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A new proof of Delsarte, Goethals and Mac Williams theorem on minimal weight codewords of generalized Reed-Muller codes

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Abstract

We give a new proof of Delsarte, Goethals and Mac williams theorem on minimal weight codewords of generalized Reed-Muller codes published in 1970. To prove this theorem, we consider intersection of support of minimal weight codewords with affine hyperplanes and we proceed by recursion.

1 Introduction

In the appendix of [1], Delsarte, Goethals and Mac Williams prove the theorem below. However, at the beginning of their proof, they point out that "it would be very desirable to find a more sophisticated and shorter proof".

In this paper, we give a new proof of this theorem that we hope is simpler.

Let \( q = p^n \), \( p \) being prime number.
We identify the \( F_q \)-algebra \( B_q^m = F_q[X_1, \ldots, X_m]/(X_1^q - X_1, \ldots, X_m^q - X_m) \) to the \( F_q \)-algebra of the functions from \( F_q^m \) to \( F_q \) through the isomorphism \( P \mapsto (x \mapsto P(x)) \).

For \( f \in B_q^m \), let \( S_f = \{ x \in F_q^m, f(x) \neq 0 \} \) the support of \( f \), and \( |f| = \text{Card}(S_f) \) the weight of \( f \). The Hamming distance in \( B_q^m \) is denoted by \( d(\ldots) \).

For \( 0 \leq r \leq m(q-1) \), the \( r \)th order generalized Reed-Muller code of length \( q^m \) is

\[ R_q(r, m) = \{ P \in B_q^m, \deg(P) \leq r \} \]

where \( \deg(P) \) is the degree of the representative of \( P \) with degree at most \( q-1 \) in each variable.

The affine group \( \text{GA}_m(F_q) \) acts on \( R_q(r, m) \) by its natural action. The minimum weight of \( R_q(r, m) \) is \( (q-s)q^{m-t-1} \), where \( r = t(q-1) + s, 0 \leq s \leq q-2 \) (see [2]).

The following theorem gives the codeword of minimum weight of \( R_q(r, m) \)
Theorem 1 Let \( r = t(q-1) + s < m(q-1) \). The minimal weight codewords of \( R_q(r,m) \) are codewords of \( R_q(r,m) \) whose support is the union of \((q-s)\) distinct parallel affine subspaces of codimension \( t+1 \) included in an affine subspace of codimension \( t \).

Remark:
1. Clearly, codewords of this form are of minimal weight.
2. Using lemma 2 and corollary 3 below, this theorem means that codewords of minimal weight are equivalent, under the action of the affine group, to a codeword of the following form:

\[
 f(x) = c \prod_{i=1}^{t} (x_i^{q-1} - 1) \prod_{j=1}^{s} (x_{i+1} - b_j)
\]

where \( c \in \mathbb{F}_q^* \), \( b_j \) are distinct elements of \( \mathbb{F}_q \).

2 Proof of Theorem 1

In this paper, we use freely the two following lemmas and their corollary proved in [1] p. 435.

Lemma 2 If \( P(x) = 0 \) whenever \( x_1 = a \), then \( P(x) = (x_1 - a)Q(x) \) where \( \deg_{x_1}(Q) \leq \deg_{x_1}(P) - 1 \).

Remark: In [1], the lemma A1.1 says that the exponent of \( x_1 \) in \( Q \) is at most \( q-2 \), but they actually prove the above lemma.

Corollary 3 If \( P(x) = 0 \) unless \( x_1 = b \), then \( P(x) = (1-(x_1-b)^{q-1})Q(x_2, \ldots, x_m) \).

Lemma 4 Let \( S \) be a subset of \( \mathbb{F}_q^m \), such that \( \text{Card}(S) = q^m < q^m \), \( 0 < t < q \). Assume that for any hyperplane of \( \mathbb{F}_q^m \), either \( \text{Card}(S \cap H) = 0 \) or \( \text{Card}(S \cap H) \geq q^{m-1} \). Then there exists an affine hyperplane of \( \mathbb{F}_q^m \) which does not meet \( S \).

Now, we prove a lemma that will be crucial in our proof of theorem 1.

Lemma 5 Let \( r = t(q-1) + s, 0 \leq s \leq q-2, f \) be a minimal weight codeword of \( R_q(r,m) \) and \( S = S_f \). If \( H \) is an hyperplane of \( \mathbb{F}_q^m \), such that \( S \cap H \neq \emptyset \) and \( S \cap H \neq S \), then either \( S \) meets all hyperplanes parallel to \( H \) or \( S \) meets \( q - s \) hyperplanes parallel to \( H \) in \( q^{m-t-1} \) points.

Proof: Since an affine transformation does not change weight, we can assume that \( H = \{ x, x_1 = 0 \} \).

Now assume that \( S \) does not meet \( k \) hyperplanes parallel to \( H \), \( k \geq 1 \). As \( S \cap H \neq \emptyset \) and \( S \cap H \neq H \), we have \( k \leq q-2 \). By lemma 2, we can write

\[
 f(x) = (x_1 - b_1)^{a_1} \ldots (x_1 - b_k)^{a_k} P(x)
\]
where $S_P$ meets all hyperplanes parallel to $H$.

Let $d = \sum_{i=1}^{k} \alpha_i \leq q - 1$. We want to prove that $d = s$.

- First, assume that $d > s$. Then the degree of $P$ is $(t-1)(q-1) + q - 1 + s - d$ and $0 \leq q - 1 + s - d \leq q - 2$. For $c \not\in \{b_1, \ldots, b_k\}$, we consider $Q_c = (1 - (x_1 - c)^{q-1})P$. The degree of $Q_c$ is $t(q - 1) + q - 1 + s - d$. So

$$(q - s)q^{m-t-1} = \text{Card}(S) = \sum_{c \not\in \{b_1, \ldots, b_k\}} \text{Card}(S_P \cap \{x_1 = c\})$$

$$\geq (q - k)(q - (q - 1 + s - d))q^{m-t-1}$$

$$= (q - k)(d - s + 1)q^{m-t-1}$$

and we obtain, since $d \geq k$, 

$$(q - 1 - k)(d - s) \leq 0,$$

which is impossible, since $d > s$ and $k < q - 1$.

- Now we have $d \leq s$.

So $\text{deg}(P) = t(q - 1) + s - d$ and we get

$$(q - s)q^{m-t-1} = \text{Card}(S) = \sum_{c \not\in \{b_1, \ldots, b_k\}} \text{Card}(S_P \cap \{x_1 = c\})$$

$$\geq (q - k)(q - s + d)q^{m-t-2}$$

which gives

$$(d - k)q + k(s - d) \leq 0.$$

Hence, since $d \geq k$, $k \geq 1$ and $s \geq d$, necessarily $d = k = s$ and $\text{Card}(S_P \cap \{x_1 = c\}) = q^{m-t-1}$ for $c \not\in \{b_1, \ldots, b_k\}$.

Now, we are able to prove theorem.\[3]

**Proof:** We prove first the case where $t = 0$ and $t = m - 1$.

- $t = 0$.

If $s = 0$, then $\text{deg}(f) = 0$. Thus, since $f \neq 0$, we have $f = c$, for $c \in \mathbb{F}_q^*$ and $S_f = \mathbb{F}_q^m$.

Otherwise, let $H$ be an affine hyperplane of $\mathbb{F}_q^m$, then $\text{Card}(S_f \cap H) = 0$ or $\text{Card}(S_f \cap H) \geq (q - s)q^{m-2}$.

Hence, by lemma, there exists an affine hyperplane $H_0$, such that $S_f \cap H_0 = \emptyset$. However, $\mathbb{F}_q^m$ is the union of the $q$ hyperplanes parallel to $H_0$, so there exists $H_1$, parallel to $H_0$, such that $H_1 \cap S_f \neq \emptyset$.

Furthermore, since $|f| = (q - s)q^{m-1} \geq 2q^{m-1}$, $S_f \cap H_1 \neq S_f$. So, by lemma,\[3]
since $S_f \cap H_0 = \emptyset$, $S_f$ meets $(q-s)$ hyperplanes parallel to $H_1$, say $H_1, \ldots, H_{q-s}$, in $q^{m-1}$ points, this means that $S_f = \bigcup_{i=1}^{q-s} H_i$.

- $t = m - 1$.
  Let $f \in R_q((m-1)(q-1) + s, m)$, $0 \leq s \leq q-2$, such that $|f| = q - s$. We put $S = S_f$. Let $\omega_1, \omega_2 \in S$ and $H$ be an hyperplane, such that $\omega_1, \omega_2 \in H$.
  Assume that $S \cap H \neq S$ then, by lemma $3$, either $S$ meets all hyperplanes parallel to $H$ (which is possible only if $s = 0$) or $S$ meets $(q-s)$ hyperplanes parallel to $H$ in one point. In both cases we get a contradiction, since in both cases $S$ meets each hyperplane in exactly one point and $\omega_1, \omega_2 \in H$.
  So $S$ is included in all hyperplanes $H$, such that $\omega_1, \omega_2 \in H$, this means that $S$ is included in the line through $\omega_1$ and $\omega_2$.

Now we prove the theorem for general $t$ by recursion.

- Assume that for a fixed $t$, $1 \leq t \leq m-2$ (we have already proved the case where $t = 0$) and for all $0 \leq s \leq q-2$, the support of a codeword of minimal weight in $R_q((t+1)(q-1)+s, m)$ is the union of $(q-s)$ distinct parallel affine subspaces of codimension $t+2$ included in an affine subspace of codimension $t+1$.

Let $f \in R_q((t(q-1)+s, m)$, such that $|f| = (q-s)q^{m-t-1}$.
We put $S = S_f$. Let $a \in S$ and $F = \{ab, b \in S\}$. We have:
\[
\text{Card}(F) = (q-s)q^{m-t-1} \leq q^{g_F}.
\]

Thus, since $0 \leq s \leq q-2$, we have $rg(F) \geq m - t$.
Let $\vec{v}_1, \ldots, \vec{v}_{m-t}$ be $m-t$ independent vectors of $F$ and $\vec{a}$, such that $\vec{a} \notin \text{Vect}(\vec{v}_1, \ldots, \vec{v}_{m-t})$.
Since $t \geq 1$, there exists an affine hyperplane, say $H$, such that $a + \vec{v}_1, \ldots, a + \vec{v}_{m-t} \in H$ and $a + \vec{a} \notin H$.
Assume that $S \cap H \neq S$. Then by lemma $3$, either $S$ meets all hyperplanes parallel to $H$ or $S$ meets $(q-s)$ hyperplanes parallel to $H$ in $q^{m-t-1}$ points.

- 1st case : $S$ meets $(q-s)$ hyperplanes.
  By applying an affine transformation, we can assume that the $q-s$ hyperplanes are $H_i = \{x, x_1 = a_i\}$, $a_i \in \mathbb{F}_q$. Without loss of generality, we can assume that $H = H_1$.
  Let $P = \prod_{i=2}^{q-s} (x_1 - a_i)f(x)$, $\deg(P) \leq t(q-1) + s + q - 1 - s = (t+1)(q-1)$ and $|P| = \text{Card}(S \cap \{x_1 = a_1\}) = q^{m-t-1}$. So $P$ is a codeword of minimal weight in $R_q((t+1)(q-1), m)$, and, by recursion hypothesis, $S_p = S \cap H$ is an affine subspace of codimension $t + 1$.

- 2nd case : $S$ meets all the hyperplanes.
  For all $G_a$ hyperplane of equation $(z = a)$, $a \in \mathbb{F}_q$, parallel to $H = G_0$,
\( g_a = f.(1 - (z - a)^{q-1}) \in R_q((t + 1)(q - 1) + s, m) \) and \( g_a \neq 0 \). So \( \text{Card}(S_{g_a}) \geq (q - s)q^{m-t-2} \). Since \( \text{Card}(S) = (q - s)q^{m-t-1} \) and \( S_{g_a} = S \cap G_a \),
\( \text{Card}(S_{g_a}) = (q - s)q^{m-t-2}. \)
By recursion hypothesis, \( S_{g_0} = S \cap H \) is included in an affine subspace of codimension \( t + 1 \).

In both cases, \( S \cap H \) is included in an affine subspace of codimension \( t + 1 \) which is impossible since \( a + \overrightarrow{v_1}, \ldots, a + \overrightarrow{v_{m-t}} \in S \cap H \).
So \( S \cap H = S \) and \( a + \overrightarrow{u} \notin S \), which means that \( \overrightarrow{u} \notin F \).
Hence, \( F \subseteq \text{Vect}(\overrightarrow{v_1}, \ldots, \overrightarrow{v_{m-t}}) \), i.e. \( S \) is included in an affine subspace of codimension \( t \), say \( A \).

By applying an affine transformation, we can assume that \( A = \{ x, x_1 = 0, \ldots, x_t = 0 \} \). Then by corollary 3, we can write
\[
f(x) = \prod_{i=1}^{t}(x_i^{q-1} - 1)P(x_{t+1}, \ldots, x_m)
\]
P \( \in R_q(s, m - t) \) and \( |P| = |f| = (q - s)q^{m-t-1} \), thus, by the case where \( t = 0 \), \( S_P \) is the union of \( (q - s) \) parallel hyperplanes of \( A \) which gives the result.

\( \square \)

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**References**