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# A GENERALIZATION OF A TRACE INEQUALITY FOR POSITIVE DEFINITE MATRICES. 

ELENA VERONICA BELMEGA, MARC JUNGERS, AND SAMSON LASAULCE

Abstract. In this note we generalize the trace inequality derived by [1] to the case where the number of terms of the sum (denoted by $K$ ) is arbitrary. More precisely we prove that $\mathcal{I}_{K}=\operatorname{Tr}\left\{\sum_{k=1}^{K}\left(\mathbf{A}_{k}-\mathbf{B}_{k}\right)\left[\left(\sum_{\ell=1}^{k} \mathbf{B}_{\ell}\right)^{-1}-\left(\sum_{\ell=1}^{k} \mathbf{A}_{\ell}\right)^{-1}\right]\right\} \geq 0$ for any set of positive definite matrices.

## 1. Introduction

Trace inequalities are useful in many areas like multiple input multiple output (MIMO) systems in control theory and communications. Proving the trace inequality under investigation in this note is a sufficient condition to ensure the uniqueness of a Nash equilibrium in certain MIMO communications game [2] where Rosen's diagonally strict concavity condition [3] is valid. The considered inequality has been proven by e.g., [4] for $K=1$ and by [1] for $K=2$. Here we generalize it to $K \in \mathbb{N}^{*}$. The main result of this note is as follows.

Theorem 1.1. Let $K \in \mathbb{N}^{*}$. Assume that
(i): $\mathbf{A}_{1}=\mathbf{A}_{1}^{H} \succ 0, \mathbf{B}_{1}=\mathbf{B}_{1}^{H} \succ 0$;
(ii): $\forall k \in\{2, \ldots, K\}, \mathbf{A}_{k}=\mathbf{A}_{k}^{H} \succeq 0$ and $\mathbf{B}_{k}=\mathbf{B}_{k}^{H} \succeq 0$.

Then, we have that

$$
\begin{equation*}
\mathcal{T}_{K} \triangleq \operatorname{Tr}\left\{\sum_{k=1}^{K}\left(\mathbf{A}_{k}-\mathbf{B}_{k}\right)\left[\left(\sum_{\ell=1}^{k} \mathbf{B}_{\ell}\right)^{-1}-\left(\sum_{\ell=1}^{k} \mathbf{A}_{\ell}\right)^{-1}\right]\right\} \geq 0 . \tag{1.1}
\end{equation*}
$$

## 2. Auxiliary Results

In order to prove Theorem 1.1, we will use two auxiliary lemmas which are stated here for the sake of clarity.

Lemma 2.1. [1] Let A, B be two positive definite matrices, C, D, two positive semidefinite matrices whereas $\mathbf{X}$ is only assumed to be Hermitian. Then

$$
\begin{equation*}
\operatorname{Tr}\left\{\mathbf{X A}^{-1} \mathbf{X} \mathbf{B}^{-1}\right\}-\operatorname{Tr}\left\{\mathbf{X}(\mathbf{A}+\mathbf{C})^{-1} \mathbf{X}(\mathbf{B}+\mathbf{D})^{-1}\right\} \geq 0 \tag{2.1}
\end{equation*}
$$

The proof is given in [1].
Lemma 2.2. Let $\mathbf{A}, \mathbf{B}$ be two positive definite matrices, $\mathbf{C}, \mathbf{D}$, two positive semi-definite matrices. Then

$$
\begin{align*}
& \operatorname{Tr}\left\{(\mathbf{A}-\mathbf{B})(\mathbf{B}+\mathbf{D})^{-1}(\mathbf{C}-\mathbf{D})(\mathbf{A}+\mathbf{C})^{-1}\right\}= \\
& \operatorname{Tr}\left\{(\mathbf{C}-\mathbf{D})(\mathbf{B}+\mathbf{D})^{-1}(\mathbf{A}-\mathbf{B})(\mathbf{A}+\mathbf{C})^{-1}\right\} \in \mathbb{R} . \tag{2.2}
\end{align*}
$$

[^0]Proof. To prove the desired result, let us define $\mathcal{E}$ by

$$
\begin{equation*}
\mathcal{E}=\operatorname{Tr}\left\{(\mathbf{C}-\mathbf{D})\left[(\mathbf{B}+\mathbf{D})^{-1}-(\mathbf{A}+\mathbf{C})^{-1}\right]\right\} \tag{2.3}
\end{equation*}
$$

and write it in two different ways.

$$
\begin{align*}
\mathcal{E}= & \operatorname{Tr}\left\{(\mathbf{C}-\mathbf{D})(\mathbf{B}+\mathbf{D})^{-1}[\mathbf{A}+\mathbf{C}-\mathbf{B}-\mathbf{D}](\mathbf{A}+\mathbf{C})^{-1}\right\} \\
= & \operatorname{Tr}\left\{(\mathbf{C}-\mathbf{D})(\mathbf{B}+\mathbf{D})^{-1}(\mathbf{C}-\mathbf{D})(\mathbf{A}+\mathbf{C})^{-1}\right\}+  \tag{2.4}\\
& \operatorname{Tr}\left\{(\mathbf{C}-\mathbf{D})(\mathbf{B}+\mathbf{D})^{-1}(\mathbf{A}-\mathbf{B})(\mathbf{A}+\mathbf{C})^{-1}\right\} . \\
\mathcal{E}= & \operatorname{Tr}\left\{(\mathbf{C}-\mathbf{D})(\mathbf{A}+\mathbf{C})^{-1}[\mathbf{A}+\mathbf{C}-\mathbf{B}-\mathbf{D}](\mathbf{B}+\mathbf{D})^{-1}\right\} \\
= & \operatorname{Tr}\left\{(\mathbf{C}-\mathbf{D})(\mathbf{A}+\mathbf{C})^{-1}(\mathbf{C}-\mathbf{D})(\mathbf{B}+\mathbf{D})^{-1}\right\}+  \tag{2.5}\\
& \operatorname{Tr}\left\{(\mathbf{C}-\mathbf{D})(\mathbf{A}+\mathbf{C})^{-1}(\mathbf{A}-\mathbf{B})(\mathbf{B}+\mathbf{D})^{-1}\right\} .
\end{align*}
$$

By using the commutation property of the trace and the two expressions of $\mathcal{E}$, we find the desired result. The only thing which needs to be proven is that $\mathcal{E}$ is real. For this purpose, observe that if we denote by $\mathbf{M}=(\mathbf{C}-\mathbf{D})(\mathbf{B}+\mathbf{D})^{-1}(\mathbf{A}-\mathbf{B})(\mathbf{A}+\mathbf{C})^{-1}$ then $\mathbf{M}^{H}=(\mathbf{A}+\mathbf{C})^{-1}(\mathbf{A}-\mathbf{B})(\mathbf{B}+\mathbf{D})^{-1}(\mathbf{C}-\mathbf{D})$ and from the result just proven we obtain that $\operatorname{Tr}\left(\mathbf{M}^{H}\right)=\operatorname{Tr}(\mathbf{M})$ and thus we have that $\operatorname{Tr}(\mathbf{M}) \in \mathbb{R}$.

## 3. Proof of Theorem 1.1

Define $\mathbf{X}_{k}=\sum_{i=1}^{k} \mathbf{A}_{i}, \mathbf{Y}_{k}=\sum_{i=1}^{k} \mathbf{B}_{i}$, for all $k \geq 1$ which are both positive definite matrices. Notice that $\mathcal{T}_{K}$ can be re-written recursively as follows:

$$
\left\{\begin{align*}
\mathcal{T}_{1}= & \operatorname{Tr}\left\{\left(\mathbf{A}_{1}-\mathbf{B}_{1}\right) \mathbf{Y}_{1}^{-1}\left(\mathbf{A}_{1}-\mathbf{B}_{1}\right) \mathbf{X}_{1}^{-1}\right\}  \tag{3.1}\\
\mathcal{I}_{K}= & \mathcal{T}_{K-1}+\operatorname{Tr}\left\{\left(\mathbf{A}_{K}-\mathbf{B}_{K}\right) \mathbf{Y}_{K}^{-1}\left(\mathbf{A}_{K}-\mathbf{B}_{K}\right) \mathbf{X}_{K}^{-1}\right\}+ \\
& \operatorname{Tr}\left\{\left(\mathbf{A}_{K}-\mathbf{B}_{K}\right) \mathbf{Y}_{K}^{-1}\left(\mathbf{X}_{K-1}-\mathbf{Y}_{K-1}\right) \mathbf{X}_{K}^{-1}\right\}
\end{align*}\right.
$$

We proceed in two steps. First, we find a lower bound for $\mathcal{I}_{K}$ and then we prove that this bound is positive. First, let us prove that, for all $K \geq 1$ :

$$
\begin{equation*}
\mathcal{T}_{K} \geq \frac{1}{2} \sum_{i=1}^{K} \operatorname{Tr}\left\{\left(\mathbf{A}_{i}-\mathbf{B}_{i}\right) \mathbf{Y}_{i}^{-1}\left(\mathbf{A}_{i}-\mathbf{B}_{i}\right) \mathbf{X}_{i}^{-1}\right\}+\frac{1}{2} \operatorname{Tr}\left\{\left(\mathbf{X}_{K}-\mathbf{Y}_{K}\right) \mathbf{Y}_{K}^{-1}\left(\mathbf{X}_{K}-\mathbf{Y}_{K}\right) \mathbf{X}_{K}^{-1}\right\} \tag{3.2}
\end{equation*}
$$

To this end we proceed by induction. For all $K \in \mathbb{N}^{*}$, define the proposition:
$\mathcal{P}_{K}: \mathcal{T}_{K} \geq \frac{1}{2} \sum_{i=1}^{K} \operatorname{Tr}\left\{\left(\mathbf{A}_{i}-\mathbf{B}_{i}\right) \mathbf{Y}_{i}^{-1}\left(\mathbf{A}_{i}-\mathbf{B}_{i}\right) \mathbf{X}_{i}^{-1}\right\}+\frac{1}{2} \operatorname{Tr}\left\{\left(\mathbf{X}_{K}-\mathbf{Y}_{K}\right) \mathbf{Y}_{K}^{-1}\left(\mathbf{X}_{K}-\mathbf{Y}_{K}\right) \mathbf{X}_{K}^{-1}\right\}$.
It is easy to check that, for $K=1, \mathcal{P}_{1}$ is true:

$$
\begin{align*}
\mathcal{T}_{1} & =\operatorname{Tr}\left\{\left(\mathbf{A}_{1}-\mathbf{B}_{1}\right) \mathbf{Y}_{1}^{-1}\left(\mathbf{A}_{1}-\mathbf{B}_{1}\right) \mathbf{X}_{1}^{-1}\right\}  \tag{3.4}\\
& =\frac{1}{2} \operatorname{Tr}\left\{\left(\mathbf{A}_{1}-\mathbf{B}_{1}\right) \mathbf{Y}_{1}^{-1}\left(\mathbf{A}_{1}-\mathbf{B}_{1}\right) \mathbf{X}_{1}^{-1}\right\}+\frac{1}{2} \operatorname{Tr}\left\{\left(\mathbf{X}_{1}-\mathbf{Y}_{1}\right) \mathbf{Y}_{1}^{-1}\left(\mathbf{X}_{1}-\mathbf{Y}_{1}\right) \mathbf{X}_{1}^{-1}\right\} .
\end{align*}
$$

Now, let us assume that $\mathcal{P}_{K-1}$ is true and then prove that this implies that $\mathcal{P}_{K}$ is also true. We have that:

$$
\begin{align*}
\mathcal{T}_{K-1} \geq & \frac{1}{2} \sum_{i=1}^{K-1} \operatorname{Tr}\left\{\left(\mathbf{A}_{i}-\mathbf{B}_{i}\right) \mathbf{Y}_{i}^{-1}\left(\mathbf{A}_{i}-\mathbf{B}_{i}\right) \mathbf{X}_{i}^{-1}\right\}+  \tag{3.5}\\
& \frac{1}{2} \operatorname{Tr}\left\{\left(\mathbf{X}_{K-1}-\mathbf{Y}_{K-1}\right) \mathbf{Y}_{K-1}^{-1}\left(\mathbf{X}_{K-1}-\mathbf{Y}_{K-1}\right) \mathbf{X}_{K-1}^{-1}\right\} .
\end{align*}
$$

From the recursive formula (3.1) we obtain:

$$
\begin{align*}
\mathcal{T}_{K} \geq & \frac{1}{2} \sum_{i=1}^{K-1} \operatorname{Tr}\left\{\left(\mathbf{A}_{i}-\mathbf{B}_{i}\right) \mathbf{Y}_{i}^{-1}\left(\mathbf{A}_{i}-\mathbf{B}_{i}\right) \mathbf{X}_{i}^{-1}\right\}+  \tag{3.6}\\
& \frac{1}{2} \operatorname{Tr}\left\{\left(\mathbf{X}_{K-1}-\mathbf{Y}_{K-1}\right) \mathbf{Y}_{K-1}^{-1}\left(\mathbf{X}_{K-1}-\mathbf{Y}_{K-1}\right) \mathbf{X}_{K-1}^{-1}\right\}+ \\
& \operatorname{Tr}\left\{\left(\mathbf{A}_{K}-\mathbf{B}_{K}\right) \mathbf{Y}_{K}^{-1}\left(\mathbf{A}_{K}-\mathbf{B}_{K}\right) \mathbf{X}_{K}^{-1}\right\}+\operatorname{Tr}\left\{\left(\mathbf{A}_{K}-\mathbf{B}_{K}\right) \mathbf{Y}_{K}^{-1}\left(\mathbf{X}_{K-1}-\mathbf{Y}_{K-1}\right) \mathbf{X}_{K}^{-1}\right\} \\
\geq & \frac{1}{2} \sum_{i=1}^{K-1} \operatorname{Tr}\left\{\left(\mathbf{A}_{i}-\mathbf{B}_{i}\right) \mathbf{Y}_{i}^{-1}\left(\mathbf{A}_{i}-\mathbf{B}_{i}\right) \mathbf{X}_{i}^{-1}\right\}+ \\
& \frac{1}{2} \operatorname{Tr}\left\{\left(\mathbf{X}_{K-1}-\mathbf{Y}_{K-1}\right) \mathbf{Y}_{K}^{-1}\left(\mathbf{X}_{K-1}-\mathbf{Y}_{K-1}\right) \mathbf{X}_{K}^{-1}\right\}+ \\
& \operatorname{Tr}\left\{\left(\mathbf{A}_{K}-\mathbf{B}_{K}\right) \mathbf{Y}_{K}^{-1}\left(\mathbf{A}_{K}-\mathbf{B}_{K}\right) \mathbf{X}_{K}^{-1}\right\}+\operatorname{Tr}\left\{\left(\mathbf{A}_{K}-\mathbf{B}_{K}\right) \mathbf{Y}_{K}^{-1}\left(\mathbf{X}_{K-1}-\mathbf{Y}_{K-1}\right) \mathbf{X}_{K}^{-1}\right\} \\
= & \frac{1}{2} \sum_{i=1}^{K-1} \operatorname{Tr}\left\{\left(\mathbf{A}_{i}-\mathbf{B}_{i}\right) \mathbf{Y}_{i}^{-1}\left(\mathbf{A}_{i}-\mathbf{B}_{i}\right) \mathbf{X}_{i}^{-1}\right\}+ \\
& \frac{1}{2} \operatorname{Tr}\left\{\left(\mathbf{X}_{K-1}-\mathbf{Y}_{K-1}\right) \mathbf{Y}_{K}^{-1}\left(\mathbf{X}_{K-1}-\mathbf{Y}_{K-1}\right) \mathbf{X}_{K}^{-1}\right\}+ \\
& \operatorname{Tr}\left\{\left(\mathbf{A}_{K}-\mathbf{B}_{K}\right) \mathbf{Y}_{K}^{-1}\left(\mathbf{A}_{K}-\mathbf{B}_{K}\right) \mathbf{X}_{K}^{-1}\right\}+\frac{1}{2} \operatorname{Tr}\left\{\left(\mathbf{A}_{K}-\mathbf{B}_{K}\right) \mathbf{Y}_{K}^{-1}\left(\mathbf{X}_{K-1}-\mathbf{Y}_{K-1}\right) \mathbf{X}_{K}^{-1}\right\}+ \\
& \frac{1}{2} \operatorname{Tr}\left\{\left(\mathbf{X}_{K-1}-\mathbf{Y}_{K-1}\right) \mathbf{Y}_{K}^{-1}\left(\mathbf{A}_{K}-\mathbf{B}_{K}\right) \mathbf{X}_{K}^{-1}\right\} \\
= & \frac{1}{2} \sum_{i=1}^{K} \operatorname{Tr}\left\{\left(\mathbf{A}_{i}-\mathbf{B}_{i}\right) \mathbf{Y}_{i}^{-1}\left(\mathbf{A}_{i}-\mathbf{B}_{i}\right) \mathbf{X}_{i}^{-1}\right\}+ \\
& \frac{1}{2} \operatorname{Tr}\left\{\left(\mathbf{X}_{K-1}-\mathbf{Y}_{K-1}\right) \mathbf{Y}_{K}^{-1}\left(\mathbf{X}_{K-1}-\mathbf{Y}_{K-1}\right) \mathbf{X}_{K}^{-1}\right\}+ \\
& \frac{1}{2} \operatorname{Tr}\left\{\left(\mathbf{A}_{K}-\mathbf{B}_{K}\right) \mathbf{Y}_{K}^{-1}\left(\mathbf{A}_{K}-\mathbf{B}_{K}\right) \mathbf{X}_{K}^{-1}\right\}+\frac{1}{2} \operatorname{Tr}\left\{\left(\mathbf{A}_{K}-\mathbf{B}_{K}\right) \mathbf{Y}_{K}^{-1}\left(\mathbf{X}_{K-1}-\mathbf{Y}_{K-1}\right) \mathbf{X}_{K}^{-1}\right\}+ \\
& \frac{1}{2} \operatorname{Tr}\left\{\left(\mathbf{X}_{K-1}-\mathbf{Y}_{K-1}\right) \mathbf{Y}_{K}^{-1}\left(\mathbf{A}_{K}-\mathbf{B}_{K}\right) \mathbf{X}_{K}^{-1}\right\}
\end{align*}
$$

$$
\begin{aligned}
= & \frac{1}{2} \sum_{i=1}^{K} \operatorname{Tr}\left\{\left(\mathbf{A}_{i}-\mathbf{B}_{i}\right) \mathbf{Y}_{i}^{-1}\left(\mathbf{A}_{i}-\mathbf{B}_{i}\right) \mathbf{X}_{i}^{-1}\right\}+ \\
& \frac{1}{2} \operatorname{Tr}\left\{\left(\mathbf{X}_{K-1}+\mathbf{A}_{K}-\mathbf{Y}_{K-1}-\mathbf{B}_{K}\right) \mathbf{Y}_{K}^{-1}\left(\mathbf{X}_{K-1}+\mathbf{A}_{K}-\mathbf{Y}_{K-1}-\mathbf{B}_{K}\right) \mathbf{X}_{K}^{-1}\right\} \\
= & \frac{1}{2} \sum_{i=1}^{K} \operatorname{Tr}\left\{\left(\mathbf{A}_{i}-\mathbf{B}_{i}\right) \mathbf{Y}_{i}^{-1}\left(\mathbf{A}_{i}-\mathbf{B}_{i}\right) \mathbf{X}_{i}^{-1}\right\}+\frac{1}{2} \operatorname{Tr}\left\{\left(\mathbf{X}_{K}-\mathbf{Y}_{K}\right) \mathbf{Y}_{K}^{-1}\left(\mathbf{X}_{K}-\mathbf{Y}_{K}\right) \mathbf{X}_{K}^{-1}\right\}
\end{aligned}
$$

The second inequality follows from applying Lemma 2.1 for the second term on the right and considering that $\mathbf{X}_{K}=\mathbf{X}_{K-1}+\mathbf{A}_{K}, \mathbf{Y}_{K}=\mathbf{Y}_{K-1}+\mathbf{B}_{K}$. The third equality follows from Lemma 2.2. Thus, we have proven the desired result.

The second step of the proof is straightforward. From (3.2), it is easy to check that $\mathcal{T}_{K} \geq 0$ (all the terms of the form $\operatorname{Tr}\left\{\mathbf{X B}^{-1} \mathbf{X A}^{-1}\right\}$ with $\mathbf{X}=\mathbf{X}^{H}, \mathbf{A} \succ \mathbf{0}, \mathbf{B} \succ \mathbf{0}$ can be re-written as $\operatorname{Tr}\left(\mathbf{N N}^{H}\right) \geq 0$ with $\left.\mathbf{N}=\mathbf{A}^{-1 / 2} \mathbf{X B}^{-1 / 2}\right)$.

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