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# A new proof of Delsarte, Goethals and Mac Williams theorem on minimal weight codewords of generalized Reed-Muller codes 

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#### Abstract

We give a new proof of Delsarte, Goethals and Mac williams theorem on minimal weight codewords of generalized Reed-Muller codes published in 1970. To prove this theorem, we consider intersection of support of minimal weight codewords with affine hyperplanes and we proceed by recursion.


## 1 Introduction

In the appendix of 11, Delsarte, Goethals and Mac Williams prove the theorem 1 below. However, at the beginning of their proof, they point out that "it would be very desirable to find a more sophisticated and shorter proof".
In this paper, we give a new proof of this theorem that we hope is simpler.

Let $q=p^{n}, p$ being prime number.
We identify the $\mathbb{F}_{q}$-algebra $B_{m}^{q}=\mathbb{F}_{q}\left[X_{1}, \ldots, X_{m}\right] /\left(X_{1}^{q}-X_{1}, \ldots, X_{m}^{q}-X_{m}\right)$ to the $\mathbb{F}_{q}$-algebra of the functions from $\mathbb{F}_{q}^{m}$ to $\mathbb{F}_{q}$ through the isomorphism $P \mapsto(x \mapsto P(x))$.
For $f \in B_{q}^{m}$, let $S_{f}=\left\{x \in \mathbb{F}_{q}^{m}, f(x) \neq 0\right\}$ the support of $f$, and
$|f|=\operatorname{Card}\left(S_{f}\right)$ the weight of $f$. The Hamming distance in $B_{m}^{q}$ is denoted by $d(.,$.$) .$
For $0 \leq r \leq m(q-1)$, the $r$ th order generalized Reed-Muller code of length $q^{m}$ is

$$
R_{q}(r, m)=\left\{P \in B_{m}^{q}, \operatorname{deg}(P) \leq r\right\}
$$

where $\operatorname{deg}(P)$ is the degree of the representative of $P$ with degree at most $q-1$ in each variable.
The affine group $\mathrm{GA}_{m}\left(\mathbb{F}_{q}\right)$ acts on $R_{q}(r, m)$ by its natural action. The minimum weight of $R_{q}(r, m)$ is $(q-s) q^{m-t-1}$, where $r=t(q-1)+s, 0 \leq s \leq q-2$ (see [1]).

The following theorem gives the codeword of minimum weight of $R_{q}(r, m)$

Theorem 1 Let $r=t(q-1)+s<m(q-1)$. The minimal weight codewords of $R_{q}(r, m)$ are codewords of $R_{q}(r, m)$ whose support is the union of $(q-s)$ distinct parallel affine subspaces of codimension $t+1$ included in an affine subspace of codimension $t$.

## Remark :

1. Clearly, codewords of this form are of minimal weight.
2. Using lemma 2 and corollary 3 below, this theorem means that codewords of minimal weight are equivalent, under the action of the affine group, to a codeword of the following form: $f(x)=c \prod_{i=1}^{t}\left(x_{i}^{q-1}-1\right) \prod_{j=1}^{s}\left(x_{t+1}-b_{j}\right)$ where $c \in \mathbb{F}_{q}^{*}, b_{j}$ are distinct elements of $\mathbb{F}_{q}$.

## 2 Proof of Theorem (1)

In this paper, we use freely the two following lemmas and their corollary proved in (1] p. 435.

Lemma 2 If $P(x)=0$ whenever $x_{1}=a$, then $P(x)=\left(x_{1}-a\right) Q(x)$ where $\operatorname{deg}_{x_{1}}(Q) \leq \operatorname{deg}_{x_{1}}(P)-1$.

Remark : In [1], the lemma A1.1 says that the exponent of $x_{1}$ in $Q$ is at most $q-2$, but they actually prove the above lemma.

Corollary 3 If $P(x)=0$ unless $x_{1}=b$, then $P(x)=\left(1-\left(x_{1}-b\right)^{q-1}\right) Q\left(x_{2}, \ldots, x_{m}\right)$.
Lemma 4 Let $S$ be a subset of $\mathbb{F}_{q}^{m}$, such that $\operatorname{Card}(S)=t q^{n}<q^{m}, 0<t<q$. Assume that for any hyperplane of $\mathbb{F}_{q}^{m}$, either $\operatorname{Card}(S \cap H)=0$ or
$\operatorname{Card}(S \cap H) \geq t q^{n-1}$. Then there exists an affine hyperplane of $\mathbb{F}_{q}^{m}$ which does not meet $S$.

Now, we prove a lemma that will be crucial in our proof of theorem
Lemma 5 Let $r=t(q-1)+s, 0 \leq s \leq q-2$, $f$ be a minimal weight codeword of $R_{q}(r, m)$ and $S=S_{f}$.
If $H$ is an hyperplane of $\mathbb{F}_{q}^{m}$, such that $S \cap H \neq \emptyset$ and $S \cap H \neq S$, then either $S$ meets all hyperplanes parallel to $H$ or $S$ meets $q-s$ hyperplanes parallel to $H$ in $q^{m-t-1}$ points.

Proof : Since an affine transformation does not change weight, we can assume that $H=\left\{x, x_{1}=0\right\}$.
Now assume that $S$ does not meet $k$ hyperplanes parallel to $H, k \geq 1$. As $S \cap H \neq \emptyset$ and $S \cap H \neq H$, we have $k \leq q-2$. By lemma 2, we can write

$$
f(x)=\left(x_{1}-b_{1}\right)^{\alpha_{1}} \ldots\left(x_{1}-b_{k}\right)^{\alpha_{k}} P(x)
$$

where $S_{P}$ meets all hyperplanes parallel to $H$.
Let $d=\sum_{i=1}^{k} \alpha_{i} \leq q-1$. We want to prove that $d=s$.

- First, assume that $d>s$. Then the degree of $P$ is $(t-1)(q-1)+q-1+s-d$ and $0 \leq q-1+s-d \leq q-2$. For $c \notin\left\{b_{1}, \ldots, b_{k}\right\}$, we consider $Q_{c}=\left(1-\left(x_{1}-c\right)^{q-1}\right) P$. The degree of $Q_{c}$ is $t(q-1)+q-1+s-d$. So

$$
\begin{aligned}
(q-s) q^{m-t-1}=\operatorname{Card}(S) & =\sum_{c \notin\left\{b_{1}, \ldots, b_{k}\right\}} \operatorname{Card}\left(S_{P} \cap\left\{x_{1}=c\right\}\right) \\
& \geq(q-k)(q-(q-1+s-d)) q^{m-t-1} \\
& =(q-k)(d-s+1) q^{m-t-1}
\end{aligned}
$$

and we obtain, since $d \geq k$,

$$
(q-1-k)(d-s) \leq 0,
$$

which is impossible, since $d>s$ and $k<q-1$.

- Now we have $d \leq s$.

So $\operatorname{deg}(P)=t(q-1)+s-d$ and we get

$$
\begin{aligned}
(q-s) q^{m-t-1}=\operatorname{Card}(S) & =\sum_{c \notin\left\{b_{1}, \ldots, b_{k}\right\}} \operatorname{Card}\left(S_{P} \cap\left\{x_{1}=c\right\}\right) \\
& \geq(q-k)(q-s+d) q^{m-t-2}
\end{aligned}
$$

which gives

$$
(d-k) q+k(s-d) \leq 0 .
$$

Hence, since $d \geq k, k \geq 1$ and $s \geq d$, necessarily $d=k=s$ and $\operatorname{Card}\left(S_{P} \cap\left\{x_{1}=c\right\}\right)=q^{m-t-1}$ for $c \notin\left\{b_{1}, \ldots, b_{k}\right\}$.

Now, we are able to prove theorem 1.

Proof : We prove first the case where $t=0$ and $t=m-1$.

- $\quad t=0$.

If $s=0$, then $\operatorname{deg}(f)=0$. Thus, since $f \neq 0$, we have $f=c$, for $c \in \mathbb{F}_{q}^{*}$ and $S_{f}=\mathbb{F}_{q}^{m}$.
Otherwise, let $H$ be an affine hyperplane of $\mathbb{F}_{q}^{m}$, then $\operatorname{Card}\left(S_{f} \cap H\right)=0$ or $\operatorname{Card}\left(S_{f} \cap H\right) \geq(q-s) q^{m-2}$.
Hence, by lemma 4 , there exists an affine hyperplane $H_{0}$, such that $S_{f} \cap H_{0}=\emptyset$. However, $\mathbb{F}_{q}^{m}$ is the union of the $q$ hyperplanes parallel to $H_{0}$, so there exists $H_{1}$, parallel to $H_{0}$, such that $H_{1} \cap S_{f} \neq \emptyset$.
Furthermore, since $|f|=(q-s) q^{m-1} \geq 2 q^{m-1}, S_{f} \cap H_{1} \neq S_{f}$. So, by lemma ${ }^{\circ}$,
since $S_{f} \cap H_{0}=\emptyset, S_{f}$ meets $(q-s)$ hyperplanes parallel to $H_{1}$, say $H_{1}, \ldots, H_{q-s}$, in $q^{m-1}$ points, this means that $S_{f}=\bigcup_{i=1}^{q-s} H_{i}$.

- $\quad t=m-1$.

Let $f \in R_{q}((m-1)(q-1)+s, m), 0 \leq s \leq q-2$, such that $|f|=q-s$. We put $S=S_{f}$. Let $\omega_{1}, \omega_{2} \in S$ and $H$ be an hyperplane, such that $\omega_{1}, \omega_{2} \in H$.
Assume that $S \cap H \neq S$ then, by lemma 5 , either $S$ meets all hyperplanes parallel to $H$ (which is possible only if $s=0$ ) or $S$ meets ( $q-s$ ) hyperplanes parallel to $H$ in one point. In both cases we get a contradiction, since in both cases $S$ meets each hyperplane in exactly one point and $\omega_{1}, \omega_{2} \in H$.
So $S$ is included in all hyperplanes $H$, such that $\omega_{1}, \omega_{2} \in H$, this means that $S$ is included in the line through $\omega_{1}$ and $\omega_{2}$.

Now we prove the theorem for general $t$ by recursion.

- Assume that for a fixed $t, 1 \leq t \leq m-2$ (we have already proved the case where $t=0$ ) and for all $0 \leq s \leq q-2$, the support of a codeword of minimal weight in $R_{q}((t+1)(q-1)+s, m)$ is the union of $(q-s)$ distinct parallel affine subspaces of codimension $t+2$ included in an affine subspace of codimension $t+1$.

Let $f \in R_{q}(t(q-1)+s, m)$, such that $|f|=(q-s) q^{m-t-1}$.
We put $S=S_{f}$. Let $a \in S$ and $F=\{\overrightarrow{a b}, b \in S\}$. We have :

$$
\operatorname{Card}(F)=(q-s) q^{m-t-1} \leq q^{r g(F)}
$$

Thus, since $0 \leq s \leq q-2$, we have $r g(F) \geq m-t$.
Let $\overrightarrow{v_{1}}, \ldots, \overrightarrow{v_{m-t}}$ be $m-t$ independent vectors of $F$ and $\vec{u}$, such that $\vec{u} \notin \operatorname{Vect}\left(\overrightarrow{v_{1}}, \ldots, \overrightarrow{v_{m-t}}\right)$.
Since $t \geq 1$, there exists an affine hyperplane, say $H$, such that
$a+\overrightarrow{v_{1}}, \ldots, a+\overrightarrow{v_{m-t}} \in H$ and $a+\vec{u} \notin H$.
Assume that $S \cap H \neq S$. Then by lemma 0 , either $S$ meets all hyperplanes parallel to $H$ or $S$ meets $(q-s)$ hyperplanes parallel to $H$ in $q^{m-t-1}$ points.

- 1 st case : $S$ meets $(q-s)$ hyperplanes.

By applying an affine transformation, we can assume that the $q-s$ hyperplanes are $H_{i}=\left\{x, x_{1}=a_{i}\right\}, a_{i} \in \mathbb{F}_{q}$. Without loss of generality, we can assume that $H=H_{1}$.
Let $P=\prod_{i=2}^{q-s}\left(x_{1}-a_{i}\right) f(x)$,
$\operatorname{deg}(P) \leq t(q-1)+s+q-1-s=(t+1)(q-1)$ and
$|P|=\operatorname{Card}\left(S \cap\left\{x_{1}=a_{1}\right\}\right)=q^{m-t-1}$. So $P$ is a codeword of minimal weight in $R_{q}((t+1)(q-1), m)$, and, by recursion hypothesis, $S_{p}=S \cap H$ is an affine subspace of codimension $t+1$.

- $2 n d$ case : $S$ meets all the hyperplanes.

For all $G_{a}$ hyperplane of equation $(z=a), a \in \mathbb{F}_{q}$, parallel to $H=G_{0}$,
$g_{a}=f .\left(1-(z-a)^{q-1}\right) \in R_{q}((t+1)(q-1)+s, m)$ and $g_{a} \neq 0$. So $\operatorname{Card}\left(S_{g_{a}}\right) \geq(q-s) q^{m-t-2}$.
Since $\operatorname{Card}(S)=(q-s) q^{m-t-1}$ and $S_{g_{a}}=S \cap G_{a}$,
$\operatorname{Card}\left(S_{g_{a}}\right)=(q-s) q^{m-t-2}$.
By recursion hypothesis, $S_{g_{0}}=S \cap H$ is included in an affine subspace of codimension $t+1$.

In both cases, $S \cap H$ is included in an affine subspace of codimension $t+1$ which is impossible since $a+\overrightarrow{v_{1}}, \ldots, a+\overrightarrow{v_{m-t}} \in S \cap H$.
So $S \cap H=S$ and $a+\vec{u} \notin S$, which means that $\vec{u} \notin F$.
Hence, $F \subset \operatorname{Vect}\left(\overrightarrow{v_{1}}, \ldots, \overrightarrow{v_{m-t}}\right)$, i.e $S$ is included in an affine subspace of codimension $t$, say A.

By applying an affine transformation, we can assume that $A=\left\{x, x_{1}=0, \ldots, x_{t}=0\right\}$. Then by corollary 3, we can write

$$
f(x)=\prod_{i=1}^{t}\left(x_{i}^{q-1}-1\right) P\left(x_{t+1}, \ldots, x_{m}\right)
$$

$P \in R_{q}(s, m-t)$ and $|P|=|f|=(q-s) q^{m-t-1}$, thus, by the case where $t=0$, $S_{P}$ is the union of $(q-s)$ parallel hyperplanes of $A$ which gives the result.

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## References

[1] P. Delsarte, J.M. Goethals, F.J. Mac Williams, On generalized Reed-Muller codes and their relatives, Information and Control, 16, 403-442 (1970)

