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# A new proof of Delsarte, Goethals and Mac Williams theorem on minimal weight codewords of generalized Reed-Muller codes

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## Abstract

We give a new proof of Delsarte, Goethals and Mac Williams theorem on minimal weight codewords of generalized Reed-Muller codes published in 1970. To prove this theorem, we consider intersection of support of minimal weight codewords with affine hyperplanes and we proceed by recursion.

## 1 Introduction

In the appendix of [1], Delsarte, Goethals and Mac Williams prove the theorem 1 below. However, at the beginning of their proof, they point out that "it would be very desirable to find a more sophisticated and shorter proof".

In this paper, we give a new proof of this theorem that we hope is simpler.

Let  $q = p^n$ ,  $p$  being prime number.

We identify the  $\mathbb{F}_q$ -algebra  $B_m^q = \mathbb{F}_q[X_1, \dots, X_m]/(X_1^q - X_1, \dots, X_m^q - X_m)$  to the  $\mathbb{F}_q$ -algebra of the functions from  $\mathbb{F}_q^m$  to  $\mathbb{F}_q$  through the isomorphism  $P \mapsto (x \mapsto P(x))$ .

For  $f \in B_m^q$ , let  $S_f = \{x \in \mathbb{F}_q^m, f(x) \neq 0\}$  the support of  $f$ , and

$|f| = \text{Card}(S_f)$  the weight of  $f$ . The Hamming distance in  $B_m^q$  is denoted by  $d(., .)$ .

For  $0 \leq r \leq m(q-1)$ , the  $r$ th order generalized Reed-Muller code of length  $q^m$  is

$$R_q(r, m) = \{P \in B_m^q, \deg(P) \leq r\}$$

where  $\deg(P)$  is the degree of the representative of  $P$  with degree at most  $q-1$  in each variable.

The affine group  $\text{GA}_m(\mathbb{F}_q)$  acts on  $R_q(r, m)$  by its natural action. The minimum weight of  $R_q(r, m)$  is  $(q-s)q^{m-t-1}$ , where  $r = t(q-1) + s$ ,  $0 \leq s \leq q-2$  (see [1]).

The following theorem gives the codeword of minimum weight of  $R_q(r, m)$

**Theorem 1** *Let  $r = t(q-1) + s < m(q-1)$ . The minimal weight codewords of  $R_q(r, m)$  are codewords of  $R_q(r, m)$  whose support is the union of  $(q-s)$  distinct parallel affine subspaces of codimension  $t+1$  included in an affine subspace of codimension  $t$ .*

**Remark :**

1. Clearly, codewords of this form are of minimal weight.
2. Using lemma 2 and corollary 3 below, this theorem means that codewords of minimal weight are equivalent, under the action of the affine group, to a codeword of the following form:  $f(x) = c \prod_{i=1}^t (x_i^{q-1} - 1) \prod_{j=1}^s (x_{t+1} - b_j)$  where  $c \in \mathbb{F}_q^*$ ,  $b_j$  are distinct elements of  $\mathbb{F}_q$ .

## 2 Proof of Theorem 1

In this paper, we use freely the two following lemmas and their corollary proved in [1] p. 435.

**Lemma 2** *If  $P(x) = 0$  whenever  $x_1 = a$ , then  $P(x) = (x_1 - a)Q(x)$  where  $\deg_{x_1}(Q) \leq \deg_{x_1}(P) - 1$ .*

**Remark :** In [1], the lemma A1.1 says that the exponent of  $x_1$  in  $Q$  is at most  $q-2$ , but they actually prove the above lemma.

**Corollary 3** *If  $P(x) = 0$  unless  $x_1 = b$ , then  $P(x) = (1 - (x_1 - b)^{q-1})Q(x_2, \dots, x_m)$ .*

**Lemma 4** *Let  $S$  be a subset of  $\mathbb{F}_q^m$ , such that  $\text{Card}(S) = tq^n < q^m$ ,  $0 < t < q$ . Assume that for any hyperplane of  $\mathbb{F}_q^m$ , either  $\text{Card}(S \cap H) = 0$  or  $\text{Card}(S \cap H) \geq tq^{n-1}$ . Then there exists an affine hyperplane of  $\mathbb{F}_q^m$  which does not meet  $S$ .*

Now, we prove a lemma that will be crucial in our proof of theorem 1 :

**Lemma 5** *Let  $r = t(q-1) + s$ ,  $0 \leq s \leq q-2$ ,  $f$  be a minimal weight codeword of  $R_q(r, m)$  and  $S = S_f$ . If  $H$  is an hyperplane of  $\mathbb{F}_q^m$ , such that  $S \cap H \neq \emptyset$  and  $S \cap H \neq S$ , then either  $S$  meets all hyperplanes parallel to  $H$  or  $S$  meets  $q-s$  hyperplanes parallel to  $H$  in  $q^{m-t-1}$  points.*

*Proof :* Since an affine transformation does not change weight, we can assume that  $H = \{x, x_1 = 0\}$ .

Now assume that  $S$  does not meet  $k$  hyperplanes parallel to  $H$ ,  $k \geq 1$ . As  $S \cap H \neq \emptyset$  and  $S \cap H \neq H$ , we have  $k \leq q-2$ . By lemma 2, we can write

$$f(x) = (x_1 - b_1)^{\alpha_1} \dots (x_1 - b_k)^{\alpha_k} P(x)$$

where  $S_P$  meets all hyperplanes parallel to  $H$ .

Let  $d = \sum_{i=1}^k \alpha_i \leq q-1$ . We want to prove that  $d = s$ .

- First, assume that  $d > s$ . Then the degree of  $P$  is  $(t-1)(q-1) + q-1 + s-d$  and  $0 \leq q-1 + s-d \leq q-2$ . For  $c \notin \{b_1, \dots, b_k\}$ , we consider  $Q_c = (1 - (x_1 - c)^{q-1})P$ . The degree of  $Q_c$  is  $t(q-1) + q-1 + s-d$ . So

$$\begin{aligned} (q-s)q^{m-t-1} = \text{Card}(S) &= \sum_{c \notin \{b_1, \dots, b_k\}} \text{Card}(S_P \cap \{x_1 = c\}) \\ &\geq (q-k)(q - (q-1 + s-d))q^{m-t-1} \\ &= (q-k)(d-s+1)q^{m-t-1} \end{aligned}$$

and we obtain, since  $d \geq k$ ,

$$(q-1-k)(d-s) \leq 0,$$

which is impossible, since  $d > s$  and  $k < q-1$ .

- Now we have  $d \leq s$ .  
So  $\deg(P) = t(q-1) + s-d$  and we get

$$\begin{aligned} (q-s)q^{m-t-1} = \text{Card}(S) &= \sum_{c \notin \{b_1, \dots, b_k\}} \text{Card}(S_P \cap \{x_1 = c\}) \\ &\geq (q-k)(q-s+d)q^{m-t-2} \end{aligned}$$

which gives

$$(d-k)q + k(s-d) \leq 0.$$

Hence, since  $d \geq k$ ,  $k \geq 1$  and  $s \geq d$ , necessarily  $d = k = s$  and  $\text{Card}(S_P \cap \{x_1 = c\}) = q^{m-t-1}$  for  $c \notin \{b_1, \dots, b_k\}$ .

□

Now, we are able to prove theorem 1.

*Proof* : We prove first the case where  $t = 0$  and  $t = m-1$ .

- $t = 0$ .

If  $s = 0$ , then  $\deg(f) = 0$ . Thus, since  $f \neq 0$ , we have  $f = c$ , for  $c \in \mathbb{F}_q^*$  and  $S_f = \mathbb{F}_q^m$ .

Otherwise, let  $H$  be an affine hyperplane of  $\mathbb{F}_q^m$ , then  $\text{Card}(S_f \cap H) = 0$  or  $\text{Card}(S_f \cap H) \geq (q-s)q^{m-2}$ .

Hence, by lemma 4, there exists an affine hyperplane  $H_0$ , such that  $S_f \cap H_0 = \emptyset$ .

However,  $\mathbb{F}_q^m$  is the union of the  $q$  hyperplanes parallel to  $H_0$ , so there exists  $H_1$ , parallel to  $H_0$ , such that  $H_1 \cap S_f \neq \emptyset$ .

Furthermore, since  $|f| = (q-s)q^{m-1} \geq 2q^{m-1}$ ,  $S_f \cap H_1 \neq S_f$ . So, by lemma 5,

since  $S_f \cap H_0 = \emptyset$ ,  $S_f$  meets  $(q-s)$  hyperplanes parallel to  $H_1$ , say  $H_1, \dots, H_{q-s}$ , in  $q^{m-1}$  points, this means that  $S_f = \bigcup_{i=1}^{q-s} H_i$ .

- $t = m - 1$ .

Let  $f \in R_q((m-1)(q-1) + s, m)$ ,  $0 \leq s \leq q-2$ , such that  $|f| = q-s$ . We put  $S = S_f$ . Let  $\omega_1, \omega_2 \in S$  and  $H$  be an hyperplane, such that  $\omega_1, \omega_2 \in H$ .

Assume that  $S \cap H \neq S$  then, by lemma 5, either  $S$  meets all hyperplanes parallel to  $H$  (which is possible only if  $s = 0$ ) or  $S$  meets  $(q-s)$  hyperplanes parallel to  $H$  in one point. In both cases we get a contradiction, since in both cases  $S$  meets each hyperplane in exactly one point and  $\omega_1, \omega_2 \in H$ .

So  $S$  is included in all hyperplanes  $H$ , such that  $\omega_1, \omega_2 \in H$ , this means that  $S$  is included in the line through  $\omega_1$  and  $\omega_2$ .

Now we prove the theorem for general  $t$  by recursion.

- Assume that for a fixed  $t$ ,  $1 \leq t \leq m-2$  (we have already proved the case where  $t = 0$ ) and for all  $0 \leq s \leq q-2$ , the support of a codeword of minimal weight in  $R_q((t+1)(q-1) + s, m)$  is the union of  $(q-s)$  distinct parallel affine subspaces of codimension  $t+2$  included in an affine subspace of codimension  $t+1$ .

Let  $f \in R_q(t(q-1) + s, m)$ , such that  $|f| = (q-s)q^{m-t-1}$ .

We put  $S = S_f$ . Let  $a \in S$  and  $F = \{\overrightarrow{ab}, b \in S\}$ . We have :

$$\text{Card}(F) = (q-s)q^{m-t-1} \leq q^{rg(F)}.$$

Thus, since  $0 \leq s \leq q-2$ , we have  $rg(F) \geq m-t$ .

Let  $\overrightarrow{v_1}, \dots, \overrightarrow{v_{m-t}}$  be  $m-t$  independent vectors of  $F$  and  $\overrightarrow{u}$ , such that  $\overrightarrow{u} \notin \text{Vect}(\overrightarrow{v_1}, \dots, \overrightarrow{v_{m-t}})$ .

Since  $t \geq 1$ , there exists an affine hyperplane, say  $H$ , such that  $a + \overrightarrow{v_1}, \dots, a + \overrightarrow{v_{m-t}} \in H$  and  $a + \overrightarrow{u} \notin H$ .

Assume that  $S \cap H \neq S$ . Then by lemma 5, either  $S$  meets all hyperplanes parallel to  $H$  or  $S$  meets  $(q-s)$  hyperplanes parallel to  $H$  in  $q^{m-t-1}$  points.

- *1st case* :  $S$  meets  $(q-s)$  hyperplanes.

By applying an affine transformation, we can assume that the  $q-s$  hyperplanes are  $H_i = \{x, x_1 = a_i\}$ ,  $a_i \in \mathbb{F}_q$ . Without loss of generality, we can assume that  $H = H_1$ .

$$\text{Let } P = \prod_{i=2}^{q-s} (x_1 - a_i) f(x),$$

$$\deg(P) \leq t(q-1) + s + q - 1 - s = (t+1)(q-1) \text{ and}$$

$|P| = \text{Card}(S \cap \{x_1 = a_1\}) = q^{m-t-1}$ . So  $P$  is a codeword of minimal weight in  $R_q((t+1)(q-1), m)$ , and, by recursion hypothesis,  $S_p = S \cap H$  is an affine subspace of codimension  $t+1$ .

- *2nd case* :  $S$  meets all the hyperplanes.

For all  $G_a$  hyperplane of equation  $(z = a)$ ,  $a \in \mathbb{F}_q$ , parallel to  $H = G_0$ ,

$g_a = f \cdot (1 - (z - a)^{q-1}) \in R_q((t+1)(q-1) + s, m)$  and  $g_a \neq 0$ . So  
 $\text{Card}(S_{g_a}) \geq (q-s)q^{m-t-2}$ .  
 Since  $\text{Card}(S) = (q-s)q^{m-t-1}$  and  $S_{g_a} = S \cap G_a$ ,  
 $\text{Card}(S_{g_a}) = (q-s)q^{m-t-2}$ .  
 By recursion hypothesis,  $S_{g_0} = S \cap H$  is included in an affine subspace of  
 codimension  $t+1$ .

In both cases,  $S \cap H$  is included in an affine subspace of codimension  $t+1$  which  
 is impossible since  $a + \vec{v}_1, \dots, a + \vec{v}_{m-t} \in S \cap H$ .  
 So  $S \cap H = S$  and  $a + \vec{u} \notin S$ , which means that  $\vec{u} \notin F$ .  
 Hence,  $F \subset \text{Vect}(\vec{v}_1, \dots, \vec{v}_{m-t})$ , i.e.  $S$  is included in an affine subspace of codi-  
 mension  $t$ , say  $A$ .

By applying an affine transformation, we can assume that  
 $A = \{x, x_1 = 0, \dots, x_t = 0\}$ . Then by corollary 3, we can write

$$f(x) = \prod_{i=1}^t (x_i^{q-1} - 1) P(x_{t+1}, \dots, x_m)$$

$P \in R_q(s, m-t)$  and  $|P| = |f| = (q-s)q^{m-t-1}$ , thus, by the case where  $t=0$ ,  
 $S_P$  is the union of  $(q-s)$  parallel hyperplanes of  $A$  which gives the result.

□

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## References

- [1] P. Delsarte, J.M. Goethals, F.J. Mac Williams, On generalized Reed-Muller codes and their relatives, Information and Control, 16, 403-442 (1970)