty\}$. For $l \in \mathbb{R}$ the scalar product and the norm in (2.1) can be written alternatively as

$$(u, v)_{D(A^l)} = \sum_{j=1}^{\infty} \lambda_j^{2l} (u, w_j)(v, w_j), \quad \|u\|_{D(A^l)} = \left\{ \sum_{j=1}^{\infty} \lambda_j^{2l} (u, w_j)^2 \right\}^{\frac{1}{2}} \quad (2.2)$$

and for $u \in D(A^l)$ we can write

$$A^l u = \sum_{j=1}^{\infty} \lambda_j^l (u, w_j) w_j.$$

Let us now define the trilinear form $b(., ., .)$ associated with the inertia terms

$$b(u, v, w) = \sum_{i,j=1}^3 \int_{\Omega} u_i \frac{\partial v_j}{\partial x_i} w_j dx.$$

The continuity property of the trilinear form enables us to define (using Riesz representation Theorem) a bilinear continuous operator $B(u, v); V_2 \times V_2 \rightarrow V_2'$ will be defined by

$$\langle B(u, v), w \rangle = b(u, v, w), \quad \forall w \in V_2. \quad (2.3)$$

Recall that for u satisfying $\nabla \cdot u = 0$ we have

$$b(u, u, u) = 0 \text{ and } b(u, v, w) = -b(u, w, v). \quad (2.4)$$

We recall some well known inequalities that we will be using in what follows.

Young's inequality

$$ab \leq \frac{\sigma}{p} a^p + \frac{1}{q\sigma^{\frac{q}{p}}} b^q, \quad a, b, \sigma > 0, \quad p > 1, \quad q = \frac{p}{p-1}. \quad (2.5)$$

Poincaré's inequality

$$\lambda_1 \|u\|^2 \leq \|A^{\frac{1}{2}} u\|^2 \text{ for all } u \in V_0. \quad (2.6)$$

3. LEVEL SETS OF THE VORTICITY FUNCTION

Using the operators defined above, we can write the modified system (1.1) in the evolution form

$$\begin{aligned} \frac{du}{dt} + \varepsilon A^l u + \nu Au + B(u, u) &= f(x), \text{ in } \Omega \times (0, \infty) \\ u_0(x) &= u_0, \quad \text{in } \Omega. \end{aligned} \quad (3.1)$$

The existence and uniqueness results for initial value problem (1.1) can be found in [22]. The following theorem collects the main result in this work

Theorem 3.1. For $l \geq \frac{d+2}{4}$, d is the space dimension, for $\varepsilon > 0$ fixed, $f \in L^2(0, T; V_0')$ and $u_0 \in V_0$ be given. There exists a unique weak solution of (1.1) which satisfies $u \in L^2(0, T; V_l) \cap L^\infty(0, T; V_0)$, $\forall T > 0$.

The vorticity, $\omega = \nabla \times u$ satisfies the equation

$$\left(\frac{d}{dt} + u \cdot \nabla + \nu \Delta + \varepsilon (-\Delta)^l\right) \omega = \omega \cdot \nabla u + F \quad (3.2)$$

where $F = \nabla \times f$.

Denoting

$$\|u\|_{G(t)}^2 = \|e^{tA^{\frac{1}{2}}} u\|^2 \quad \text{and} \quad (u, v)_{G(t)} = (e^{tA^{\frac{1}{2}}} u, e^{tA^{\frac{1}{2}}} v).$$

The set $D(e^{\alpha A})$ is called the Gevrey class of operator of order $\alpha \geq 0$ [10]. Our use of Gevrey classes shall be based on the following consideration.

Denote with $N_h^0 = \{x \in \Omega : h(x) = 0\}$ the zero (nodal) set of a function h in a set Ω , and let \mathcal{H}^2 be the two-dimensional Hausdorff measure operating on subsets of \mathbb{R}^3 (area in this case).

Theorem 3.2. [17] Suppose that a nonzero function $h \in V_1$ satisfies

$$\|e^{\alpha A} h\|_1 \leq M \|h\|_1$$

Then

$$\mathcal{H}^2(N_h^0) \leq C_1 L^2 (1 + \log M) e^{C_2 L/\alpha}.$$

Hereafter, C_i for $i \in \mathbb{N}$, stand for universal constants. The above statement is a special case of [17, Theorem 2.1]. It will be used in conjunction with the following statement:

Lemma 3.3. [15] Let $u \in V_0$, and let ω and c be its vorticity. If

$$\|A^{\frac{1}{2}} e^{\alpha A^{\frac{1}{2}}} u\| \leq M \|A^{\frac{1}{2}} u\| \quad (3.3)$$

for some $M > 0$, then, for every $c \in \mathbb{R}$

$$\|e^{\alpha A^{\frac{1}{2}}} (\omega - c)\| \leq M \|\omega - c\|. \quad (3.4)$$

For the rest of the paper, let $u(t)$ be an arbitrary solution of the the modified Navier Stokes system (1.1) with $u(0) = u_0$.

Theorem 3.4. Let $\|u\| \geq \beta \|f\|$ for any $\alpha \leq \frac{\nu \lambda_1^{\frac{1}{2}}}{4}$ and $\beta \leq \frac{\nu \lambda_1}{2\sqrt{2}}$. Then There exists a universal constant C_3 such that if $\|A^{\frac{1}{2}} u\| \leq C_3$, then

$$\|A^{\frac{1}{2}} e^{\alpha t A^{\frac{1}{2}}} u\| \leq 2 \|A^{\frac{1}{2}} u_0\|, \quad t \geq 0. \quad (3.5)$$

Proof. For any $\alpha, t \geq 0$, We take the inner product of (3.1) with u , to obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|A^{\frac{1}{2}} u\|_{G(t)}^2 &= \alpha \|A^{\frac{3}{4}} u\|_{G(t)}^2 + (A\dot{u}, u)_{G(t)} \\ &= \alpha \|A^{\frac{3}{4}} u\|_{G(t)}^2 - \varepsilon \|A^{\frac{l+1}{2}} u\|_{G(t)}^2 - \nu \|Au\|_{G(t)}^2 - b(u, u, Au)_{G(t)} + (f, Au)_{G(t)}. \end{aligned} \quad (3.6)$$

then using the Young's inequality (2.5) we have

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|A^{\frac{1}{2}} u\|_{G(t)}^2 &\leq -\varepsilon \|A^{\frac{l+1}{2}} u\|_{G(t)}^2 + \frac{\nu}{4} \|Au\|_{G(t)}^2 + \frac{\alpha^2}{\nu} \|A^{\frac{1}{2}} u\|_{G(t)}^2 - \nu \|Au\|_{G(t)}^2 \\
&\quad + \left(\frac{\nu}{2} \|Au\|_{G(t)}^2 + \frac{1}{2\nu} \|f\|_{G(t)}^2\right) + b(u, u, Au)_{G(t)} \\
&\leq -\varepsilon \|A^{\frac{l+1}{2}} u\|_{G(t)}^2 + \frac{\nu}{4} \|Au\|_{G(t)}^2 + \frac{\alpha^2}{\nu} \|A^{\frac{1}{2}} u\|_{G(t)}^2 - \nu \|Au\|_{G(t)}^2 \\
&\quad + \left(\frac{\nu}{2} \|Au\|_{G(t)}^2 + \frac{\beta^2}{2\lambda_1\nu} \|A^{\frac{1}{2}} u\|_{G(t)}^2\right) + b(u, u, Au)_{G(t)} \\
&\leq -\varepsilon \|A^{\frac{l+1}{2}} u\|_{G(t)}^2 + \frac{-\lambda_1\nu}{4} \|A^{\frac{1}{2}} u\|_{G(t)}^2 + \frac{\alpha^2}{\nu} \|A^{\frac{1}{2}} u\|_{G(t)}^2 \\
&\quad + \frac{\beta^2}{2\lambda_1\nu} \|A^{\frac{1}{2}} u\|_{G(t)}^2 + b(u, u, Au)_{G(t)}.
\end{aligned}$$

We get for $\beta^2 = 2\lambda_1\alpha^2$

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|A^{\frac{1}{2}} u\|_{G(t)}^2 &\leq -\varepsilon \|A^{\frac{l+1}{2}} u\|_{G(t)}^2 - \frac{\lambda_1\nu}{4} \|A^{\frac{1}{2}} u\|_{G(t)}^2 \\
&\quad + \frac{2\alpha^2}{\nu} \|A^{\frac{1}{2}} u\|_{G(t)}^2 + b(u, u, Au)_{G(t)}.
\end{aligned}$$

We use the following inequality from [10] and [15, Section 4]

$$b(u, u, Au)_{G(t)} \leq C_4 \|A^{\frac{1}{2}} u\|_{G(t)}^2 \|Au\|_{G(t)} (1 + \log \frac{\|Au\|_{G(t)}^2}{\lambda_1 \|A^{\frac{1}{2}} u\|_{G(t)}^2})^{\frac{1}{2}} \quad (3.7)$$

to obtain

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|A^{\frac{1}{2}} u\|_{G(t)}^2 &\leq -\varepsilon \|A^{\frac{l+1}{2}} u\|_{G(t)}^2 - \frac{\lambda_1\nu}{4} \|A^{\frac{1}{2}} u\|_{G(t)}^2 + \frac{2\alpha^2}{\nu} \|A^{\frac{1}{2}} u\|_{G(t)}^2 \\
&\quad + C_4 \|A^{\frac{1}{2}} u\|_{G(t)}^2 \|Au\|_{G(t)} (1 + \log \frac{\|Au\|_{G(t)}^2}{\lambda_1 \|A^{\frac{1}{2}} u\|_{G(t)}^2})^{\frac{1}{2}}.
\end{aligned}$$

To establish (3.5) we use the estimate [15]

$$a \mu (1 + \log \frac{\mu^2}{b^2})^{\frac{1}{2}} \leq d\mu^2 + \frac{a^2}{d^2} \log \frac{2a}{bd} \quad a, d > 0, \mu \geq b > 0. \quad (3.8)$$

By applying the Poincaré's inequality (2.6), we have that for $\mu = \|Au\|_{G(t)}$ and $d = \frac{\nu}{8}$

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|A^{\frac{1}{2}} u\|_{G(t)}^2 + \varepsilon \|A^{\frac{l+1}{2}} u\|_{G(t)}^2 &\leq -\frac{\lambda_1\nu}{8} \|A^{\frac{1}{2}} u\|_{G(t)}^2 + \frac{2\alpha^2}{\nu} \|A^{\frac{1}{2}} u\|_{G(t)}^2 \\
&\quad + C_5 \|A^{\frac{1}{2}} u\|_{G(t)}^4 \log \frac{C_6 \|A^{\frac{1}{2}} u\|_{G(t)}}{\lambda_1^{\frac{1}{2}}}.
\end{aligned}$$

Letting $\alpha \leq \frac{\nu\lambda_1^{\frac{1}{2}}}{4}$ we have for $\beta \leq \frac{\nu\lambda_1}{2\sqrt{2}}$ that

$$\frac{1}{2} \frac{d}{dt} \|A^{\frac{1}{2}} u\|_{G(t)}^2 \leq C_5 \|A^{\frac{1}{2}} u\|_{G(t)}^4 \log \frac{C_6 \|A^{\frac{1}{2}} u\|_{G(t)}}{\lambda_1^{\frac{1}{2}}}. \quad (3.9)$$

If $\|A^{\frac{1}{2}} u_0\| < \frac{\lambda_1^{\frac{1}{2}}}{C_6} = C_3$, the term with a logarithm in (3.9) is negative at $t = 0$, and thus (3.9) implies that $\|A^{\frac{1}{2}} u\|_{G(t)}$ is a decreasing function of t . \square

Theorem 3.4. implies that, for any solution $u(t)$ of (1.1), the space analyticity radius of $u(t)$ goes to infinity as $t \rightarrow \infty$.

Let Ω be a periodic box, for simplicity assume $\Omega = (0, L)^3$, A has eigenvalues $0 < \lambda_1 < \lambda_2 < \dots$ with corresponding eigenspaces E_1, E_2, \dots . Let P_m be the projection on the eigenspaces $E_1 \oplus E_2 \oplus \dots \oplus E_m$ and let $Q_m = I - P_m$ we have $\|u\|^2 = \|P_m u\|^2 + \|Q_m u\|^2$ and we also have from (2.2) that

$$\|A^l u\| \leq \lambda_m^l \|u\| \text{ for every } l \geq 0 \text{ and } u \in D(A^l). \quad (3.10)$$

For any $t \geq 0$, let $\omega(t)$ be the vorticity of $u(t)$. We shall, for any fixed $t > 0$, estimate the quantity

$$l(\omega(t)) = \sup_{c \in \mathbb{R}} \mathcal{H}_t^2(N_\omega^c). \quad (3.11)$$

Recall that for a function $h : \Omega \rightarrow \mathbb{R}$, $N_h^0 = \{x \in \Omega : h(x) = 0\}$. We need the following fact

Lemma 3.5. *Let $\|u\| \geq \beta \|f\|$ for any $\beta \geq 0$. Then*

$$\|u(t)\| \geq \|u(0)\| \exp(\eta t) \text{ for every } t \geq 0. \quad (3.12)$$

With $\eta = -(\varepsilon \lambda_m^{\frac{1}{2}} + \frac{1+\beta^2}{2\beta^2})$.

Proof. Taking the scalar product of both sides of (1.1) by $u(t)$ and using (2.4), we have that

$$\frac{1}{2} \frac{d}{dt} \|u\|^2 + \nu \|A^{\frac{1}{2}} u\|^2 + \varepsilon \|A^{\frac{1}{2}} u\|^2 = (f, u) \text{ for } t \geq 0. \quad (3.13)$$

Using (3.10) and the following inequality

$$(f, u) \geq -\frac{1}{2} \|f\|^2 - \frac{1}{2} \|u\|^2 \quad (3.14)$$

Because the increasing sequence $0 \leq \lambda_1 \leq \lambda \leq \lambda_m$ we have

$$\frac{1}{2} \frac{d}{dt} \|u\|^2 \geq (-\nu \lambda \|A^{\frac{1}{2}} u\|^2 - (\varepsilon \lambda_m^{\frac{1}{2}} + \frac{1+\beta^2}{2\beta^2}) \|u\|^2) \quad (3.15)$$

note that since

$$\|A^{\frac{1}{2}} u\| \leq C_3$$

we have that

$$\frac{1}{2} \frac{d}{dt} \|u\|^2 \geq -C_3 - (\varepsilon \lambda_m^{\frac{1}{2}} + \frac{1+\beta^2}{2\beta^2}) \|u\| \quad (3.16)$$

if we set $\eta = -(\varepsilon \lambda_m^{\frac{1}{2}} + \frac{1+\beta^2}{2\beta^2})$ then we have from (3.16) that

$$\frac{1}{2} \frac{d}{dt} \|u\|^2 \geq -C_3 + \eta \|u\|.$$

Integrating the above inequality from 0 to t , we get

$$\|u\|^2 \geq \frac{-C_3}{\eta} (1 - \exp(\eta t)) + \|u(0)\|^2 \exp(\eta t) \quad (3.17)$$

or, since

$$\frac{-C_3}{\eta} (1 - \exp(\eta t)) \geq 0.$$

Thus, we have the inequality (3.12). \square

Proposition 3.6. *Let $\|u\| \geq \beta \|f\|$ and $u_0 \neq 0$, and suppose that $\|u_0\| \leq C_3 \nu \lambda_1^{\frac{1}{2}}$. Then we have that*

$$l(\omega(t)) \leq C_1 L \left(1 + \frac{1}{2} \text{Log} \frac{\lambda_m}{\lambda_1} + \left(\varepsilon \lambda_m^{\frac{1}{2}} + \frac{1 + \beta^2}{2\beta^2}\right)t\right) e^{\frac{C_2 L}{\alpha t}} \text{ for } t \geq 0, \quad (3.18)$$

for any $\alpha \leq \frac{\nu \lambda_1^{\frac{1}{2}}}{4}$ and $\beta \leq \frac{\nu \lambda_1}{2\sqrt{2}}$.

Proof. By Theorem 3.4 and Lemma 3.5, we get for $t \geq 0$ the following

$$\begin{aligned} \|A^{\frac{1}{2}} e^{\alpha t A^{\frac{1}{2}}} u\| &\leq 2 \|A^{\frac{1}{2}} u_0\| \\ &\leq 2 \lambda^{\frac{1}{2}} \|u_0\| \end{aligned} \quad (3.19)$$

and use the inequality (3.12) to get

$$\begin{aligned} \|A^{\frac{1}{2}} e^{\alpha t A^{\frac{1}{2}}} u\| &\leq 2 \lambda^{\frac{1}{2}} \|u(t)\| \exp\left(\varepsilon \lambda_m^{\frac{1}{2}} + \frac{1 + \beta^2}{2\beta^2}\right)t \\ &\leq 2 \left(\frac{\lambda}{\lambda_1}\right)^{\frac{1}{2}} \|A^{\frac{1}{2}} u(t)\| \exp\left(\varepsilon \lambda_m^{\frac{1}{2}} + \frac{1 + \beta^2}{2\beta^2}\right)t. \end{aligned} \quad (3.20)$$

The rest follows by combining (3.20) with Lemma 3.3 and Theorem 3.2. \square

The foundational result for our two-dimensional Hausdorff measure estimates of $N_\omega^0 = \{x \in \Omega \subset \mathbb{R}^3 / \omega(x, t) = 0\}$ the level sets of the vorticity ω of solutions is

Theorem 3.7. *Let $\|u\| \geq \beta \|f\|$ and $u_0 \neq 0$, and suppose that $\|u_0\| \leq C_3 \nu \lambda_1^{\frac{1}{2}}$. Then*

$$l(\omega(t)) \leq C_7 \lambda_m^{\frac{1}{2}} \text{ for } t \geq t_0, \quad (3.21)$$

with $t_0 = \frac{2C_2 L}{\nu \lambda_1^{\frac{1}{2}}}$ for any $\alpha \leq \frac{\nu \lambda_1^{\frac{1}{2}}}{4}$ and $\beta \leq \frac{\nu \lambda_1}{2\sqrt{2}}$.

Proof. With $t \geq \frac{2C_2 L}{\nu \lambda_1^{\frac{1}{2}}}$ the inequality (3.18) implies

$$l(\omega(t)) \leq C_1 e \left(2 + \frac{1}{2} \text{Log} \frac{\lambda_m}{\lambda_1} + \left(\varepsilon \lambda_m^{\frac{1}{2}} + \frac{1 + \beta^2}{2\beta^2}\right) \frac{C_2 L}{\alpha}\right) \text{ for } t \geq 0. \quad (3.22)$$

Since $\lambda_m \geq \lambda_1$ (3.22) follows directly from the above inequality. \square

The estimate of the Hausdorff measure \mathcal{H}_l^2 grows in m due to the term $\frac{\lambda_m}{\lambda_1}$ but at a rate lower than $\frac{1}{3}$.

Proposition 3.8. *Let $\|u\| \geq \beta \|f\|$ and $u_0 \neq 0$, and suppose that $\|u_0\| \leq C_3 \nu \lambda_1^{\frac{1}{2}}$. Then*

$$\sup_{t \rightarrow \infty} l(\omega(t)) \leq C_8 m^{\frac{1}{3}} \text{ for } t \geq t_0. \quad (3.23)$$

for any $\alpha \leq \frac{\nu \lambda_1^{\frac{1}{2}}}{4}$ and $\beta \leq \frac{\nu \lambda_1}{2\sqrt{2}}$.

Proof. Note that in the 3D case we have $\lambda_j \geq C_9 L^{-2} j^{\frac{2}{3}}$ for some positive universal constant (see, for example [24, Lemma VI 2.1]). Therefore, Since $\lambda_m \sim \lambda_1 m^{\frac{2}{3}}$ the growth in m of the Hausdorff measure (3.23) is less than $m^{\frac{1}{3}}$. \square

If we impose the condition $\lambda_m \leq (\frac{1}{l_\epsilon})^2$ or $\frac{\lambda_m}{\lambda_1} \leq (\frac{l_0}{l_\epsilon})^2$ where $l_0 = \lambda_1^{-\frac{1}{2}}$ represents characteristic macroscopic length, and l_ϵ is the Kolmogorov length scale, i.e. $l_\epsilon = \frac{\nu^3}{\epsilon}$ where ϵ is Kolmogorov's mean rate of dissipation of energy in turbulent flow (see e.g. [1, 9, 13, 24], and the references contained therein) is defined as

$$\epsilon = \lambda_1^{\frac{3}{2}} \nu \limsup_{T \rightarrow \infty} \int_0^T \|A^{\frac{1}{2}}\|_2^2 ds.$$

Substituting this in (3.21) gives

$$\sup_{t \rightarrow \infty} l(\omega(t)) \leq C_{10} \left(\frac{l_0}{l_\epsilon}\right)^{\frac{1}{3}}. \quad (3.24)$$

Since the (dimensionless) Grashoff number $G = \frac{\sup_{t>0} \|f\|_2^2}{\nu^3 \lambda_1^{\frac{3}{2}}}$ in 3D (see e.g. [1, 9, 24]) is an upper bound for $(\frac{l_0}{l_\epsilon})^2$. Hence, we obtain for the Hausdorff measure of the equation (1.1) the following estimate in terms of the Grashoff number G .

Proposition 3.9. *Let $\|u\| \geq \beta \|f\|$ and $u_0 \neq 0$, and suppose that $\|u_0\| \leq C_3 \nu \lambda_1^{\frac{1}{2}}$.*

Then for any $\alpha \leq \frac{\nu \lambda_1^{\frac{1}{2}}}{4}$ and $\beta \leq \frac{\nu \lambda_1}{2\sqrt{2}}$ we have

$$\sup_{t \rightarrow \infty} l(\omega(t)) \leq C_{11} G^{\frac{1}{6}} \text{ for } t \geq t_0. \quad (3.25)$$

This result holds independently of m , with C_{11} independent of m . The estimate grows in l at a rate lower than $\frac{1}{6}$. If we impose the condition $l = 1$, the estimates become $\sup_{t \rightarrow \infty} l(\omega(t)) \leq C_{11} G^{\frac{1}{6}}$. This result recover the usual 3D Navier–Stokes equations estimates, for the two-dimensional Hausdorff measure $\mathcal{H}_1^2(N_\omega^0)$ estimates of the level sets associated with the vorticity. Here again our results indicate that under certain conditions the upper bounds for $\mathcal{H}_1^2(N_\omega^0)$ converge to the associated upper bounds of the two-dimensional Hausdorff measure $\mathcal{H}_1^2(N_\omega^0)$ estimates for the usual 3D Navier–Stokes equations with potential force.

4. Conclusion

Proving global regularity for the 3D Navier–Stokes equations is one of the most challenging outstanding problems in nonlinear analysis. The main difficulty in establishing this result lies in controlling certain norms of vorticity. More specifically, the vorticity stretching term in the 3D vorticity equation forms the main obstacle to achieving this control, C. Foias [8] and estimates on the number of degrees of freedom for the Navier–Stokes equations and its closure models are a measure of the complexity of the system J. Avrin [1]. This paper proposed another interesting way to estimate this complexity through bounding the size of the nodal set for the vorticity and expressing this estimate in terms of G .

We provide upper bounds for the two-dimensional Hausdorff measure \mathcal{H}_l^2 of the level sets associated with the vorticity of modified three dimensional Navier–Stokes equations this bounds depend on m and l , this dependence is a fractional power of l . Thus answering a question raised by J. Avrin [1].

Our future investigations will be concerned with the corresponding problem for the three dimensional case without any restriction on the general force.

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