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► **To cite this version:**

Salvatore Monaco, Dorothee Normand-Cyrot. Linearization by Output Injection under Approximate Sampling. European Journal of Control, 2009, 15 (2), pp.205-217. 10.3166/ejc.15.205-217. hal-00447177

HAL Id: hal-00447177

<https://hal.science/hal-00447177>

Submitted on 21 Jun 2022

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Linearization by Output Injection under Approximate Sampling

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Linearization by output injection has been studied in both continuous time and discrete-time contexts. In this paper we discuss the possible preservation of such a property under sampling. It is shown that it can be maintained under approximate sampling up to the order of the system itself.

Keywords: Observer forms, nonlinear sampled-data systems, discrete-time systems, nonlinear observer design.

1. Introduction

Linearization by coordinates change and output injection has been firstly studied in a continuous-time context in [16], [17], and further extended and developed as a basic tool for observer design (we refer to [13] for survey on the subject). Even less popular, the same problem has been investigated in discrete time pursuing different approaches (see [7, 11, 15, 19, 20, 29]). While techniques and results are similar when dealing with maps or vector fields, that is for continuous-time or discrete-time uncontrolled dynamics, specialized studies are necessary when controlled equations are investigated.

In this paper linearization by output injection is revised as the dual problem of linearization under state feedback ([5, 9, 14, 18, 21]). The geometric conditions ensuring linearization through coordinates

change and feedback or output injection admit similar formulations for both continuous-time and discrete-time systems. Such a similarity is possible making use of an alternate representation of controlled discrete-time dynamics as two coupled differential/difference equations rather than a map parameterized by the control. In this context the authors showed how results on structural and control properties admit similar formulations [23]. In this formalism, feedback linearization was studied in [24], starting from the geometric conditions to the computations of structural invariants and controller normal forms, possibly through successive approximations of increasing degree. Following the same lines, normal forms associated with linearization under coordinates change and output injection, have been recently computed making use of successive transformations of increasing degree in [3, 26].

These analogies in the results' formulation legitimate the question of their preservation under sampling, i. e. when applied to the discrete-time model issued from the sampling of the continuous system: a problem which has been widely investigated in the literature. Different sampling procedures give rise to sampled equivalent models which may exhibit different characteristics w. r. to the preservation of some continuous time properties. For, multirate or higher order holding procedures have been introduced to maintain properties under sampling ([10], [27]). As far as feedback linearization is concerned, the problem has been firstly addressed starting from [1] and then studied in an approximate context in [2] and [25]. In these papers, more strictly linked to the present

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contribution, we showed that feedback linearization can be preserved under sampling up to the system order making use of a new filtered output mapping. Moreover, multirate control strategies or multirate sampling procedures have been proposed to enlarge the degree of approximation or to achieve exact solutions [10].

On these bases, we show in this paper that equivalence through coordinates change to the observer form can be preserved under sampling up to the system order, say n , the result is constructive for the coordinates change. It relies on the existence of a fictitious vector field with respect to which the given output mapping has relative degree equal to n . It is immediately understood that this result is the dual of the one in [2] where feedback linearization is achieved through the existence of a fictitious output mapping which has relative degree equal to n . The effects of sampling over the observer design have been investigated in [8] and [28]; in particular in [8], the authors proved that for systems over R^2 , linearization by output injection under sampling is in fact equivalent to linear equivalence. The present work shows that preservation under time-sampling is in fact possible over R^n when considering the approximate sampled model at the order n in the sampling time δ . This result suggests to use the approximate model of order n for designing the sampled-data observer since increasing the accuracy of the sampled model by considering higher order terms would only bring to additional computational complexity even if linearization by output injection could be assured. The present work is based on [6] regarding geometric necessary and sufficient conditions for equivalence to the linear observer form through coordinates change and to [25] where a complete description of the sampled equivalent to an input-affine system is given. This paper represents an extended version of [27].

The paper is organized as follows. Section 2 sets the problem and recalls the geometric conditions ensuring linearization through coordinates change and output injection in the continuous-time and discrete-time contexts. Linearization by output injection under sampling is set in the differential/difference context in section 3 where the conditions of section 2 are specified on the sampled equivalent model. In section 4 the main result is given and a constructive procedure for computing the coordinates change which solves the problem is detailed for unforced and forced dynamics with extra conditions on the output injection. The algorithm is worked out on an example: an elementary application in state estimation of the duffing oscillator. Some conclusions end the paper.

Some standard notations : Given τ , a vector field over R^n , L_τ denotes the associated formal Lie derivative which acts over real valued functions $h: R^n \rightarrow R$ as a first order differential operator $L_\tau h := dh\tau$; the Lie bracket $\text{ad}_{\tau_1}\tau_2 := [\tau_1, \tau_2]$ between two vector fields over R^n is described by the non commuting product $L_{[\tau_1, \tau_2]} = L_{\tau_1}L_{\tau_2} - L_{\tau_2}L_{\tau_1}$; the Lie series associated with L_τ is defined by its exponential expansion $e^{L_\tau} := 1 + \sum_{i \geq 1} \frac{L_\tau^i}{i!}$, usually denoted as e^τ , which gives $e^\tau h(x) = h(e^\tau x)$; given a diffeomorphism Φ on R^n , $\text{Ad}_\Phi \tau$ denotes the transport of τ along Φ satisfying $\text{Ad}_\Phi \tau := ([J_x \Phi] \tau)|_{\Phi^{-1}} = \left(\frac{d\Phi}{dx} \cdot \tau\right)|_{\Phi^{-1}}$. All maps and vector fields, possibly parameterized, are assumed analytic. The manipulations performed over the asymptotic series expansions are formal ones, no convergence studies are performed.

2. Linearization by Output Injection

2.1. In Continuous Time

Let Σ_c denote a SISO input-affine system

$$\begin{aligned} \dot{x}(t) &= f(x(t)) + g(x(t))u(t) \\ y(t) &= h(x(t)) \end{aligned} \quad (1)$$

where $x \in R^n$, $y \in R$, f and g are analytic vector fields and h an analytic mapping. Assuming, without any loss of generality, that $(0, 0)$ is an equilibrium pair, $f(0) = 0$, $h(0) = 0$, *linearization by output injection* stands for the equivalence under coordinates change to the *observer canonical form* COF

$$\begin{aligned} \dot{z}(t) &= A_O z(t) + \psi(y(t), u(t)) \\ y(t) &= C_O z(t) \end{aligned} \quad (2)$$

with (A_O, C_O) in the Brunovsky observability form

$$A_O = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ 1 & \ddots & & & \vdots \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix},$$

$$C_O = (0 \quad \cdots \quad 0 \quad 1)$$

and $\psi(y, u) : R \times R \rightarrow R^n$, an analytic map characterizing the input-output injection: i.e. the injection on the dynamics which should render linear the dynamics itself.

The problem, referred to as *linearization by output injection* as opposed to *feedback linearization*, is of interest in the design of nonlinear observers. It results from (2) that

$$\dot{\hat{z}}(t) = A_O \hat{z}(t) + \psi(y(t), u(t)) + K(y - C_O \hat{z})$$

estimates the state with a linear error dynamics:

$$\dot{e}(t) = (A_O - KC_O)e(t).$$

Conditions ensuring local equivalence of Σ_c to (2) are well known.

Theorem 2.1: [16, 17] Σ_c is locally equivalent to the observer canonical form (2) if and only if

- $A_{c1} : \rho(\mathrm{d}h, \mathrm{d}L_f h, \dots, \mathrm{d}L_f^{n-1} h)(0) = n;$
- $A_{c2} : [r_{ci}, r_{cj}] = 0 \quad \text{for } i+j \in [2, 2n-1];$
- $A_{c3} : [g, r_{ci}] = 0 \quad \text{for } i \in [1, n-1];$

where r_{c1} is the vector field solution of

$$\mathbf{dH}r_{c1} := \begin{pmatrix} \mathrm{d}h \\ \vdots \\ \mathrm{d}L_f h \\ \mathrm{d}L_f^{n-1} h \end{pmatrix} r_{c1} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \quad (3)$$

and $r_{ci} := -\mathrm{ad}_f r_{c(i-1)} = (-1)^{i-1} \mathrm{ad}_f^{i-1} r_{c1}$ for $i \in [2, n]$. Moreover, the coordinates change $\Phi(x)$ satisfies

$$[\mathbf{J}_x \Phi] \times (r_{c1} \quad \dots \quad r_{cn}) = I_{dn}$$

with I_{dn} the $n \times n$ -identity matrix.

Remark: Applying the Jacobian identity, condition A_{c2} is recognized to be equivalent to $[r_{c1}, r_{ci}] = 0$ for $i \in [2, 2n-2]$.

Remark: In the uncontrolled case ($g=0$) in (1), under the stronger condition $A'_{c2} : [r_{ci}, r_{cj}] = 0$ for $i+j \in [2, 2n+1]$ (equivalently $[r_{c1}, r_{ci}] = 0$ for $i \in [2, 2n]$), one gets linear equivalence under linear output injection.

Remark: The ‘‘fictitious’’ controlled dynamics $\dot{x}(t) = f(x) + ur_{c1}(x)$ with output mapping $y = h(x)$ exhibits a relative degree equal to n , ($L_{r_{c1}} h = 0, L_{r_{c2}} h = 0, \dots, L_{r_{c(n-1)}} h = 0$) with moreover $L_{r_{cn}} h = 1$.

2.2. In Discrete Time

A discrete-time dynamics is usually represented as a map

$$x_k \rightarrow x_{k+1} = F(x_k, u_k) \quad (4)$$

where $x \in R^n, u \in R$ and $F(\cdot, u)$ is an analytic map, analytically parameterized by u . In [22], we proposed

an alternate representation as two differential/difference equations - DDR

$$x^+ = F_0(x) \quad (5)$$

$$\frac{\mathrm{d}x^+(u)}{\mathrm{d}u} = G(x^+(u), u) \quad \text{with } x^+(0) = x^+ \quad (6)$$

where $x^+(u)$ indicates a curve in R^n parameterized by u . (4) and (5–6) describe the same discrete-time dynamics provided that $G(x, u)$ satisfies

$$G(F(x, u), u) := \frac{\partial F(x, u)}{\partial u}. \quad (7)$$

Indeed, the integration of (6) with respect to u between 0 and u_k with initial condition specified by (5), $x^+(0) = F_0(x_k) := F(x_k, 0)$, returns x_{k+1} in (4), i.e.

$$x^+(u_k) = x^+(0) + \int_0^{u_k} G(x^+(\nu), \nu) \mathrm{d}\nu = F(x_k, u_k).$$

In the same way w.r. to an output mapping $y = h(x)$, one has

$$\begin{aligned} y_{k+1} &= h(x^+(0)) + \int_0^{u_k} L_{G(\cdot, \nu)} h(x^+(\nu)) \mathrm{d}\nu \\ &= h(x^+(u_k)) = h(F(x_k, u_k)). \end{aligned}$$

The Taylor expansion of $G(x, u)$ in powers of u

$$G(x, u) = G_0(x) + \sum_{i \geq 1} \frac{u^i}{i!} G_i(x) \quad (8)$$

gets the analytic vector fields $G_i(x)$, $i \geq 0$, which play a fundamental role in the geometric characterization of the properties under study.

Let Σ_d denote the DDR (5-6) with output map $y = h(x)$, and define the *canonical observer differential/difference representation (CO-DDR)* as the DDR of the COF (2). Denoting by ψ_i the i -th component of ψ , and assuming, without loss of generality, that $\frac{\partial \psi_1(y, u)}{\partial y} \Big|_{(0,0)} \neq 0$, which can be achieved through a possible preliminary linear output injection. One has

Proposition 2.1: *The discrete-time observer canonical form*

$$\begin{aligned} z_{k+1} &= A_O z_k + \psi(y_k, u_k) \\ y &= C_O z \end{aligned} \quad (9)$$

admits the DDR

$$z^+ = A_O z + \psi(y) \quad (10)$$

$$\frac{dz^+(u)}{du} = G(z_1^+(u), u); \quad z^+(0) = z^+ \quad (11)$$

$$y = C_O z$$

with (A_O, C_O) in the Brunovsky form, $\psi(y) = \psi(y, 0)$ and

$$G(z_1^+(u), u) = \frac{\partial (\cdot, u)}{\partial u} \Big|_{\psi_1^{-1}(z_1^+(u), u)}$$

where $\psi_1^{-1}(\cdot, u)$ indicates the inverse of $\psi_1(\cdot, u)$, the first component of $\psi(\cdot, u)$.

Proof: The result is an immediate consequence of the definition of the differential equation and the particular form of the matrix A_O so getting with $z_1^+(0) = 1(y)$

$$(y, u) = \psi(y) + \int_0^u G(z_1^+(v), v) dv. \quad \triangleleft$$

Remark: The input-output injection term (11) takes the form

$$\beta(y, u) = \frac{\partial (y, u)}{\partial u} = G(z_1^+(u), u) = G(1(y, u), u).$$

Theorem 2.2: [6] Σ_d is locally equivalent to the discrete-time observer canonical form if and only if

- $A_{d1} : \rho(dh, d(h \circ F_0), \dots, d(h \circ F_0^{n-1}))(0) = n;$
- $A_{d2} : [r_{d1}, r_{di}] = 0; \quad i \in [2, n]$
- $A_{d3} : [G_p, r_{di}] = 0 \quad \text{for } i \in [2, n], \quad p \geq 0;$

where the vector field r_{d1} satisfies

$$d\mathbf{H}_d r_{d1} := \begin{pmatrix} dh \\ \vdots \\ dh \circ F_0^{n-2} \\ dh \circ F_0^{n-1} \end{pmatrix} r_{d1} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \quad (12)$$

and for $i \in [2, n]$, r_{di} 's are given by

$$r_{di} = \text{Ad}_{F_0} r_{d(i-1)} = \text{Ad}_{F_0^{i-1}} r_{d1}.$$

Moreover the coordinates change, Φ_d , satisfies

$$[J_x \Phi_d](r_{d1} \dots r_{dn}) = I_{dn}.$$

Condition A_{d1} is the observability rank condition which ensures the existence and uniqueness of r_{d1} satisfying (12); A_{d2} requires first order nilpotency of the distribution generated by the vector fields (r_{d1}, \dots, r_{dn}) and guarantees the existence of a coordinates change as well as the specific structure of (10); A_{d3} guarantees the specific structure (11).

It is worthy to note the strict analogy between statements of Theorem 2.1 and Theorem 2.2.

Remark: The fictitious controlled dynamics

$$x^+(0) = F_0(x)$$

$$\frac{dx^+(u)}{du} = r_{d1}(x^+(u))$$

with output mapping $y = h(x)$ has relative degree equal to n ; i.e. using the DDR formalism ($L_{r_{d1}} h = 0, L_{r_{d2}} h = 0, \dots, L_{r_{dn-1}} h = 0$) with moreover $L_{r_{dn}} h = 1 \neq 0$.

3. Linearization by Output Injection Under Sampling

Assuming the control variable $u(t)$ constant over small time intervals of amplitude δ , with value u_k over $[k\delta, (k+1)\delta]$ for $k \geq 0$, the sampled equivalent system, Σ_s^δ , is the discrete-time system which reproduces the state and output evolutions of Σ_c at the sampling instants for any $x_0 = x(t=0)$. As well known the discrete-time dynamics is drift invertible for small δ , i.e. $F_0^\delta(x) = e^{\delta f}(x)$ and $(F_0^\delta)^{-1}(x) = e^{-\delta f}(x)$. As a consequence $G^\delta(x, u)$ satisfying (7) exists and it is uniquely defined by $G^\delta(x, u) := \frac{\partial F^\delta(x, u)}{\partial u} \Big|_{x=F^{-\delta}(x, u)}$ with $F^\delta(x, u) = e^{\delta f + u \delta g}(x)$. It is a matter of computations to verify that the DDR

$$x^+ = F_0^\delta(x) \quad (13)$$

$$\frac{dx^+(u)}{du} = G^\delta(x^+(u), u) \quad \text{with } x^+(0) = x^+ \quad (14)$$

$$y = h(x) \quad (15)$$

describes the sampled equivalent Σ_s^δ of Σ_c . Its input-state or output-state evolutions match, at the sampling instants $t = k\delta$ those of Σ_c . Combinatoric relations between the continuous-time model and its sampled equivalent are detailed in [25], we just recall what is necessary in the present context. First, $G^\delta(\cdot, u)$ is described by its asymptotic expansion in δ

$$G^\delta(\cdot, u) = \int_0^\delta e^{-s(\text{ad}_f + u \text{ad}_g)} g ds = G_0^\delta + \sum_{i \geq 1} \frac{u^i}{i!} G_i^\delta \quad (16)$$

as well as each $G_i^\delta = \sum_{p \geq i+1} \frac{\delta^p}{p!} G_{i,p}$ for $i \geq 0$ so getting for the first ones

$$G_0^\delta = \int_0^\delta e^{-s \text{ad}_f} g ds := \sum_{p \geq 1} \frac{(-1)^{p-1} \delta^p}{p!} \text{ad}_f^{p-1} g \quad (17)$$

$$G_1^\delta = \sum_{p \geq 2} \frac{(-1)^{p-1} \delta^{p+1}}{(p+1)!} \sum_{k=0}^{p-1} C_p^k [\text{ad}_f^k g, \text{ad}_f^{p-k-1} g] \quad (18)$$

where $C_p^k := \frac{p!}{k!(p-k)!}$ denotes the binomial.

We note that the term in δ^p in G_0^δ described by (17) is the Lie bracket $\text{ad}_f^{p-1}g$ which contains the vector field g one-time and the vector field f , $(p-1)$ -times. G_1^δ in (18) involves Lie brackets of the first order in the ad_f^jg of the form $[\text{ad}_f^k g, \text{ad}_f^{p-k-1}g]$ in δ^{p+1} for $p \geq 1$. Such a term contains g , two-times and f , $(p-1)$ -times. More in general, G_i^δ can be iteratively deduced according to a combinatoric rule; it involves Lie brackets of order i in the ad_f^jg and it contains g , $(i+1)$ -times. The transport of any G_i^δ along F_0^δ is denoted by $\text{Ad}_{F_0^\delta}G_i^\delta$ and its transport along F_0^δ , q -times is defined as $\text{Ad}_{F_0^\delta}G_i^\delta$ because $F_0^\delta \circ \dots \circ F_0^\delta = F_0^{q\delta}$.

Theorem 3.1: [27] Σ_s^δ is locally equivalent to the discrete-time observer canonical form (10-11) if and only if there exists $T > 0$ such that for any $\delta \in [0, T]$, the conditions below hold true

- $A_{s1} : \rho(\text{d}h, \text{d}e^{\delta f}h, \dots, \text{d}e^{(n-1)\delta f}h)(0) = n$;
- $A_{s2} : [r_{s1}, r_{si}] = 0$ for $i \in [2, n]$;
- $A_{s3} : [G_p^\delta, r_{si}] = 0$ for $i \in [2, n], p \geq 0$;

where the vector field r_{s1} satisfies:

$$\text{dH}^\delta r_{s1} := \begin{pmatrix} \text{d}h \\ \vdots \\ \text{d}e^{(n-2)\delta f}h \\ \text{d}e^{(n-1)\delta f}h \end{pmatrix} r_{s1} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \delta^{n-1} \end{pmatrix} \quad (19)$$

and r_{si} is the transport of r_{s1} along F_0^δ , $r_{si} := e^{-\delta \text{ad}_f} r_{s1} = e^{-(i-1)\delta \text{ad}_f} r_{s1}$ for $i \in [2, n]$. Moreover, the coordinates change Φ^δ satisfies

$$[\text{J}_x \Phi^\delta](r_{s1} \dots r_{sn}) = \delta^{n-1} I_{dn}.$$

When it is referred to a sampled model, the observer canonical form maintains the discrete-time structure (9-10-11) with δ -dependent output injection ψ^δ and vector field G^δ .

Assuming Σ_c locally equivalent to the observer canonical form (2), the present paper discusses the possible preservation of such a property under approximated sampling. The loss of linearization by output injection under sampling, as noted in [8], is the natural counterpart of what occurs when investigating the dual problem of feedback linearization under sampling. As a matter of fact, preservation of linearization by feedback under sampling should imply linear equivalence as conjectured in [1] and proved for $n=2$. However, some approximate result can be proven. As proposed in [2], feedback linearization can be maintained till an approximation order with respect to the sampling period equal to the state dimension. The same idea is here developed with respect to linearization by output injection. Our result

stands in proving that the conditions above can be maintained in an approximate meaning. More precisely, these conditions, reformulated as equalities between asymptotic expansions in δ , hold true up to a certain degree of approximation. With this in mind the following definition is mandatory.

Definition 3.1: Assuming Σ_c linearizable by output-injection, the property is maintained under sampling if there exists $T > 0$ such that for any $\delta \in [0, T]$ the sampled equivalent model satisfies the property. Approximate solution at order p stands for the existence of a solution on the approximate sampled model at order p in δ (error in $O(\delta^{p+1})$).

The problem has been approached in [27] where we showed that linearization by output injection could be maintained under sampling thanks to the computation of a ‘‘fictitious’’ δ -dependent vector field with respect to which the relative degree is maintained equal to n at order $n-1$ (i.e. up to an error in $O(\delta^n)$). As a matter of fact, a stronger result can be proven.

Theorem 3.2: For Σ_c , locally linear equivalent by output injection, there exists $T > 0$ such that for any $\delta \in [0, T]$ a vector field r_{s1} can be computed to satisfy (19) at any degree of approximation in δ .

Proof: The proof works out showing first that

$$\text{dH}^\delta = H(\delta)\text{dH} + O(\delta^n) \quad (20)$$

with $H(\delta) \in R^{n \times n}$, invertible by construction for any $\delta \in]0, T]$

$$H(\delta) = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & \delta & \delta^2/2 & \ddots & \vdots \\ 1 & 2\delta & 2\delta^2 & \dots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & (n-1)\delta & \frac{(n-1)^2\delta^2}{2} & \dots & \frac{(n-1)^{n-1}\delta^{n-1}}{(n-1)!} \end{pmatrix}.$$

Then, given r_{c1} satisfying (3), we set in (19)

$$r_{s1} = \sum_{i=1}^{n-1} \delta^{i-1} \alpha_{i0} r_{ci} = (r_{c1} \dots r_{cn}) \begin{pmatrix} \alpha_{10} \\ \delta \alpha_{20} \\ \vdots \\ \delta^{n-2} \alpha_{n-10} \\ 0 \end{pmatrix}.$$

Since r_{s1} must solve (19) in $O(\delta^n)$, due to (20), we have

$$\begin{pmatrix} \alpha_{10} \\ \vdots \\ \alpha_{n-10} \end{pmatrix} = \mathcal{M}_0^{-1} \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix} = (\mathcal{M}_0^{-1})_{lc} \quad (21)$$

where $(\mathcal{M}_0^{-1})_{lc}$ is the last column of $\mathcal{M}_0 \in R^{(n-1) \times (n-1)}$, invertible by construction; i.e.

$$\left(\mathcal{M}_0 \begin{pmatrix} 0 \\ \alpha_{10} \\ \vdots \\ \alpha_{n-10} \end{pmatrix} \right) = \frac{H(\delta)}{\delta^{n-1}} \begin{pmatrix} 0 & \cdots & 1 \\ \vdots & 1 & \vdots \\ 1 & \cdots & 0 \end{pmatrix} \begin{pmatrix} \alpha_{10} \\ \vdots \\ \delta^{n-2} \alpha_{n-10} \\ 0 \end{pmatrix}$$

so getting

$$\mathcal{M}_0 := \begin{pmatrix} \frac{1}{(n-1)!} & \frac{1}{(n-2)!} & \cdots & \frac{1}{2} & 1 \\ \frac{2^{n-1}}{(n-1)!} & \frac{2^{n-2}}{(n-2)!} & \cdots & 4 & 2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \frac{(n-1)^{n-1}}{(n-1)!} & \frac{(n-1)^{n-2}}{(n-2)!} & \cdots & \frac{(n-1)^2}{2} & n-1 \end{pmatrix}.$$

Setting now in (19), $r_{s1} = \sum_{i=1}^{n-1} \delta^{i-1} (\alpha_{i0} + \delta \alpha_{i1}) r_{ci}$, i. e.

$$r_{s1} = (r_{c1} \quad \cdots \quad r_{cn}) \begin{pmatrix} \alpha_{10} + \delta \alpha_{11} \\ \delta \alpha_{20} + \delta^2 \alpha_{21} \\ \vdots \\ \delta^{n-2} \alpha_{n-10} + \delta^{n-1} \alpha_{n-11} \\ 0 \end{pmatrix}$$

and solving equality (19) in $O(\delta^{n+1})$,

$$H(\delta) d\mathbf{H}.r_{s1} + \frac{\alpha_{10} \delta^n}{n!} \begin{pmatrix} 0 \\ 1 \\ 2^n \\ \vdots \\ (n-1)^n \end{pmatrix} d\mathbf{L}_f^n h.r_{c1} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \delta^{n-1} \end{pmatrix}$$

leads to

$$\begin{pmatrix} \alpha_{11}(x) \\ \vdots \\ \alpha_{n-11}(x) \end{pmatrix} = -\frac{d_n(x)}{n!} \mathcal{M}_0^{-1} \begin{pmatrix} 1 \\ 2^n \\ \vdots \\ (n+1)^n \end{pmatrix}$$

with $d_n(x) := d\mathbf{L}_f^n h(x).r_{c1}$ and $\alpha_{10} = 1$ computed from (21). Iterating the procedure and setting $r_{s1} = \sum_{i=1}^{n-1} \delta^{i-1} (\alpha_{i0} + \delta \alpha_{i1} + \frac{\delta^2}{2!} \alpha_{i2} + \dots) r_{ci}$, a solution to (19) up to increasing approximation order in δ

exists thanks to the invertibility of \mathcal{M}_0 . The proof is constructive for the successive α_{ip} which are functions of x for $p \geq 1$. \triangleleft

Starting from the so computed vector field r_{s1} , truncated at a certain approximation in δ , special assumptions on the output injection must be taken into account to satisfy conditions A_{s2} , A_{s3} in Theorem 3.1 so maintaining the observer canonical structure under approximate sampling. This is the case in [27] where the output injection is assumed piecewise constant. A more general result is proposed in the sequel where we describe a procedure for computing a δ -dependent coordinates transformation Φ^δ under which the approximate sampled model exhibits the sampled observer canonical form - sampled COF.

4. Approximate Sampled COF, an Algorithm

We first describe an algorithm for computing a coordinates transformation under which the COF structure is recovered under n -th order approximate sampling; i.e. starting from a continuous-time COF, an approximate sampled COF is described. The algorithm is proposed below for unforced systems $\Sigma_c(0)$, and then extended to forced ones $\Sigma_c(u)$ under some extra conditions on the output injection.

4.1. The Algorithm for $\Sigma_c(0)$

Starting from the continuous-time COF over R^n

$$\begin{aligned} \dot{x}_1 &= -x_1(x_n) \\ \dot{x}_2 &= x_1 + x_2(x_n) \\ &\dots \\ \dot{x}_n &= x_{n-1} + x_n(x_n) \\ y &= x_n \end{aligned} \tag{22}$$

we look for a coordinates change $z = \mathbf{T}^\delta(x)$ such that in the new coordinates, the sampled equivalent model, truncated at order n in δ exhibits a sampled COF. The proof is constructive for \mathbf{T}^δ .

- Initialization - set $z_n := x_n = y$.
- Step 1 - compute the evolution of y , truncated at order n in δ (n -th order approximation)

$$y(k+1) = y + \delta \dot{y} + \frac{\delta^2}{2} \ddot{y} + \cdots + \frac{\delta^n}{n!} y^{(n)}$$

and put in evidence in the r.h.s. the terms which depend on y only, denoted by $\alpha_n(y, \delta)$, from the terms which depend on the state variables $x_i, i \in [1, n-1]$ denoted by $\beta_n(x, \delta)$. One gets

$$y(k+1) = z_n(k+1) := \alpha_n(z_n, \delta) + \beta_n(x, \delta) \quad (23)$$

with

$$\beta_n(x, \delta) = \delta\beta_{n,1}(x) + \dots + \frac{\delta^n}{n!}\beta_{n,n}(x)$$

and by construction $\beta_{n,1}(x) = x_{n-1}$. Set

$$z_{n-1} := \beta_n(x, \delta) + (-1)^1 C_{n-1}^1 z_n \quad (24)$$

with $C_n^p := \frac{n!}{p!(n-p)!}$. By construction, the z_n -dynamics takes the form

$$\begin{aligned} z_n(k+1) &= z_{n-1} + C_{n-1}^1 z_n + \alpha_n(z_n, \delta) \\ &= z_{n-1} + \psi_n^\delta(y) \end{aligned}$$

so recovering the desired structure over the n -th equation where $\psi_n^\delta(y) := C_{n-1}^1 y + \alpha_n(y, \delta)$ specifies the n -th component of the output injection.

- Step 2 – compute the n -th order approximate evolution of z_{n-1} . From (23 – 24) and dropping the (x, δ) -dependency for simplicity, one gets

$$\begin{aligned} z_{n-1}(k+1) &= \beta_n + \delta\dot{\beta}_n + \dots + \frac{\delta^{n-1}}{(n-1)!}\beta_n^{(n-1)} \\ &+ (-1)^1 C_{n-1}^1 (\alpha_n(z_n, \delta) + \beta_n) \\ &= (1 - C_{n-1}^1)\beta_n + \beta_{n-1} + \alpha_{n-1}(z_n, \delta) \end{aligned} \quad (25)$$

where $\alpha_{n-1}(z_n, \delta)$ contains the terms in the r.h.s. which depend on z_n only; $\beta_{n-1}(x, \delta)$ is a new term containing all the terms in $\delta\dot{\beta}_n(x, \delta) + \frac{\delta^2}{2!}\ddot{\beta}_n(x, \delta) + \dots + \frac{\delta^{n-1}}{(n-1)!}\beta_n^{(n-1)}(x, \delta)$ which do not depend on z_n only, so getting

$$\beta_{n-1}(x, \delta) = \delta^2\beta_{n-1,2} + \dots + \frac{\delta^n}{(n-1)!}\beta_{n-1,n}$$

with $\beta_{n-1,2} = x_{n-2}$. Set

$$z_{n-2} = (1 - C_{n-1}^1)\beta_n + \beta_{n-1} + C_{n-1}^2 z_n \quad (26)$$

so that, from (25–26), the z_{n-1} -dynamics takes the form

$$\begin{aligned} z_{n-1}(k+1) &:= z_{n-2} - C_{n-1}^2 z_n + \alpha_{n-1}(z_n, \delta) \\ &= z_{n-2} + \psi_{n-1}^\delta(y) \end{aligned}$$

so recovering the desired structure with output injection term, $\psi_{n-1}^\delta(y) := -C_{n-1}^2 y + \alpha_{n-1}(y, \delta)$.
... Iterate the steps so characterizing the $(i+1)$ -th step as follows.

- Step $(i+1)$ – compute the n -th order approximate evolution of z_{n-i}

$$\begin{aligned} z_{n-i}(k+1) &= \sum_{p=0}^i (-1)^p C_{n-1}^p \beta_n \\ &+ \sum_{p=0}^{i-1} (-1)^p C_{n-2}^p \beta_{n-1} + \sum_{p=0}^{i-2} (-1)^p C_{n-3}^p \beta_{n-2} + \dots \\ &+ \sum_{p=0}^1 (-1)^p C_{n-i}^p \beta_{n-i-1} + \beta_{n-i}(x, \delta) + \alpha_{n-i}(z_n, \delta) \end{aligned}$$

where α_{n-i} regroups terms in the r.h.s. which depend on z_n only while β_n and $\beta_{n-1}, \dots, \beta_{n-i}$ regroups terms which do not depend on z_n only. One gets

$$\beta_{n-i}(x, \delta) = \delta^{i+1}\beta_{n-i,i+1} + \dots + \frac{\delta^n}{(n-i)!}\beta_{n-i,n}$$

with $\beta_{n-i,i+1} = x_{n-i-1}$. Set

$$\begin{aligned} z_{n-i-1} &= \sum_{p=0}^i (-1)^p C_{n-1}^p \beta_n + \sum_{p=0}^{i-1} (-1)^p C_{n-2}^p \beta_{n-1} + \dots \\ &+ \sum_{p=0}^1 (-1)^p C_{n-i}^p \beta_{n-i-1} + \beta_{n-i} + (-1)^i C_{n-1}^i z_n \end{aligned}$$

and verify that the z_{n-i} -dynamics takes the form

$$z_{n-i}(k+1) = z_{n-i-1}(k) + \psi_{n-i}^\delta(y)$$

so describing the output-injection term $\psi_{n-i}^\delta(y)$.

...

- Step $(n-1)$ – compute the approximate z_2 -evolution at order n

$$\begin{aligned} z_2(k+1) &= (1 + (-1)^1 C_{n-1}^1 + \dots + (-1)^{n-2} C_{n-1}^{n-2})\beta_n \\ &+ (1 + (-1)^1 C_{n-2}^1 + \dots + (-1)^{n-3} C_{n-2}^{n-3})\beta_{n-1} \\ &+ \dots + (1 + (-1)^1 C_2^1)\beta_3(x, \delta) + \beta_2 + \alpha_2(z_n, \delta) \\ &= (-1)^{n-2}\beta_n + (-1)^{(n-3)}\beta_{n-1} + \dots + \beta_2 + \alpha_2(z_n, \delta) \end{aligned}$$

where $\alpha_2(z_n, \delta)$ regroups terms in the r.h.s. which depend on z_n only and $\beta_2(x, \delta)$ those which do not depend on z_n only so getting

$$\beta_2 = \delta^{n-1}\beta_{2,n-1} + \dots + \frac{\delta^n}{2!}\beta_{2,n}$$

with $\beta_{2,n-1} = x_1$. Set

$$\begin{aligned} z_1 &= (-1)^{n-2}\beta_n + (-1)^{(n-3)}\beta_{n-1} + \dots \\ &- \beta_3 + \beta_2 + (-1)^{n-1} C_{n-1}^{n-1} z_n \end{aligned}$$

so computing the z_1 -dynamics

$$\begin{aligned} z_1(k+1) &= (-1)^{n-2}(\beta_n + \delta\dot{\beta}_n + \dots + \frac{\delta^{n-1}}{(n-1)!}\beta_n^{(n-1)}) \\ &\quad + (-1)^{(n-3)}(\beta_{n-1} + \delta\dot{\beta}_{n-1} + \dots + \frac{\delta^{n-2}}{(n-2)!}\beta_{n-1}^{(n-2)}) \\ &\quad + \dots - (\beta_3 + \delta\dot{\beta}_3 + \frac{\delta^2}{2!}\ddot{\beta}_3) + (\beta_2 + \delta\dot{\beta}_2) \\ &\quad + (-1)^{n-1}(\alpha_n(z_n) + \beta_n(x, \delta)) = \alpha_1(z_n) + \delta\dot{\beta}_2 \end{aligned}$$

rewritten as $z_1(k+1) = \psi_1^\delta(y)$ with $\psi_1^\delta(y) := \alpha_1(z_n) + \delta\dot{\beta}_2$ because $\delta\dot{\beta}_2 = \delta^n \dot{x}_1 = \delta^n \psi_1(y)$ is a function of y only and $\sum_{p=0}^n (-1)^p C_n^p = 0$.

4.2. Approximate Sampled COF for Unforced Dynamics

The result of the previous algorithm is summarized below.

Theorem 4.1: *Linearization by output injection of $\Sigma_c(0)$ is maintained under approximate sampling at order n through the coordinates transformation*

$$z = \Phi^\delta(x) = \mathbf{T}^\delta(\Phi(x)) \quad (27)$$

where Φ denotes the coordinates change which transforms $\Sigma_c(0)$ into its COF (22) and \mathbf{T}^δ is the coordinates change computed through the algorithm.

Proof: Taking in mind that the algorithm has been worked out on the approximate sampled equivalent, $S(\Phi(\Sigma_{c0}))$ of the continuous-time COF, $\Phi(\Sigma_{c0})$ and that the procedures of applying a coordinates transformation and sampling commute $S \circ \Phi = \Phi \circ S$, we can deduce Φ^δ from the composition (27). More precisely, setting $\Phi^\delta = \mathbf{T}^\delta \circ \Phi$, one has $\Phi^\delta \circ S = \mathbf{T}^\delta \circ \Phi \circ S = \mathbf{T}^\delta \circ S \circ \Phi$ so proving that Φ^δ transforms directly the approximate sampled equivalent model $S(\Sigma_c(0))$ of $\Sigma_c(0)$ into the sampled canonical normal form issued from the algorithm, $\mathbf{T}^\delta(S(\Phi(\Sigma_{c0}))$ provided homogeneous approximations in δ up to order n are performed. \triangleleft

Remark: In [8], it was conjectured (proven for $n=2$) that preservation under sampling of linear equivalence through output-injection should imply linear equivalence of Σ_{c0} . Our result shows that preservation holds true for any n but up to order n in δ .

Example: Let us illustrate the computation over R^2 . Consider the continuous-time uncontrolled COF

$$\begin{aligned} \dot{x}_1 &= -x_1(x_2) \\ \dot{x}_2 &= x_1 + x_2(x_2) \\ y &= x_2 \end{aligned}$$

and its approximate sampled equivalent at order 2 in δ

$$\begin{aligned} x_1(k+1) &= x_1 + \delta \dot{x}_1 + \frac{\delta^2}{2!} \ddot{x}_1(x_1 + \psi_2) \\ x_2(k+1) &= x_2 + \delta(x_1 + \psi_2) + \frac{\delta^2}{2!} (\psi_1 + \dot{\psi}_2(x_1 + \psi_2)) \\ y &= x_2 \end{aligned}$$

which clearly does not preserve the observer structure; $(\cdot)'$, $(\cdot)''$, \dots indicate the successive derivatives w.r.t. its arguments of the function into the parentheses. According to the algorithm, one sets $z_2 = x_2$ and puts in evidence in $z_2(k+1) = y(k+1)$ the parts which depend on z_2 only from the remaining ones, so getting

$$\begin{aligned} z_2(k+1) &= \alpha_2(z_2) + \beta_2(x, \delta) \\ &= x_2 + \delta\psi_2 + \frac{\delta^2}{2!} (\dot{x}_1 + \dot{\psi}_2) \\ &\quad + \delta x_1 + \frac{\delta^2}{2!} \psi_2' x_1. \end{aligned}$$

Setting $z_1 = \beta_2(x, \delta) - z_2$, one recovers

$$z_2(k+1) = z_1(k) + z_2(k) + \alpha_2(z_2(k)) = z_1 + \psi_2^\delta(y)$$

so defining ψ_2^δ as $\psi_2^\delta := z_2 + \delta\psi_2 + \frac{\delta^2}{2!} (\psi_1 + \dot{\psi}_2)$. The z_1 -dynamics

$$\begin{aligned} z_1(k+1) &= \delta(x_1 + \delta\psi_1) + \frac{\delta^2}{2!} \dot{x}_1 - z_2(k+1) \\ &= -x_2 - \delta\psi_2 + \frac{\delta^2}{2!} (x_1 - \frac{\delta^2}{2!} \dot{\psi}_2) := \phi_1^\delta(y) \end{aligned}$$

depends on $z_2 = x_2 = y$ only as ψ_2 depends on x_2 only. In conclusion, the coordinates change, $z = \mathbf{T}^\delta(x)$ described by

$$z_1 = \delta x_1 + \frac{\delta^2}{2!} \dot{x}_1 - x_2; \quad z_2 = x_2 \quad (28)$$

transforms the approximate sampled equivalent to the COF into the 2nd order sampled COF over R^2

$$\begin{aligned} z_1(k+1) &= -z_2 - \delta \dot{z}_2 + \frac{\delta^2}{2!} (\psi_1 - \dot{z}_2) \\ z_2(k+1) &= z_1 + 2z_2 + \delta \dot{z}_2 + \frac{\delta^2}{2!} (\psi_1 + \dot{z}_2) \\ y &= z_2. \end{aligned}$$

To complete the discussion, it is a matter of computations to verify that the Jacobian of \mathbf{T}^δ satisfies the condition set in Theorem 2.3; i.e.

$$[J_x \mathbf{T}^\delta](r_1^\delta \ r_2^\delta) = \delta I_{d_2} + O(\delta^3)$$

where the vector field r_1^δ is computed to satisfy (19) up to an error in $O(\delta^3)$ and $r_2^\delta := e^{-\delta \text{ad}_f} r_1^\delta$ so getting up to δ^2

$$r_1^\delta = r_{c1} - \frac{\delta}{2} d_2(x) r_{c1}; \quad r_2^\delta = r_{c1} - \frac{\delta}{2} d_2(x) r_{c1} + \delta r_{c2}$$

with $d_2(x) = L_{r_{c3}} h(x)$. In the present case, because $r_{c1} = (1, 0)^T$, $r_{c2} = (1, 0)^T$ and $d_2(x) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, one gets in $O(\delta^3)$

$$\begin{pmatrix} \delta + \frac{\delta^2}{2!} & \begin{pmatrix} 1 \\ 2 \end{pmatrix} & -1 \\ 0 & 1 & \end{pmatrix} \begin{pmatrix} 1 - \frac{\delta}{2} \psi_2' & 1 - \frac{\delta}{2} \psi_2' \\ 0 & \delta \end{pmatrix} = \begin{pmatrix} \delta & 0 \\ 0 & \delta \end{pmatrix}.$$

The specific structure of the n -th order sampled COF over R^n can be described in terms of the continuous-time COF according to the same procedure.

4.3. Approximate Sampled COF for Controlled Dynamics

The result stated can be extended to controlled systems $\Sigma_c(u)$ admitting a COF (2) with output injection satisfying the condition

$$\frac{\partial^2 \psi_i(y, u)}{\partial y \partial u} = 0 \quad \text{for } i \in [2, n] \quad (29)$$

which is equivalent to assume that the derivatives $\psi_i'(y)$ do not depend on u except $\psi_1'(y, u)$. The following result holds true.

Theorem 4.2: *Linearization by output injection of $\Sigma_c(u)$ is maintained under approximate sampling at order n through the coordinates transformation (27) if the conditions (29) are satisfied.*

Proof: It is readily understood that under (29), the output injection in (2) takes the form

$$\Psi(y, u) = (\psi_1(y, u), b_2 u + \psi_2(y), \dots, b_n u + \psi_n(y))^T.$$

Linearity in u of the input-output injection $\psi_{i \geq 2}(y, u)$ guarantees that the algorithm proposed above works out identically yielding to a coordinates change which does not depend on the control variable which is assumed constant over the sampling intervals. Relaxing (29) yields to a coordinates change which depends on u . \triangleleft

Remark: An interesting case is represented by controlled system $\Sigma_c(u)$ having relative degree n , ($L_g L_f^k h = 0$; $k = 0, \dots, n-2$; $L_g L_f^{n-1} h \neq 0$).

Example: Let us illustrate the procedure on the controlled COF over R^3 with input-output injection satisfying (29)

$$\begin{aligned} \dot{x}_1 &= \psi_1(x_3, u) \\ \dot{x}_2 &= x_1 + \psi_2(x_3) \\ \dot{x}_3 &= x_2 + \psi_3(x_3) \\ y &= x_3 \end{aligned}$$

with approximate sampled equivalent at order 3 in δ

$$\begin{aligned} x_1(k+1) &= x_1 + \delta \psi_1(u) + \frac{\delta^2}{2!} \psi_1'(u)(x_2 + \psi_3) \\ &\quad + \frac{\delta^3}{3!} (\psi_1''(u)(x_1 + \psi_2) \\ &\quad + \psi_1'''(u)(x_2 + \psi_3)^2 \\ &\quad + \psi_1^{(4)}(u) \psi_3'(x_2 + \psi_3)) \\ x_2(k+1) &= x_2 + \delta(x_1 + \psi_2) + \frac{\delta^2}{2!} (\psi_1(u) \\ &\quad + \psi_2'(x_2 + \psi_3)) \\ &\quad + \frac{\delta^3}{3!} (\psi_1'(u)(x_2 + \psi_3) + \psi_2''(x_1 + \psi_2) \\ &\quad + \psi_2' \psi_3'(x_2 + \psi_3) + \psi_2'''(x_2 + \psi_3)^2) \\ x_3(k+1) &= x_3 + \delta(x_2 + \psi_3) \\ &\quad + \frac{\delta^2}{2!} (x_1 + \psi_2 + \psi_3'(x_2 + \psi_3)) \\ &\quad + \frac{\delta^3}{3!} (\psi_1(u) + \psi_2'(x_2 + \psi_3) \\ &\quad + \psi_3''(x_2 + \psi_3)^2 \\ &\quad + \psi_3'(x_1 + \psi_2) + \psi_3^{(2)}(x_2 + \psi_3)) \\ y &= x_3 \end{aligned}$$

where the u -dependency is in ψ_1 only. Setting $z_3 := x_3$, one deduces from $z_3(k+1)$

$$\begin{aligned} \alpha_3(z_3, \delta, u) &:= z_3 + \delta \psi_3 + \frac{\delta^2}{2!} (\psi_2 + \psi_3' z_3) \\ &\quad + \frac{\delta^3}{3!} (\psi_1(u) + \psi_2' z_3 + \psi_3'' z_3^2 \\ &\quad + \psi_3' z_2 + \psi_3^{(2)} z_3) \\ \beta_3(x, \delta) &:= \delta x_2 + \frac{\delta^2}{2!} (x_1 + \psi_3' x_2) \\ &\quad + \frac{\delta^3}{3!} (\psi_2' x_2 + \psi_3'' x_2^2 \\ &\quad + \psi_3' x_1 + 2 \psi_3'' \psi_3 x_2 + \psi_3^{(2)} x_2). \end{aligned}$$

Setting $z_2 := \beta_3(x, \delta) - 2z_3$, one computes

$$\begin{aligned} z_3(k+1) &= z_2 + 2z_3 + \alpha_3(z_3, \delta) \\ &= z_2 + \psi_3^\delta(z_3, u). \end{aligned}$$

Moreover, by computing

$$\begin{aligned} z_2(k+1) &= -2\alpha_3(z_3, \delta, u) - \beta_3(z_3, \delta) \\ &\quad + \delta\dot{\beta}_3(z_3, \delta) + \frac{\delta^2}{2!}\ddot{\beta}_3(z_3, \delta) \end{aligned}$$

with

$$\begin{aligned} \delta\dot{\beta}_3 &= \delta^2(x_1 + \psi_2) + \frac{\delta^3}{2}(\psi_1(u) \\ &\quad + {}_3''x_2(x_2 + \psi_3) + {}_3'(x_1 + \psi_2)) \\ \frac{\delta^2}{2!}\ddot{\beta}_3 &= \delta^3({}_1(u) + {}_2'(x_2 + \psi_3)) \end{aligned}$$

we get

$$\begin{aligned} z_2(k+1) &= \alpha_2(z_3, \delta, u(k)) \\ &\quad - \beta_3(z_3, \delta) + \beta_2(z_3, \delta) \end{aligned}$$

with

$$\begin{aligned} \alpha_2(z_3, \delta, u) &:= -2\alpha_3 + \delta^2 {}_2 + \frac{\delta^3}{2}(2\psi_1(u) \\ &\quad + {}_2' {}_3 + {}_3' {}_2) \\ \beta_2(x, \delta) &:= \delta^2 x_1 + \frac{\delta^3}{2}({}_3''x_2^2 + \psi_3''\psi_3 x_2 \\ &\quad + {}_2'x_2 + {}_3'x_1). \end{aligned}$$

Setting $z_1 := -\beta_3 + \beta_2 + z_3$, one has

$$\begin{aligned} z_2(k+1) &= z_1 - z_3 + \alpha_2(z_3, \delta, u) \\ &= z_1 + \psi_2^\delta(z_3, u) \end{aligned}$$

with $\psi_2^\delta(z_3, u) := -z_3 + \alpha_2(z_3, \delta, u)$ and also

$$\begin{aligned} z_1(k+1) &= \alpha_3 + \beta_3 - \beta_3 - \delta\dot{\beta}_3 \\ &\quad - \frac{\delta^2}{2!}\ddot{\beta}_3 + \beta_2 + \delta\dot{\beta}_2 \\ &= z_3 + \delta\psi_3 + \frac{\delta^2}{2}(-\psi_2' + {}_3' {}_3) \\ &\quad + \frac{\delta^3}{3!}(\psi_1(u) - 2\psi_2' {}_3 - 2\psi_3' {}_2 + {}_3''\psi_3^2 + {}_3' {}_3) \\ &:= \psi_1^\delta(z_3, u). \end{aligned}$$

In conclusion, the coordinates change, $z = \mathbf{T}^\delta(x)$

$$\begin{aligned} z_1 &= -\delta x_2 + \frac{\delta^2}{2!}(x_1 - {}_3'x_2) \\ &\quad + \frac{\delta^3}{3!}(2\psi_3''x_2^2 + {}_3''\psi_3 x_2 + 3\psi_2'x_2 \\ &\quad + 2\psi_3'x_1 - {}_3'x_2) + x_3 \\ z_2 &= \delta x_2 + \frac{\delta^2}{2!}(x_1 + {}_3'x_2) \\ &\quad + \frac{\delta^3}{3!}({}_2'x_2 + {}_3''x_2^2 + \psi_3'x_1 \\ &\quad + 2\psi_3''\psi_3 x_2 + {}_3'x_2) - 2x_3 \\ z_3 &= x_3 = y \end{aligned}$$

transforms the controlled continuous-time COF over R^3 into the controlled approximate sampled COF at order 3 in δ

$$\begin{aligned} z_1(k+1) &= \psi_1^\delta(z_3(k), u(k)) \\ z_2(k+1) &= z_1(k) + \psi_2^\delta(z_3(k), u(k)) \\ z_3(k+1) &= z_2(k) + \psi_3^\delta(z_3(k), u(k)) \\ y &= z_3 \end{aligned}$$

with the $\psi_i^\delta(z_3, u)$ described above for $i=[1,3]$.

4.4. The Duffing Oscillator

A very simple example is the duffing oscillator which admits the COF

$$\begin{aligned} \dot{x}_1 &= -x_2 - x_2^3 \\ \dot{x}_2 &= x_1 \\ y &= x_2 \end{aligned}$$

with $\psi_1 = -x_2 - x_2^3$ and $\psi_2 = 0$. Under approximate sampling at order 2, the COF structure is lost

$$\begin{aligned} x_1(k+1) &= x_1 + \delta(-x_2 - x_2^3) \\ &\quad + \frac{\delta^2}{2}(-x_1 - 3x_2^2x_1) \\ x_2(k+1) &= x_2 + \delta x_1 + \frac{\delta^2}{2}(-x_2 - x_2^3) \\ y &= x_2. \end{aligned}$$

It is straightforward to verify that the coordinates change (28) takes the very simple linear form

$$\begin{aligned} z_1 &= \delta x_1 - x_2 \\ z_2 &= x_2 = y \end{aligned}$$

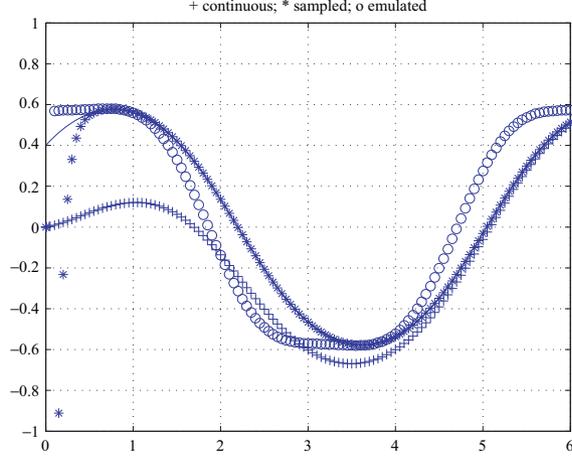


Fig. 1. $\delta = 0.05$.

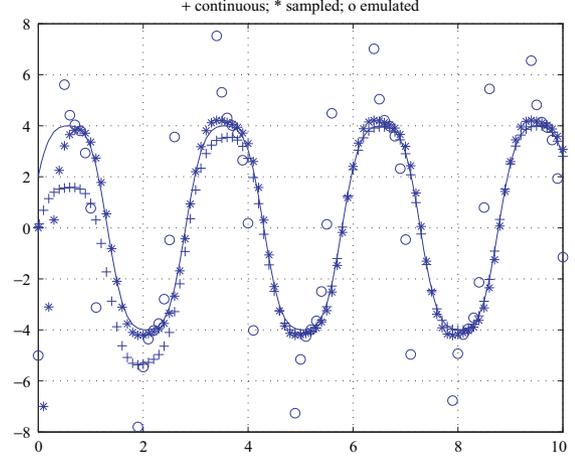


Fig. 3. $\delta = 0.1$.

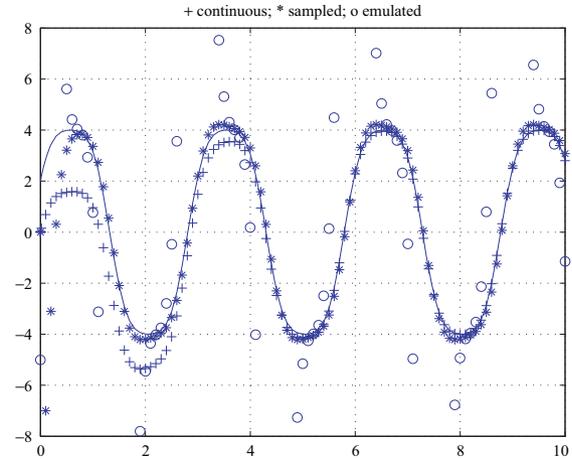


Fig. 2. $\delta = 0.05$.

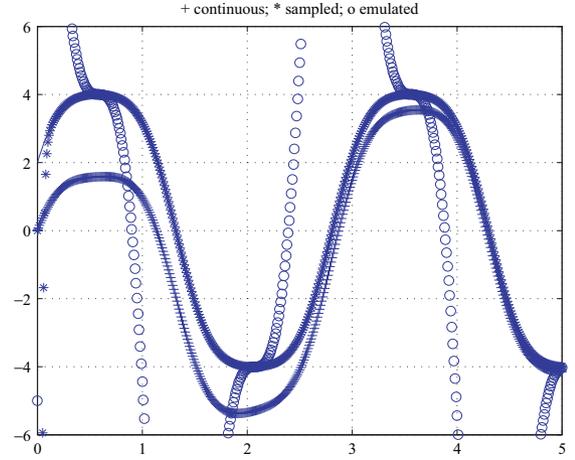


Fig. 4. $\delta = 0.1$.

so getting under transformation and up to an error in $O(\delta^3)$ the Sampled COF

$$\begin{aligned} z_1(k+1) &= -z_2 + \frac{\delta^2}{2}(-z_2 - z_2^3) \\ z_2(k+1) &= z_1 + 2z_2 + \frac{\delta^2}{2}(-z_2 - z_2^3) \\ y &= z_2. \end{aligned}$$

It is a simple exercise to verify that the discrete-time observer

$$\begin{aligned} \hat{z}_1(k+1) &= -y + \frac{\delta^2}{2}(-y - y^3) + k_1(y - \hat{z}_2) \quad (30) \\ \hat{z}_2(k+1) &= \hat{z}_1 + 2y + \frac{\delta^2}{2}(-y - y^3) + k_2(y - \hat{z}_2) \\ y &= z_2 \end{aligned}$$

yields to a linear error dynamics $e_1(k+2) + k_1 e_1(k) + k_2 e_1(k+1) = 0$ for $e_1 = z_1 - \hat{z}_1$.

In Figs. 1–4, the behavior of the approximate sampled observer (30) with eigenvalues at e^{-1} is compared with the sampled values of the continuous-time observer, which has linear error dynamics with eigenvalues $\lambda_1 = \lambda_2 = -1$, and with the approximate sampled observer at the first order in the output injection with eigenvalues at zero, named *emulated*, i. e. given by

$$\begin{aligned} \tilde{z}_1(k+1) &= \tilde{z}_1 - \delta y - \delta y^3 - \tilde{k}_1(y - \tilde{z}_2) \quad (31) \\ \tilde{z}_2(k+1) &= \tilde{z}_2 + \delta \tilde{z}_1 - \tilde{k}_2(y - \tilde{z}_2) \\ y &= z_2 \end{aligned}$$

In all the simulations the observer dynamics is initialized at zero and the evolution of x_1 is represented together with its estimates. It results from the figures that to a fast convergence, which is typical of a discrete-time device, the proposed observer associates very good steady state performances. Figs 1 and 2 show how the

performances of the emulated observer degraded with the initial error: the system evolves starting from $x_0 = (0.5, -0.5)^T$ and $x_0 = (1, -1)^T$, respectively. Figs. 3 and 4 put in light the performances of the proposed sampled observer also when increasing the sampling interval and the initial error $x_0 = (0.5, -2)^T$, while the convergence of the emulated observer is lost even by reducing the sampling interval.

5. Conclusions

In this paper we have shown that linearization through coordinates change and input-output injection of uncontrolled dynamics can be preserved under sampling up to approximations at order n , the state dimension. The proof is constructive for the coordinates change so exhibiting n -th order approximate sampled counterparts of COF over R^n . The result is extended to controlled dynamics with reference to a specific structure of the input-output injection. A global version of these results, with respect to the x -dependency, could be given assuming that the conditions hold globally and assuming completeness of the vector fields describing the continuous-time dynamics. Work is progressing to relax the extra conditions set on the output injection by considering multi-output injections; i.e. depending also on time instants internal to the sampling intervals, a concept some how similar to multirate control.

Acknowledgment

This work was supported by the University Italy-France/France-Italy-UIF/UFI under the programme Galileo/Galilée.

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