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ON STABILITY OF SETS FOR SAMPLED-DATA NONLINEAR INCLUSIONS VIA THEIR APPROXIMATE DISCRETE-TIME MODELS AND SUMMABILITY CRITERIA*

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Abstract. This paper consists of two main parts. In the first part, we provide a framework for stabilization of arbitrary (not necessarily compact) closed sets for sampled-data nonlinear differential inclusions via their approximate discrete-time models. We generalize [19, Theorem 1] in several different directions: we consider stabilization of arbitrary closed sets, plants described as sampled-data differential inclusions and arbitrary dynamic controllers in the form of difference inclusions. Our result does not require the knowledge of a Lyapunov function for the approximate model, which is a standing assumption in [21] and [19, Theorem 2]. We present checkable conditions that one can use to conclude semi-global asymptotic (SPA) stability, or global exponential stability (GES), of the sampled-data system via appropriate properties of its approximate discrete-time model.

In the second part, we present sufficient conditions for stability of parameterized difference inclusions that involve various summability criteria on trajectories of the system to conclude global asymptotic stability (GAS), or GES, and they represent discrete-time counterparts of results given in [32]. These summability criteria are not Lyapunov based and they are tailored to be used within our above mentioned framework for stabilization of sampled-data differential inclusions via their approximate discrete-time models. We believe that these tools will be a useful addition to the toolbox for controller design for sampled-data nonlinear systems via their approximate discrete-time models.

Key words. Sampled-data systems, stability, difference inclusions

AMS subject classifications.

1. Introduction.

Although most controllers are nowadays implemented digitally using sample and hold devices, sampled data nonlinear control has received much less attention than continuous time nonlinear control. The controller design problem for sampled-data systems can be carried out in three essentially different ways: (i) emulation (design continuous time controller and then discretize the controller); (ii) discrete-time design (discretize the plant and design a discrete-time controller directly on the discrete time model); (iii) sampled-data design (use the real model of the sampled-data system that includes the inter-sample behavior to design the controller). For nonlinear systems, some results on emulation can be found in [16], while we are not aware of any results on sampled data design for nonlinear systems (details on the sampled data method for linear systems can be found in [4] and references cited therein).

For nonlinear plants, the discrete-time design is frustrated by the fact that it is typically not possible to analytically find the *exact* discrete-time model of the plant and in such situation an *approximate* discrete-time model is the only alternative to use for controller design. However, it was shown already in [19] and later in [21] that

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there are situations where a controller stabilizes an approximate discrete-time plant model for all small sampling periods but at the same *destabilizes* the exact discrete-time plant model for all small sampling periods. This has lead to a range of different results that provide sufficient conditions on the approximate model, controller and the continuous-time plant model that guarantee stabilizing properties of controllers designed via approximate discrete-time plant models. Such results have been proved for stabilization [19, 21], input-to-state stabilization [26], integral input-to-state stabilization in [20] and observer design in [1]. Also, these results imply stability of the sampled-data systems under mild conditions [18].

We note that this framework is *prescriptive* and *not constructive*. In other words, the results in above cited references tell us what conditions the controller and approximate model need to satisfy for the design to be successful but *they do not tell us how to design such controllers*. Hence, one needs to develop tools that would guarantee the type of stability properties required by the framework. A range of tools has been developed to aid the controller design within this framework: construction of appropriate strict Lyapunov functions via change of supply rates techniques [15, 26, 27], stability of cascaded systems [23, 24] and Matrosov theorem [22]. These results were used, for instance, to construct controllers based on approximate models using backstepping [25], optimization based stabilization [7], model predictive control [5], nonholonomic systems [13] and port controlled Hamiltonian systems [14]. Simulation comparisons in these references invariably show that controllers designed within our framework perform better than appropriate emulated controllers, see e.g. [25].

The purpose of this paper is twofold. First, we contribute novel results on the *framework* for stabilization via approximate discrete-time models. In particular, our Theorem 3.3 is a generalization of [19, Theorem 1] in several directions: we consider semi-global practical (SPA) stability of arbitrary (not necessarily compact) sets, plants modelled as differential inclusions and arbitrary dynamic controllers modelled as difference inclusions. Motivation for considering such general stability properties, classes of plants and controllers is given in [21, 32]. Our Theorem 3.6 provides stronger conditions under which one can conclude global exponential stability (GES) for the exact discrete-time model and we are not aware of similar results even in the simpler setting of [19]. We emphasize that these results are different from the main results in [21] that assume existence of an appropriate Lyapunov function for the approximate model. Proofs in this paper are purely trajectory based and they do not need such Lyapunov functions. Second, we provide a range stability analysis *tools* that involve summability type conditions on trajectories of the system to conclude the right type of stability properties for the approximate model in absence of a Lyapunov function. We present results for global asymptotic stability (GAS) and GES of arbitrary sets for families of difference inclusions. These results are discrete-time counterparts of continuous-time results in [32], [28, Appendix B] and generalize the main results in [17] for systems described by continuous difference equations. The main technical issues lay in stating appropriate (natural) definitions and showing that they lead to the right type of stability properties required by the above design framework. Moreover, some conditions for discrete-time systems are different when compared to their continuous-time counterparts in [32]. We believe that these tools are a useful addition to the toolbox for controller design via approximate discrete-time models and in our future work we will use them to construct controllers for classes of nonlinear sampled-data systems. Finally, we note that our results can be adapted to the case when the exact discrete time model of the plant is known. However, in such cases the proofs are

quite different (more straightforward) and can be carried out under different (weaker) assumptions and are not reported here for space reasons.

The paper is organized as follows. Section 2 contains mathematical preliminaries and the description of the mathematical set-up that we use. Trajectory based results that relate stability properties of sampled-data inclusions and stability properties of their approximate discrete time models are presented in Section 3. Section 4 contains summability criteria for GAS and GES for parameterized difference inclusions. In Section 5 we illustrate how results of Section 4 can be used to check stability of some classes of systems via the method of "output injection". All proofs are presented in Section 6 and Conclusions are given in the last section.

2. Preliminaries. Sets of real and natural numbers are respectively denoted as R and N. A function $\gamma : R_{\geq 0} \to R_{\geq 0}$ is said to be of class $\mathcal{K}$ if it is continuous, $\gamma(0) = 0$ and strictly increasing. γ is said to be of class $\mathcal{K}_\infty$, denoted as $\gamma \in \mathcal{K}_\infty$, if $\gamma \in \mathcal{K}$ and it is unbounded. Class $\mathcal{K}_\infty$ functions are globally invertible. A continuous function $\beta : R_{\geq 0} \times R_{\geq 0} \to R_{\geq 0}$ is said to be of class $\mathcal{KL}$, denoted as $\beta \in \mathcal{KL}$, if for each fixed $t \geq 0$ we have that $\beta(\cdot, t) \in \mathcal{K}$ and for each fixed $s \geq 0$ we have that $\lim_{t\to\infty} \beta(s, t) = 0$. For arbitrary positive L, T we define:

$$\ell_{L,T} := \left\lfloor \frac{L}{T} \right\rfloor \ ,$$

where for arbitrary $x \in R$ we have that $\lfloor x \rfloor := \max\{z \in N : z \leq x\}$. Given a closed (not necessarily compact) set $\mathcal{A} \subset R^n$, we denote the distance of a point $x \in R^n$ to the set as:

$$|x|_{\mathcal{A}} := \inf_{z\in\mathcal{A}} |z - x| \ .$$

We often use the well known fact that $|\cdot|_{\mathcal{A}}$ is globally Lipschitz with the Lipschitz constant equal to one, that is for all $x, y \in R^n$ we have:

$$||x|_{\mathcal{A}} - |y|_{\mathcal{A}}| \leq |x - y| \ .$$

We consider nonlinear control systems of the form

$$\dot{x}_p \in F(x_p, u) \ , \qquad y \in H(x_p) \tag{2.1}$$

where $x_p \in R^{n_p}$, $y \in R^p$ and $u \in R^m$. It is assumed that $u(t) = const., \forall t \in [kT, (k + 1)T)$ where $T > 0$ is the sampling period and $k \in N$. The set-valued map $F(\cdot, u)$ is assumed to have enough regularity to guarantee existence of solutions:

Assumption 1. *For each $u \in R^m$, the set-valued map $F(\cdot, u)$ satisfies the following basic conditions: 1) it is upper semi-continuous, i.e., for each $x_p \in R^{n_p}$ and each $\varepsilon > 0$ there exists $\delta > 0$ such that, for all $\xi \in R^{n_p}$ satisfying $|\xi - x_p| \leq \delta$ we have $F(\xi, u) \subseteq F(x_p, u) + \varepsilon\overline{\mathcal{B}}_{n_p}$, where $\overline{\mathcal{B}}_{n_p}$ denotes the closed unit ball in R^{n_p}, 2) for each $x_p \in R^{n_p}$ the set $F(x_p, u)$ is nonempty, compact and convex.* We will use $\mathcal{S}(x_p, u)$ to denote the set of solutions to (2.1) starting at x_p with constant input u. For a given $t > 0$ and $(x_p, u) \in R^{n_p} \times R^m$ we use the following notation $F^e_t(x_p, u) := \{\xi \in R^n : \xi = \phi(t, x_p, u),\ \phi \in \mathcal{S}(x_p, u)\}$.

The exact discrete-time model of the sampled-data system is given by:

$$x_p^+ \in F^e_T(x_p, u) \ , \qquad y \in H(x_p) \tag{2.2}$$

where $F_T^e(x_p, u)$ is the set of values the solutions to (2.1) can take at time T when starting at x_p and with the constant input u applied. The parameter $T > 0$ represents the sampling period. We will assume that for each fixed u and each initial condition x_p there exists at least one solution to (2.1) for all $t \in [0, T]$, where T is the sampling period, i.e. for each x_p and u we have that $F_T^e(x_p, u)$ is non-empty. We will consider the case where the sampling period T can be adjusted to arbitrarily small positive values. Hence, (2.2) represents a family of systems. We note that since F in (2.1) is in general nonlinear, it is not possible to analytically determine F_T^e in (2.2). Instead, we assume that the family of approximate discrete-time models

$$x_p^+ \in F_T^a(x_p, u) \ , \qquad y \in H(x_p) \tag{2.3}$$

which approximates the exact discrete-time model (2.2), is used in the control design. In particular, we assume that a family of, possibly discontinuous, discrete-time controllers

$$x_c^+ \in G_T(x_c, H(x_p)); \qquad u \ \in \ U_T(x_c, H(x_p)) \ , \tag{2.4}$$

where $x_c \in R^{n_c}$, has been designed to (approximately) asymptotically stabilize a nonempty closed set $\mathcal{A} \subset R^n$, where $n := n_c + n_p$, for the family (2.3). Our object of study is the stability of the system (2.3), (2.4) or (2.2), (2.4) with respect to a nonempty closed set $\mathcal{A} \subset R^n$. To shorten notation, we introduce $x = (x_p^T \ x_c^T)^T$, $\mathcal{H}_{\mathcal{A}}(\delta, \Delta) := \{x \in R^n : \delta \leq |x|_{\mathcal{A}} \leq \Delta\}$ and

$$\mathcal{F}_T^a(x) := \begin{pmatrix} F_T^a(x_p, U_T(x_c, H(x_p))) \\ G_T(x_c, H(x_p)) \end{pmatrix} \ , \quad \mathcal{F}_T^e(x) := \begin{pmatrix} F_T^e(x_p, U_T(x_c, H(x_p))) \\ G_T(x_c, H(x_p)) \end{pmatrix} \ .$$

Then, we write

$$x^+ \in \mathcal{F}_T^\star(x) \tag{2.5}$$

and denote as $\mathcal{S}_T^\star(x_\circ)$ the set of all solutions $\phi_T^\star(k, x_\circ)$ initialized at $x_\circ$. The symbol $\star = e$ is used for the exact closed loop (2.2), (2.4) and $\star = a$ for the approximate closed loop (2.3), (2.4).

REMARK 1. *In general, it is possible to consider more complex classes of approximate discrete-time models of the form $x^+ \in F_{T,h}^a(x, u)$, where T is the sampling period and h is a modelling parameter that can be used to reduce the mismatch between the approximate and exact models (usually, it is an integration period of the numerical integration scheme). The case when $T \neq h$ is useful in situations when the structure of the underlying approximate model is not exploited in controller design, such as in model predictive control. When $T = h$, then we write $x^+ \in F_{T,T}^a(x, u) =: F_T^a(x, u)$ and such situations typically lead to approximate models with simpler structure that are amenable to constructive nonlinear control techniques. Our stability results in the second part of the paper are tailored to such situations hence, we concentrate only on the case $T = h$,* i.e.*, we concentrate on the approximate models of the form (2.3). For more details on the Lyapunov based approach to stability of the general case, see [21].* In the sequel, we need the following definition[1]:

DEFINITION 2.1. [**Uniform forward completeness**] *Consider the family of systems (2.5), where $\star \in \{a, e\}$. The family of systems (2.5) is said to be uniformly*

[1]This property is used in the proof of Proposition 4.4.

forward complete if there exist strictly positive numbers T^*, c *and* $\sigma_1, \sigma_2 \in \mathcal{K}_\infty$ *such that for all* $T \in (0, T^*)$ *and* $x_\circ \in R^n$ *we have that all solutions* $\phi_T^\star \in \mathcal{S}_T^\star(x_\circ)$ *of the family (2.5) satisfy:*

$$|\phi_T(k, x_\circ)| \leq \sigma_1(|x_\circ|) + \sigma_2(kT) + c \qquad \forall k \geq 0 \ . \tag{2.6}$$

REMARK 2. *Note that if (2.5) with* $\star = e$ *is uniformly forward complete, then this rules out finite escape times for the sampled-data system consisting of the plant (2.1) and the controller (2.4). The definition of uniform forward completeness given above was first used in [24] to treat stability of time-varying discrete-time parameterized cascaded systems. Lyapunov like sufficient conditions that guarantee uniform forward completeness in the sense of Definition 2.1 can be found in [24].*

3. Stabilization via approximate discrete-time models. In this section, we pose and answer the following question:

> If there exists a (not necessarily compact) set $\mathcal{A}$ such that the system (2.3), (2.4) is asymptotically/exponentially stable with respect to $\mathcal{A}$ for all small T, then under which conditions is the family of exact discrete-time models (2.2), (2.4) also (approximately) asymptotically/exponentially stable with respect to the set $\mathcal{A}$ for sufficiently small values T?

The above question was answered in [19] for a less general set up and in [21] for the same set up like in this paper but with the assumption that an appropriate family of strict Lyapunov functions can be constructed for the family (2.3), (2.4). Constructing such families of Lyapunov functions is in general hard and the question arises whether one can answer the above question without knowledge of appropriate Lyapunov functions for (2.3), (2.4). We present several such non-Lyapunov based results in this section.

3.1. SPA stability via approximate discrete-time models. In order to state the main result of this subsection we first need to define an appropriate stability property and a consistency property that quantifies the mismatch between the approximate and exact closed loop systems.

DEFINITION 3.1. **[SPA stability]** *Consider the family of systems (2.5), where* $\star \in \{a, e\}$. *Let a nonempty closed set* $\mathcal{A} \subset R^n$ *be given. The family of systems (2.5) is said to be (*β*,*$\mathcal{A}$*)-semi-globally practically asymptotically (SPA) stable if the system is uniformly forward complete and there exists* $\beta \in \mathcal{KL}$ *such that for any pair of strictly positive numbers* (Δ, ν) *there exists* $T^* > 0$ *such that for all* $T \in (0, T^*)$, *all* $x_\circ \in \mathcal{H}_\mathcal{A}(0, \Delta)$ *and all solutions* $\phi_T^\star(\cdot, x_\circ)$ *of the family (2.5) we have:*

$$|\phi_T^\star(k, x_\circ)|_\mathcal{A} \leq \beta(|x_\circ|_\mathcal{A}, kT) + \nu, \qquad \forall k \in N \ . \tag{3.1}$$

Moreover, if the system is forward complete and there exists $T^* > 0$ *such that for all* $T \in (0, T^*)$ *we have that (3.1) holds for all* $x_\circ \in R^n$ *and with* $\nu = 0$, *then we say that the system (2.5) is (*β*,* $\mathcal{A}$*)-globally asymptotically stable (GAS).* The following definition of multi-step consistency is a generalization to differential inclusions of the multi-step consistency property in [19] that was given for differential equations only.

DEFINITION 3.2. **[Multi-step upper consistency]** *The family* $\mathcal{F}_T^a$ *is said to be* $\mathcal{A}$*-multi-step upper semi-consistent with* $\mathcal{F}_T^e$ *if, for each triple of strictly positive real*

numbers (L,η,Δ) *there exist a function* $\alpha : R_{\geq 0}\times R_{\geq 0}\to R_{\geq 0}\cup\{\infty\}$ *and* $T^*>0$ *such that, for all* $T\in(0,T^*)$ *we have*

$$\{x,y\in\mathcal{H}_{\mathcal{A}}(0,\Delta)\ ,\ |x-y|\leq\delta\}\qquad\Longrightarrow\qquad\mathcal{F}_T^e(x)\subseteq\mathcal{F}_T^a(y)+\alpha(\delta,T)\overline{\mathcal{B}}_n \tag{3.2}$$

and[2]

$$k\in[0,\ell_{L,T}]\qquad\Longrightarrow\qquad\alpha^k(0,T):=\overbrace{\alpha\left(\cdots\alpha\left(\alpha\right.\right.}^{k}\left.\left.(0,T),T\right)\cdots,T\right)\leq\eta\ . \tag{3.3}$$

In the sequel we may refer to this property simply as "multi-step consistency".

REMARK 3. *We present sufficient conditions for multi-step upper semi-consistency in Subsection 3.3. We emphasize that this property can be checked without knowing the exact discrete-time model. The notion of consistency is adapted from numerical analysis literature [30, 33] and was already used in [19, 21].*

With these definitions, we can state the main result of this subsection:

THEOREM 3.3. *Let* $\beta\in\mathcal{KL}$ *and let a nonempty set* $\mathcal{A}\subset R^n$ *be given. If the following holds:*

1. $\mathcal{F}_T^a$ *is multi-step upper semi-consistent with* $\mathcal{F}_T^e$*;*
2. *The approximate closed loop system (2.3),(2.4) is* $(\beta,\mathcal{A})$*-SPA stable (or* $(\beta,\mathcal{A})$*-GAS)*

then, the family of exact closed loop systems (2.4), (2.2) is $(\beta,\mathcal{A})$*-SPA stable.*

REMARK 4. *We note that stability of the exact discrete-time model implies under mild and reasonable assumptions also the stability of the sampled-data system (including the inter-sample behaviour), see e.g. [18].*

REMARK 5. *Theorem 3.3 presents stability conditions that can be verified without the knowledge of the exact discrete-time model. Indeed, we already noted that the consistency property can be checked without knowing the exact discrete-time model of the system (item 1). Hence, we only need to verify an appropriate stability of the approximate model (item 2) to conclude a corresponding stability property of the exact discrete-time system. Note that in general stability for the exact closed loop can be guaranteed only for sufficiently small sampling periods* T*.*

REMARK 6. *Several examples in [19] and [21] illustrate that if the item 1 in Theorem 3.3 does not hold while the item 2 holds, it may happen that the exact discrete-time model can not be stabilized by sufficiently reducing* T*. Also, it is trivial to see that we do need the item 2 to state the result. Hence, while our conditions are only sufficient, they are tight since if one of them does not hold there are examples for which the conclusion does not hold.*

REMARK 7. *The paper [21] presents Lyapunov conditions that can be used to verify SPA stability (or GAS) of arbitrary sets for parameterized inclusions of the form (2.5). In Section 4, we present new non-Lyapunov results that use different types of summability conditions to conclude GAS in the sense of Definition 3.1. These results constitute a toolbox for controller design for sampled-data nonlinear systems via their approximate discrete-time models.*

REMARK 8. *Theorem 3.3 generalizes [19, Theorem 1] in several different directions: it covers differential inclusions, it is given for stability with respect to arbitrary*

[2]Note that, for $k=0$, we define $\alpha^0(0,T):=0$.

sets and the controllers are allowed to be dynamic. We note that Theorem 3.3 differs from results presented in [21] because we do not use a family of Lyapunov functions for the approximate model to state the result. In particular, results in [21] generalize [19, Theorem 2] whereas Theorem 3.3 generalizes [19, Theorem 1].

REMARK 9. *We note that Lyapunov based result in [21] and [19, Theorem 2] use a different notion of the so-called one-step consistency. It was shown in [19] that one-step consistency and an appropriate local Lipschitz condition of (2.3), (2.4) imply multi-step consistency. However, it was shown that the two consistency properties are genuinely different and without some extra conditions neither implies another.*

3.2. GES via approximate discrete-time models. In some cases, it is possible to establish stronger global exponential stability for the family of approximate models and it is natural to look for appropriate consistency conditions that will guarantee that the family of exact closed loops will also be globally exponentially stable. We summarize such a result (Theorem 3.6) in this section. We use the following definitions:

DEFINITION 3.4. [**GES stability**] *Consider the family of systems (2.5), where* $\star \in \{a, e\}$. *Let a nonempty closed set* $\mathcal{A} \subset R^n$ *be given. The family of systems (2.5) is said to be (K,λ,A)- globally exponentially stable (GES) if the system is forward complete and there exist positive numbers* K, λ *and* T^* *such that for all* $T \in (0, T^*)$, *all* $x_\circ \in R^n$ *and all solutions* $\phi_T^\star(\cdot, x_\circ)$ *of the family (2.5) we have:*

$$|\phi_T^\star(k, x_\circ)|_{\mathcal{A}} \leq K \exp(-\lambda k T)\, |x_\circ|_{\mathcal{A}}\,, \qquad \forall k \in N\ . \tag{3.4}$$

DEFINITION 3.5. [**Linear gain multi-step upper consistency**] *The family* $\mathcal{F}_T^a$ *is said to be linear gain* $\mathcal{A}$*-multi-step upper semi-consistent with* $\mathcal{F}_T^e$ *if for each pair of positive numbers* (L, η) *there exists* $T^* > 0$ *and a function* $\alpha : R_{\geq 0} \times R_{\geq 0} \times R_{\geq 0} \to R_{\geq 0} \cup \{\infty\}$ *such that, for all* $T \in (0, T^*)$ *and all* $\Delta > 0$ *we have*

$$\{x, y \in \mathcal{H}_{\mathcal{A}}(0, \Delta)\ ,\ |x - y| \leq \delta\} \qquad \Longrightarrow \qquad \mathcal{F}_T^e(x) \subseteq \mathcal{F}_T^a(y) + \alpha(\delta, T, \Delta)\overline{\mathcal{B}}_n \tag{3.5}$$

and[3]

$$\{k \in N,\ kT \leq L\} \qquad \Longrightarrow \qquad \alpha^k(0, T, \Delta) := \overbrace{\alpha\left(\cdots \alpha\left(\alpha\right.\right.}^{k}(0, T, \Delta), T, \Delta) \cdots, T, \Delta) \leq \eta \cdot \Delta\ . \tag{3.6}$$

The main result of this section is stated next.

THEOREM 3.6. *Let positive* K, λ *and a nonempty set* $\mathcal{A} \subset R^n$ *be given. If the following holds:*

1. $\mathcal{F}_T^a$ *is linear-gain* $\mathcal{A}$*-multi-step upper semi-consistent with* $\mathcal{F}_T^e$*;*
2. *The approximate closed loop system (2.3),(2.4) is* $(K, \lambda, \mathcal{A})$*-GES.*

Then, there exist positive K_1, λ_1 *such that the family of exact closed loop systems (2.4), (2.2) is* $(K_1, \lambda_1, \mathcal{A})$*-GES.*

REMARK 10. *Theorem 3.6 can be used to conclude stronger stability property (GES) of the exact discrete-time model if the approximate discrete-time model is GES and a stronger linear gain multi-step upper semi-consistency holds.*

[3] We define $\alpha^0(0, T, \Delta) := 0$.

REMARK 11. *We are not aware whether Theorem 3.6 has been proved even in the case of sampled-data differential equations, static state feedback controllers and stability of the origin.*

3.3. Sufficient conditions for multi-step consistency. In this subsection we present several different conditions to guarantee the consistency properties that we used in Theorems 3.3 and 3.6. We emphasize that all these conditions can be checked without knowing the exact discrete-time model of the system. The proofs are appropriate generalizations of proofs in [19] that were given only for differential equations. First, we present sufficient conditions for the consistency property needed in Theorem 3.3.

PROPOSITION 3.7. *If, for each $\Delta > 0$, there exist $K > 0$, $\rho \in \mathcal{K}_\infty$ and $T^* > 0$ such that for all $T \in (0, T^*)$ and all $x, y \in \mathcal{H}_{\mathcal{A}}(0, \Delta)$ we have*

$$\mathcal{F}_T^e(x) \subseteq \mathcal{F}_T^a(y) + [(1 + KT)\,|x - y| + T\rho(T)]\overline{\mathcal{B}}_n \tag{3.7}$$

then $\mathcal{F}_T^a$ is $\mathcal{A}$-multi-step upper semi-consistent with $\mathcal{F}_T^e$.

Proof. Let (L, η, Δ) be given. From the assumption of the lemma, let Δ generate $K > 0$, $\rho \in \mathcal{K}_\infty$ and $T_1^* > 0$. Define

$$\alpha(\delta, T) := (1 + KT)\delta + T\rho(T); \qquad T^* := \min\left\{T_1^*, \rho^{-1}\left(\frac{\eta K}{\exp(KL) - 1}\right)\right\}\ . \tag{3.8}$$

With these definitions, the condition (3.2) is satisfied. Also note that for all k such that $kT \le L$ we have:

$$\begin{aligned}\alpha^k(0, T) = T\rho(T)\sum_{j=0}^{k-1}(1 + KT)^j &= \frac{\rho(T)}{K}\left[(1 + KT)^k - 1\right] \\ &\le \frac{\rho(T)}{K}\left[\exp(KTk) - 1\right] \le \frac{\rho(T)}{K}\left[\exp(KL) - 1\right]\end{aligned}$$

and so (3.3) is satisfied.

REMARK 12. *It should be noted that one can state sufficient conditions for multi-step upper semi-consistency in terms of another (one step) consistency condition that characterizes the mismatch between the open loop exact $F_T^e(x, u)$ and approximate $F_T^a(x, u)$ plant models, a Lipschitz property on F_T^a and uniform boundedness of the control law (2.4). Such conditions can be found in [19] and [21] and are omitted for space reasons.*

Next, we present sufficient conditions for the consistency property used in Theorem 3.6.

PROPOSITION 3.8. *If there exist positive numbers K and T^* and $\rho \in \mathcal{K}$ such that, for all $T \in (0, T^*)$ and all $x, y \in R^n$ we have*

$$\mathcal{F}_T^e(x) \subseteq \mathcal{F}_T^a(y) + [(1 + KT)|x - y| + T\rho(T)\max\{|x|_{\mathcal{A}}, |y|_{\mathcal{A}}\}]\overline{\mathcal{B}}_n\ , \tag{3.9}$$

then $\mathcal{F}_T^a$ is linear-gain multi-step upper semi-consistent with $\mathcal{F}_T^e$.

Proof. Let (L, η) be given. Let $K > 0$, $\rho \in \mathcal{K}_\infty$ and $T_1^* > 0$ come from the conditions in the lemma. Define

$$\alpha(\delta, T, \Delta) := (1 + KT)\delta + T\rho(T)\Delta; \qquad T^* := \min\left\{T_1^*, \rho^{-1}\left(\frac{\eta K}{\exp(KL) - 1}\right)\right\}\ . \tag{3.10}$$

With these definitions, the condition (3.2) is satisfied. Also note that for all k such that $kT \leq L$ we have:

$$\begin{aligned}\alpha^k(0,T,\Delta) &= T\rho(T)\Delta\sum_{j=0}^{k-1}(1+KT)^j \\ &= \frac{\rho(T)\Delta}{K}\left[(1+KT)^k-1\right] \\ &\leq \frac{\rho(T)\Delta}{K}\left[\exp(KTk)-1\right]\frac{\rho(T)\Delta}{K}\left[\exp(KL)-1\right] \leq \eta\cdot\Delta\end{aligned}$$

and so (3.6) is satisfied.

4. Summability conditions for stability. In this section, we consider stability properties of the family of parameterized discrete-time inclusions:

$$x^+ \in \mathcal{F}_T(x)\ . \tag{4.1}$$

We present summability type conditions that can be used to verify that a parameterized family of difference inclusions is GAS (Theorem 4.5) or GES (Theorem 4.11). These results are discrete-time counterparts of results in [32] and are tailored carefully to be used within the framework that Theorems 3.3 and 3.6 provide for controller design via approximate discrete-time models. Results of this section are useful in situations when one can not find a strict Lyapunov function for the family of approximate closed-loop systems, *i.e.* when one can not use results from [21] –this is a common situation, for instance, in analysis of adaptive control systems, see [28] for examples in continuous time. We consider only GAS and GES for space reasons but appropriate versions of results that guarantee SPA stability can be stated in a similar manner.

4.1. Summability conditions for GAS. All of the below definitions are stated for the system (4.1).

Definition 4.1. *The closed set $\mathcal{A}$ is globally stable (GS) if the system (4.1) is uniformly forward complete and there exist $\rho \in \mathcal{K}_\infty$ and $T^* > 0$ such that for all $x_\circ \in R^n$, $T \in (0,T^*)$ and $\phi_T \in \mathcal{S}_T(x_\circ)$ we have:*

$$|\phi_T(k,x_\circ)|_{\mathcal{A}} \leq \rho(|x_\circ|_{\mathcal{A}}) \qquad \forall k \geq 0\ . \tag{4.2}$$

The set $\mathcal{A}$ is GS with linear gain if ρ is of the form $\rho(s) = \bar{\rho}\cdot s$ for some $\bar{\rho} > 0$. Next, we state a definition of GAS that is equivalent to Definition 3.1 but that is easier to use in the proofs.

Definition 4.2. *The closed set $\mathcal{A}$ is globally asymptotically stable (GAS) if it is GS and there exists $T^* > 0$ such that for any $r > 0, \epsilon > 0$ there exists $\tau > 0$ such that for all $x_\circ \in R^n$, $T \in (0,T^*)$ and $\phi_T \in \mathcal{S}_T(x_\circ)$ we have*

$$|x_\circ|_{\mathcal{A}} \leq r, \qquad k \geq \ell_{\tau,T} \implies |\phi_T(k,x_\circ)|_{\mathcal{A}} \leq \epsilon\ . \tag{4.3}$$

Remark 13. *Using similar arguments to [12, Proposition 2.5] it can be shown that GAS implies existence of $\beta \in \mathcal{KL}$ and $T^* > 0$ such that for all $T \in (0,T^*)$, all $x_\circ$ and all $\phi_T \in \mathcal{S}_T(x_\circ)$ we have:*

$$|\phi_T(k,x_\circ)|_{\mathcal{A}} \leq \beta(|x_\circ|_{\mathcal{A}}, kT) \qquad \forall k \geq 0\ .$$

DEFINITION 4.3. *The closed set $\mathcal{A}$ is said to be globally sliding time stable (GSTS) if the system (4.1) is uniformly forward complete and there exist class $\mathcal{K}_\infty$ functions $\tau(\cdot)$ and $\rho(\cdot)$ and $T^* > 0$ such that for all $x_\circ \in R^n$, $T \in (0, T^*)$ and $\phi_T \in \mathcal{S}_T(x_\circ)$ we have*

$$\tau := \tau(r), \qquad k \in [0, \ell_{\tau,T}], \qquad |x_\circ|_{\mathcal{A}} \leq r \implies |\phi_T(k, x_\circ)|_{\mathcal{A}} \leq \rho(r) \ . \tag{4.4}$$

PROPOSITION 4.4. *Suppose that:*

1. *The closed set $\mathcal{A}$ is compact.*
2. *For every $\Delta > 0$, there exist $M > 0$ and $T^* > 0$ such that for all $T \in (0, T^*)$ we have $\sup_{x \in \mathcal{H}_{\mathcal{A}}(0,\Delta), w \in \mathcal{F}_T(x)} |w - x| \leq TM$.*

Then, the set $\mathcal{A}$ is GSTS for the system (4.1) if and only if the system is uniformly forward complete.

The main result of this subsection is presented next.

THEOREM 4.5. *For the system (4.1), the following statements are equivalent:*

1. *The set $\mathcal{A}$ is GAS.*
2. *(a) The set $\mathcal{A}$ is GSTS.*
   (b) There exist $\alpha \in \mathcal{K}$, $\gamma \in \mathcal{K}_\infty$ and $T^* > 0$ such that for all $T \in (0, T^*)$, $x_\circ \in R^n$ and all $\phi_T \in \mathcal{S}_T(x_\circ)$ we have:

$$T \sum_{k=0}^{\infty} \alpha\left(|\phi_T(k, x_\circ)|_{\mathcal{A}}\right) \leq \gamma(|x_\circ|_{\mathcal{A}}) \tag{4.5}$$

3. *(a) The set $\mathcal{A}$ is GS.*
   (b) There exists $T^* > 0$ such that for all $T \in (0, T^*)$ the following holds: for any $0 < \delta \leq \Delta$ there exists a continuous function $\omega_{\delta,\Delta} : R^n \to R_{\geq 0}$ and strictly positive $\omega_m, \gamma > 0$ such that
      i. for any $T \in (0, T^*)$ and $x \in \mathcal{H}_{\mathcal{A}}(\delta, \Delta)$ we have:

$$\omega_{\delta,\Delta}(x) \geq \omega_m \ . \tag{4.6}$$

      ii. for all $T \in (0, T^*)$, $x_\circ \in \mathcal{H}_{\mathcal{A}}(\delta, \Delta)$, $\phi_T \in \mathcal{S}_T(x_\circ)$ and all $\tau > 0$

$$T \sum_{k=0}^{\ell_{\tau,T}} \omega_{\delta,\Delta}(\phi_T(k, x_\circ)) \leq \gamma \ . \tag{4.7}$$

REMARK 14. *We note that Theorem 4.5 can be used to conclude GAS in cases when it is not easy to find a strict Lyapunov function. Such situations arise when the first difference of a Lyapunov function candidate is negative semi-definite instead of negative definite. Since such situations are quite common, Theorem 4.5 is an important tool in establishing stability properties of the inclusion (4.1).*

REMARK 15. *It is instructive to compare and discuss the conditions 2 and 3 in Theorem 4.5. First, note that the condition 2(a) is weaker than the condition 3(a) but at the same time the condition 2(b) is stronger than the condition 3(b). Also, it*

is worthwhile to point out that if $\alpha \in \mathcal{K}_\infty$ in the condition 2(b), then we can relax the condition 2(a) by requiring only uniform forward completeness instead of GSTS.

Theorem 4.5 can be combined with Theorem 3.3 to conclude SPA stability of the exact discrete time model of the system via its approximate model. Indeed, we can state:

COROLLARY 4.6. *Let a nonempty set $\mathcal{A} \subset R^n$ be given. If the following holds:*

1. *$\mathcal{F}_T^a$ is multi-step upper semi-consistent with $\mathcal{F}_T^e$;*
2. *One of the items in Theorem 4.5 holds for the approximate closed loop system (2.3),(2.4).*

Then, there exists $\beta \in \mathcal{KL}$ such that the family of exact closed loop systems (2.2), (2.4) is $(\beta, \mathcal{A})$-SPA stable.

REMARK 16. *We note that Theorems 4.5 and 4.11 are tailored for situations when the exact discrete time model of the plant is not known and the analysis and controller design are carried out via an approximate discrete time model. On the other hand, similar results can be proved for the non-parameterized difference inclusions of the form:*

$$x^+ \in \mathcal{F}(x) \ ,$$

which is useful in (rare) situations when the exact discrete-time model of the system is known to the designer. However, in this case the assumptions can be relaxed and proofs greatly simplified. Since differences between these two cases are substantial we will report these results in our future work.

Next, we present a result that can be used to check GAS via Lyapunov like functions.

PROPOSITION 4.7. *Suppose for the system (4.1) that the set $\mathcal{A}$ is GS. The set $\mathcal{A}$ is GAS if there exists $T^* > 0$ such that for $T \in (0, T^*)$ there exists a family of functions $V_T : R^n \to R$, a function $\kappa : R^n \to R$ and for each positive δ, Δ satisfying $0 < \delta < \Delta$, there exist positive real numbers ψ_1, ψ_2, ω_m and a continuous function $\omega_{\delta,\Delta} : R^n \to R$ such that:*

1. *for all $x \in \mathcal{H}_{\mathcal{A}}(\delta, \Delta)$ we have $\omega_{\delta,\Delta}(x) \geq \omega_m$;*
2. *for all $x \in \mathcal{H}_{\mathcal{A}}(0, \Delta)$ and $T \in (0, T^*)$ we have:*

   (a)

$$\max \left\{ \left| \sup_{w \in \mathcal{F}_T(x)} V_T(w) \right|, |V_T(x)| \right\} \leq \psi_1 \ ;$$

   (b)

$$\frac{\sup_{w \in \mathcal{F}_T(x)} V_T(w) - V_T(x)}{T} \leq -\omega_{\delta,\Delta}(x) + \kappa(x) \ ;$$

   (c) for each $\tau > 0$ we have:

$$T \sum_{k=0}^{\ell_{\tau,T}} \kappa(\phi_T(k, x)) \leq \psi_2 \ .$$

REMARK 17. *Note that Proposition 4.7 does not require any continuity properties of V_T, which was needed to prove its continuous-time counterpart (c.f. [32, Lemma 2]). Also, the function κ does not need to be continuous which was required in [32, Lemma 2]. On the other hand, we still require continuity of the function $\omega_{\delta,\Delta}$, which was needed in the proof of Theorem 4.5.*

REMARK 18. *We note that the integral lemmas for continuous time systems were used in [32] to establish a generalization of the Matrosov theorem. In a similar fashion, Theorem 4.5 could be used to prove a generalized Matrosov theorem for parameterized difference inclusions. However, we do not present such a result since a generalized Matrosov theorem was investigated in detail in [22].*

4.2. Summability conditions for GES. We present now a result that uses summability type conditions to conclude global exponential stability of (4.1). This result can be used in conjunction with Theorem 3.6 to conclude GES of the exact discrete-time model via an approximate discrete-time model.

DEFINITION 4.8. *The closed set $\mathcal{A}$ is said to be globally fixed time stable (GFTS) with linear gain if the system (4.1) is forward complete and there exist $\bar{\rho} > 0$, $\tau > 0$ and $T^* > 0$ such that for all $x_\circ \in R^n$, $T \in (0, T^*)$ and $\phi_T \in \mathcal{S}_T(x_\circ)$ we have*

$$k \in [0, \ell_{\tau,T}] \implies |\phi_T(k, x_\circ)|_{\mathcal{A}} \leq \bar{\rho}\, |x_\circ|_{\mathcal{A}} \ . \tag{4.8}$$

DEFINITION 4.9. *The system (4.1) has the unboundedness observability property through $|\cdot|_{\mathcal{A}}$ if the following holds: if there exist $\tau > 0$, $x_\circ \in R^n$ and $\phi_T \in \mathcal{S}_T(x_\circ)$ such that*

$$\lim_{k \to \ell_{\tau,T}, T \to 0} |\phi_T(k, x_\circ)| = \infty \tag{4.9}$$

then, the following holds:

$$\lim_{k \to \ell_{\tau,T}, T \to 0} |\phi_T(k, x_\circ)|_{\mathcal{A}} = \infty \ . \tag{4.10}$$

Sufficient conditions for GFTS are presented next:

PROPOSITION 4.10. *Suppose that: (i) the system (4.1) has the unboundedness observability property through $|\cdot|_{\mathcal{A}}$; (ii) there exist strictly positive numbers c, T^* such that for all $T \in (0, T^*)$ and $x \in R^n$ we have:*

$$\frac{\sup_{w \in \mathcal{F}_T(x)} |w|_{\mathcal{A}} - |x|_{\mathcal{A}}}{T} \leq c\, |x|_{\mathcal{A}} \ . \tag{4.11}$$

Then, the set $\mathcal{A}$ is GFTS with linear gain.

The main result of this subsection is given below. It provides summability type conditions that guarantee GES of arbitrary sets for parameterized inclusions of the form (4.1).

THEOREM 4.11. *For the system (4.1), the following statements are equivalent:*

1. *The set $\mathcal{A}$ is GES.*
2. *(a) The set $\mathcal{A}$ is GFTS with linear gain.*

(b) There exist strictly positive real numbers c, p and T^* such that for all $x_\circ \in R^n$, $T \in (0, T^*)$ and $\phi_T \in \mathcal{S}_T(x_\circ)$ we have:

$$T \sum_{k=0}^{\infty} |\phi_T(k, x_\circ)|_{\mathcal{A}}^p \leq c\, |x_\circ|_{\mathcal{A}}^p \ . \tag{4.12}$$

We can combine results of Theorems 3.6 and 4.11 to conclude GES of the exact discrete-time model via an approximate model that is consistent with the former.

COROLLARY 4.12. *Let a nonempty set $\mathcal{A} \subset R^n$ be given. If the following holds:*

1. *$\mathcal{F}_T^a$ is linear gain $\mathcal{A}$-multi-step upper semi-consistent with $\mathcal{F}_T^e$;*
2. *One of the items in Theorem 4.11 holds for the approximate closed loop system (2.3),(2.4).*

Then, there exists $K, \lambda > 0$ such that the family of exact closed-loop systems (2.2), (2.4) is $(K, \lambda, \mathcal{A})$-GES.

Sufficient conditions for GES via Lyapunov like functions are given below:

PROPOSITION 4.13. *Suppose for the system (4.1) that the set $\mathcal{A}$ is GS with linear gain. The set $\mathcal{A}$ is GES if there exists $T^* > 0$ such that for $T \in (0, T^*)$ there exists a family of functions $V_T : R^n \to R$, a function $\kappa : R^n \to R$ and positive real numbers ψ_1, ψ_2, ψ_3 such that*

1. *for all $x \in R^n$ and $T \in (0, T^*)$:*

$$\max \left\{ \left| \sup_{w \in \mathcal{F}_T(x)} V_T(w) \right|, |V_T(x)| \right\} \leq \psi_1\, |x|_{\mathcal{A}}^p \ ;$$

2. *for all $x \in R^n$ and $T \in (0, T^*)$:*

$$\frac{\sup_{w \in \mathcal{F}_T(x)} V_T(w) - V_T(x)}{T} \leq -\psi_2\, |x|_{\mathcal{A}}^p + \kappa(x) \ ;$$

3. *for each $\tau > 0$, $T \in (0, T^*)$, $x_\circ \in R^n$ and $\phi_T \in \mathcal{S}_T(x_\circ)$ we have:*

$$T \sum_{k=0}^{\ell_{\tau,T}} \kappa(\phi_T(k, x_\circ)) \leq \psi_3\, |x_\circ|_{\mathcal{A}}^p \ .$$

5. Results for systems under output injection. In nonlinear stability analysis we often analyze stability properties of a system via stability properties of another auxiliary system that is easier to analyze (e.g. known to be stable). In particular, summability based stability results of Theorems 4.5 and 4.11 can be used in the following manner. Suppose that we want to analyze stability properties of the system (4.1) and it is known that a set $\mathcal{A}$ is GS for this system. Suppose that there exists a continuous function $K : R^n \to R_{\geq 0}$ is such that for all $x \in R^n$ and all $\phi_T \in \mathcal{S}_T(x)$ we have that the function $K(\phi_T(k, x))$ is summable in an appropriate sense (Definition 5.1) and, moreover, for the inclusion

$$x^+ \in \tilde{\mathcal{F}}_T(x) \ , \tag{5.1}$$

with

$$\mathcal{F}_T(x) \subseteq \tilde{\mathcal{F}}_T(x) + TK(x)\bar{B}_n \qquad \forall x \in R^n \ , \tag{5.2}$$

we have that the set $\mathcal{A}$ is GAS. Then, we can conclude via Theorem 4.5 that the set $\mathcal{A}$ is GAS for the system (4.1). Similar results can be stated for GES and they are related to results on stability under output injection (see [32, Section 6]). In particular, we use the following definition of summability for the function $K(\cdot)$.

DEFINITION 5.1. *The continuous function $K : R^n \to R_{\geq 0}$ is said to be weakly uniformly summable for the system (4.1) if there exists $T^* > 0$ for each $\epsilon > 0$ there exists a number $\beta > 0$ such that for all $T \in (0, T^*)$, $x \in R^n$, all $\phi_T \in \mathcal{S}_T(x)$, $\tau > 0$ we have*

$$T \sum_{k=0}^{\ell_{\tau,T}} K(\phi_T(k, x)) \leq \beta + \epsilon T \ell_{\tau,T} \tag{5.3}$$

Checkable sufficient conditions for weak uniform summability are presented next.

PROPOSITION 5.2. *Suppose that the set $\mathcal{A}$ is GS for the inclusion (4.1). If there exist $T^* > 0$, a continuous function $h : R^n \to R^m$, nondecreasing functions $\kappa, k_1, k_2 : R_{\geq 0} \to R_{\geq 0}$ a continuous positive definite function $\gamma : R_{\geq 0} \to R_{\geq 0}$ and $k \in \mathcal{K}_\infty$ such that for all $T \in (0, T^*)$, $x \in R^n$, $\phi_T \in \mathcal{S}_T(x)$ we have:*

1. $T \sum_{k=0}^{\infty} \gamma(|h(\phi_T(k, x))|) \leq \kappa(|x|_{\mathcal{A}})$;
2. $K(x) \leq k_1(|x|_{\mathcal{A}}) \cdot k(|h(x)|)$;
3. $|h(x)| \leq k_2(|x|_{\mathcal{A}})$;

then, the function $K(\cdot)$ is weakly uniformly summable for the inclusion (4.1).

Next, we state the main result of this section:

PROPOSITION 5.3. *(GAS under output injection) Suppose that the following conditions hold:*

1. *The set $\mathcal{A}$ is GS for the system (4.1);*
2. *There exists $T^\star > 0$ such that for all $T \in (0, T^\star)$ we have $\mathcal{F}_T(x) \subseteq \tilde{\mathcal{F}}_T(x) + TK(x)\bar{B}_n$;*
3. *There exists T^* and for $T \in (0, T^*)$ there exists a family of functions $V_T : R^n \to R_{\geq 0}$, $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, a positive definite function $\alpha_3 : R_{\geq 0} \to R_{\geq 0}$ such that for all $x \in R^n$, $T \in (0, T^*)$ we have:*

$$\alpha_1(|x|_{\mathcal{A}}) \ \leq \ V_T(x) \leq \alpha_1(|x|_{\mathcal{A}}) \tag{5.4}$$

$$\frac{\sup_{w \in \tilde{\mathcal{F}}_T(x)} V_T(w) - V_T(x)}{T} \leq -\alpha_3(|x|_{\mathcal{A}}) \tag{5.5}$$

4. *For any $\Delta > 0$ there exist $T^*, L > 0$ such that for all $x, y \in \mathcal{H}_{\mathcal{A}}(0, \Delta)$, $T \in (0, T^*)$ we have*

$$|V_T(x) - V_T(y)| \leq L\,|x - y| \tag{5.6}$$

5. *The function $K(\cdot)$ is uniformly weakly summable for the system (4.1).*

Then, the set $\mathcal{A}$ is GAS for the system (4.1).

REMARK 19. *One can state a similar result for GES via output injection but we do not include it for space reasons.*

6. Proofs of main results.

6.1. Proof of Theorem 3.3. To prove Theorem 3.3, we first need the following.

LEMMA 6.1. *If $\mathcal{F}_T^a$ is multi-step upper semi-consistent with $\mathcal{F}_T^e$, then for each strictly positive triple (L, η, Δ) there exist $T^* > 0$ such that, for each $\xi \in R^n$ having the property that each solution of the approximate closed-loop system (2.3), (2.4) starting at ξ satisfies*

$$\phi_T^a(k, \xi) \in \mathcal{H}_{\mathcal{A}}(0, \Delta) \quad \forall k \in [0, \ell_{L,T}] \,,\ T \in (0, T^*), \tag{6.1}$$

and any solution ϕ_T^e of the exact closed-loop system (2.4), (2.2) starting at ξ, there exists a solution ϕ_T^a such that

$$|\phi_T^e(k, \xi) - \phi_T^a(k, \xi)| \leq \eta \qquad \forall k \in [0, \ell_{L,T}] \,. \tag{6.2}$$

Proof. Let (L, η, Δ) be given. Define $\Delta_1 := \Delta + \eta$. Since $\mathcal{F}_T^a$ is multi-step upper semi-consistent with $\mathcal{F}_T^e$, there exist a function $\alpha(\cdot, \cdot)$ and a strictly positive real number T^* such that (3.2) and (3.3) are satisfied for the triple (L, η, Δ_1). We now prove the result by induction. First for $k = 0$ we have $|\phi_T^e(0, \xi) - \phi_T^a(0, \xi)| = |\xi - \xi| = 0 = \alpha^0(0, T) \leq \eta$. Next, suppose that for every $\phi_T^e(k, \xi)$ there exists $\phi_T^a(k, \xi)$ such that $|\phi_T^e(k, \xi) - \phi_T^a(k, \xi)| \leq \alpha^k(0, T) \leq \eta$ and $k + 1 \in [0, \ell_{L,T}]$. Since $\phi_T^a(k, \xi) \in \mathcal{H}_{\mathcal{A}}(0, \Delta)$, it follows from the definition of Δ_1 that all solutions of exact and approximate closed loops satisfy $\phi_T^a(k, \xi), \phi_T^e(k, \xi) \in \mathcal{H}_{\mathcal{A}}(0, \Delta_1)$. Then, it follows from (3.2) that for any solution $\phi_T^e(k + 1, \xi)$ there exists $\phi_T^a(k + 1, \xi)$ such that $|\phi_T^e(k + 1, \xi) - \phi_T^a(k + 1, \xi)| \leq \alpha(\alpha^k(0, T), T) = \alpha^{k+1}(0, T)$. Since $k + 1 \in [0, \ell_{L,T}]$ it follows from (3.3) that $\alpha^{k+1}(0, T) \leq \eta$.

Now we prove the theorem. Let (Δ, ν) be given. Let β come from the item 2 of the theorem. Let $\eta > 0$ and $\epsilon \in (0, 1)$ be such that[4]:

$$\beta(2\eta + \epsilon\nu, 0) + 2\eta + \epsilon\nu \leq \nu \tag{6.3}$$

$$2\eta + \epsilon\nu \leq \Delta \,. \tag{6.4}$$

Let $L > 1$ be such that

$$\beta(\Delta, t) \leq \eta \qquad \forall t \geq L - 1 \,. \tag{6.5}$$

Let

$$\Delta_1 := \beta(\Delta, 0) + \nu \,. \tag{6.6}$$

Let (L, η, Δ_1) generate $T_1^* > 0$ via the item 1 of the the theorem, let $(\Delta_1, \epsilon\nu)$ generate $T_2^* > 0$ via the item 2 and let (L, η, Δ_1) generate T_3^* via Lemma 1. Let

$$T^* := \min\{T_1^*, T_2^*, T_3^*, 1\} \tag{6.7}$$

and $T \in (0, T^*)$. From the item 2 and the choice of T, we have that:

$$\xi \in \mathcal{H}_{\mathcal{A}}(0, \Delta) \implies \phi_T^a(k, \xi) \in \mathcal{H}_{\mathcal{A}}(0, \Delta_1) \qquad \forall k \in N \,. \tag{6.8}$$

[4]Since $\beta(s, 0) \in \mathcal{K}$, it is always possible to find such numbers.

Using the item 1 of the theorem and Lemma 6.1 we have that for all $\xi \in \mathcal{H}_{\mathcal{A}}(0,\Delta)$, $T \in (0,T^*)$ and any solution $\phi_T^e(k,\xi)$ there exists a solution $\phi_T^a(k,\xi)$ such that for all k with $k \in [0,\ell_{L,T}]$ we have:

$$|\phi_T^e(k,\xi) - \phi_T^a(k,\xi)| \leq \eta \ . \tag{6.9}$$

Thus, for any such $\phi_T^e(k,\xi)$ there exists $\phi_T^a(k,\xi)$ such that

$$\begin{aligned} |\phi_T^e(k,\xi)|_{\mathcal{A}} &\leq |\phi_T^a(k,\xi)|_{\mathcal{A}} + |\phi_T^e(k,\xi) - \phi_T^a(k,\xi)| \\ &\leq \beta\left(|\xi|_{\mathcal{A}}, kT\right) + \epsilon\nu + \eta \qquad \forall k \in [0,\ell_{L,T}] \ . \end{aligned} \tag{6.10}$$

Since (6.3) implies that $\epsilon\nu + \eta < \nu$, we have that the desired bound (3.1) holds for all k such that $k \in [0,\ell_{L,T}]$. Now we need to prove that the desired bound holds for all $k \geq 0$. Then, since $T < T^* \leq 1$, we can write:

$$\ell_{L,T} T > L - T > L - 1 \ . \tag{6.11}$$

Define $k_i := i \cdot \ell_{L,T}$ for $i = 1, 2, \ldots$. Thus, using the definition of L, η and ϵ, we get from (6.4) that for all $\xi \in \mathcal{H}_{\mathcal{A}}(0,\Delta)$ we have:

$$|\phi_T^e(k_1,\xi)|_{\mathcal{A}} \leq \beta(\Delta, L-1) + \epsilon\nu + \eta \leq 2\eta + \epsilon\nu \leq \Delta \ . \tag{6.12}$$

Now consider those k such that $k \in [k_1,k_2]$. We have, using time-invariance, (6.9), (6.12) and the fact that $\phi_T^e(k_1,\xi) \in \mathcal{H}_{\mathcal{A}}(0,\Delta)$, that for each ϕ_T^e there exists ϕ_T^a such that

$$\begin{aligned} |\phi_T^e(k,\xi)|_{\mathcal{A}} &= |\phi_T^e(k-k_1, \phi_T^e(k_1,\xi))|_{\mathcal{A}} \\ &\leq |\phi_T^a(k-k_1, \phi_T^e(k_1,\xi))|_{\mathcal{A}} \\ &\qquad + |\phi_T^e(k-k_1, \phi_T^e(k_1,\xi)) - \phi_T^a(k-k_1, \phi_T^e(k_1,\xi))| \\ &\leq \beta(2\eta + \epsilon\nu, (k-k_1)T) + \epsilon\nu + \eta \end{aligned} \tag{6.13}$$

from which it follows (using (6.3)) that for all $k \in [k_1,k_2]$,

$$|\phi_T^e(k,\xi)|_{\mathcal{A}} \leq \beta(2\eta + \epsilon\nu, 0) + \epsilon\nu + \eta < \nu \tag{6.14}$$

and, using the definition of k_i and (6.4) we have that

$$|\phi_T^e(k_2,\xi)|_{\mathcal{A}} \leq \beta(2\eta + \epsilon\nu, L-1) + \epsilon\nu + \eta \leq 2\eta + \epsilon\nu \leq \Delta \ . \tag{6.15}$$

The result then follows by induction.

6.2. Proof of Theorem 3.6. In terms of trajectory error over "continuous-time" intervals with length of order one, linear gain $\mathcal{A}$-multi-step consistency gives the following:

LEMMA 6.2. *Suppose that $\mathcal{F}_T^a$ is linear-gain multi-step consistent with $\mathcal{F}_T^e$ and there exist positive T_1^*, B such that that for each $L > 0$ and for all $T \in (0,T_1^*)$ all solutions of the approximate closed loop satisfy*

$$|\phi_T^a(k,\xi)|_{\mathcal{A}} \leq B \cdot |\xi|_{\mathcal{A}} \quad \forall k \in [0,\ell_{L,T}]. \tag{6.16}$$

Then, for each strictly positive pair (L,η) there exists $T^*>0$ such that for all $T\in(0,T^*)$ and for any solution of the exact closed loop (2.4), (2.2), there exists a solution of the approximate closed loop (2.4), (2.3) such that

$$|\phi_T^e(k,\xi)-\phi_T^a(k,\xi)|\leq\eta\cdot|\xi|_{\mathcal{A}}\qquad\forall k\in[0,\ell_{L,T}]\ . \tag{6.17}$$

Proof. Let (L,η) be given. Let L, B and T_1^* be such that (6.16) holds. Let $B_1:=B+1$ and $\eta_1:=\frac{1}{B_1}\min\{\eta,1\}$. Let (L,η_1) generate $T_2^*>0$ and $\alpha(\cdot,\cdot,\cdot)$ via the linear multi-step upper semi-consistency. Let $T^*:=\min\{T_1^*,T_2^*\}$ and $T\in(0,T^*)$ and define $\Delta:=B_1|\xi|_{\mathcal{A}}$. The proof is completed by induction. First we have $|\phi_T^e(0,\xi)-\phi_T^a(0,\xi)|=|\xi-\xi|=0=\alpha^0(0,T,\Delta)\leq\eta_1\Delta=\eta_1\cdot B_1\cdot|\xi|_{\mathcal{A}}\leq\eta\,|\xi|_{\mathcal{A}}$ (which follows from the definition of Δ and η_1). Next, suppose that for every $\phi_T^e(k,\xi)$ there exists $\phi_T^a(k,\xi)$ such that $|\phi_T^e(k,\xi)-\phi_T^a(k,\xi)|\leq\alpha^k(0,T,\Delta)\leq\eta_1\cdot\Delta$ and $k+1\in[0,\ell_{L,T}]$. Since $|\phi_T^a(k,\xi)|\leq B\,|\xi|_{\mathcal{A}}$ by assumption, it follows from the definition of B_1 and $\eta_1\leq1$ that all solutions of exact and approximate closed loops satisfy $\max\{|\phi_T^a(k,\xi)|_{\mathcal{A}},|\phi_T^e(k,\xi)|_{\mathcal{A}}\}\leq B_1\,|\xi|_{\mathcal{A}}$. It then follows from (3.5) that for any solution $\phi_T^e(k+1,\xi)$ there exists $\phi_T^a(k+1,\xi)$ such that $|\phi_T^e(k+1,\xi)-\phi_T^a(k+1,\xi)|\leq\alpha(\alpha^k(0,T,\Delta),T,\Delta)=\alpha^{k+1}(0,T,\Delta)$. Since $k+1\in[0,\ell_{L,T}]$ it follows from (3.6) that $\alpha^{k+1}(0,T,\Delta)\leq\eta_1\Delta=\eta_1\cdot B_1|\xi|_{\mathcal{A}}\leq\eta|\xi|_{\mathcal{A}}$.

Now we prove the theorem. Let $c\in(0,1)$ and $\delta\in(0,c)$ be arbitrary. Let K,λ,T_1^* come from the item 2. Let $L:=\frac{1}{\lambda}\ln\left(\frac{K}{c-\delta}\right)$. Define $L_1:=L+1$ and let L_1 and δ generate T_2^* via the item 1. Let $T^*:=\min\{T_1^*,T_2^*,1\}$ and let $T\in(0,T^*)$ be arbitrary. Note from the definitions and the fact that $T<T^*\leq1$, we have:

$$L=L_1-1\leq\ell_{L_1,T}\cdot T\leq L_1\ . \tag{6.18}$$

Define $k_i:=i\cdot\ell_{L_1,T}$. From item 1 (with Lemma 6.2), we can write that for every $x_\circ$ and every $\phi_T^e\in\mathcal{S}_T^e(x_\circ)$ there exists $\phi_T^a\in\mathcal{S}_T^a(x_\circ)$ such that:

$$\begin{aligned}|\phi_T^e(k_{i+1},x_\circ)|_{\mathcal{A}}&=|\phi_T^e(k_{i+1}-k_i,\phi_T^e(k_i,x_\circ))|_{\mathcal{A}}\\&\leq|\phi_T^a(k_{i+1}-k_i,\phi_T^e(k_i,x_\circ))|_{\mathcal{A}}\\&\qquad+|\phi_T^e(k_{i+1}-k_i,\phi_T^e(k_i,x_\circ))-\phi_T^a(k_{i+1}-k_i,\phi_T^e(k_i,x_\circ))|\\&\leq|\phi_T^a(k_{i+1}-k_i,\phi_T^e(k_i,x_\circ))|_{\mathcal{A}}+\delta\cdot|\phi_T^e(k_i,x_\circ)|_{\mathcal{A}}\ .\end{aligned} \tag{6.19}$$

Using the item 2, (6.19), (6.18) and definitions of L_1 and L we can write:

$$\begin{aligned}|\phi_T^e(k_{i+1},x_\circ)|_{\mathcal{A}}&\leq K\exp(-\lambda(k_{i+1}-k_i)T)\,|\phi_T^e(k_i,x_\circ)|_{\mathcal{A}}+\delta\cdot|\phi_T^e(k_i,x_\circ)|_{\mathcal{A}}\\&=[K\exp(-\lambda\ell_{L_1,T}T)+\delta]\cdot|\phi_T^e(k_i,x_\circ)|_{\mathcal{A}}\\&\leq[K\exp(-\lambda L)+\delta]\cdot|\phi_T^e(k_i,x_\circ)|_{\mathcal{A}}\\&\leq[(c-\delta)+\delta]\cdot|\phi_T^e(k_i,x_\circ)|_{\mathcal{A}}\\&=c\cdot|\phi_T^e(k_i,x_\circ)|_{\mathcal{A}}\ .\end{aligned} \tag{6.20}$$

From (6.20), we conclude that for all $x_\circ$ and all $\phi_T^e\in\mathcal{S}_T^e(x_\circ)$ we have

$$|\phi_T^e(k_i,x_\circ)|_{\mathcal{A}}\ \leq\ c^i\cdot|x_\circ|_{\mathcal{A}}=\exp(-\tilde{\lambda}_1 i)\,|x_\circ|_{\mathcal{A}}\ , \tag{6.21}$$

for $\tilde{\lambda}_1:=\ln(\frac{1}{c})>0$. Using the definitions of k_i and $\ell_{L_1,T}$ and (6.18) we can write:

$$-\tilde{\lambda}_1 i=-\tilde{\lambda}_1\frac{k_i}{\ell_{L_1,T}}\leq-\frac{\tilde{\lambda}_1}{L_1}k_iT=-\lambda_1k_iT\ ,$$

where $\lambda_1 := \frac{\tilde{\lambda}_1}{L_1}$, and using (6.21), we obtain:

$$|\phi_T^e(k_i, x_\circ)|_{\mathcal{A}} \leq \exp(-\lambda_1 k_i T)\, |x_\circ|_{\mathcal{A}} \qquad \forall i = 0, 1, \ldots \tag{6.22}$$

Again, using item 1 with Lemma 6.1 and 2, we have for all $x_\circ$ and $\phi_T^e \in \mathcal{S}_T^e(x_\circ)$ that there exists $\phi_T^a \in \mathcal{S}_T^a(x_\circ)$ such that for all $k \in [k_i, k_{i+1}]$ we have:

$$\begin{aligned} |\phi_T^e(k, x_\circ)|_{\mathcal{A}} &= |\phi_T^e(k - k_i, \phi_T^e(k_i, x_\circ))|_{\mathcal{A}} \\ &\leq |\phi_T^a(k - k_i, \phi_T^e(k_i, x_\circ))|_{\mathcal{A}} \\ &\quad + |\phi_T^e(k - k_i, \phi_T^e(k_i, x_\circ)) - \phi_T^a(k - k_i, \phi_T^e(k_i, x_\circ))| \\ &\leq (K + \delta)\, |\phi_T^e(k_i, x_\circ))|_{\mathcal{A}} \ . \end{aligned} \tag{6.23}$$

Finally, using (6.22) and (6.23), we obtain:

$$\begin{aligned} |\phi_T^e(k, x_\circ)|_{\mathcal{A}} &\leq (K + \delta)\, |\phi_T^e(k_i, x_\circ))|_{\mathcal{A}} \\ &\leq (K + \delta) \exp(-\lambda_1 k_i T)\, |x_\circ|_{\mathcal{A}} \\ &\leq (K + \delta) \exp(\lambda_1 \ell_{L_1, T} T) \exp(-\lambda_1 k T)\, |x_\circ|_{\mathcal{A}} \\ &\leq (K + \delta) \exp(\lambda_1 L_1) \exp(-\lambda_1 k T)\, |x_\circ|_{\mathcal{A}} \\ &=: K_1 \exp(-\lambda_1 k T)\, |x_\circ|_{\mathcal{A}} \end{aligned} \tag{6.24}$$

which completes the proof.

6.3. Proof of Proposition 4.4. The proof follows the steps of [32, Lemma 1]. Note that GSTS implies forward completeness by definition. Hence, we need to prove the sufficiency part: that forward completeness implies GSTS when $\mathcal{A}$ is compact. Let $T^*, c, \sigma_1, \sigma_2$ come from forward completeness and let $T \in (0, T^*)$ be arbitrary. Since $\mathcal{A}$ is compact, then $\mathcal{H}_{\mathcal{A}}(0, 2)$ is compact. Let $M > 0$ be such that $\sup_{x \in \mathcal{H}_{\mathcal{A}}(0,2), w \in \mathcal{F}_T(x)} |w - x| \leq MT$. Then, we have that for every $x_\circ \in \mathcal{H}_{\mathcal{A}}(0, 1)$, $\phi_T \in \mathcal{S}_T(x_\circ)$ we have:

$$k \in \left[0, \ell_{M^{-1}, T}\right] \implies |\phi_T(k, x_\circ) - x_\circ| \leq MkT \ .$$

Since $|\cdot|_{\mathcal{A}}$ is globally Lipschitz with constant one, we can write:

$$|\phi_T(k, x_\circ)|_{\mathcal{A}} \leq |\phi_T(k, x_\circ) - x_\circ| + |x_\circ|_{\mathcal{A}} \ . \tag{6.25}$$

Hence, for all $r \in [0, \min\{1, M^{-1}\}]$ we have:

$$k \in [0, \ell_{r,T}],\ |x_\circ|_{\mathcal{A}} \leq r \implies |\phi_T(k, x_\circ)|_{\mathcal{A}} \leq (M + 1)r \ . \tag{6.26}$$

Denote $\nu := \max_{x \in \mathcal{A}} |x| < \infty$. Then, we have that $|x| \leq |x|_{\mathcal{A}} + \nu$ and $|x|_{\mathcal{A}} \leq |x| + \nu$ for any $x \in R^n$. Consider now arbitrary $r > 0$ and we have from forward completeness that

$$k \in [0, \ell_{r,T}],\ |x_\circ|_{\mathcal{A}} \leq r \implies |\phi_T(k, x_\circ)| \leq \sigma_1(r + \nu) + \sigma_2(r) + c =: \chi(r) \ , \tag{6.27}$$

which implies

$$k \in [0, \ell_{r,T}],\ |x_\circ|_{\mathcal{A}} \leq r \implies |\phi_T(k, x_\circ)|_{\mathcal{A}} \leq |\phi_T(k, x_\circ)| + \nu \leq \chi(r) + \nu =: \bar{\chi}(r). \tag{6.28}$$

Next, we define $\rho(s) := s(M+1) + b(s) \cdot \bar{\chi}(s)$ where $b : R \to [0,1]$ is an increasing continuous function such that $b(0) = 0$ and $b(s) = 1$, for all $s \geq \min\{1, M^{-1}\}$. Hence, we have $\rho \in \mathcal{K}_\infty$ and from (6.26) and (6.28) we have that for any $r \geq 0$, $T \in (0, T^*)$, $x_\circ \in R^n$ and $\phi_T \in \mathcal{S}_T(x_\circ)$:

$$k \in [0, \ell_{r,T}],\ |x_\circ|_{\mathcal{A}} \leq r \implies |\phi_T(k, x_\circ)|_{\mathcal{A}} \leq \rho(r)\ ,$$

which completes the proof.

6.4. Proof. of Theorem 4.5. In the sequel we refer to the function β defined in Remark 13.

1 $\implies$ 2(a) We have forward completeness from GAS and by defining $\tau(r) := r$ and $\rho(s) \geq \beta(s, 0), \forall s \geq 0$ we have that GSTS holds since:

$$\tau := \tau(r),\ k \in [0, \ell_{\tau,T}],\ |x_\circ|_{\mathcal{A}} \leq r \implies |\phi_T(k, x_\circ)|_{\mathcal{A}} \leq \beta(|x_\circ|_{\mathcal{A}}, 0) \leq \rho(|x_\circ|_{\mathcal{A}})\ .$$

1 $\implies$ 2(b) Let $T_1^* > 0$ and $\beta \in \mathcal{KL}$ come from GAS. Let $T_2^* > 0$ be such that

$$\frac{T}{1 - \exp(-T)} \leq 2, \qquad \forall T \in (0, T_2^*)\ . \tag{6.29}$$

Let $T \in (0, T^*)$, with $T^* := \min\{T_1^*, T_2^*\}$. From Sontag's Lemma [29, Lemma 8] there exist $\alpha, \gamma_1 \in \mathcal{K}_\infty$ such that

$$\alpha(\beta(s,t)) \leq \gamma_1(s)\exp(-t), \qquad \forall s, t \geq 0\ .$$

Hence, for GAS we can write for all $x_\circ$, $\phi_T \in \mathcal{S}_T(x_\circ)$:

$$\alpha(|\phi_T(k, x_\circ)|_{\mathcal{A}}) \leq \alpha(\beta(|x_\circ|_{\mathcal{A}}, kT)) \leq \gamma_1(|x_\circ|_{\mathcal{A}})\exp(-kT)$$

Summing both sides of the above equation for $k \geq 0$ and multiplying with T, we obtain:

$$\begin{aligned} T\sum_{k=0}^{\infty} \alpha(|\phi_T(k, x_\circ)|_{\mathcal{A}}) &\leq \gamma_1(|x_\circ|_{\mathcal{A}}) T \sum_{k=0}^{\infty} \exp(-kT) \\ &= \gamma_1(|x_\circ|_{\mathcal{A}}) \frac{T}{1 - \exp(-T)} \leq 2\gamma_1(|x_\circ|_{\mathcal{A}}) =: \gamma(|x_\circ|_{\mathcal{A}})\ , \end{aligned} \tag{6.30}$$

where the last inequality follows from the definition of T_2^*.

2 $\implies$ 3(a) Let T_1^*, ρ, τ come from item 2(a). Let T_2^*, α, γ come from the item 2(b). Let $T^* := \min\{T_1^*, T_2^*\}$ and $T \in (0, T^*)$. Let $\kappa \in \mathcal{K}_\infty$ be such that

$$\kappa^{-1}(s) \leq \min\left\{s, \gamma^{-1}\left(\frac{1}{2}\tau(s) \cdot \alpha(s)\right)\right\}$$

(this function always exists, see equation (17) in [32]). From the definition of κ, it follows that for all $s \geq 0$ we have $\kappa(s) \geq s$ and

$$\gamma(s) \leq \frac{1}{2}\tau \circ \kappa(s) \cdot \alpha \circ \kappa(s)\ . \tag{6.31}$$

We show that for all $x_\circ$ and all $\phi_T \in \mathcal{S}_T(x_\circ)$ we have:

$$|\phi_T(k, x_\circ)|_{\mathcal{A}} \leq \rho \circ \kappa(|x_\circ|_{\mathcal{A}}) \qquad \forall k \geq 0\ . \tag{6.32}$$

If $|x_\circ|_{\mathcal{A}} = 0$, then it follows from item 2(b) that $|\phi_T(k, x_\circ)| = 0, \forall k \geq 0$ hence, (6.32) holds. Suppose now that $|x_\circ|_{\mathcal{A}} > 0$. Also, for the purpose of showing contradiction suppose there exist $T \in (0, T^*)$, $x_\circ$ and $\phi_T \in \mathcal{S}_T(x_\circ)$ and $k_1 > 0$ such that

$$|\phi_T(k_1, x_\circ)|_{\mathcal{A}} > \rho \circ \kappa(|x_\circ|_{\mathcal{A}}) \ . \tag{6.33}$$

Since we have that $|\phi_T(0, x_\circ)|_{\mathcal{A}} = |x_\circ|_{\mathcal{A}} \leq \kappa(|x_\circ|_{\mathcal{A}})$, we have that $k_1 > 0$ and there exists $k_0 \in [0, k_1)$ such that

$$|\phi_T(k_0, x_\circ)|_{\mathcal{A}} \leq \kappa(|x_\circ|_{\mathcal{A}}) \tag{6.34}$$

$$|\phi_T(k, x_\circ)|_{\mathcal{A}} > \kappa(|x_\circ|_{\mathcal{A}}) \qquad \forall k \in [k_0 + 1, k_1] \ , \tag{6.35}$$

and (6.33) holds. From the item 2(a) we have:

$$T(k_1 - k_0) > \tau \circ \kappa(|x_\circ|_{\mathcal{A}}) \ . \tag{6.36}$$

Using (6.36), (6.34), (6.35) and the item 2(b), we can write:

$$\alpha \circ \kappa(|x_\circ|_{\mathcal{A}}) \cdot \tau \circ \kappa(|x_\circ|_{\mathcal{A}}) < T \sum_{k=k_0}^{k_1} \alpha(|\phi_T(k, x_\circ)|_{\mathcal{A}}) \leq \gamma(|x_\circ|_{\mathcal{A}}) \ ,$$

which contradicts (6.31).

2 $\implies$ 3(b)i. Let T_1^*, α, γ come from item 2(b). Let T_2^* come from (6.29) and define $T^* := \min\{T_1^*, T_2^*\}$. Let $T \in (0, T^*)$ and let arbitrary $0 < \delta \leq \Delta$ be given. Define $\omega_{\delta,\Delta}(x) := \alpha(|x|_{\mathcal{A}})$, $\omega_m := \alpha(\delta)$. Hence, we have:

$$x \in \mathcal{H}_{\mathcal{A}}(\delta, \Delta) \implies \omega_{\delta,\Delta}(x) = \alpha(|x|_{\mathcal{A}}) \geq \alpha(\delta) = \omega_m \ .$$

2 $\implies$ 3(b)ii. Using the above definitions of $T^*, \omega_m, \omega_{\delta,\Delta}$ and $\gamma := 2\gamma_1(\Delta)$ we have for all $T \in (0, T^*)$, $x_\circ \in \mathcal{H}_{\mathcal{A}}(\delta, \Delta)$, $\phi_T \in \mathcal{S}_T(x_\circ)$ and any $\tau > 0$:

$$T \sum_{k=0}^{\ell_{\tau,T}} \omega_{\delta,\Delta}(\phi_T(k, x_\circ)) \leq T \sum_{k=0}^{\infty} \alpha(|\phi_T(k, x_\circ)|_{\mathcal{A}}) \leq 2\gamma_1(|x_\circ|_{\mathcal{A}}) \leq 2\gamma_1(\Delta) = \gamma \ ,$$

where the second last inequality follows using the definition of T_2^* and similar arguments as to obtain (6.30).

3 $\implies$ 1 Note that GS is assumed and we only need to prove uniform attractivity. Let T_1^*, ρ come from item 3(a) and let T_2^* come from item 3(b). Let $T^* := \min\{T_1^*, T_2^*, 1\}$ and $T \in (0, T^*)$. From the item 3(a) we have that for all $x_\circ$ and all $\phi_T \in \mathcal{S}_T(x_\circ)$ the following holds:

$$|\phi_T(k, x_\circ)|_{\mathcal{A}} \leq \rho(|x_\circ|_{\mathcal{A}}) \qquad k \geq 0 \ . \tag{6.37}$$

Fix $r, \epsilon > 0$ and define $\Delta := \rho(r)$, $\delta := \min\{\Delta, \rho^{-1}(\epsilon)\}$. Let Δ, δ generate $\omega_{\delta,\Delta}(\cdot), \omega_m, \gamma$ and let $\tau := \frac{2\gamma}{\omega_m} + 1$. We claim that for all $x_\circ$, $\phi_T \in \mathcal{S}_T(x_\circ)$ there exists $k_1 \in [0, \ell_{\tau,T}]$ such that[5]

$$|\phi_T(k_1, x_\circ)|_{\mathcal{A}} \leq \rho^{-1}(\epsilon) \ .$$

[5]Note that because of (6.37), this is enough to conclude uniform attractivity.

For the purpose of showing contradiction, suppose that this is not true, that is, there exists $x_\circ$ and $\phi_T \in \mathcal{S}_T(x_\circ)$ such that

$$|\phi_T(k, x_\circ)|_{\mathcal{A}} > \rho^{-1}(\epsilon) \qquad \forall k \in [0, \ell_{\tau,T}] \ .$$

From the item 3(a) and definition of δ we have

$$\phi_T(k, x_\circ) \in \mathcal{H}_{\mathcal{A}}(\delta, \Delta) \qquad \forall k \in [0, \ell_{\tau,T}] \ ,$$

and using item 3(b)i, we have

$$\omega_{\delta,\Delta}(\phi_T(k, x_\circ)) \geq \mathcal{H}_{\mathcal{A}}(\delta, \Delta) \qquad \forall k \in [0, \ell_{\tau,T}] \ .$$

Hence, we can write:

$$T \sum_{k=0}^{\ell_{\tau,T}} \omega_{\delta,\Delta}(\phi_T(k, x_\circ)) \geq T\ell_{\tau,T}\omega_m \geq 2\gamma \ ,$$

which contradicts the item 3(b)ii (in the second last inequality we use the fact that since $T < 1$, then $T\ell_{\tau,T} > \tau - 1 = \frac{2\gamma}{\omega_m}$).

6.5. Proof of Proposition 4.7. Let $T_1^* > 0$ come from GS and $T_2^* > 0$ from the conditions of the proposition. Let $T^* := \min\{T_1^*, T_2^*\}$ and $T \in (0, T^*)$. Let $\delta \leq \Delta$ be arbitrary and let $\tilde{\Delta} := \rho(\Delta)$ where ρ comes from GS. Let $\delta, \tilde{\Delta}$ generate the numbers ω_m, ψ_1, ψ_2 and $\omega_{\delta,\tilde{\Delta}}$ via the conditions of the proposition (we can write $\omega_{\delta,\tilde{\Delta}} = \tilde{\omega}_{\delta,\Delta}$ since $\tilde{\Delta}$ depends on Δ). Note first that for any arbitrary $x_\circ \in \mathcal{H}_{\mathcal{A}}(0, \Delta)$ and $\phi_T \in \mathcal{S}_T(x_\circ)$ we have

$$\phi_T(k, x) \in \mathcal{H}_{\mathcal{A}}(0, \tilde{\Delta}) \qquad \forall k \geq 0 \ .$$

Hence, from the item 2(b) we can write for all $x_\circ \in \mathcal{H}_{\mathcal{A}}(\delta, \Delta)$ and $\phi_T \in \mathcal{S}_T(x_\circ)$ and $k \geq 0$:

$$T\tilde{\omega}_{\delta,\Delta}(\phi_T(k, x_\circ)) \leq V_T(\phi_T(k, x_\circ)) - \sup_{w \in \mathcal{F}_T(\phi_T(k,x_\circ))} V_T(w) + T\kappa(\phi_T(k, x_\circ)) \ . \tag{6.38}$$

Moreover, since $\phi(k, x_\circ) \in \mathcal{F}_T(\phi_T(k-1, x_\circ))$ for $k \geq 1$ we can also write using (6.38),

$$T\tilde{\omega}_{\delta,\Delta}(\phi_T(k, x_\circ)) \leq \sup_{w \in \mathcal{F}_T(\phi_T(k-1,x_\circ))} V_T(w) - \sup_{w \in \mathcal{F}_T(\phi_T(k,x_\circ))} V_T(w) + T\kappa(\phi_T(k, x_\circ)) \ . \tag{6.39}$$

Consider an arbitrary $\tau > 0$ and add both sides of the inequality (6.39) from $k = 1$ to $\ell_{\tau,T}$ to the inequality (6.38) with $k = 0$. Then, using the items 2(a) and 2(c) we have

$$\begin{aligned}
T \sum_{k=0}^{\ell_{\tau,T}} \tilde{\omega}_{\delta,\Delta}(\phi_T(k, x_\circ)) &\leq V_T(x_\circ) + \sum_{k=0}^{\ell_{\tau,T}-1} \sup_{w \in \mathcal{F}_T(\phi_T(k,x_\circ))} V_T(w) - \sum_{k=0}^{\ell_{\tau,T}} \sup_{w \in \mathcal{F}_T(\phi_T(k,x_\circ))} V_T(w) \\
&\quad + T \sum_{k=0}^{\ell_{\tau,T}} \kappa(\phi_T(k, x_\circ)) \\
&\leq V_T(x_\circ) - \sup_{w \in \mathcal{F}_T(\phi_T(\ell_{\tau,T},x_\circ))} V_T(w) + T \sum_{k=0}^{\ell_{\tau,T}} \kappa(\phi_T(k, x_\circ)) \\
&\leq |V_T(x_\circ)| + \left| \sup_{w \in \mathcal{F}_T(\phi_T(\ell_{\tau,T},x_\circ))} V_T(w) \right| + T \sum_{k=0}^{\ell_{\tau,T}} \kappa(\phi_T(k, x_\circ)) \\
&\leq 2\psi_1 + \psi_2 =: \gamma \ .
\end{aligned} \tag{6.40}$$

The conclusion follows from the proof **3** $\Longrightarrow$ **1** in Theorem 3.3.

6.6. Proof of Proposition 4.10. Let $T^* > 0$ come from the item (ii). Let $T \in (0, T^*)$ and note that the item (ii) implies that for all $x \in R^n$ we have:

$$\sup_{w \in \mathcal{F}_T(x)} |w|_{\mathcal{A}} \leq (1 + cT)\, |x|_{\mathcal{A}} \ .$$

By induction, this implies that for all $T \in (0, T^*)$, $x_\circ \in R^n$ and $\phi_T \in \mathcal{S}_T(x_\circ)$ we have:

$$|\phi_T(k, x_\circ)|_{\mathcal{A}} \leq (1 + cT)^k\, |x_\circ|_{\mathcal{A}} \leq \exp(ckT)\, |x_\circ|_{\mathcal{A}} \qquad \forall k \geq 0 \ . \tag{6.41}$$

Hence, the bound (4.8) holds with $\bar{\rho} = \exp(c)$ and $\tau = 1$. We only need to show that the system is forward complete. For the purpose of showing contradiction, suppose it is not. Then, it is not hard to see that there must exists $x_\circ$ and $\tau > 0$ such that (4.9) holds. From the item (i) then we also have that (4.10) holds but this then contradicts (6.41), which completes the proof.

6.7. Proof of Theorem 4.11. $1 \Longrightarrow 2$: Let λ, K and T_1^* come from the item 1. Let $T_2^* > 0$ be such that

$$\frac{T}{1 - \exp(-\lambda p T)} \leq \frac{2}{\lambda p} \qquad \forall T \in (0, T_2^*) \ .$$

Let $T^* := \min\{T_1^*, T_2^*\}$ and $T \in (0, T^*)$. It is immediate that $|\phi_T(k, x_\circ)|_{\mathcal{A}} \leq K\, |x_\circ|_{\mathcal{A}}$ hence, $\mathcal{A}$ is GFTS with linear gain. Moreover, for any $p > 0$, we can write:

$$|\phi_T(k, x_\circ)|_{\mathcal{A}}^p \leq K^p\, |x_\circ|_{\mathcal{A}}^p \exp(-\lambda p k T) \ ,$$

hence, we have

$$\begin{aligned} T \sum_{k=0}^{\infty} |\phi_T(k, x_\circ)|_{\mathcal{A}}^p &\leq T K^p\, |x_\circ|_{\mathcal{A}}^p \sum_{k=0}^{\infty} \exp(-\lambda p k T) \\ &= K^p \frac{T}{1 - \exp(-\lambda p T)}\, |x_\circ|_{\mathcal{A}}^p \ \leq \ \frac{2K^p}{\lambda p}\, |x_\circ|_{\mathcal{A}}^p \ . \end{aligned}$$

$2 \Longrightarrow 1$: Let τ, $\bar{\rho}$ come from 2(a) and c, p come from 2(b). Let T_1^* and T_2^* come respectively from items 2(a) and 2(b) of the theorem. Let $T^* := \min\{T_1^*, T_2^*\}$ and $T \in (0, T^*)$. Define the function

$$\kappa(s) := \max\left\{1, \left(\frac{2c}{\tau}\right)^{1/p}\right\} \cdot s \ .$$

Note that for all $s \geq 0$ and using the definition of T^*we have that:

$$\kappa(s) \geq s \tag{6.42}$$

$$c \cdot s^p \leq \frac{1}{2} \tau \cdot \kappa(s)^p \ . \tag{6.43}$$

We first show that for all $x_\circ \in R^n$ and $\phi_T(k, x_\circ) \in \mathcal{S}_T(x_\circ)$ we have

$$|\phi_T(k, x_\circ)|_{\mathcal{A}} \leq \bar{\rho} \cdot \kappa\left(|x_\circ|_{\mathcal{A}}\right) = \bar{\rho} \cdot \max\left\{1, \left(\frac{2c}{\tau}\right)^{1/p}\right\} \cdot |x_\circ|_{\mathcal{A}} =: \rho \cdot |x_\circ|_{\mathcal{A}} \ , \tag{6.44}$$

that is, the set $\mathcal{A}$ is GS. For the purpose of showing contradiction, suppose that there exist $T_1 \in (0, T^*)$, $x_\circ \in R^n$ and $k_1 \in N$ such that

$$|\phi_{T_1}(k_1, x_\circ)|_{\mathcal{A}} > \bar{\rho} \cdot \kappa(|x_\circ|_{\mathcal{A}})$$

Note that $\bar{\rho} \geq 1$ and as a result we have $|\phi_{T_1}(k_1, x_\circ)|_{\mathcal{A}} > \kappa(|x_\circ|_{\mathcal{A}})$. Define

$$k_0 := \min\{k \in [0, k_1] : |\phi_{T_1}(i, x_\circ)|_{\mathcal{A}} > \kappa(|x_\circ|_{\mathcal{A}}), \forall i \in [k, k_1]\} \ .$$

Because of (6.42) we have $|\phi_T(0, x_\circ)|_{\mathcal{A}} = |x_\circ|_{\mathcal{A}} \leq \kappa(|x_\circ|_{\mathcal{A}})$ and hence we have $k_1, k_0 > 0$ (it may happen that $k_0 = k_1$!). To summarize, we have:

$$|\phi_{T_1}(i, x_\circ)|_{\mathcal{A}} > \kappa(|x_\circ|_{\mathcal{A}}) \qquad \forall i \in [k_0, k_1] \tag{6.45}$$

$$|\phi_{T_1}(k_1, x_\circ)|_{\mathcal{A}} > \bar{\rho} \cdot \kappa(|x_\circ|_{\mathcal{A}}) \tag{6.46}$$

$$|\phi_{T_1}(k_0 - 1, x_\circ)|_{\mathcal{A}} \leq \kappa(|x_\circ|_{\mathcal{A}}) \ . \tag{6.47}$$

From (6.46) and (6.47) and the item 2(a) (i.e. the definition of GFTS), we have that

$$T_1(k_1 - k_0 + 1) \geq \tau \ . \tag{6.48}$$

Next, using the item 2(b) and (6.48), we have:

$$\begin{aligned} \tau\kappa(|x_\circ|_{\mathcal{A}})^p &\leq T_1(k_1 - k_0 + 1)\kappa(|x_\circ|_{\mathcal{A}})^p \\ &\leq T_1 \sum_{i=k_0}^{k_1} |\phi_{T_1}(i, x_\circ)|_{\mathcal{A}}^p \\ &\leq c \cdot |x_\circ|_{\mathcal{A}}^p \ , \end{aligned}$$

which contradicts (6.43).

Let $\lambda \in (0, 1)$ and define $\Delta := \frac{\rho^p c}{\lambda^p}$ where ρ comes from (6.44) and c comes from (4.12). Define $\Delta_T := \lfloor \frac{\Delta}{T} \rfloor$, where $z = \lfloor s \rfloor$ is the largest integer that is smaller than $s \in R$. First, we show that for all $T \in (0, T^*)$, $x_\circ$ and all $\phi_T \in \mathcal{S}_T(x_\circ)$ we have:

$$|\phi_T(\Delta_T, x_\circ)|_{\mathcal{A}} \leq \lambda \cdot |x_\circ|_{\mathcal{A}} \ . \tag{6.49}$$

Note that because of (6.44), it is enough to show that under above conditions there exists $k' \in [0, \Delta_T]$ such that

$$|\phi_T(k', x_\circ)|_{\mathcal{A}} \leq \frac{\lambda}{\rho} |x_\circ|_{\mathcal{A}} \ .$$

For the purpose of showing contradiction, assume the opposite. That is, there exists $x_\circ$ and $T \in (0, T^*)$ and $\phi_T \in \mathcal{S}(x_\circ)$ such that $|\phi_T(k, x_\circ)|_{\mathcal{A}} > \frac{\lambda}{\rho} |x_\circ|_{\mathcal{A}}$ for all $k \in [0, \Delta_T]$. Then, we have

$$T \sum_{i=0}^{\infty} |\phi_T(i, x_\circ)|_{\mathcal{A}}^p \geq T \sum_{i=0}^{\Delta_T} |\phi_T(i, x_\circ)|_{\mathcal{A}}^p > T(\Delta_T + 1) \frac{\lambda^p}{\rho^p} |x_\circ|_{\mathcal{A}}^p \geq \Delta \frac{\lambda^p}{\rho^p} |x_\circ|_{\mathcal{A}}^p = c \cdot |x_\circ|_{\mathcal{A}}^p \ ,$$

which contradicts (4.12). Hence, (6.49) holds. Define $\gamma := -\frac{1}{\Delta} \ln(\lambda)$ and note that for each $N \in N$, we have

$$\lambda^N = e^{-\gamma N \Delta} \tag{6.50}$$

We claim that for all $x_\circ$, $T \in (0, T^*)$ and $\phi_T \in \mathcal{S}_T(x_\circ)$, we have for all $k \geq 0$:

$$|\phi_T(k, x_\circ)|_{\mathcal{A}} \leq \frac{\rho}{\lambda} |x_\circ|_{\mathcal{A}} \exp(-\gamma k T) \ . \tag{6.51}$$

For $k \in [0, \Delta_T]$, this follows from (6.44) and (6.50) with $N = 1$: we have for all $k \in [0, \Delta_T]$ that $\exp(-\gamma k T) \geq \exp(-\gamma \Delta_T T) \geq \exp(-\gamma \Delta) = \lambda$. For $k \geq \Delta_T$, let $N \geq 1$ be the largest integer such that $k \geq N \Delta_T$. Using (6.49), time invariance of the system, (6.51) for $k \in [0, \Delta_T]$, $k - \lfloor N\Delta/T \rfloor \in [0, \Delta_T]$ and (6.50), we can write:

$$\begin{aligned} |\phi_T(k, x_\circ)|_{\mathcal{A}} &\leq \lambda^N |\phi_T(k - \lfloor N\Delta/T \rfloor, x_\circ)|_{\mathcal{A}} \\ &\leq \lambda^N \frac{\rho}{\lambda} |x_\circ|_{\mathcal{A}} \exp(-\gamma T (k - \lfloor N\Delta/T \rfloor)) \\ &\leq \lambda^N \exp(\gamma N \Delta) \frac{\rho}{\lambda} |x_\circ|_{\mathcal{A}} \exp(-\gamma T k) \\ &\leq \frac{\rho}{\lambda} |x_\circ|_{\mathcal{A}} \exp(-\gamma T k) \ , \end{aligned}$$

which completes the proof.

6.8. Proof of Proposition 4.13. In a similar way as in the proof of Proposition 4.7, we can obtain from item 2 that for arbitrary $\tau > 0$, $T \in (0, T^*)$, $x_\circ \in R^n$ and $\phi_T \in \mathcal{S}_T(x_\circ)$ we have:

$$\psi_2 T \sum_{k=0}^{\ell_{\tau,T}} |\phi_T(k, x_\circ)|_{\mathcal{A}}^p \leq \left| \sup_{w \in \mathcal{F}_T(\phi_T(\ell_{\tau,T}, x_\circ))} V_T(w) \right| + |V_T(x)| + T \sum_{k=0}^{\ell_{\tau,T}} \kappa(\phi_T(k, x_\circ)) \ .$$

Then, using items 1 and 3 and GES with linear gain property, we have:

$$T \sum_{k=0}^{\ell_{\tau,T}} |\phi_T(k, x_\circ)|_{\mathcal{A}}^p \leq c \, |x_\circ|_{\mathcal{A}}^p \ ,$$

where $c = \frac{\psi_1 + \psi_1 \bar{\rho}^p + \psi_3}{\psi_2}$. The conclusion follows from Theorem 4.11.

6.9. Proof of Proposition 5.2. Follows exactly the same steps as the proof of [32, Lemma 3] by using sums instead of integrals.

6.10. Proof of Proposition 5.3. We prove the result by showing that all conditions of Proposition 4.7 hold.

Let $\rho \in \mathcal{K}_\infty$ and $T_1^* > 0$ come from the GS assumption. Let $T_2^* > 0$, $V_T(\cdot)$ and $\alpha_i(\cdot)$ come from item 3. Let arbitrary $0 < \delta \leq \Delta$ be given. Let $\Delta_1 := \rho(\Delta)$ and let $M > 0$ and $T_3^* \in (0, T^\star)$ be such that for all $x \in \mathcal{H}_{\mathcal{A}}(0, \Delta_1)$ and $T \in (0, T_3^*)$ we have

$$\sup_{w \in \tilde{\mathcal{F}}_T(x) + TK(x)\bar{B}_n} |w| \leq M$$

Let $\tilde{\Delta} := \max\{M, \Delta_1\}$ generate L, T_4^* via the item 4 and let the item 5 generate T_5^*. Let $T^* := \min\{T_1^*, T_2^*, T_3^*, T_4^*, T_5^*\}$ and $T \in (0, T^*)$.

Let $\omega_m > 0$ be such that

$$\alpha_3(s) \geq 2\omega_m \qquad \forall s \in [\delta, \Delta] \ .$$

and define $\omega_{\delta,\Delta}(x) := \alpha_3(|x|_{\mathcal{A}}) - \omega_m$. Note that the definition of ω_m implies that the item 1 of Proposition 4.7 holds. Moreover, from the item 3, we have that for all $T \in (0, T^*)$ and $x \in \mathcal{H}_{\mathcal{A}}(0, \Delta)$:

$$\max\left\{\left|\sup_{w\in\tilde{\mathcal{F}}_T(x)} V_T(w)\right|, |V_T(x)|\right\} \leq \alpha_2(\tilde{\Delta}) =: \psi_1 \ ,$$

which implies that the item 2(a) of Proposition 4.7 holds. Let $\epsilon := \frac{\omega_m}{L}$ and let ϵ and Δ generate β via the item 5. We also define $\kappa(\cdot) := L(K(\cdot) - \epsilon)$. Using these definitions and the items 3 and 4, we can write for all $x \in \mathcal{H}_{\mathcal{A}}(0, \Delta)$ and all $T \in (0, T^*)$:

$$\begin{aligned}
\sup_{w\in\mathcal{F}_T(x)} V_T(w) &\leq \sup_{w\in\tilde{\mathcal{F}}_T(x)+TK(x)\bar{B}_n} V_T(w) \\
&\leq \sup_{w\in\tilde{\mathcal{F}}_T(x)} V_T(w) + \sup_{w\in\tilde{\mathcal{F}}_T(x)+TK(x)\bar{B}_n} V_T(w) - \sup_{w\in\tilde{\mathcal{F}}_T(x)} V_T(w) \\
&\leq -T\alpha_3(|x|_{\mathcal{A}}) + V_T(x) + TLK(x) \\
&\leq -[\alpha_3(|x|_{\mathcal{A}}) - \omega_m] + V_T(x) + TL\left[K(x) - \frac{\omega_m}{L}\right] \\
&\leq -T\omega_{\delta,\Delta}(x) + V_T(x) + T\kappa(x) \ ,
\end{aligned}$$

which implies that the item 2(b) of Proposition 4.7 holds. Finally, from the item 5 we have that item 2(c) of Proposition 4.7 trivially holds.

7. Conclusions. We presented a framework for stabilization of arbitrary closed (not necessarily compact) sets for nonlinear sampled-data differential inclusions. Our main results (Theorem 3.3 and 3.6) present stability conditions that guarantee SPA stability or GES of an arbitrary closed set for the exact discrete-time model of the sampled-data inclusion that can be checked without knowing the exact discrete-time model. Theorem 3.3 generalizes [19, Theorem 1] in several directions: we consider sampled-data differential inclusions, arbitrary dynamic controllers represented as discrete-time difference inclusions and we consider stability of arbitrary closed sets. We are not aware whether Theorem 3.6 has been published previously in the literature even in the simpler case of sampled-data differential equations, static controllers and stability of the origin. These results are proved via trajectory based techniques and they do not use the knowledge of a Lyapunov function for the approximate discrete-time model, which was a standing assumption in [21].

In the second part of the paper we presented several non-Lyapunov based conditions for achieving GAS or GES of the family of approximate closed loops. These results are discrete-time versions of results in [32] and they are an important addition to the toolbox that the control designer can use to design controllers for sampled-data nonlinear systems via their approximate discrete-time models, especially in cases when it is not easy to construct a strict Lyapunov function for the family of approximate discrete-time models.

REFERENCES

[1] M. Arcak and D. Nešić, "A framework for nonlinear sampled-data observer design via approximate discrete time models an emulation", Automatica, vol. 40 (2004), pp. 1931-1938.
[2] J-P. Aubin and H. Frankowska, *Set-valued analysis*. Birkhäuser: Boston, 1990.
[3] J.S. Baras and N.S. Patel, *Robust control of set-valued discrete-time dynamical systems*, IEEE Trans. Automat. Contr., 43 (1998), pp. 61-75.

[4] T. Chen and B. A. Francis, *Optimal sampled-data control systems*. Springer-Verlag: London, 1995.
[5] Gyurkovics E, Elaiw, "Stabilization of sampled-data nonlinear systems by receding horizon control via discrete-time approximations", Automatica, vol. 40 (2004), pp. 2017-2028.
[6] A. F. Filippov, *Differential equations with discontinuous right-hand side*. Kluwer Academic, Dordrecht, 1988.
[7] L. Grüne and D. Nešić, "Optimization based stabilization of sampled-data nonlinear systems via their approximate discrete-time models", SIAM J. Opt. Contr., vol. 42 (2003), pp. 98-122.
[8] L. Grüne, *Asymptotic behavior of dynamical and control systems under perturbation and discretization*, Habilitationsschrift, J. W. Goethe University, 2001.
[9] H. K. Khalil, *Nonlinear systems*. Prentice-Hall: New Jersey, 1996.
[10] P.E. Kloeden and J. Lorenz, "Stable attracting sets in dynamical systems and their one-step discretizations", SIAM J. Num. Anal., 23 (1986), pp. 986-995.
[11] P.E. Kloeden and B. Schmalfuss, "Asymptotic behavior of non autonomous difference inclusions", Syst. Contr. Lett., 33 (1998), pp. 275-280.
[12] Y. Lin, E.D. Sontag and Y. Wang, "A smooth converse Lyapunov theorem for robust stability", SIAM J. Contr. Optimiz., vol. 34 (1996), pp. 124-160.
[13] D. S. Laila and A. Astolfi, "Input-to-state stability for discrete-time time-varying systems with applications to robust stabilization of systems in power form", Automatica, vol. 41, (2005), pp. 1891-1903.
[14] D. S. Laila and A. Astolfi, "On the construction of discrete-time model for port-controlled Hamiltonian systems with applications", Syst. Contr. Lett., vol. 55 (2004), pp. 673-680.
[15] D.S.Laila and D. Nešić, "Changing supply rates for input-output to state stable discrete-time nonlinear systems with applications", Automatica, vol. 39 (2001), pp. 821-835.
[16] D.S. Laila, D. Nešić and A.R. Teel, "Open and closed loop dissipation inequalities under sampling and controller emulation", Europ. J. Contr., vol. 18 (2002), pp. 109-125.
[17] A. Loría F. Lamnabhi-Lagarrigue, and D. Nešić, "Summation-type conditions for uniform asymptotic convergence in discrete-time systems: applications in identification," in *Proc. 44th. IEEE Conf. Decision Contr.*, pp. 6591–6595, 2005.
[18] D. Nešić, A. R. Teel and E.D.Sontag, "Formulas relating $\mathcal{KL}$ stability estimates of discrete-time and sampled-data nonlinear systems", Syst. Contr. Lett., vol. 38 (1999), pp. 49-60.
[19] D. Nešić, A.R. Teel and P. V. Kokotović, "Sufficient conditions for stabilization of sampled-data nonlinear systems via discrete-time approximations", Sys. Contr. Lett., vol. 38 (1999), pp. 259-270.
[20] D. Nešić and D. Angeli, "Integral versions of ISS for sampled-data nonlinear systems via their approximate discrete-time models", IEEE Trans. Automat. Contr. Vol. 47 (2002), pp. 2033-2038.
[21] D. Nešić and A.R. Teel, "A framework for stabilization of nonlinear sampled-data systems based on their approximate discrete-time models", IEEE Trans. Automat. Contr., vol. 49 (2004), pp. 1103-1034.
[22] D. Nešić and A.R. Teel, "Matrosov theorem for parameterized families of discrete-time systems", Automatica, vol. 40 (2004), pp. 1025-1034.
[23] D. Nešić and A. Loria, "On uniform asymptotic stability of time-varying parameterized discrete-time cascades", IEEE Trans. Automt. Contr., vol. 49 (2004), pp. 875-887.
[24] A. Loria and D. Nešić, "On uniform boundedness of parameterized discrete-time cascades with decaying inputs: applications to cascades", Syst. Contr. Lett., vol. 49 (2003), pp. 163-174.
[25] D. Nešić and A.R. Teel, "Stabilization of sampled-data nonlinear systems via backstepping on their Euler approximate model", Automatica, vol. 42 (2006), pp. 1801-1808.
[26] D. Nešić and D.S.Laila, "A note on input-to-state stabilization for nonlinear sampled-data systems", IEEE Trans. Automat. Contr., vol. 47 (2002), pp. 1153-1158.
[27] D. Nešić and A. R. Teel, "Changing supply rates in input-to-state stable systems: the discrete-time case", IEEE Trans. Automt. Contr., vol. 46 (2001), pp. 960-962.
[28] E. Panteley, A. Loría and A. Teel, "Relaxed persistency of excitation for uniform asymptotic stability," *IEEE Trans. on Automat. Contr.*, vol. 46, no. 12, pp. 1874–1886, 2001.
[29] E. Sontag. *Comments on integral variants of ISS.* Systems & Control Letters, 34 (1998), 93-100.
[30] A. M. Stuart and A. R. Humphries, *Dynamical systems and numerical analysis*. Cambridge University Press: New York, 1996.
[31] A. R. Teel, L. Moreau and D. Nešić, "A unification of time-scale methods for systems with disturbances", IEEE Trans. Automat. Contr, vol.48 (2003), pp. 1526-1544.
[32] A. R. Teel, E. Panteley and A. Loria, "Integral characterizations of uniform asymptotic stability

and exponential stability with applications", Math. Contr. Sig. Sys., vol. 15 (2002), pp. 177–201.

[33] A. Donchev and F. Lempio, "Difference methods for differential inclusions: a survey", SIAM Review, vol. 34 (1992), pp. 263-294.