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Semiparametric inference for the recurrent event process by means of a single-index model

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Abstract

In this paper, we introduce new parametric and semiparametric regression techniques for a recurrent event process subject to random right censoring. We develop models for the cumulative mean function and provide asymptotically normal estimators. Our semiparametric model which relies on a single-index assumption can be seen as a reduction dimension technique that, contrary to a fully nonparametric approach, is not stroke by the curse of dimensionality when the number of covariates is high. We discuss data-driven techniques to choose the parameters involved in the estimation procedures and provide a simulation study to support our theoretical results.

Key words: asymptotic normality, dimension reduction, empirical processes, recurrent events, right-censoring, single-index model.

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1 Introduction

The modeling of recurrent events has become a crucial issue in various application fields of statistical inference such as clinical and epidemiological studies, insurance or actuarial science. Among many examples, one can mention the modeling of asthma, of epileptic seizures or of repeated warranty claims. In these settings, regression models are a valuable tool for predicting or identifying the causes which influence the number of such events occurring during a given time period. A natural way to measure the impact of covariates on the recurrent event process consists of estimating the conditional cumulative mean function. In this paper, our aim consists of developing both parametric and semiparametric inference for this conditional cumulative mean function. To that aim, we introduce new estimators and study their asymptotic behavior. We also discuss the data-driven way of calibrating the parameters involved in the estimation procedures.

In the literature, various authors have studied Cox regression models adapted to the recurrent event context. For example, in the absence of dependent death, Prentice *et al.* (1981) considered Cox-type regression models which allow the intensity of the recurrent event process to depend on the individual's prior failure history through stratification. Allowing for independent censoring and time-dependent covariates, Andersen & Gill (1982) carried out Cox-type regression analysis for the intensity of the recurrent process which is assumed to be a time-transformed Poisson process. Andersen *et al.* (1993) also adopted modeling techniques based on the intensity process in the presence of censoring under a non-homogeneous Markov assumption. Lin *et al.* (2000) provided asymptotic distribution theory for the fitting of Cox-type marginal models without the Poisson assumption. Lawless & Nadeau (1995) proposed a semiparametric regression model where the conditional cumulative mean function is proportional to an unknown baseline function through a coefficient that depends parametrically on the covariates. More recently, Ghosh & Lin (2003) performed semiparametric regression with a scale-change model that formulates the marginal distributions of the recurrent event process and death as two joint accelerated failure time models while leaving the dependence structure unspecified.

The main advantage of these kinds of models stands in the simplicity of the regression function. But they unfortunately face the disadvantage (with respect to a purely nonparametric approach) to rely on strong modeling assumptions that may not hold in practice.

In this work, we first study a general parametric regression model for the recurrent event process. We then study a semiparametric generalization which relies on a single-

index assumption. We propose a new procedure to estimate both the index and the conditional cumulative mean regression function and provide a detailed asymptotic study of the proposed estimators. This single-index model can be seen as a compromise between a parametric approach and a nonparametric one. In particular, while allowing full flexibility, the nonparametric approach is known to fail when the number of covariates is high (greater than 3 in practice) which is the so-called “curse-of-dimensionality”. It turns out that single-index models rely on a dimension reduction assumption which allows to achieve better convergence rates and still ensures enough flexibility to be adapted to a large number of practical cases. This model can also be seen as a generalization of Cox regression model. Compared to uncensored single-index models adapted to mean-regression, see e.g. Ichimura (1993), in the specific setting of recurrent events, the presence of censoring usually deteriorates the quality of estimation in the tail of the distribution. Therefore, in our approach, we introduce a weight function designed to compensate the lack of information induced by censoring. The main novelty of our procedure stands in the fact that this weight function may be chosen using data-driven techniques.

The paper is organized as follows. In Section 2, we define the parametric and semi-parametric models and explain the general methodology. Asymptotic results are presented in Section 3. Simulation studies are carried out in Section 4 to investigate on the performance of our methods for finite sample size. Technical results are postponed to the Appendix in Section 6.

2 Model assumptions and methodology

In this section, we present the general setting. Specifically, Section 2.1 introduces the different regression models. Section 2.2 presents the estimation procedures. They are based on a least-squares type criterion and on a rescaled process defined in Section 2.2.1 which permits to correct the impact of censoring.

2.1 Regression models for recurrent events

Consider the recurrent event process $N^*(t)$ which denotes the number of recurrent events occurring in the time interval $[0, t]$. This process can be seen as a piecewise constant function with jump only on $[0, D]$ where D can be random. In clinical applications, this time D may stand for the death time of a patient. For insurance applications, D can represent the warranty length (which can be random if the client has the possibility of

breaking the contract) or the lifetime of the insured good. In this paper, we aim to infer on the cumulative conditional mean function given for $t \in [0, D]$ by

$$\mu(t|z) = E [N^*(t)|Z = z],$$

where $Z \in \mathcal{Z} \subset \mathbb{R}^d$ is a d -dimensional vector of covariates.

We now present the two different models for μ that are studied throughout this paper.

Model 1 : parametric case.

$$\mu(t|z) = \mu_0(t, z; \theta_0), \tag{2.1}$$

where $\theta_0 \in \Theta \subset \mathbb{R}^d$ is unknown and μ_0 is a known function.

Model 2 : semiparametric case.

$$\mu(t|z) = \mu_{\theta_0}(t, \theta'_0 z), \tag{2.2}$$

where $\theta_0 \in \Theta \subset \mathbb{R}^d$, $\mu_{\theta}(t, u) = E[N^*(t)|\theta'Z = u]$ and the family of functions $\mathcal{F} = \{\mu_{\theta} : \theta \in \Theta\}$ is unknown. We impose that the first component of θ_0 is 1 to identify this parameter. Another equivalent condition could consist of imposing that θ_0 is of norm 1 for some given norm on \mathbb{R}^d .

The appealing feature of the first model stands in the simplicity of the regression function. However, like every parametric procedure, it relies on strong assumptions which have few chances to hold in practice. On the opposite, a fully nonparametric procedure requires fewer assumptions but suffers from the so-called ‘‘curse of dimensionality’’ when the number of covariates is high. Therefore, the second model appears as a good compromise between the parametric approach and the nonparametric one. Indeed it is more flexible than a fully parametric one but is not stroke by the curse of dimensionality since it relies on a dimension reduction assumption. Moreover, model 2 can be seen as a generalization of widely studied models. For example, the models $\mu(t|z) = \mu_0(t) \exp(\theta'_0 z)$ and $\mu(t|z) = \mu_0(t \exp(\theta'_0 z))$ where μ_0 is an unknown baseline function correspond respectively to the popular Cox regression model and to the accelerated failure time model and are covered by model 2 as special cases.

One does not generally observe N^* on the whole time interval $[0, D]$ because the random variable D is subject to right-censoring. Let C be a positive random variable

standing for the censoring time. The observation time T is then given by $T = D \wedge C$. Hence, instead of observing $N^*(t)$ for $t \in [0, D]$, one only observes $N(t) = N^*(t \wedge T)$ for $t \geq 0$. Letting $\delta = I(D \leq C)$, the observations consist of n i.i.d. replications $(T_i, \delta_i, Z_i, N_i(\cdot))_{1 \leq i \leq n}$ of $(T, \delta, Z, N(\cdot))$. Let us introduce the distribution functions of the observed variables in the censored data model:

$$\begin{cases} H(t) = P(T \leq t), \\ F(t) = P(D \leq t), \\ G(t) = P(C \leq t). \end{cases}$$

We also define $\tau_H = \inf(t : H(t) = 1)$ the right endpoint of the support of the random variable T . In the sequel, we use some assumptions needed to identify these distribution functions.

Assumption 1. *Assume that*

$$\begin{cases} P(dN^*(C) \neq 0) = 0, \\ P(D = C) = 0. \end{cases}$$

This is a common assumption in the context of recurrent events which prevents us from ties between the occurrence times of death, censoring and recurrent events.

Assumption 2. *Assume that*

$$\begin{cases} C \perp\!\!\!\perp (N^*, D), \\ P(C \leq t | N^*, Z, D) = P(C \leq t | N^*, D) \text{ for } t \in [0, \tau_H]. \end{cases}$$

Assumption 2 holds in the particular case where C is independent of (N^*, D, Z) but is more general since it does not require the independence between C and Z . Similar kinds of assumptions are often considered in the literature on the Kaplan-Meier estimator for the survival distribution function, see e.g. Stute (1993).

2.2 Estimation procedure

2.2.1 The rescaled process

One of the difficulties we face when estimating the conditional expectation of N^* is that the process N^* is not directly observed. Hence, the most natural criteria we may like

to use can not be computed since they rely on N^* . Therefore, we introduce a rescaled process Y designed to compensate the censoring effects. We define, for $t \in [0, D]$

$$Y(t) = \int_0^t \frac{dN(s)}{1 - G(s-)}. \quad (2.3)$$

In the definition (2.3), the denominator is decreasing when s grows to infinity. This means that we allow more weight to the events we observe when s is large. This compensates the lack of observations due to censoring for s large. Under Assumptions 1 and 2, we have for $s \in [0, D]$

$$\begin{aligned} E[dN(s)|Z] &= E[dN^*(s \wedge C)|Z] \\ &= E[dN^*(s)I(s \leq C)|Z] \\ &= E[dN^*(s)|Z](1 - G(s-)), \end{aligned} \quad (2.4)$$

so that

$$E[Y(t)|Z] = E[N^*(t)|Z].$$

However, the rescaled process Y can not be computed in practice since it relies on the distribution function G which is usually unknown. But the process $Y(t)$ can be estimated, for $t \in [0, D]$ by

$$\hat{Y}(t) = \int_0^t \frac{dN(s)}{1 - \hat{G}(s-)}, \quad (2.5)$$

where \hat{G} denotes the Kaplan-Meier estimator of G given by

$$\hat{G}(t) = 1 - \prod_{i: T_i \leq t} \left(1 - \frac{1}{\sum_{j=1}^n I(T_j \geq T_i)} \right)^{1 - \delta_i}.$$

2.2.2 The parametric case

Going back to the definition of the conditional expectation, it is quite natural to perform estimation of θ_0 in the parametric model using minimization of a least-squares-type criterion. Once again, since N^* is unavailable, we consider a criterion based on the estimated rescaled process Y .

Let w denote a measure such that $w([0, \infty)) < \infty$ and define

$$M_w(\theta, \mu_0) = \int_0^{\tau_H} E[\mu_0(t, Z; \theta)^2] dw(t) - 2 \int_0^{\tau_H} E[Y(t)\mu_0(t, Z; \theta)] dw(t).$$

By definition of the conditional expectation, the true parameter value θ_0 satisfies

$$\theta_0 = \arg \min_{\theta \in \Theta} M_w(\theta, \mu_0). \quad (2.6)$$

To estimate θ_0 , it is natural to replace the function M_w by an empirical version, that is

$$M_{n,w}(\theta, \mu_0) = \frac{1}{n} \sum_{i=1}^n \int_0^{T_{(n)}} \mu_0(t, Z_i; \theta)^2 dw(t) - \frac{2}{n} \sum_{i=1}^n \int_0^{T_{(n)}} \hat{Y}_i(t) \mu_0(t, Z; \theta) dw(t),$$

where $T_{(n)}$ is the greatest order statistics associated to the sample T_1, \dots, T_n . Then we define an estimator of θ_0 as

$$\hat{\theta}(w) = \arg \min_{\theta \in \Theta} M_{n,w}(\theta, \mu_0). \quad (2.7)$$

In the above definition, we emphasize the fact that this estimator depends on the choice of the measure w . This measure w is an important feature of our procedure. First, in some situations, the statistician may wish to give more weight to some time intervals which are of higher importance. Moreover, the measure w is also useful to control the rescaled process. Indeed, in equation (2.5), the denominator goes to zero when s grows large and w can be precisely designed to avoid the practical problems caused by these too small denominators. Therefore, the finite sample behavior of our estimation procedure strongly relies on a wise choice of the measure w .

In Section 3.3, we obtain an asymptotic representation of $\hat{\theta}(w)$ as a process indexed by w which holds uniformly in $w \in \mathcal{W}$ where \mathcal{W} is a set of measures in which the statistician plans to choose w . We then discuss in Section 3.5 the adaptive choice of w .

2.2.3 The semiparametric case

In the semiparametric case, the family of functions μ_θ is unknown. However, the criterion used for the parametric case can be slightly modified to estimate θ_0 . We can write

$$\theta_0 = \arg \min_{\theta \in \Theta} M_w(\theta, \mu_\theta),$$

where

$$M_w(\theta, \mu_\theta) = \int_0^{\tau_H} E[\mu_\theta(t, \theta' Z)^2] dw(t) - 2 \int_0^{\tau_H} E[Y(t) \mu_\theta(t, \theta' Z)] dw(t).$$

Using a family of nonparametric estimators $\hat{\mu}_\theta$ of μ_θ , we define the estimator of θ_0 as

$$\hat{\theta}(w) = \arg \min_{\theta \in \Theta} M_{n,w}(\theta, \hat{\mu}_\theta), \quad (2.8)$$

where

$$M_{n,w}(\theta, \hat{\mu}_\theta) = n^{-1} \sum_{i=1}^n \int_0^{T(n)} \hat{\mu}_\theta(t, \theta' Z_i)^2 dw(t) - 2n^{-1} \sum_{i=1}^n \int_0^{T(n)} \hat{Y}_i(t) \hat{\mu}_\theta(t, \theta' Z_i) dw(t).$$

In Section 3.4, we derive an asymptotic representation of $\hat{\theta}(w)$ (see Theorem 3.3) regardless of the type of nonparametric estimators $\hat{\mu}_\theta$ used in the computation and provided these nonparametric estimators satisfy a list of uniform convergence conditions. Nevertheless, let us give a precise example of $\hat{\mu}_\theta$ using kernel estimators. The convergence properties of this type of estimator is derived in Section 6.2.

Using the same arguments as in (2.4), we have from the identifiability Assumptions 1 and 2,

$$\mu_\theta(t, u) = \int_0^t \frac{E[dN(s)|\theta'Z = u]}{1 - G(s-)} ds. \quad (2.9)$$

We estimate the numerator in (2.9) using a kernel estimator and the denominator by the Kaplan-Meier estimator \hat{G} , leading to

$$\hat{\mu}_{\theta,h}(t, u) = \int_0^t \frac{\sum_{i=1}^n K\left(\frac{\theta'Z_i - u}{h}\right) dN_i(s)}{\sum_{j=1}^n K\left(\frac{\theta'Z_j - u}{h}\right) (1 - \hat{G}(s-))} ds, \quad (2.10)$$

where K is a kernel function and h a bandwidth sequence tending to zero. In Section 6.2, we list some conditions on K and h . How to choose the bandwidth from the data in practice is considered at the end of Section 3.7.

3 Asymptotic results

In this section, we provide asymptotic properties for our estimators. In Section 3.1, we first expose and briefly discuss a list of technical assumptions on the model and on the different elements needed for the estimation procedures. In Section 3.2, we expose our main lemma, which is the keystone of our theoretical results. In the next two sections we give asymptotic representations of $\hat{\theta}(w)$ for the parametric and semiparametric models. We then discuss the adaptive choice of the measure w in order to improve the performance of our procedure in Section 3.5. The variance of the limiting process is estimated in Section 3.6 and the choice of the bandwidth h in (2.10) is highlighted in Section 3.7.

3.1 Exposition and discussion of assumptions

In order to obtain our asymptotic results, we first need to impose some conditions on different classes of functions.

Let us introduce some notations about the covering number. Let \mathcal{F} be a class of functions with envelope \bar{F} . Define, for a probability measure Q , the norm $\|\cdot\|_{p,Q}$ as the norm of $L^p(Q)$. The covering number of the class \mathcal{F} for the measure Q denoted by $N(\varepsilon, \mathcal{F}, \|\cdot\|_{p,Q})$ is the smaller number of $L^p(Q)$ -balls of radius ε needed to cover the set \mathcal{F} . The uniform covering number is defined as $N(\varepsilon, \mathcal{F}, \|\cdot\|_p) = \sup_Q N(\varepsilon, \mathcal{F}, \|\cdot\|_{p,Q})$ where the supremum is taken over all probability measures. In what follows, we say that a class of functions \mathcal{F} is a $\|\cdot\|_p$ -VC-class of functions if there exists two positive constants γ and c such that $N(\varepsilon, \mathcal{F}, \|\cdot\|_p) \leq c\varepsilon^{-\gamma}$.

A class of functions \mathcal{F} is said to satisfy one of the following property if the corresponding condition holds.

Property 1. For a class of functions $\mathcal{F} = \{f : (t, z) \in [0, \tau_H] \times \mathcal{Z} \mapsto f(t, z)\}$ and for any $\tau < \tau_H$, define

$$\mathcal{F}_\tau = \{f(t, \cdot), t \in [0, \tau]\},$$

which is a set of functions defined on \mathcal{Z} . Then, for any $\tau < \tau_H$, \mathcal{F}_τ is a VC-class of functions.

Property 2. For a class of functions $\mathcal{F} = \{f : (t, z) \in [0, \tau_H] \times \mathcal{Z} \mapsto f(t, z)\}$, the family of functions defined by $\{(z, y) \mapsto \int_0^{\tau_H} y(t) f(t, z) dw(t), f \in \mathcal{F}, w \in \mathcal{W}\}$ is Glivenko Cantelli.

In Section 6.3.3 in the appendix, we give a general type of sufficient conditions to fulfill these properties. It is easy to check that these technical assumptions are verified when the following conditions hold altogether:

- \mathcal{F} is a class of polynomial functions $f(t, z)$ (with bounded coefficients),
- $dE[Y(t)] = g(t)dt$ for some polynomial function $g(t)$,
- the class of measures is of the form $\mathcal{W} = \{w : dw(t) = W_0(t)d\tilde{w}(t)\}$ where $W_0(t)$ is a decreasing function (of order t^{-k} for k sufficiently high or exponential) and where \tilde{w} belongs to a class of monotone positive uniformly bounded functions sufficiently small (for example, piecewise constant bounded functions with a finite number of jumps).

Property 3. Let $\mathcal{F} = \{f_\theta : (t, z) \in [0, \tau_H] \times \mathcal{Z} \mapsto f_\theta(t, z), \theta \in \Theta\}$ be a family of functions indexed by θ . For any $f_{\theta_1}, f_{\theta_2} \in \mathcal{F}$ and $z \in \mathcal{Z}$, we have

$$\sup_{w \in \mathcal{W}} \int_0^{\tau_H} \|f_{\theta_1}(t, z) - f_{\theta_2}(t, z)\| dw(t) \leq c\|\theta_1 - \theta_2\|,$$

where c is a positive constant.

We now introduce the assumptions needed to derive the asymptotic normality of $\hat{\theta}$ in the parametric and semiparametric models.

Assumptions for the parametric model.

In the estimation procedures, we consider integrated versions of the rescaled process with respect to a measure w belonging to a class of measures \mathcal{W} . Detailed comments on this family and its role in the statistical procedure are discussed in Section 3.5. We need the following assumption for this class of measures.

Assumption 3. *Assume there exists some probability measure w_0 and a positive constant c_0 such that, for any $w \in \mathcal{W}$,*

$$\int_t^{\tau_H} dw(s) \leq c_0 W_0(t),$$

where $W_0(t) = \int_t^{\tau_H} dw_0(s)$ can be written as

$$W_0(t) = W_1(t)W_2(t)$$

where W_1 and W_2 are two positive and non-increasing functions satisfying

$$(1) \int_0^{\tau_H} W_1^2(t)(1 - F(t-))^{-1}(1 - G(t-))^{-2}dG(t) < \infty,$$

$$(2) \int_0^{\tau_H} W_2(t)E[dN^*(t)] < \infty,$$

$$(3) \lim_{t \rightarrow \tau_H} W_2(t) = 0.$$

In particular, Assumption 3 holds when all the measures w have their support included in a common compact subspace strictly included in $[0, \tau_H]$. On the other hand, since the function W_1 controls $1 - \hat{G}(s-)$ in $\hat{Y}(s)$ for s in the vicinity of the tail of the distribution, Assumption 3 also allows to consider measures w which are supported in the whole interval $[0, \tau_H]$. Taking $W_1(t) = (1 - H(t-))^{1/2}(1 - G(t-))^\varepsilon$ for some $\varepsilon > 0$ would be sufficient to obtain (1). Moreover, in the case where $\tau_H = \infty$, if we suppose that, for $\beta_1 > 0$, we have $E[N^*(t)] \sim \beta_1 t$ when $t \rightarrow \infty$, we could take for example $W_2(t) = t^{-\beta_2}$ for $\beta_2 > 1$ to fulfill (2) and (3).

We also need the following Hölder condition on the process N . This is a technical assumption used in the proof of our main lemma.

Assumption 4. Suppose there exists $\gamma > 0$ such that

$$E \left[\sup_{t \leq \tau, t' \leq \tau} \frac{|N(t) - N(t')|}{|t - t'|^\gamma} \right] < \infty.$$

Let $\nabla_{\theta} \mu_0(s, z; \theta_1)$ (resp. $\nabla_{\theta}^2 \mu_0(s, z; \theta_1)$) denote the vector of partial derivatives (resp. the Hessian matrix) of $\mu_0(s, z; \theta)$ with respect to all the components of θ evaluated at θ_1 . The following assumption can be understood as a regularity assumption on the regression model.

Assumption 5. Assume that, for all $w \in \mathcal{W}$, the matrix

$\Sigma_{w,p} = \int_0^{\tau_H} E[\nabla_{\theta} \mu_0(t, Z, \theta_0) \nabla_{\theta} \mu_0(t, Z, \theta_0)'] dw(t)$ is invertible. Moreover, assume that the classes of functions $\{\mu_0(\cdot, \cdot; \theta), \theta \in \Theta\}$, $\{\nabla_{\theta} \mu_0(\cdot, \cdot; \theta), \theta \in \Theta\}$ and $\{\nabla_{\theta}^2 \mu_0(\cdot, \cdot; \theta), \theta \in \Theta\}$ satisfy Properties 1, 2 and 3.

Additional assumptions for the semiparametric model.

The following assumption is similar to Assumption 5. Here, $\nabla_{\theta} \mu_{\theta_1}(s, z)$ (resp. $\nabla_{\theta}^2 \mu_{\theta_1}(s, z)$) denotes the vector of partial derivatives (resp. the Hessian matrix) of $\mu_{\theta}(s, \theta'z)$ with respect to all the components of θ evaluated at θ_1 . Note that the gradient vector $\nabla_{\theta} \mu_{\theta_1}(s, z)$ does not only depend on $\theta'z$ but also depends on the whole vector z . We give an explicit expression of this gradient in Lemma 6.5.

Assumption 6. Assume that, for all $w \in \mathcal{W}$, the matrix

$\Sigma_{w,sp} = \int_0^{\tau_H} E[\nabla_{\theta} \mu_{\theta_0}(t, Z) \nabla_{\theta} \mu_{\theta_0}(t, Z)'] dw(t)$ is invertible. Moreover, assume that the classes of functions $\{\mu_{\theta}(\cdot, \cdot), \theta \in \Theta\}$, $\{\nabla_{\theta} \mu_{\theta}(\cdot, \cdot), \theta \in \Theta\}$ and $\{\nabla_{\theta}^2 \mu_{\theta}(\cdot, \cdot), \theta \in \Theta\}$ satisfy Properties 1, 2 and 3.

As announced, we need uniform convergence properties for the nonparametric estimators $\hat{\mu}_{\theta}$.

Assumption 7. Define $\bar{\mu}_{\theta}(t, u) = \sup(\mu_{\theta}(t, u), 1)$.

(1) Assume that

$$\begin{aligned} \sup_{t \leq T_{(n)}, \theta \in \Theta, z \in \mathcal{Z}} \left| \frac{\hat{\mu}_{\theta}(t, \theta'z) - \mu_{\theta}(t, \theta'z)}{\bar{\mu}_{\theta_0}(t, \theta'_0 z)^{\lambda_1 + \lambda_2}} \right| &= o_P(1), \\ \sup_{t \leq T_{(n)}, \theta \in \Theta, z \in \mathcal{Z}} \left\| \frac{\nabla_{\theta} \hat{\mu}_{\theta}(t, z) - \nabla_{\theta} \mu_{\theta}(t, z)}{\bar{\mu}_{\theta_0}(t, \theta'_0 z)^{\lambda_1 + \lambda_2}} \right\| &= o_P(1), \\ \sup_{t \leq T_{(n)}, \theta \in \Theta, z \in \mathcal{Z}} \left\| \frac{\nabla_{\theta}^2 \hat{\mu}_{\theta}(t, z) - \nabla_{\theta}^2 \mu_{\theta}(t, z)}{\bar{\mu}_{\theta_0}(t, \theta'_0 z)^{\lambda_1 + \lambda_2}} \right\| &= o_P(1), \end{aligned}$$

where λ_1, λ_2 are such that $\lambda_1 + \lambda_2 \geq 1$.

(2) Assume also that

$$\sup_{t \leq T_{(n)}, z \in \mathcal{Z}} \left| \frac{\hat{\mu}_{\theta_0}(t, \theta'_0 z) - \mu_{\theta_0}(t, \theta'_0 z)}{\bar{\mu}_{\theta_0}(t, \theta'_0 z)^{\lambda_1 + \lambda_2}} \right| = O_P(\varepsilon_n),$$

$$\sup_{t \leq T_{(n)}, z \in \mathcal{Z}} \left\| \frac{\nabla_{\theta_0} \hat{\mu}_{\theta_0}(t, z) - \nabla_{\theta} \mu_{\theta_0}(t, z)}{\bar{\mu}_{\theta_0}(t, \theta'_0 z)^{\lambda_1 + \lambda_2}} \right\| = O_P(\varepsilon'_n),$$

where $\varepsilon_n \varepsilon'_n = o_P(n^{-1/2})$.

Assumption 8. Assume that

$$\sup_{z \in \mathcal{Z}} \int_0^{\tau_H} \mu_{\theta_0}(t, \theta'_0 z)^{2(\lambda_1 + \lambda_2)} dw(t) < \infty$$

where λ_1, λ_2 were defined in Assumption 7.

The following assumption is essential to the empirical process theory used in our proofs. We assume that the nonparametric estimators and μ_{θ_0} belong to some Donsker classes of functions.

Assumption 9. Assume that there exists some Donsker classes of functions \mathcal{G} and \mathcal{H} such that for all $w \in \mathcal{W}$

$$(z, y) \mapsto \int_0^{\tau_H} (\mu_{\theta_0}(t, \theta'_0 z) - y(t)) \nabla_{\theta} \mu_{\theta_0}(t, z) dw(t) \in \mathcal{G},$$

$$z \mapsto \int_0^{\tau_H} \mu_{\theta_0}(t, \theta'_0 z) \nabla_{\theta} \mu_{\theta_0}(t, z) dw(t) \in \mathcal{H}.$$

Moreover, assume that, almost surely for n large enough,

$$(z, y) \mapsto \int_0^{\tau_H} (\mu_{\theta_0}(t, \theta'_0 z) - y(t)) \nabla_{\theta} \hat{\mu}_{\theta_0}(t, z) dw(t) \in \mathcal{G},$$

$$z \mapsto \int_0^{\tau_H} \hat{\mu}_{\theta_0}(t, \theta'_0 z) \nabla_{\theta} \mu_{\theta_0}(t, z) dw(t) \in \mathcal{H}.$$

To give examples of such kind of classes, consider \mathcal{F} and \mathcal{W} as defined in the discussion following Property 2 and suppose, in addition, that the functions $(t, u) \rightarrow W_0(t) f(t, u)$ for $f \in \mathcal{F}$ (f is defined on \mathbb{R}^2 since $\theta'_0 z \in \mathbb{R}$) are twice continuously differentiable with bounded derivatives up to order 2. It follows from the results of Section 6.3.3 and from the decomposition of the gradient vector $\nabla_{\theta} \mu_{\theta_0}(t, z)$ obtained in Lemma 6.5 that we can consider $\mathcal{H} = \mathcal{G} = \mathcal{F}' + z\mathcal{F}'$ where $\mathcal{F}' = \{(u, y) \rightarrow \int_0^{\tau_H} (f_1(t, u) - y(t)) f_2(t, u) dw(t), w \in \mathcal{W}, f_1, f_2 \in \mathcal{F}\}$.

3.2 The main lemma

From a theoretical viewpoint, the main issue stands in studying the difference between Y and its estimated version. The following lemma provides an asymptotic representation for a class of empirical sums in which the process \hat{Y} is involved.

Such kind of asymptotic representations have become very valuable tools for inference in survival analysis, since they allow to transform a non i.i.d. problem on a quantity that can be easily studied using the central limit theorem. See e.g. Stute (1995), Van Keilegom & Akritas (1999), Sánchez Sellero *et al.* (2005) or Lopez (2009) for some similar results in other frameworks.

Lemma 3.1. *Let \mathcal{F} be a class of functions with bounded envelope \bar{F} satisfying Property 1 and assume that Assumptions 3 and 4 hold. Define, for any function $f \in \mathcal{F}$,*

$$S_n(f, w) = \frac{1}{n} \sum_{i=1}^n \int_0^{\tau_H} Y_i(t) f(t, Z_i) dw(t)$$

and

$$\hat{S}_n(f, w) = \frac{1}{n} \sum_{i=1}^n \int_0^{T_i} \hat{Y}_i(t) f(t, Z_i) dw(t).$$

(1) *Assume that $\sup_{w \in \mathcal{W}} E[S_n(\bar{F}, w)] < \infty$. Then, for all $f \in \mathcal{F}$,*

$$\hat{S}_n(f, w) - S_n(f, w) = \frac{1}{n} \sum_{i=1}^n \int_0^{\tau_H} \int_0^t \eta_{s-}(T_i, \delta_i) E[f(t, Z) d\mu(s|Z)] dw(t) + R_n(f, w),$$

where

$$\eta_t(T, \delta) = \frac{(1 - \delta)I(T \leq t)}{1 - H(T-)} - \int_0^t \frac{I(T \geq s) dG(s)}{[1 - H(s-)][1 - G(s-)]}$$

and where

$$\sup_{w \in \mathcal{W}, f \in \mathcal{F}} |R_n(f, w)| = o_P(n^{-1/2}).$$

Moreover, if the measures w are all supported in $[0, \tau]$ for some $\tau < \tau_H$, then

$$\sup_{w \in \mathcal{W}, f \in \mathcal{F}} |R_n(f, w)| = O_P(n^{-1} \log n).$$

(2) *If \hat{f} denotes a family of nonparametric estimators of functions $f \in \mathcal{F}$ satisfying $\sup_{f \in \mathcal{F}} \|\hat{f} - f\|_\infty = o_P(1)$, then*

$$\sup_{w \in \mathcal{W}} |\hat{S}_n(\hat{f}, w) - S_n(\hat{f}, w)| = o_P(n^{-1/2}).$$

Moreover, if the measures w are all supported in $[0, \tau]$ for some $\tau < \tau_H$, then

$$\sup_{w \in \mathcal{W}} |\hat{S}_n(\hat{f}, w) - S_n(\hat{f}, w)| = O_P(n^{-1} \log n).$$

The proof is postponed to Section 6.1. With the estimated rescaled process \hat{Y} at hand, we can now propose M -estimation procedures to estimate the regression function in both the parametric and semiparametric cases.

3.3 Asymptotic normality of $\hat{\theta}$ in the parametric case

Let \implies denote the weak convergence.

Theorem 3.2. *Assume that (2.1) holds. Under Assumptions 1 to 5, the estimator in (2.7) admits the following asymptotic representation*

$$\begin{aligned} \hat{\theta}(w) - \theta_0 &= \Sigma_{w,p}^{-1} \left\{ \frac{1}{n} \sum_{i=1}^n \left(\int_0^{\tau_H} [Y_i(t) - \mu_0(t, Z_i; \theta_0)] \nabla_{\theta} \mu_0(t, Z_i; \theta_0) dw(t) \right. \right. \\ &\quad \left. \left. + \int_0^{\tau_H} \int_0^t \eta_{s-}(T_i, \delta_i) E[\nabla_{\theta} \mu_0(t, Z; \theta_0) d\mu_0(s, Z; \theta_0)] dw(t) \right) \right\} + R_n(w), \end{aligned}$$

where $\sup_{w \in \mathcal{W}} |R_n(w)| = o_P(n^{-1/2})$. As a consequence, for any $w \in \mathcal{W}$,

$$\sqrt{n}(\hat{\theta}(w) - \theta_0) \implies \mathcal{N}(0, V_{w,p}),$$

where $V_{w,p} = \Sigma_{w,p}^{-1} \Delta_{w,p} \Sigma_{w,p}^{-1}$ and $\Delta_{w,p}$ is the covariance matrix of each term of the i.i.d. sum in the asymptotic expansion.

Proof. Write

$$M_{n,w}(\theta, \mu_0) = -2\hat{S}_n(\mu_0(\cdot, \cdot; \theta), w) + n^{-1} \sum_{i=1}^n \int_0^{T_i} \mu_0(t, Z_i; \theta)^2 dw(t).$$

Then, use the asymptotic representation of Lemma 3.1. Uniform consistency of $\hat{\theta}(w)$ follows from the uniform convergence of $M_{n,w}(\theta, \mu_0)$ which is obtained from Properties 2 and 3 for the classes of functions $\{\mu_0(\cdot, \cdot; \theta), \theta \in \Theta\}$ and $\{\nabla_{\theta} \mu_0(\cdot, \cdot; \theta), \theta \in \Theta\}$ (see Assumption 5).

To obtain the uniform CLT property for $\hat{\theta}(w)$, use a Taylor expansion of $\nabla_{\theta} M_{n,w}(\theta, \mu_0)$ around θ_0 :

$$\nabla_{\theta} M_{n,w}(\hat{\theta}, \mu_0) = \nabla_{\theta} M_{n,w}(\theta_0, \mu_0) + \nabla_{\theta}^2 M_{n,w}(\tilde{\theta}, \mu_0)(\hat{\theta} - \theta_0), \quad (3.1)$$

for some $\tilde{\theta}$ between $\hat{\theta}$ and θ_0 . The left-hand side of (3.1) is zero by definition of $\hat{\theta}$. Moreover, the matrix $\nabla_{\theta}^2 M_{n,w}(\tilde{\theta}, \mu_0)$ is almost surely invertible for n large enough under Assumption 5 since $\hat{\theta}$ (and consequently $\tilde{\theta}$) tends to θ_0 almost surely. This leads to

$$\hat{\theta} - \theta_0 = -\nabla_{\theta}^2 M_{n,w}^{-1}(\tilde{\theta}, \mu_0) \nabla_{\theta} M_{n,w}(\theta_0, \mu_0).$$

Write

$$\begin{aligned} \nabla_{\tilde{\theta}}^2 M_{n,w}(\tilde{\theta}, \mu_0) = & -2 \left[\hat{S}_n(\nabla_{\tilde{\theta}}^2 \mu_0(\cdot, \cdot; \tilde{\theta}), w) - \frac{1}{n} \sum_{i=1}^n \int_0^{\tau_H} \left(\nabla_{\tilde{\theta}} \mu_0(t, Z_i; \tilde{\theta}) \nabla_{\tilde{\theta}} \mu_0(t, Z_i; \tilde{\theta})' \right. \right. \\ & \left. \left. + \mu_0(t, Z_i; \tilde{\theta}) \nabla_{\tilde{\theta}}^2 \mu_0(t, Z_i; \tilde{\theta}) \right) dw(t) \right] + R_n(\theta, w) \end{aligned}$$

where $R_n(\theta, w)$ comes from the change in the integration bounds of $[0, T_{(n)}]$ by $[0, \tau_H]$ and can be seen to tend uniformly to zero from Lebesgue's dominated convergence since the term inside the integral is bounded. From Lemma 3.1, the almost sure convergence of $\tilde{\theta}$ and the fact that $\{\nabla_{\theta}^2 \mu_0(\cdot, \cdot, \theta), \theta \in \Theta\}$ satisfies Property 3 (see Assumption 5), we get that $\hat{S}_n(\nabla_{\tilde{\theta}}^2 \mu_0(\cdot, \cdot; \tilde{\theta}), w)$ converges to $\int_0^{\tau_H} E[Y(t) \nabla_{\theta}^2 \mu_0(t, Z; \theta_0)] dw(t)$ uniformly in w . The second part converges uniformly to its expectation over Θ as a consequence of the Glivenko-Cantelli property of classes of functions satisfying Property 3. This shows that

$$\sup_w |\nabla_{\tilde{\theta}}^2 M_{n,w}^{-1}(\tilde{\theta}, \mu_0) - \nabla_{\theta}^2 M_w^{-1}(\theta_0, \mu_0)| = o_P(1).$$

On the other hand, we write

$$\begin{aligned} \nabla_{\theta} M_{n,w}(\theta_0, \mu_0) = & -2 \left[\hat{S}_n(\nabla_{\theta} \mu_0(\cdot, \cdot; \theta_0), w) - \frac{1}{n} \sum_{i=1}^n \int_0^{\tau_H} \mu_0(t, Z; \theta_0) \nabla_{\theta} \mu_0(t, Z; \theta_0) dw(t) \right] \\ & + \frac{2}{n} \sum_{i=1}^n \int_{T_{(n)}}^{\tau_H} \mu_0(t, Z; \theta_0) \nabla_{\theta} \mu_0(t, Z; \theta_0) dw(t). \end{aligned}$$

Using Lebesgue's dominated convergence theorem, the last term tends uniformly to zero at a $n^{-1/2}$ rate. Finally, the asymptotic representation follows from Lemma 3.1. \square

3.4 Asymptotic normality of $\hat{\theta}$ in the semiparametric case

Theorem 3.3. *Assume that (2.2) holds. Under Assumptions 1 to 4 and 6 to 9, the estimator in (2.8) admits the following asymptotic representation*

$$\begin{aligned} \hat{\theta}(w) - \theta_0 = & \Sigma_{w,sp}^{-1} \left\{ \frac{1}{n} \sum_{i=1}^n \left(\int_0^{\tau_H} [Y_i(t) - \mu_{\theta_0}(t, \theta_0' Z_i)] \nabla_{\theta} \mu_{\theta_0}(t, Z_i) dw(t) \right. \right. \\ & \left. \left. + \int_0^{\tau_H} \int_0^t \eta_{s-}(T_i, \delta_i) E[\nabla_{\theta} \mu_{\theta_0}(t, \theta_0' Z) d\mu_{\theta_0}(s, \theta_0' Z)] dw(t) \right) \right\} + R_n(w), \end{aligned}$$

where $\sup_{w \in \mathcal{W}} |R_n(w)| = o_P(n^{-1/2})$. As a consequence, for any $w \in \mathcal{W}$,

$$\sqrt{n}(\hat{\theta}(w) - \theta_0) \implies \mathcal{N}(0, V_{w,sp}),$$

where $V_{w,sp} = \Sigma_{w,sp}^{-1} \Delta_{w,sp} \Sigma_{w,sp}^{-1}$ and $\Delta_{w,sp}$ is the covariance matrix of each term of the i.i.d. sum in the asymptotic expansion.

Proof. The consistency of the preliminary estimator can be proved in the same way as in the proof of Theorem 3.2, using now the second part of Lemma 3.1 and the uniform consistency of $\hat{\mu}_\theta$ (Assumption 7). Asymptotic normality comes from the fact that

$$\hat{\theta} - \theta_0 = -\nabla_{\tilde{\theta}}^2 M_{n,w}^{-1}(\tilde{\theta}, \hat{\mu}_{\tilde{\theta}}) \nabla_{\theta} M_{n,w}(\theta_0, \hat{\mu}_{\theta_0}).$$

The fact that

$$\sup_w \|\nabla_{\tilde{\theta}}^2 M_{n,w}^{-1}(\tilde{\theta}, \hat{\mu}_{\tilde{\theta}}) - \nabla_{\tilde{\theta}}^2 M_w^{-1}(\theta_0, \mu_{\theta_0})\| = o_P(1)$$

can be shown in the same way as in the proof of Theorem 3.2 using now the second part of Lemma 3.1. The big issue consists of proving the asymptotic representation of $\nabla_{\theta} M_{n,w}(\theta_0, \hat{\mu}_{\theta_0})$. Write

$$\nabla_{\theta} M_{n,w}(\theta_0, \hat{\mu}_{\theta_0}) = -2 \left[\hat{S}_n(\nabla_{\theta} \hat{\mu}_{\theta_0}(\cdot, \theta'_0 \cdot), w) - \frac{1}{n} \sum_{i=1}^n \int_0^{T(n)} \hat{\mu}_{\theta_0}(t, \theta'_0 Z_i) \nabla_{\theta} \hat{\mu}_{\theta_0}(t, Z_i) dw(t) \right].$$

Using the second part of Lemma 3.1, this can be rewritten as

$$\begin{aligned} & \nabla_{\theta} M_{n,w}(\theta_0, \hat{\mu}_{\theta_0}) \\ &= \nabla_{\theta} M_{n,w}(\theta_0, \mu_{\theta_0}) \\ & \quad - \frac{2}{n} \sum_{i=1}^n \int_0^{\tau_H} \bar{\mu}_{\theta_0}(t, \theta'_0 Z_i)^{\lambda_1 + \lambda_2} (\mu_{\theta_0}(t, \theta'_0 Z_i) - Y_i(t)) \frac{\nabla_{\theta} \mu_{\theta_0}(t, Z_i) - \nabla_{\theta} \hat{\mu}_{\theta_0}(t, Z_i)}{\bar{\mu}_{\theta_0}(t, \theta'_0 Z_i)^{\lambda_1 + \lambda_2}} dw(t) \\ & \quad + \frac{2}{n} \sum_{i=1}^n \int_0^{\tau_H} \frac{\hat{\mu}_{\theta_0}(t, \theta'_0 Z_i) - \mu_{\theta_0}(t, \theta'_0 Z_i)}{\bar{\mu}_{\theta_0}(t, \theta'_0 Z_i)^{\lambda_1 + \lambda_2}} \bar{\mu}_{\theta_0}(t, \theta'_0 Z_i)^{\lambda_1 + \lambda_2} \nabla_{\theta} \mu_{\theta_0}(t, Z_i) dw(t) \\ & \quad + \frac{2}{n} \sum_{i=1}^n \int_0^{\tau_H} \frac{(\hat{\mu}_{\theta_0}(t, \theta'_0 Z_i) - \mu_{\theta_0}(t, \theta'_0 Z_i)) (\nabla_{\theta} \hat{\mu}_{\theta_0}(t, Z_i) - \nabla_{\theta} \mu_{\theta_0}(t, Z_i))}{\bar{\mu}_{\theta_0}(t, \theta'_0 Z_i)^{2(\lambda_1 + \lambda_2)} \bar{\mu}_{\theta_0}(t, \theta'_0 Z_i)^{-2(\lambda_1 + \lambda_2)}} dw(t) \\ & \quad + R_{4n}(w) \\ &= \nabla_{\theta} M_{n,w}(\theta_0, \mu_{\theta_0}) + R_{1n}(w) + R_{2n}(w) + R_{3n}(w) + R_{4n}(w), \end{aligned}$$

where $R_{4n}(w)$ comes from Lemma 3.1 and the change in the integration bound of $[0, T(n)]$ by $[0, \tau_H]$. Using the same arguments as in the proof of Theorem 3.2, we deduce that $\sup_w \|R_{4n}(w)\| = o_P(n^{-1/2})$. Using the uniform convergence rates of $\hat{\mu}_{\theta_0}$ and of $\nabla_{\theta} \hat{\mu}_{\theta_0}$, we get straightforwardly that $\sup_w \|R_{3n}(w)\| = o_P(n^{-1/2})$. Using the uniform convergence of

$\nabla_{\theta} \hat{\mu}_{\theta_0}$, we see that the term R_{1n} can be decomposed into

$$R_{1n}(w) = n^{-1} \sum_{i=1}^n (f_w(Z_i, Y_i) - f_{n,w}(Z_i, Y_i))$$

where f_w and $f_{n,w}$ both belong (almost surely for n large enough) to the class \mathcal{G} defined in Assumption 9 and with $\sup_w \|f_w - f_{n,w}\|_{\infty} \rightarrow 0$ a.s. Therefore, using the asymptotic equicontinuity of the Donsker class \mathcal{G} (see e.g. Section 2.1.2 in Van der Vaart & Wellner (1996)), this shows that

$$\sup_w \|R_{1n}(w) - \int (f_w(z, y) - f_{n,w}(z, y)) dP_{Z,Y}(z, y)\| = o_P(n^{-1/2}).$$

Moreover, it is clear that $\int (f_w(z, y) - f_{n,w}(z, y)) dP_{Z,Y}(z, y) = 0$ using the fact that $\nabla_{\theta} \mu_{\theta_0}(t, z) - \nabla_{\theta} \hat{\mu}_{\theta_0}(t, z)$ is a function of z only and that $E[\mu_{\theta_0}(t, \theta'_0 Z_i) - Y_i(t) | Z_i] = 0$.

The term $R_{2n}(w)$ can be handled in the same way using now the Donsker class \mathcal{H} in Assumption 9, observing that $\hat{\mu}_{\theta_0}(t, \theta'_0 z) - \mu_{\theta_0}(t, \theta'_0 z)$ is a function of $\theta'_0 z$ only and getting from Lemma 6.5 that $E[\nabla_{\theta} \mu_{\theta_0}(t, Z) | \theta'_0 Z] = 0$. \square

3.5 Adaptive choice of w

The representations of Theorems 3.2 and 3.3 hold uniformly in $w \in \mathcal{W}$. Therefore, the asymptotic normality of our estimators of the parameter remains valid if we replace w by a data-driven measure \hat{w} that converges to a specific optimal measure w_0 . We give some indications on a method to obtain such kind of data-driven measure adapted to our estimation problem.

The empirical measure \hat{w} will be defined as the minimizer of some criterion. Since it is generally impossible to perform minimization on the functional space \mathcal{W} , we minimize over a growing subset \mathcal{W}_n . The adaptive procedure we propose consists of first estimating the asymptotic covariance matrix $V_{w,sp}$ (or $V_{w,p}$ in the parametric case) for any $w \in \mathcal{W}_n$. From the asymptotic variance estimators, we derive the estimation of the mean squared error $E[\|\hat{\theta}(w) - \theta_0\|^2]$. We then take \hat{w} as the element of \mathcal{W}_n such that the estimated mean squared error is minimal over \mathcal{W}_n . Then, our final estimator is

$$\hat{\theta} = \hat{\theta}(\hat{w}).$$

The uniform convergence of the remainder term in the representations of Theorems 3.2 and 3.3 provides the asymptotic normality of $\hat{\theta}$ in the case where $\Sigma_{\hat{w}} \rightarrow \Sigma_{w_0}$ a.s. for some $w_0 \in \mathcal{W}$.

3.6 Estimation of the variance

We show how to estimate the variance in the representation of Theorem 3.3 and we propose an estimator of the mean squared error of θ_0 . Denote by $\xi_{n,w}$ the term between brackets in the representation of Theorem 3.3 so that

$$\hat{\theta}(w) - \theta_0 = \Sigma_{w,sp}^{-1} \xi_{n,w} + R_n(w),$$

where $\sup_{w \in \mathcal{W}} |R_n(w)| = o_P(n^{-1/2})$. The quantity $\xi_{n,w}$ can be estimated in the following way

$$\hat{\xi}_{n,w} = \frac{1}{n} \sum_{i=1}^n \hat{\psi}(\delta_i, Z_i, T_i, Y_i; w),$$

where

$$\begin{aligned} \hat{\psi}(\delta, Z, T, Y; w) &= \int_0^{T^{(n)}} (Y(t) - \hat{\mu}_{\hat{\theta}}(t, \hat{\theta}' Z)) \nabla_{\theta} \hat{\mu}_{\hat{\theta}}(t, Z) dw(t) \\ &\quad + \int_0^{T^{(n)}} \int_0^t \hat{\eta}_{s-}(T, \delta) n^{-1} \sum_{i=1}^n (\nabla_{\theta} \hat{\mu}_{\hat{\theta}}(t, \hat{\theta}' Z_i) d\hat{\mu}_{\hat{\theta}}(s, \hat{\theta}' Z_i)) dw(t), \end{aligned}$$

$$\hat{\eta}_t(T, \delta) = \frac{(1 - \delta)I(T \leq t)}{1 - \hat{H}(T-)} - \int_0^t \frac{I(T \geq s) d\hat{G}(s)}{(1 - \hat{H}(s-))(1 - \hat{G}(s-))}$$

and \hat{H} is the empirical estimator of H . Therefore, the quantity $\Delta_{w,sp}$ can be estimated by

$$\hat{\Delta}_{w,sp} = \frac{1}{n} \sum_{i=1}^n \left(\hat{\psi}(\delta, Z, T, Y; w) - \frac{1}{n} \sum_{i=1}^n \hat{\psi}(\delta, Z, T, Y; w) \right)^{\otimes 2},$$

where $\otimes 2$ denotes the product of the matrix with its transpose. To consistently estimate $\Sigma_{w,sp}$, we use

$$\hat{\Sigma}_{w,sp} = \frac{1}{n} \sum_{i=1}^n \int_0^{T^{(n)}} \nabla_{\theta} \hat{\mu}_{\hat{\theta}}(t, Z_i) \nabla_{\theta} \hat{\mu}_{\hat{\theta}}(t, Z_i)'$$

A consistent estimator of $V_{w,sp}$ can then be computed from $\hat{V}_{w,sp} = \hat{\Sigma}_{w,sp}^{-1} \hat{\Delta}_{w,sp} \hat{\Sigma}_{w,sp}^{-1}$. Finally, we take $\hat{E}_{\tau}^2 = \hat{\xi}_{n,w}' \hat{\Sigma}_{w,sp}^{-1} \hat{\Sigma}_{w,sp}^{-1} \hat{\xi}_{n,w}$ as the mean squared error estimate.

3.7 Estimation of the nonparametric part

In the semiparametric model, estimation of the finite dimensional parameter θ_0 is only the first step of the method. With our estimator $\hat{\theta}$ at hand, we wish to estimate the conditional mean function $\mu(t|z)$. Different strategies can be proposed to perform this estimation. For this final estimator, there is no theoretical need to use the same kind of

nonparametric estimator as in the computation of $\hat{\theta}$. Proposition 3.4 below states that, under some convergence assumptions for the nonparametric estimator used in this second step, the asymptotic behavior of the final semiparametric estimator of μ is identical to the asymptotic behavior of a purely nonparametric estimator in the case where θ_0 is exactly known.

Proposition 3.4. *Let Θ^* be some neighborhood of θ_0 , and let \mathcal{T} be a set on which $\sup_{\theta \in \Theta^*, t \in \mathcal{T}, z \in \mathcal{Z}} \|\nabla_{\theta} \mu_{\theta_0}(t, z)\| < \infty$. Let $\hat{\mu}_{\theta}$ be a family of nonparametric estimators of μ_{θ} satisfying the assumption*

$$\sup_{\theta \in \Theta^*, t \in \mathcal{T}, z \in \mathcal{Z}} \|\nabla_{\theta} \hat{\mu}_{\theta}(t, z) - \nabla_{\theta} \mu_{\theta}(t, z)\| = o_P(1). \quad (3.2)$$

Then, we have

$$\sup_{t \in \mathcal{T}, z \in \mathcal{Z}} |\hat{\mu}_{\hat{\theta}}(t, \hat{\theta}'z) - \hat{\mu}_{\theta_0}(t, \theta_0'z)| = O_P(n^{-1/2}).$$

Proof. This is a direct consequence of a Taylor expansion of $\hat{\mu}_{\hat{\theta}}$ around θ_0 . From Theorem 3.3 we have $\hat{\theta} - \theta_0 = O_P(n^{-1/2})$. Then, the boundedness of $\nabla_{\theta} \mu_{\theta_0}(t, z)$ and the uniform convergence in assumption (3.2) give the result. \square

In the kernel estimator example of equation (2.10), a crucial issue stands in the choice of the bandwidth which strongly influences the performance of the nonparametric estimation. A first method to define our final estimator of μ_{θ_0} consists of using an arbitrary sequence of bandwidth h to compute $\hat{\theta}$, then of using cross-validation techniques to select a bandwidth \hat{h} . The final estimator is finally set as $\hat{\mu}_{\hat{\theta}, \hat{h}}(t, \hat{\theta}'z)$. However, it seems more appealing to us to define a procedure which can be seen as an extension of the adaptive choice of bandwidth proposed by Härdle *et al.* (1993) and Delecroix *et al.* (2006). An interesting feature of this technique is that it selects an adaptive bandwidth \hat{h} and a direction $\hat{\theta}$ at the same time. Indeed, define

$$(\hat{\theta}, \hat{h}) = \arg \min_{\theta \in \Theta, h \in \mathcal{H}} M_{n,w}(\theta, \hat{\mu}_{\theta,h}). \quad (3.3)$$

The uniform in bandwidth consistency of the kernel estimators we use (see Section 6.2) ensures us that $\hat{\theta}$ has the same asymptotic properties as in Theorem 3.3. On the other hand, Proposition 3.5 below shows that the adaptive bandwidth \hat{h} is asymptotically equivalent to the bandwidth we could obtain using a classical cross-validation technique in the case where the parameter θ_0 is exactly known.

Proposition 3.5. For some positive constants a, b and α , let $\mathcal{H} = [an^{-\alpha}, bn^{-\alpha}]$ be a set of bandwidth satisfying Assumption 10 and let

$$h_0 = \arg \min_{h \in \mathcal{H}} M_{n,w}(\theta_0, \hat{\mu}_{\theta_0, h}).$$

Under the assumptions of Theorem 3.3 and provided that $\sup_{h \in \mathcal{H}, t \in \mathbb{R}_+, z \in \mathcal{Z}} |\hat{\mu}_{\theta, h}(t, \theta' z) - \mu_{\theta, h}(t, \theta' z)| = o_P(1)$, we have

$$\hat{h}/h_0 \rightarrow 1 \text{ a.s.}$$

Proof. Define $\phi(h/h_0) = M_{n,w}(\theta_0, \hat{\mu}_{\theta_0, h})$ and $\phi_n(h/h_0) = \arg \min_{\theta \in \Theta} M_{n,w}(\theta, \hat{\mu}_{\theta, h})$. By definition of h_0 and \hat{h} we have $\arg \min_{x \in [a, b]} \phi(x) = 1$ and $\hat{h}/h_0 = \arg \min_{x \in [a, b]} \phi_n(x)$. Now write

$$\begin{aligned} \phi_n(x) &= \phi(x) - \frac{2}{n} \sum_{i=1}^n \int_0^{\tau_H} \hat{Y}_i(t) (\hat{\mu}_{\theta, x h_0}(t, \theta' Z_i) - \hat{\mu}_{\theta, h_0}(t, \theta' Z_i)) dw(t) \\ &\quad + \frac{1}{n} \sum_{i=1}^n \int_0^{\tau_H} (\hat{\mu}_{\theta, x h_0}(t, \theta' Z_i)^2 - \hat{\mu}_{\theta, h_0}(t, \theta' Z_i)^2) dw(t) + M_{n,w}(\theta, \hat{\mu}_{\theta, h_0}). \end{aligned}$$

Using Lemma 3.1 and the uniform in bandwidth consistency of $\hat{\mu}_{\theta, h}$, the second and third terms in the decomposition tend to zero uniformly in x . On the other hand, the last term does not depend on x . This shows that $\hat{h}/h_0 \rightarrow 1$ a.s. \square

4 Simulations

We present here some empirical evidence of the good behavior of our semiparametric estimation procedure for finite sample sizes.

In our simulation study, we consider the case where, conditionally on Z_i , the process N^* is an homogeneous Poisson process with intensity $\theta'_0 Z_i + 5$, that is

$$E[N^*(t)|Z_i] = (\theta'_0 Z_i + 5)t, \quad i = 1, \dots, n.$$

We take $\theta_0 = (1, 1.6, 1.25, 0.7)'$ and we consider 4-dimensional covariates $Z_i \sim \otimes^4 \mathcal{U}[1, 2]$ for $i = 1, \dots, n$. The variables D_i for $i = 1, \dots, n$ are generated according to a Weibull distribution with parameters (10, 1.09). The censoring distribution is selected to be Weibull with parameters (4, λ). Taking $\lambda = 1.38$ or $\lambda = 1$ leads respectively to 30% or 50% of censoring and an average of 20 or 18 recurrent events per sample. In our results, we emphasize the impact of the two parameters involved in our semiparametric procedure, namely the bandwidth of the nonparametric kernel estimators and the measure w .

First, we consider the case of a fixed bandwidth and show how the adaptive choice of \hat{w} can improve the estimation performance of the parameter θ_0 . The nonparametric estimators are kernel estimators computed using the Epanechnikov kernel and a bandwidth $h_0 = 0.2$. We consider a set of discrete measures supported on $\mathcal{I} = \{0.1, 0.2, \dots, 1.2\}$. Hence, for any function f , the integral with respect to w reduces to a finite sum. Indeed, we have

$$\int f(t)dw(t) = \sum_{k \in \mathcal{I}} f(k)w(\{k\}).$$

Moreover, we consider only a finite number of choices for the weights $w(\{k\})$, that is

$$\begin{cases} w(\{k\}) = 1 & \text{for } k = 0.1, \dots, 0.8 \\ w(\{k\}) \in \{0.25, 0.5, 0.75, 1\} & \text{for } k = 0.9, 1, 1.1, 1.2. \end{cases}$$

The intuition is that our procedure should allocate smaller weights to large values of T_i since the behavior of the Kaplan-Meier estimator is known to be less effective in this part of the distribution (and contributes significantly to the variance). Our estimator $\hat{\theta} = \hat{\theta}(\hat{w}, h_0)$ is then compared to the estimator $\tilde{\theta}$ obtained for the measure w_0 which puts mass 1 at every point of \mathcal{I} .

In the table below, we report our results over 100 simulations of samples of size 100 for two different rates of censoring ($p = 30\%$ and $p = 50\%$). Recalling that the first component of θ_0 is imposed to be one, we only have to estimate the three other components. For each estimator, the Mean Squared Error (MSE) $E(\|\hat{\theta} - \theta_0\|^2)$ is decomposed into bias and variance.

$p = 30\%$	Bias	Variance	MSE
$\tilde{\theta}$	$\begin{pmatrix} -0.322 \\ -0.198 \\ -0.02 \end{pmatrix}$	$\begin{pmatrix} 0.452 & 0.111 & 0.041 \\ 0.111 & 0.42 & 0.009 \\ 0.041 & 0.009 & 0.249 \end{pmatrix}$	1.264
$\hat{\theta}$	$\begin{pmatrix} -0.129 \\ -0.162 \\ -0.042 \end{pmatrix}$	$\begin{pmatrix} 0.2 & 0.062 & 0.047 \\ 0.062 & 0.272 & -0.004 \\ 0.047 & -0.004 & 0.168 \end{pmatrix}$	0.685

We also computed the average weights of \hat{w} for the last four points of \mathcal{I} . For 30% of censoring, we have : $E[\hat{w}(\{0.9\})] = 0.777$, $E[\hat{w}(\{1\})] = 0.652$, $E[\hat{w}(\{1.1\})] = 0.607$ and $E[\hat{w}(\{1.2\})] = 0.535$ and for 50% of censoring, $E[\hat{w}(\{0.9\})] = 0.782$, $E[\hat{w}(\{1\})] =$

$p = 50\%$	Bias	Variance	MSE
$\tilde{\theta}$	$\begin{pmatrix} -0.428 \\ -0.324 \\ -0.05 \end{pmatrix}$	$\begin{pmatrix} 0.478 & 0.129 & 0.156 \\ 0.129 & 0.386 & 0.034 \\ 0.156 & 0.034 & 0.335 \end{pmatrix}$	1.49
$\hat{\theta}$	$\begin{pmatrix} -0.276 \\ -0.287 \\ -0.096 \end{pmatrix}$	$\begin{pmatrix} 0.242 & 0.035 & 0.033 \\ 0.035 & 0.234 & 0.023 \\ 0.033 & 0.023 & 0.199 \end{pmatrix}$	0.843

0.682, $E[\hat{w}(\{1.1\})] = 0.575$ and $E[\hat{w}(\{1.2\})] = 0.487$. Clearly, choosing the measure from the data improves both the bias and the variance of our estimator. Moreover the weights of \hat{w} get smaller for large values of k , especially when the proportion of censored data is high. Consequently, the adaptive measure seems to have a significant impact on the quality of the estimation of θ_0 .

Next, we show how the choice of the parameter h influences the quality of estimation. We consider the fixed measure w_0 which puts the same weights 1 at each point. The bandwidth \hat{h} is chosen adaptively in a regular grid of length 0.05 in the set $[0.05, 0.3]$. The performance of the resulting estimator presented below is compared with the estimator $\tilde{\theta}$ of the previous table and shows significant improvement of its MSE.

	Bias	Variance	MSE
$\hat{\theta}_{w_0, \hat{h}}, p = 30\%$	$\begin{pmatrix} -0.19 \\ -0.155 \\ 0.084 \end{pmatrix}$	$\begin{pmatrix} 0.216 & 0.08 & -0.08 \\ 0.08 & 0.351 & -0.009 \\ -0.08 & -0.009 & 0.174 \end{pmatrix}$	0.967
$\hat{\theta}_{w_0, \hat{h}}, p = 50\%$	$\begin{pmatrix} -0.281 \\ -0.309 \\ -0.114 \end{pmatrix}$	$\begin{pmatrix} 0.244 & -0.056 & -0.081 \\ -0.056 & 0.256 & 0.027 \\ -0.081 & 0.027 & 0.17 \end{pmatrix}$	1.126

5 Conclusion

We proposed a new procedure to estimate the conditional cumulative mean function of the recurrent event process. We considered both parametric and semiparametric models for the conditional cumulative mean function. Our semiparametric single-index model

can be seen as a generalization of both the Cox model and the accelerated failure time model. Moreover, a new feature of our procedure stands in the measure w involved in our estimators which is designed to prevent us from problems in the tail of the distribution due to the presence of censoring. Then, we proposed a data-driven method to choose this measure adaptively. Our criterion is based on the minimization of the mean squared error for the estimation of θ_0 but our procedure is flexible enough to allow the use of any other criteria more adapted to the context. For example, we could consider a criterion directly based on the error of the estimation of μ .

In this work, we mainly focused on kernel estimators for estimating the nonparametric part of our model, providing methods to choose the smoothing parameters from the data. Nevertheless, all our results are still valid for a general class of nonparametric estimators and do only rely on convergence properties. Hence, other kinds of estimators may be used provided they satisfy these conditions.

References

- Andersen, P., Borgan, O., Gill, R. & Keiding, N. (1993). *Statistical models based on counting processes*. Springer-Verlag, New-York.
- Andersen, P. & Gill, R. (1982). Cox's regression model for counting processes: a large sample study. *Ann. Statist.* **10**, 1100–1120.
- Billingsley, P. (1999). *Convergence of probability measures*. Wiley Series in Probability and Statistics. John Wiley & Sons Inc., New-York. A Wiley-Interscience Publication.
- Bosq, D. & Lecoutre, J.-P. (1997). *Théorie de l'estimation fonctionnelle*, Volume 3 of *Economie et statistiques avancées*. Economica, Paris.
- Delecroix, M., Hristache, M. & Patilea, V. (2006). On semiparametric M -estimation in single-index regression. *J. Statist. Plann. Inference* **136**(3), 730–769.
- Einmahl, U. & Mason, D.M. (2005). Uniform in bandwidth consistency of kernel-type function estimators. *Ann. Statist.* **33**(3), 1380–1403.
- Ghosh, D. & D. Lin (2003). Semiparametric analysis of recurrent events in the presence of dependent censoring. *Biometrics* **59**, 877–885.
- Gijbels, I. & Veraverbeke, N. (1991). Almost sure asymptotic representation for a class of functionals of the Kaplan-Meier estimator. *Ann. Statist.* **19**(3), 1457–1470.
- Gill, R. (1983). Large sample behavior of the product-limit estimator on the whole line. *Ann. Statist.* **11**, 49–58.

- Härdle, W., Hall, P. & Ichimura, H. (1993). Optimal smoothing in single-index models. *Ann. Statist.* **21**(1), 157–178.
- Ichimura, H. (1993). Semiparametric least squares (SLS) and weighted SLS estimation of single-index models. *J. Econometrics* **58**(1-2), 71–120.
- Lawless, J. & Nadeau, C. (1995). Some simple robust methods for the analysis of recurrent events. *Technometrics* **37**, 158–168.
- Lin, D., Wei, L., Yang, I. & Ying, Z. (2000). Semiparametric regression for the mean and rate functions of recurrent events. *J. Roy. Statist. Soc. B* **62**, 711–730.
- Lopez, O. (2009). Single-index regression models with right-censored responses. *J. Statist. Plann. Inference* **139**(3), 1082–1097.
- Nolan, D. & Pollard, D. (1987). U -processes: rates of convergence. *Ann. Statist.* **15**(2), 780–799.
- Prentice, L., Williams, B. & Peterson, A. (1981). On the regression analysis of multivariate time data. *Biometrika* **68**, 373–379.
- Sánchez Sellero, C., González Manteiga, W. & Van Keilegom, I. (2005). Uniform representation of product-limit integrals with applications. *Scand. J. Statist.* **32**(4), 563–581.
- Sherman, R. P. (1994). Maximal inequalities for degenerate U -processes with applications to optimization estimators. *Ann. Statist.* **22**(1), 439–459.
- Stute, W. (1993). Consistent estimation under random censorship when covariables are present. *J. Multivariate Anal.* **45**(1), 89–103.
- Stute, W. (1995). The central limit theorem under random censorship. *Ann. Statist.* **23**(2), 422–439.
- Talagrand, M. (1994). Sharper bounds for Gaussian and empirical processes. *Ann. Probab.* **22**(1), 28–76.
- Van Keilegom, I., & Akritas, M.G. (1999). Transfer of tail information in censored regression models. *Ann. Statist.* **27**(5), 1745–1784.
- Van der Vaart, A. W. & Wellner, J.A. (1996). *Weak convergence and empirical processes*. Springer Series in Statistics. Springer-Verlag, New York. With applications to statistics.

6 Appendix

6.1 Proof of Lemma 3.1

Let

$$S_n^{T(n)}(f, w) = \frac{1}{n} \sum_{i=1}^n \int_0^{T(n)} Y_i(t) f(Z_i, t) dw(t).$$

Write

$$\begin{aligned} \hat{S}_n(f, w) &= S_n^{T(n)}(f, w) + \frac{1}{n} \sum_{i=1}^n \int_0^{T(n)} f(Z_i, t) \int_0^t \frac{(\hat{G}(s-) - G(s-)) dN_i(s)}{(1 - G(s-))(1 - \hat{G}(s-))} dw(t) \\ &= S_n^{T(n)}(f, w) + R_n(f, w). \end{aligned}$$

Decompose f into its positive and negative parts denoted respectively by f^+ and f^- . The expectations of the two resulting sums $S_n^{T(n)}(f^+, w)$ and $S_n^{T(n)}(f^-, w)$ go to zero faster than $n^{-1/2}$ using Lebesgue's dominated convergence. This entails that

$$\sup_{f \in \mathcal{F}, w \in \mathcal{W}} |S_n^{T(n)}(f, w) - S_n(f, w)| = o_P(n^{-1/2}).$$

Let $\tau < \tau_H$ and define $w_\tau(t) = w(t)I(t \leq \tau)$. On $[0, \tau]$, we use the asymptotic i.i.d. expansion of the Kaplan-Meier estimator \hat{G} proposed by Gijbels & Veraverbeke (1991) which can also be deduced from Stute (1995):

$$\frac{\hat{G}(t) - G(t)}{1 - G(t)} = \frac{1}{n} \sum_{j=1}^n \eta_t(T_j, \delta_j) + \tilde{R}_n(t),$$

where $\sup_{t \leq \tau} |\tilde{R}_n(t)| = O_P(n^{-1} \log n)$ and

$$\eta_t(T, \delta) = \frac{(1 - \delta)I(T \leq t)}{1 - H(T-)} - \int_0^t \frac{I(T \geq s) dG(s)}{(1 - H(s-))(1 - G(s-))}.$$

Moreover, recall that $\sup_{t \leq \tau} |\hat{G}(t) - G(t)| = O_P(n^{-1/2})$ (see Gill (1983), Theorem 2.1) and that $\sup_{t \leq \tau} (1 - G(t))(1 - \hat{G}(t))^{-1} = O_P(1)$ (see Gill (1983), Lemma 2.6). Then, write

$$R_n(f, w_\tau) = \frac{1}{n^2} \sum_{i,j} \int_0^{T(n)} f(Z_i, t) \int_0^t \frac{\eta_{s-}(T_j, \delta_j) dN_i(s)}{1 - G(s-)} dw_\tau(t) + R'_n(f, w_\tau).$$

Using the fact that \mathcal{F} is an uniformly bounded class, that $\int dw_\tau \leq c_0$ from Assumption 3 and that $E[N_i(\tau)] < \infty$ for all τ , we deduce that $\sup_{f,w} |R'_n(f, w_\tau)| = O_P(n^{-1})$. The first term in $R_n(f, w_\tau)$ can be rewritten as

$$\frac{1}{n} \sum_{j=1}^n \int_0^{T(n)} \int_0^t \eta_{s-}(T_j, \delta_j) E[f(Z, t) d\mu(s|Z)] dw_\tau(t) + \int \left(\frac{1}{n^2} \sum_{i,j} \psi^{f,t}(Z_i, N_i, T_j, \delta_j) \right) dw_\tau(t),$$

where

$$\psi^{f,t}(Z_i, N_i, T_j, \delta_j) = \int_0^t \eta_{s-}(T_j, \delta_j) \left\{ \frac{f(Z_i, t) dN_i(s)}{1 - G(s-)} - E[f(Z, t) d\mu(s|Z)] \right\}.$$

Observe that, with probability tending to one, the upper bound $T(n)$ in the integrals can be replaced by $\tau < \tau_H$. Let $f, f' \in \mathcal{F}$ and $t, t' \in [0, \tau]$. We have

$$|\psi^{f,t}(Z_i, N_i, T_j, \delta_j) - \psi^{f',t'}(Z_i, N_i, T_j, \delta_j)| \leq c_\tau \left(\|f - f'\|_\infty N_i(\tau) + |t - t'|^\gamma \sup_{t,t' \leq \tau} \frac{N_i(t) - N_i(t')}{|t - t'|^\gamma} \right), \quad (6.1)$$

where $c_\tau < \infty$ and $\gamma > 0$. Let \mathcal{H}_τ denote the set of all functions $\psi^{f,t}$ when f ranges \mathcal{F} and t ranges $[0, \tau]$. It follows from (6.1) and Assumption 4 that \mathcal{H}_τ is a $\|\cdot\|_2$ -VC-class of functions. From this, using the Glivenko-Cantelli property of \mathcal{H}_τ ,

$$\sup_{f,t \leq \tau} \left| \frac{1}{n^2} \sum_{i=1}^n \psi^{f,t}(Z_i, N_i, T_i, \delta_i) \right| = O_P(n^{-1})$$

and

$$\sup_{f,t \leq \tau} \left| \frac{1}{n^2} \sum_{i \neq j} \psi^{f,t}(Z_i, N_i, T_j, \delta_j) \right| = O_P(n^{-1}),$$

since this can be seen as the supremum of a second order degenerate U -process indexed by \mathcal{H}_τ (see Sherman (1994)). This leads to the i.i.d. representation for $\hat{S}_n(f, w_\tau)$ for any $\tau < \tau_H$.

Similarly, write

$$\hat{S}_n(\hat{f}, w_\tau) = S_n^{T(n)}(\hat{f}, w_\tau) + R_n(\hat{f} - f, w_\tau) + R_n(f, w_\tau)$$

and using the fact that $\sup_{f \in \mathcal{F}} \|f - \hat{f}\|_\infty = o_P(1)$ and that $\sup_{t \leq \tau} |\hat{G}(t) - G(t)| = O_P(n^{-1/2})$, we deduce that $\sup_{f,w} |R_n(\hat{f} - f, w_\tau)| = o_P(n^{-1/2})$. The representation for $\hat{S}_n(\hat{f}, w_\tau)$ follows.

Now, we make τ tend to τ_H . Let $\hat{P}_n^\tau(f, w) = \hat{S}_n(f, w) - S_n^{T(n)}(f, w)$ and $P_n^\tau(f, w) = \hat{S}_n(f, w_\tau) - S_n^{T(n)}(f, w_\tau)$. Since the class \mathcal{F} is uniformly bounded, we get

$$\begin{aligned} |\hat{P}_n^\tau(f, w) - P_n^\tau(f, w)| &\leq \frac{M}{n} \sum_{i=1}^n \int_\tau^{T(n)} \int_0^t \frac{|\hat{G}(s-) - G(s-)|}{(1 - G(s-))(1 - \hat{G}(s-))} dN_i(s) dw(t) \\ &\leq \frac{M'}{n} \sum_{i=1}^n \int_0^{T(n)} \frac{W_0(s \vee \tau) |\hat{G}(s-) - G(s-)| dN_i(s)}{(1 - G(s-))(1 - \hat{G}(s-))}, \end{aligned}$$

where the last inequality is obtained from Fubini's theorem and Assumption 3. From Theorem 1.2 in Gill (1983), Assumption 3 and the fact that $\sup_{t \leq T(n)} (1 - G(t-))(1 - \hat{G}(t-))^{-1} = O_P(1)$ (see again Gill, 1983), we get that

$$|\hat{P}_n^\tau(f, w) - P_n^\tau(f, w)| \leq \frac{A_n}{n} \sum_{i=1}^n \int_0^{T(n)} \frac{W_2(s \vee \tau) dN_i(s)}{1 - G(s-)},$$

where $A_n = O_P(n^{-1/2})$. The result follows from Lemma 6.6.

6.2 Uniform convergence of the nonparametric estimators

In this section, we show that the kernel estimator $\hat{\mu}_{\theta, h}$ defined by (2.10) satisfies the convergence rates required by Assumption 7. Introduce the quantity

$$\tilde{\mu}_{\theta, h}(t, u) = \sum_{i=1}^n \int_0^t \frac{K\left(\frac{\theta' Z_i - u}{h}\right) dN_i(s)}{\sum_{j=1}^n K\left(\frac{\theta' Z_j - u}{h}\right) (1 - G(s-))}.$$

We first study the convergence rate of the difference between $\tilde{\mu}_{\theta, h}$ and μ_θ and their derivatives. Since no Kaplan-Meier functions are involved in this expression, we can use classical results on uniform convergence of kernel estimators, mainly from Einmahl & Mason (2005).

Assumption 10. *Assume that*

- (1) K has a compact support, say $[-1, 1]$, $\int_{\mathbb{R}} K(s) ds = 1$ and $\sup_x |K(x)| < \infty$,
- (2) K is a twice differentiable and two order kernel with derivatives of order 0, 1 and 2 of bounded variation,
- (3) $\mathcal{K} := \{K((x - \cdot)/h) : h > 0, x \in \mathbb{R}^d\}$ is a pointwise measurable class of functions,
- (4) $h \in \mathcal{H}_n \subset [an^{-\alpha}, bn^{-\alpha}]$ with $a, b > 0$ and $\alpha \in (1/8, 1/5)$.

We also introduce a trimming function in order to prevent from denominator close to zero in the definition of $\hat{\mu}_{\theta,h}$. Indeed, to ensure uniform consistency of our estimator, we need to bound this denominator away from zero. We use the same methodology as in Delecroix *et al.* (2006). Let $f_{\theta'_0 Z}$ denote the density of $\theta'_0 Z$ and define the “ideal” trimming function $J_{\theta_0}(\theta'_0 Z, c) = I(\theta'_0 Z \in B_0)$ where $B_0 = \{u : f_{\theta'_0 Z}(u) \geq c\}$ for some constant $c > 0$. As in Delecroix *et al.* (2006) (see also Lopez (2009)), we first assume that we know some set B on which $\inf\{f_{\theta'_0 Z}(\theta'z) : z \in B, \theta \in \Theta\} > c$ where c is a strictly positive constant. In a preliminary step, we can use this set B to compute the preliminary trimming $J_B(z) = I(z \in B)$. Using this trimming function and a deterministic sequence of bandwidth h_0 satisfying (4) in Assumption 10, we define a preliminary estimator θ_n of θ_0 as

$$\theta_n(w) = \arg \min_{\theta \in \Theta} M_{n,w}(\theta, \hat{\mu}_\theta) J_B(z).$$

Given this preliminary consistent estimator of θ_0 , we use the following trimming $J_n(\theta'_n Z, c) = I(\hat{f}_{\theta'_n Z}(\theta'_n Z) \geq c)$ which appears to be asymptotically equivalent to $J_{\theta_0}(\theta'_0 Z, c)$ (see e.g. Lopez (2009)). Then, our final estimator consists of

$$\hat{\theta}(w) = \arg \min_{\theta \in \Theta_n} M_{n,w}(\theta, \hat{\mu}_\theta) J_n(\theta'_n z, c),$$

where Θ_n is a shrinking neighborhood of θ_0 accordingly to our preliminary estimator θ_n .

As announced, the next proposition gives the rates of convergence of $\tilde{\mu}_{\theta,h}$ and its derivatives. Since we need a convergence over $\theta \in \Theta$, the trimming we need to use is $J_\theta(\theta'Z, c) := I(\hat{f}_{\theta'Z}(\theta'Z) \geq c)$. But notice that $J_{\theta_0}(\theta'_0 Z, c)$ can be replaced by $J_\theta(\theta'Z, c/2)$ on shrinking neighborhoods of θ_0 .

Proposition 6.1. *Under Assumption 10, for z such that $J_\theta(\theta'z, c) > 0$, we have*

$$\sup_{t \leq T_{(n)}, \theta, z, h} \sqrt{\frac{nh}{\log n}} \left| \frac{\tilde{\mu}_\theta(t, \theta'z) - \mu_\theta(t, \theta'z)}{\bar{\mu}_{\theta_0}(t, \theta'_0 z)^{\lambda_1 + \lambda_2}} \right| = O_P(1), \quad (6.2)$$

$$\sup_{t \leq T_{(n)}, \theta, z, h} \sqrt{\frac{nh^3}{\log n}} \left\| \frac{\nabla_\theta \tilde{\mu}_\theta(t, z) - \nabla_\theta \mu_\theta(t, z)}{\bar{\mu}_{\theta_0}(t, \theta'_0 z)^{\lambda_1 + \lambda_2}} \right\| = O_P(1), \quad (6.3)$$

$$\sup_{t \leq T_{(n)}, \theta, z, h} \sqrt{\frac{nh^5}{\log n}} \left\| \frac{\nabla_\theta^2 \tilde{\mu}_\theta(t, z) - \nabla_\theta^2 \mu_\theta(t, z)}{\bar{\mu}_{\theta_0}(t, \theta'_0 z)^{\lambda_1 + \lambda_2}} \right\| = O_P(1). \quad (6.4)$$

Proof. The proofs of (6.2)-(6.4) are all similar. The most delicate term to handle, coming

from (6.4), is

$$\hat{A}_\theta^{n,h}(t, z) := \frac{1}{nh^3} \sum_{i=1}^n \frac{(Z_i - z)^2}{\bar{\mu}_{\theta_0}(t, \theta'_0 z)^{\lambda_1 + \lambda_2}} K'' \left(\frac{\theta' Z_i - \theta' z}{h} \right) \int_0^t \frac{dN_i(s)}{1 - G(s-)}.$$

Consider the class of functions \mathcal{K} introduced in Assumption 10. From Nolan & Polard (1987), it can easily be seen that, using a kernel K satisfying Assumption 10, for some $c' > 0$ and $\nu > 0$, we have $N(\varepsilon, \mathcal{K}, \|\cdot\|_\infty) \leq c'\varepsilon^{-\nu}$, $0 < \varepsilon < 1$.

Then, concerning the uniformity with respect to θ , Lemma 22 (ii) of Nolan & Polard (1987) shows that the family of functions $\left\{ (Z, N) \mapsto \hat{A}_\theta^{n,h}(t, z) \right\}$ satisfies the assumptions of Proposition 1 in Einmahl & Mason (2005).

Define

$$\begin{aligned} \tilde{A}_\theta^h(t, z) &:= \frac{1}{h^3} E \left[\frac{(Z - z)^2}{\bar{\mu}_{\theta_0}(t, \theta'_0 z)^{\lambda_1 + \lambda_2}} K'' \left(\frac{\theta' Z - \theta' z}{h} \right) \int_0^t \frac{dN(s)}{1 - G(s-)} \right], \\ A_\theta^h(t, z) &:= \frac{\partial^2}{\partial u^2} \left\{ E \left[\frac{(Z - z)^2}{\bar{\mu}_{\theta_0}(t, \theta'_0 z)^{\lambda_1 + \lambda_2}} \int_0^t \frac{dN(s)}{1 - G(s-)} \middle| \theta' Z = u \right] f_{\theta' Z}(u) \right\} \Bigg|_{u=\theta' z} \end{aligned}$$

and apply Talagrand's inequality (see Talagrand (1994), see also Einmahl & Mason (2005)) to obtain that

$$\sup_{t \leq T_{(n)}, \theta, z, h} |\hat{A}_{n,h}(t, z) - \tilde{A}_{n,h}(t, z)| = O_P(n^{-1/2} h^{-5/2} (\log n)^{1/2}).$$

For the bias term, classical kernel arguments (see for instance Bosq & Lecoutre (1997)) show that

$$\sup_{t \leq T_{(n)}, \theta, z, h} |\tilde{A}_{n,h}(t, z) - A_{n,h}(t, z)| = O(h^2).$$

□

It remains to study $\hat{\mu}_{\theta,h} - \tilde{\mu}_{\theta,h}$. The following lemma gives some precision on the difference between the Kaplan Meier weights of $\tilde{\mu}_{\theta,h}$ and the "ideal" weights involving the true function G in $\tilde{\mu}_{\theta,h}$.

Lemma 6.2. *Let $\hat{\Lambda}_G(s) = (1 - \hat{G}(s-))^{-1}$, $\tilde{\Lambda}_G(s) = (1 - G(s-))^{-1}$ and*

$$C_G(t) = \int_0^t \frac{dG(s)}{(1 - G(s-))(1 - H(s-))}.$$

(1) *We have*

$$\sup_{t \leq T_{(n)}} \frac{1 - G(t)}{1 - \hat{G}(t)} = O_P(1).$$

(2) For all $0 \leq \beta \leq 1$ and $\varepsilon > 0$, we have

$$|\hat{\Lambda}_G(s) - \tilde{\Lambda}_G(s)| \leq R_n(s) \tilde{\Lambda}_G(s) C_G(s)^{\beta(1/2+\varepsilon)},$$

where $\sup_{s \leq T_{(n)}} R_n(s) = O_P(n^{-\beta/2})$.

Proof. (1) This result is a consequence of Lemma 2.6 in Gill (1983).

(2) For $0 \leq \beta \leq 1$ and $\varepsilon > 0$, write

$$\hat{\Lambda}_G(s) - \tilde{\Lambda}_G(s) = \tilde{\Lambda}_G(s) C_G(s)^{\beta(1/2+\varepsilon)} (R_G(s) C_G(s)^{-1/2-\varepsilon})^\beta (R_G(s))^{1-\beta} \frac{1 - G(s-)}{1 - \hat{G}(s-)},$$

where $R_G(s) = (\hat{G}(s-) - G(s-))(1 - G(s-))^{-1}$. Since $\int_0^{\tau_H} C_G(s)^{-1-2\varepsilon} dC_G(s) < \infty$, apply Theorem 1 in Gill (1983) and use the first part of the current lemma to conclude the proof. □

From the definition of our estimator, problems arise when studying $\hat{\mu}_{\theta,h}$ for t in the tail of the distribution. This is a common problem when studying Kaplan-Meier estimators but it can be circumvented by some moment conditions on the response and censoring distribution. For instance, in the classical censored framework, Stute (1995) used the function C_G to compensate the bad behavior of Kaplan Meier estimator in the tail of the distribution. The following assumption gives a similar moment condition but adapted to our recurrent event context.

Assumption 11. Assume that, for some $\varepsilon > 0$,

$$\sup_{t,u} \frac{C_G(t)^{7/20+\varepsilon}}{\bar{\mu}_{\theta_0}(t,u)^{\lambda_1}} < \infty$$

and

$$\sup_{t,u} \frac{\int_0^t (1 - G(s-)) E[N^*(s) dN^*(s)]}{(1 - G(t-))^2 \bar{\mu}_{\theta_0}(t,u)^{2\lambda_2}} < \infty,$$

where λ_1 and λ_2 are defined in Assumption 3.

Therefore, these conditions allow us to consider a process N^* and variables D and C that are supported on the whole interval $[0, \tau_H]$. However they will hold true only if there is enough information on the recurrent event process in the tails of the distribution. For further illustration take, for $k = 1$ and 2 , $\mu_{\theta_0}^k(t, u) \sim c_k (1 - G(t))^{-\beta_1}$ for t in a neighborhood of τ_H , $u \rightarrow \infty$ and where c_1, c_2, β_1 are three positive constants. Take also, for $c_3 > 0$ and

$\beta_2 > 0$, $1 - F(t) \sim c_3(1 - G(t))^{\beta_2}$ for t in a neighborhood of τ_H . Then it can be shown that these conditions are verified for example in the case $\beta_1 > 1$, $\beta_2 = 1$ and $\lambda_1 = \lambda_2 = 1$.

The next proposition gives the convergence rate of $\hat{\mu}_{\theta,h} - \tilde{\mu}_{\theta,h}$. Notice that if w is supported on a compact interval, we only need this result on a compact subset of $[0, T_n]$ and in this case Assumption 11 is automatically fulfilled.

Proposition 6.3. *Under Assumptions 10 and 11, for z such that $J_\theta(\theta'z, c) > 0$, we have*

$$\sup_{t \leq T_{(n)}, \theta, z, h} \left| \frac{\hat{\mu}_\theta(t, \theta'z) - \tilde{\mu}_\theta(t, \theta'z)}{\bar{\mu}_{\theta_0}(t, \theta'_0 z)^{\lambda_1 + \lambda_2}} \right| = O_P(n^{-7/20}), \quad (6.5)$$

$$\sup_{t \leq T_{(n)}, \theta, z, h} h \left\| \frac{\nabla_\theta \hat{\mu}_\theta(t, z) - \nabla_\theta \tilde{\mu}_\theta(t, z)}{\bar{\mu}_{\theta_0}(t, \theta'_0 z)^{\lambda_1 + \lambda_2}} \right\| = O_P(n^{-7/20}), \quad (6.6)$$

$$\sup_{t \leq T_{(n)}, \theta, z, h} h^2 \left\| \frac{\nabla_\theta^2 \hat{\mu}_\theta(t, z) - \nabla_\theta^2 \tilde{\mu}_\theta(t, z)}{\bar{\mu}_{\theta_0}(t, \theta'_0 z)^{\lambda_1 + \lambda_2}} \right\| = O_P(n^{-7/20}). \quad (6.7)$$

Proof. We only prove (6.7) since (6.5) and (6.6) can be handled similarly. Let us consider the following term involving the second derivative of K

$$\frac{1}{nh^3} \sum_{i=1}^n (Z_i - z)^2 K'' \left(\frac{\theta' Z_i - \theta' z}{h} \right) (\bar{\mu}_{\theta_0}(t, \theta'_0 z)^{\lambda_1 + \lambda_2} f_{\theta'Z}(\theta'z))^{-1} \int_0^t (\hat{\Lambda}(s) - \tilde{\Lambda}(s)) dN_i(s).$$

From Lemma 6.2, this term can be bounded by

$$O_P(n^{-\beta/2} h^{-2}) \left| \frac{1}{nh} \sum_{i=1}^n K'' \left(\frac{\theta' Z_i - \theta' z}{h} \right) \bar{\mu}_{\theta_0}(t, \theta'_0 z)^{-(\lambda_1 + \lambda_2)} \int_0^t \tilde{\Lambda}(s) C_G(s)^{\beta(1/2 + \varepsilon)} dN_i(s) \right| \quad (6.8)$$

where the O_P -rate does not depend on t, θ, z nor h . Now, consider the family of functions indexed by t, θ, z and h ,

$$\left\{ (Z, N) \mapsto K'' \left(\frac{\theta' Z - \theta' z}{h} \right) \bar{\mu}_{\theta_0}(t, \theta'_0 z)^{-(\lambda_1 + \lambda_2)} \int_0^t \tilde{\Lambda}(s) C_G(s)^{\beta(1/2 + \varepsilon)} dN(s) \right\}.$$

This family is Euclidian (see Nolan & Pollard (1987)) for an envelope

$$\sup_{t, z} \frac{\tilde{\Lambda}(t) C_G^{\beta(1/2 + \varepsilon)}(t) N(t)}{\bar{\mu}_{\theta_0}(t, \theta'_0 z)^{\lambda_1 + \lambda_2}}$$

which is, for $\beta = 7/10$, square integrable from Assumption 11. Then, using the results of Sherman (1994), the second part of (6.8) is $O_P(1)$ uniformly in t, θ, z and h .

□

Finally, combination of Propositions 6.1 and 6.3 leads to the following result.

Corollary 6.4. *Under Assumptions 10 and 11, for z such that $J_\theta(\theta'z, c) > 0$,*

$$\sup_{t \leq T_{(n)}, \theta, z, h} \frac{|\hat{\mu}_\theta(t, \theta'z) - \mu_\theta(t, \theta'z)| \cdot \|\nabla_\theta \hat{\mu}_\theta(t, z) - \nabla_\theta \mu_\theta(t, z)\|}{|\bar{\mu}_{\theta_0}(t, \theta'_0 z)|^{2(\lambda_1 + \lambda_2)}} = o_P(n^{-1/2}).$$

6.3 Technical lemmas

6.3.1 Gradient vector in the single-index model

Lemma 6.5. *If the function $\theta \mapsto \mu_\theta(t|\theta'z)$ is differentiable, we have*

$$\nabla_\theta \mu_{\theta_0}(t|Z) = \mu'_{\theta_0}(t|\theta'_0 Z)(Z - E(Z|\theta'_0 Z)),$$

where $\mu'_{\theta_0}(t|u) = \frac{\partial}{\partial u} \mu_{\theta_0}(t|u)$. As a consequence

$$E[\nabla_\theta \mu_{\theta_0}(t|Z)|\theta'_0 Z] = 0. \quad (6.9)$$

Proof. Observe that $\mu_\theta(t|\theta'Z) = E[\mu_{\theta_0}(t|\theta'_0 Z)|\theta'Z]$ and let $\zeta(Z, \theta) = \theta'_0 Z - \theta'Z$ for $\theta \in \Theta$. We have

$$\mu_\theta(t|\theta'Z) = E[\mu_{\theta_0}(t|\zeta(Z, \theta) + \theta'_0 Z)|\theta'Z].$$

Defining $\Gamma(\theta_1, \theta_2) = E[\mu_{\theta_0}(t|\zeta(Z, \theta_1) + \theta'_2 Z)|\theta'_2 Z]$, we have $\Gamma(\theta, \theta) = \mu_\theta(t|\theta'Z)$, which leads to

$$\begin{aligned} \nabla_{\theta_1} \Gamma(\theta_1, \theta_0)|_{\theta_1=\theta_0} &= -\mu'_{\theta_0}(t|\theta'_0 Z)E[Z|\theta'_0 Z], \\ \nabla_{\theta_2} \Gamma(\theta_0, \theta_2)|_{\theta_2=\theta_0} &= Z\mu'_{\theta_0}(t|\theta'_0 Z). \end{aligned}$$

□

6.3.2 Auxiliary lemma for tightness conditions

Lemma 6.6. *Let \mathcal{F} be a class of functions. Let $P_n(t, f)$ be a process on $[0, \tau_H] \times \mathcal{F}$. Define, for any $\tau \in [0, \tau_H]$, $R_n(\tau, f) = P_n(\tau_H, f) - P_n(\tau, f)$. Assume that for any τ such that $\tau < \tau_H$*

$$P_n(t, f) \implies \mathbb{W}(V_f(t)) \in \mathcal{D}([0, \tau]), f \in \mathcal{F},$$

where $\mathbb{W}(V_f(t))$ is a centered Gaussian process with covariance function V_f and \mathcal{D} denotes the set of càdlàg functions.

Assume that, for a sequence of random variables (X_n) and two functions Γ and Γ_n , the following conditions hold

- (1) $\lim_{\tau \rightarrow \tau_H} V_f(\tau) = V_f(\tau_H)$ with $\sup_{f \in \mathcal{F}} |V_f(\tau_H)| < \infty$,
- (2) $|R_n(\tau', f)| \leq X_n \times \Gamma_n(\tau)$ for all $\tau < \tau' < \tau_H$,
- (3) $X_n = O_P(1)$,

(4) $\Gamma_n(\tau) \rightarrow \Gamma(\tau)$ in probability,

(5) $\lim_{\tau \rightarrow \tau_H} \Gamma(\tau) = 0$.

Then $P_n(\tau_H, f) \implies \mathcal{N}(0, V_f(\tau_H))$.

Proof. From Theorem 13.5 in Billingsley (1999) and condition (1), it suffices to show that, for all $\varepsilon > 0$

$$\lim_{\tau \rightarrow \tau_H} \limsup_{n \rightarrow \infty} P \left(\sup_{\tau \leq t \leq \tau_H, f \in \mathcal{F}} |R_n(t, f)| > \varepsilon \right) = 0. \quad (6.10)$$

Using condition (2), the probability in equation (6.10) is bounded, for all $M > 0$, by

$$P(|\Gamma_n(\tau) - \Gamma(\tau)| > \varepsilon/M - \Gamma(\tau)) + P(X_n > M). \quad (6.11)$$

Moreover, from condition (4), we can state that

$$\limsup_{n \rightarrow \infty} P(|\Gamma_n(\tau) - \Gamma(\tau)| > \varepsilon/M - \Gamma(\tau)) = I(\varepsilon/M - \Gamma(\tau) \geq 0).$$

Since $\Gamma(\tau) \rightarrow 0$ (condition (5)), we can deduce that

$$\lim_{\tau \rightarrow \tau_H} \limsup_{n \rightarrow \infty} P(|\Gamma_n(\tau) - \Gamma(\tau)| > \varepsilon/M - \Gamma(\tau)) = 0.$$

As a consequence,

$$\lim_{\tau \rightarrow \tau_H} \limsup_{n \rightarrow \infty} P \left(\sup_{\tau \leq t \leq \tau_H, f \in \mathcal{F}} |R_n(t, f)| > \varepsilon \right) \leq \lim_{M \rightarrow \infty} \limsup_{n \rightarrow \infty} P(X_n > M) = 0,$$

using condition (3). □

6.3.3 Covering number results

In this section, we determine the covering numbers of some particular classes of functions. From these computations, we can easily deduce sufficient conditions to check Property 2 and Assumption 9.

Proposition 6.7. *Let \mathcal{F} be a class of functions $f(t, z)$ with envelope \bar{F} defined on $\mathbb{R} \times \mathbb{R}^d$ with continuous derivative with respect to the first component. Let \tilde{F} be the envelope of the class of functions $\partial f(s, z)/\partial s$. Let $W_0(t)$ be a positive bounded decreasing function and set $\mathcal{W} = \{w : dw(t) = W_0(t)d\tilde{w}(t), \tilde{w} \in \tilde{\mathcal{W}}\}$ where $\tilde{\mathcal{W}}$ is a class of monotone positive functions with envelope function \tilde{W} .*

Assume that $E[(\int_0^{\tau_H} \bar{F}(t, z)W_0(t)dY(t))^2] < \infty$, $E[(\int_0^{\tau_H} \bar{F}(t, z)Y(t)dW_0(t))^2] < \infty$ and $E[(\int_0^{\tau_H} \tilde{F}(t, z)W_0(t)Y(t)dt)^2] < \infty$.

Then, the class of functions $\mathcal{H} = \{(z, y) \rightarrow \int_0^{\tau_H} f(t, z)y(t)dw(t), f \in \mathcal{F}, w \in \mathcal{W}\}$ has a uniform covering number satisfying, for some constant C

$$N(\varepsilon, \mathcal{H}, \|\cdot\|_2) \leq CN(\varepsilon, W_0\mathcal{F}, \|\cdot\|_2)N(\varepsilon, \tilde{\mathcal{W}}, \|\cdot\|_2).$$

Proof. Let Q be a probability measure and introduce

$N_1 = \sup_Q N(\varepsilon\|W_0\bar{F}\|_Q, W_0\mathcal{F}, \|\cdot\|_{2,Q})$ and $N_2 = \sup_Q N(\varepsilon\|\tilde{W}\|_Q, \tilde{\mathcal{W}}, \|\cdot\|_{2,Q})$. Let $\{f_i^{W_0}\}_{1 \leq i \leq N_1}$ (respectively $\{\tilde{w}_j\}_{1 \leq j \leq N_2}$) be the center of the $\varepsilon - \|\cdot\|_{2,Q}$ balls needed to cover $W_0\mathcal{F}$ (respectively $\tilde{\mathcal{W}}$). Writing $dw(t) = W_0(t)d\tilde{w}(t)$, we have for any $1 \leq i \leq N_1$ and $1 \leq j \leq N_2$

$$\begin{aligned} & \left| \int_0^{\tau_H} Y(t)f(t, z)W_0(t)d\tilde{w}(t) - \int_0^{\tau_H} Y(t)f_i^{W_0}(t, z)d\tilde{w}_j(t) \right| \\ & \leq \left| \int_0^{\tau_H} Y(t)(f(t, z)W_0(t) - f_i^{W_0}(t, z))d\tilde{w}_j(t) \right| + \left| \int_0^{\tau_H} Y(t)f(t, z)W_0(t)(d\tilde{w} - d\tilde{w}_j)(t) \right|. \end{aligned}$$

For any $f \in \mathcal{F}$, there exists some i such that the first term is seen to be less than $c_1\varepsilon$ in $L^2(Q)$ -norm. For the second term, there also exists some j such that this is smaller than $c_2\varepsilon$, which can be seen using integration by parts. The result follows. \square