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To cite this version:

Elodie Leducq. A new proof of Delsarte, Goethals and Mac Williams theorem on minimal weight codewords of generalized Reed-Muller code. 2010. hal-00446913

HAL Id: hal-00446913
https://hal.science/hal-00446913
Preprint submitted on 13 Jan 2010

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A new proof of Delsarte, Goethals and Mac Williams theorem on minimal weight codewords of generalized Reed-Muller codes

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January 13, 2010

Abstract

We give a new proof of Delsarte, Goethals and Mac Williams theorem on minimal weight codewords of generalized Reed-Muller codes published in 1970. To prove this theorem, we consider intersection of support of minimal weight codewords with affine hyperplanes and we proceed by recursion.

1 Introduction

In the appendix of [1], Delsarte, Goethals and Mac Williams prove the theorem below. However, at the beginning of their proof, they point out that "it would be very desirable to find a more sophisticated and shorter proof". In this paper, we give a new proof of this theorem that we hope is simpler.

Let \( q = p^n \), \( p \) being prime number. We identify the \( \mathbb{F}_q \)-algebra \( B_q^n = \mathbb{F}_q[X_1, \ldots, X_m]/(X_1^q - X_1, \ldots, X_m^q - X_m) \) to the \( \mathbb{F}_q \)-algebra of the functions from \( \mathbb{F}_q^m \) to \( \mathbb{F}_q \) through the isomorphism \( P \mapsto x \mapsto P(x) \).

For \( f \in B_q^n \), let \( S_f = \{ x \in \mathbb{F}_q^m, f(x) \neq 0 \} \) the support of \( f \), and \( |f| = \text{Card}(S_f) \) the weight of \( f \). The Hamming distance in \( B_q^n \) is denoted by \( d(\ldots) \).

For \( 0 \leq r \leq m(q-1) \), the \( r \)th order generalized Reed-Muller code of length \( q^m \) is

\[ R_q(r, m) = \{ P \in B_q^n, \text{deg}(P) \leq r \} \]

where \( \text{deg}(P) \) is the degree of the representative of \( P \) with degree at most \( q-1 \) in each variable.

The affine group \( \text{GA}_m(\mathbb{F}_q) \) acts on \( R_q(r, m) \) by its natural action. The minimum weight of \( R_q(r, m) \) is \( (q-s)q^{m-t-1} \), where \( r = t(q-1) + s, 0 \leq s \leq q-2 \) (see [1]).

The following theorem gives the codeword of minimum weight of \( R_q(r, m) \)
Theorem 1 Let \( r = t(q-1) + s < m(q-1) \). The minimal weight codewords of \( R_q(r, m) \) are codewords of \( R_q(r, m) \) whose support is the union of \( (q-s) \) distinct parallel affine subspaces of codimension \( t+1 \) included in an affine subspace of codimension \( t \).

Remark:
1. Clearly, codewords of this form are of minimal weight.
2. Using lemma 2 and corollary 3 below, this theorem means that codewords of minimal weight are equivalent, under the action of the affine group, to a codeword of the following form:

\[
    f(x) = c \prod_{i=1}^{t} (x_i^{q-1} - 1) \prod_{j=1}^{s} (x_{t+1} - b_j)
\]

where \( c \in \mathbb{F}_q^* \), \( b_j \) are distinct elements of \( \mathbb{F}_q \).

2 Proof of Theorem 1

In this paper, we use freely the two following lemmas and their corollary proved in [1] p. 435.

Lemma 2 If \( P(x) = 0 \) whenever \( x_1 = a \), then \( P(x) = (x_1 - a)Q(x) \) where \( \deg_{x_1}(Q) \leq \deg_{x_1}(P) - 1 \).

Remark: In [1], the lemma A1.1 says that the exponent of \( x_1 \) in \( Q \) is at most \( q-2 \), but they actually prove the above lemma.

Corollary 3 If \( P(x) = 0 \) unless \( x_1 = b \), then \( P(x) = (1-(x_1-b)^{q-1})Q(x_2, \ldots, x_m) \).

Lemma 4 Let \( S \) be a subset of \( \mathbb{F}_q^m \), such that \( \text{Card}(S) = tq^n < q^m \), \( 0 < t < q \). Assume that for any hyperplane of \( \mathbb{F}_q^m \), either \( \text{Card}(S \cap H) = 0 \) or \( \text{Card}(S \cap H) \geq tq^{n-1} \). Then there exists an affine hyperplane of \( \mathbb{F}_q^m \) which does not meet \( S \).

Now, we prove a lemma that will be crucial in our proof of theorem 1.

Lemma 5 Let \( r = t(q-1) + s \), \( 0 \leq s \leq q-2 \), \( f \) be a minimal weight codeword of \( R_q(r, m) \) and \( S = S_f \). If \( H \) is an hyperplane of \( \mathbb{F}_q^m \), such that \( S \cap H \neq \emptyset \) and \( S \cap H \neq S \), then either \( S \) meets all hyperplanes parallel to \( H \) or \( S \) meets \( q-s \) hyperplanes parallel to \( H \) in \( q^{m-t}-1 \) points.

Proof: Since an affine transformation does not change weight, we can assume that \( H = \{ x, x_1 = 0 \} \).

Now assume that \( S \) does not meet \( k \) hyperplanes parallel to \( H \), \( k \geq 1 \). As \( S \cap H \neq \emptyset \) and \( S \cap H \neq H \), we have \( k \leq q-2 \). By lemma 3, we can write

\[
    f(x) = (x_1 - b_1)^{a_1} \ldots (x_1 - b_k)^{a_k} P(x)
\]
where $SP$ meets all hyperplanes parallel to $H$.

Let $d = \sum_{i=1}^{k} \alpha_i \leq q - 1$. We want to prove that $d = s$.

- First, assume that $d > s$. Then the degree of $P$ is $(t-1)(q-1) + q-1+s-d$ and $0 \leq q-1+s-d \leq q-2$. For $c \not\in \{b_1, \ldots, b_k\}$, we consider $Q_c = (1-(x_1-c)^{q-1})P$. The degree of $Q_c$ is $t(q-1) + q-1+s-d$. So

$$ \begin{align*}
(q-s)q^{m-t-1} &= \text{Card}(S) = \sum_{c \not\in \{b_1, \ldots, b_k\}} \text{Card}(SP \cap \{x_1 = c\}) \\
&\geq (q-k)(q-(q-1+s-d))q^{m-t-1} \\
&= (q-k)(d-s+1)q^{m-t-1}
\end{align*} $$

and we obtain, since $d \geq k$,

$$(q-1-k)(d-s) \leq 0,$$

which is impossible, since $d > s$ and $k < q-1$.

- Now we have $d \leq s$.

So deg$(P) = t(q-1) + s - d$ and we get

$$ \begin{align*}
(q-s)q^{m-t-1} &= \text{Card}(S) = \sum_{c \not\in \{b_1, \ldots, b_k\}} \text{Card}(SP \cap \{x_1 = c\}) \\
&\geq (q-k)(q-s+d)q^{m-t-2}
\end{align*} $$

which gives

$$(d-k)q + k(s-d) \leq 0.$$

Hence, since $d \geq k$, $k \geq 1$ and $s \geq d$, necessarily $d = k = s$ and Card$(SP \cap \{x_1 = c\}) = q^{m-t-1}$ for $c \not\in \{b_1, \ldots, b_k\}$.

Now, we are able to prove theorem 1.

**Proof**: We prove first the case where $t = 0$ and $t = m - 1$.

- $t = 0$.

If $s = 0$, then deg$(f) = 0$. Thus, since $f \not= 0$, we have $f = c$, for $c \in \mathbb{F}_q^*$ and $S_f = \mathbb{F}_q^m$.

Otherwise, let $H$ be an affine hyperplane of $\mathbb{F}_q^m$, then Card$(S_f \cap H) = 0$ or Card$(S_f \cap H) \geq (q-s)q^{m-2}$.

Hence, by lemma 3, there exists an affine hyperplane $H_0$, such that $S_f \cap H_0 = \emptyset$.

However, $\mathbb{F}_q^m$ is the union of the $q$ hyperplanes parallel to $H_0$, so there exists $H_1$, parallel to $H_0$, such that $H_1 \cap S_f \not= \emptyset$.

Furthermore, since $|f| = (q-s)q^{m-1} \geq 2q^{m-1}$, $S_f \cap H_1 \not= S_f$. So, by lemma 3, 

\[3\]
since $S_f \cap H_0 = \emptyset$, $S_f$ meets $(q-s)$ hyperplanes parallel to $H_1$, say $H_1, \ldots, H_{q-s}$, in $q^{m-1}$ points, this means that $S_f = \bigcup_{i=1}^{q-s} H_i$.

- $t = m - 1$.

Let $f \in R_q((m-1)(q-1)+s,m)$, $0 \leq s \leq q-2$, such that $|f| = q-s$. We put $S = S_f$. Let $\omega_1, \omega_2 \in S$ and $H$ be an hyperplane, such that $\omega_1, \omega_2 \in H$. Assume that $S \cap H \neq S$ then, by lemma $B$ either $S$ meets all hyperplanes parallel to $H$ (which is possible only if $s = 0$) or $S$ meets $(q-s)$ hyperplanes parallel to $H$ in one point. In both cases we get a contradiction, since in both cases $S$ meets each hyperplane in exactly one point and $\omega_1, \omega_2 \in H$.

So $S$ is included in all hyperplanes $H$, such that $\omega_1, \omega_2 \in H$, this means that $S$ is included in the line through $\omega_1$ and $\omega_2$.

Now we prove the theorem for general $t$ by recursion.

- Assume that for a fixed $t$, $1 \leq t \leq m-2$ (we have already proved the case where $t = 0$) and for all $0 \leq s \leq q-2$, the support of a codeword of minimal weight in $R_q((t+1)(q-1)+s,m)$ is the union of $(q-s)$ distinct parallel affine subspaces of codimension $t+2$ included in an affine subspace of codimension $t+1$.

Let $f \in R_q((t(q-1)+s,m)$, such that $|f| = (q-s)q^{m-t-1}$.

We put $S = S_f$. Let $a \in S$ and $F = \{ab, b \in S\}$. We have :

$$\text{Card}(F) = (q-s)q^{m-t-1} < q^{rg(F)}.$$  

Thus, since $0 \leq s \leq q-2$, we have $rg(F) \geq m-t$.

Let $v_1, \ldots, v_{m-t}$ be $m-t$ independent vectors of $F$ and $\overrightarrow{u}$, such that $\overrightarrow{u} \notin \text{Vect}(v_1, \ldots, v_{m-t})$.

Since $t \geq 1$, there exists an affine hyperplane, say $H$, such that $a + v_1, \ldots, a + v_{m-t} \in H$ and $a + \overrightarrow{u} \notin H$.

Assume that $S \cap H \neq S$. Then by lemma $B$ either $S$ meets all hyperplanes parallel to $H$ or $S$ meets $(q-s)$ hyperplanes parallel to $H$ in $q^{m-t-1}$ points.

- 1st case : $S$ meets $(q-s)$ hyperplanes.

By applying an affine transformation, we can assume that the $(q-s)$ hyperplanes are $H_i = \{x, x_1 = a_i\}$, $a_i \in \mathbb{F}_q$. Without loss of generality, we can assume that $H = H_1$.

Let $P = \prod_{i=2}^{q-s} (x_1 - a_i) f(x)$, then $\deg(P) \leq t(q-1) + s + q - 1 - s = (t+1)(q-1)$ and $|P| = \text{Card}(S \cap \{x_1 = a_1\}) = q^{m-t-1}$. So $P$ is a codeword of minimal weight in $R_q((t+1)(q-1),m)$, and, by recursion hypothesis, $S_p = S \cap H$ is an affine subspace of codimension $t+1$.

- 2nd case : $S$ meets all the hyperplanes.

For all $G_a$ hyperplane of equation $(z = a)$, $a \in \mathbb{F}_q$, parallel to $H = G_0$,
\[ g_a = f(1 - (z - a)^q - 1) \in R_q((t + 1)(q - 1) + s, m) \text{ and } g_a \neq 0. \]

So 
\[
\text{Card}(S_{g_a}) \geq (q - s)q^{m-t-2}.
\]

Since \( \text{Card}(S) = (q - s)q^{m-t-1} \) and \( S_{g_a} = S \cap G_a \),
\[
\text{Card}(S_{g_a}) = (q - s)q^{m-t-2}.
\]

By recursion hypothesis, \( S_{g_0} = S \cap H \) is included in an affine subspace of codimension \( t + 1 \).

In both cases, \( S \cap H \) is included in an affine subspace of codimension \( t + 1 \) which is impossible since \( a + \overrightarrow{v_1}, \ldots, a + \overrightarrow{v_{m-t}} \in S \cap H \).

So \( S \cap H = S \) and \( a + \overrightarrow{u} \not\in S \), which means that \( \overrightarrow{u} \not\in F \).

Hence, \( F \subset \text{Vect}(\overrightarrow{v_1}, \ldots, \overrightarrow{v_{m-t}}) \), i.e \( S \) is included in an affine subspace of codimension \( t \), say \( A \).

By applying an affine transformation, we can assume that \( A = \{x, x_1 = 0, \ldots, x_t = 0\} \). Then by corollary 5, we can write
\[
f(x) = \prod_{i=1}^{t}(x_i^{q-1} - 1)P(x_{t+1}, \ldots, x_m)
\]

\( P \in R_q(s, m - t) \) and \( |P| = |f| = (q - s)q^{m-t-1} \), thus, by the case where \( t = 0 \), \( S_P \) is the union of \( (q - s) \) parallel hyperplanes of \( A \) which gives the result.

\[ \square \]

Acknowledgements: I want to thank my supervisor, Jean-François Mestre, for his very helpful remarks.

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