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Gautami Bhowmik, Jan-Christoph Schlage-Puchta. Essential singularities of Euler products. 2009. hal-00446359

**HAL Id: hal-00446359**

**<https://hal.science/hal-00446359>**

Preprint submitted on 12 Jan 2010

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# ESSENTIAL SINGULARITIES OF EULER PRODUCTS

GAUTAM IBHOMMIK AND JAN-CHRISTOPH SCHLAGE-PUCHTA

**Abstract.** We classify singularities of Dirichlet series having Euler products which are rational functions of  $p$  and  $p^{-s}$  for  $p$  a prime number and give examples of natural boundaries from zeta functions of groups and height zeta functions.

## 1. Introduction and results

Many Dirichlet series occurring in practice satisfy an Euler-product, and if they do so, the Euler-product is often the easiest way to access the series. Therefore, it is important to deduce information on the series from the Euler-product representation. One of the most important applications of Dirichlet-series, going back to Riemann, is the asymptotic estimation of the sum of its coefficients via Perron's formula, that is, the use of the equation

$$\sum_{n \leq x} a_n = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \sum_{n=1}^{\infty} \frac{a_n}{n^s} \frac{x^s}{s} ds.$$

To use this relation, one usually shifts the path of integration to the left, thereby reducing the contribution of the term  $x^s$ . This becomes possible only if the function  $D(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$  is holomorphic on the new path and therefore the question of continuation of Dirichlet-series beyond their domain of absolute convergence is a central issue in this theory. In fact, the importance of the Riemann hypothesis stems from the fact that it would allow us to move the path of integration for  $D(s) = \zeta(s)$  to the line  $\sigma = -\frac{1}{2} + it$  without meeting any singularity besides the obvious pole at 1.

Estermann [4] appears to be the first to address this problem. He showed that for an integer valued polynomial  $W(x)$  with  $W(0) = 1$  the Dirichlet-series  $D(s) = \sum_p W(p^{-s})$  can either be written as a finite product of the form  $\prod_{N=1}^N (s)^{c_N}$  for certain integers  $c_N$ , and is therefore meromorphically continuable to the whole complex plane, or is continuable to the halfplane  $\sigma > 0$ . In the latter case the line  $\sigma = 0$  is the natural boundary of the Dirichlet-series. The strategy of his proof was to show that every point on the line  $\sigma = 0$  is an accumulation point of poles or zeros of  $D$ . Note that, the Riemann-zeta function itself, does not fall among the cases under consideration, since  $W(x) = (1-x)^{-1}$  is a rational function. Dahlquist [5] generalized Estermann's work allowing  $W$  to be a function holomorphic in the unit circle with the exception of isolated singularities and in particular covering the case that  $W$  be rational. This method of proof was extended to much greater generality, interest being sparked by  $\ell$ -functions of nilpotent groups introduced by Grunewald, Segal and Smith [9] as well as height zeta functions [3].

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2010 Mathematics Subject Classification 30B50, 11M41, 30B40, 20F69, 11G50

Key words and phrases. Dirichlet series, Euler product, singularities, natural boundary, zeta functions of groups.

Functions arising in these contexts are often of the form  $D(s) = \prod_p W(p; p^{-s})$  for an integral polynomial  $W$ . Du Sautoy and Grunewald [7] gave a criterion for such a function to have a natural boundary which, in a probabilistic sense, applies to almost all polynomials. Again, it is shown that every point on the presumed boundary is an accumulation point of zeros or poles. The following conjecture, see for example [8, 1.11][7, 1.4], is believed to be true.

**Conjecture 1.** Let  $W(x; y) = \sum_{n,m} a_{n,m} x^n y^m$  be an integral polynomial with  $W(x; 0) = 1$ . Then  $D(s) = \prod_p W(p; p^{-s})$  is meromorphically continuable to the whole complex plane if and only if it is a finite product of Riemann  $\zeta$ -functions. Moreover, in the latter case if  $\sigma = \max_m \frac{n+1}{m} : m \geq 1; a_{n,m} \neq 0$ , then  $\sigma$  is the natural boundary of  $D$ .

In this paper we show that any refinement of Estermann's method is bound to fail to prove this conjecture.

If  $W(X; Y)$  is a rational function, we expand  $W$  into a power series  $W(X; Y) = \sum_{n,m} a_{n,m} X^n Y^m$ , and define  $\sigma = \sup \frac{n+1}{m} : m \geq 1; a_{n,m} \neq 0$ ,  $\sigma^* = \sup \frac{n}{m} : m \geq 1; a_{n,m} \neq 0$ . It is easy to see that the supremum is actually attained, and that the function  $\tilde{W} = 1 + \sum_{m=1}^{\infty} a_{n,m} X^n Y^m$  is again a rational function. We call  $\tilde{W}$  the main part of  $W$ , since only  $\tilde{W}$  is responsible for the convergence of the product  $D(s)$ . For  $W$  a polynomial  $\tilde{W}$  was called the ghost of  $W$  in [7]. A rational function  $W$  is called cyclotomic if it can be written as the product of cyclotomic polynomials and their inverses.

We define an obstructing point  $z$  to be a complex number with  $\sigma < z = \sigma^*$ , such that there exists a sequence of complex numbers  $z_i$ ,  $\sigma < z_i < \sigma^*$ ,  $z_i \rightarrow z$ , such that  $D$  has a pole or a zero in  $z_i$  for all  $i$ . Obviously, each obstructing point is an essential singularity for  $D$ , the converse not being true in general.

Our main result is the following.

**Theorem 1.** Let  $W(X; Y)$  be a rational function, which can be written as  $\frac{P(X; Y)}{Q(X; Y)}$ , where  $P, Q \in \mathbb{Z}[X; Y]$  satisfy  $P(X; 0) = Q(X; 0) = 1$ . Define  $\tilde{a}_{n,m}; \tilde{W}$  and  $D$  as above. Then the product representation of  $D$  converges in the half-plane  $\sigma < s < \sigma^*$ ,  $D$  can meromorphically continued into the half-plane  $\sigma < s < \sigma^*$ , and precisely one of the following holds true.

- (1)  $W$  is cyclotomic and once its unitary factors are removed,  $\tilde{W} = \tilde{W}^*$ ; in this case  $D$  is a finite product of Riemann  $\zeta$ -functions;
- (2)  $\tilde{W}$  is not cyclotomic; in this case every point of the line  $\sigma < s = \sigma^*$  is an obstructing point;
- (3)  $W \notin \tilde{W}^*$ ,  $\tilde{W}$  is cyclotomic and there are infinitely many pairs  $n; m$  with  $a_{n,m} \neq 0$  and  $\frac{n}{m} < \sigma < \frac{n+1}{m}$ ; in this case  $\sigma$  is an obstructing point;
- (4)  $W \notin \tilde{W}^*$ ,  $\tilde{W}$  is cyclotomic, there are only finitely many pairs  $n; m$  with  $a_{n,m} \neq 0$  and  $\frac{n}{m} < \sigma < \frac{n+1}{m}$ , but there are infinitely many primes  $p$  such that the equation  $W(p; p^{-s}) = 0$  has a solution  $s_0$  with  $\sigma < s_0 < \sigma^*$ ; in this case every point of the line  $\sigma < s = \sigma^*$  is an obstructing point;
- (5) None of the above; in this case no point on the line  $\sigma < s = \sigma^*$  is an obstructing point.

We remark that each of these cases actually occurs, that is, there are Euler-products for which Estermann's approach cannot work.

Notice that while in the third case we need information on the zeros of the Riemann-zeta function to know about the meromorphic continuation, in the last case we can say nothing about their continuation.

While the above classification looks pretty technical, these cases actually behave quite differently. To illustrate this point we consider a domain  $C$  with a function  $f : C \rightarrow \mathbb{C}$ , let  $N(f; z) = N(f; z, z) = N(f; z, z)$  the number of zeros and poles of  $f$  in  $C$  counted with positive multiplicity, that is, an  $n$ -fold zero or a pole of order  $n$  is counted  $n$  times. Then we have the following.

Corollary 1. Let  $W$  be a rational function, and define as above. Then one of the following two statements holds true:

- (1) For every  $\epsilon > 0$  we have  $N(f; z) = N(f; z, z) = N(f; z, z) = 1$ ;
- (2) We have  $N(f; z) = N(f; z, z) = N(f; z, z) = O(T \log T)$ .

If  $W$  is a polynomial and we assume the Riemann hypothesis as well as the  $\mathbb{Q}$ -linear independence of the imaginary parts of the non-trivial zeros of  $\zeta$ , then there exist constants  $c_1, c_2$ , such that  $N(f; z) = N(f; z, z) = N(f; z, z) = c_1 T \log T + c_2 T + O(\log T)$ .

Finally we remark that for  $\mathbb{Q}$ -functions of nilpotent groups the generalization to rational functions is irrelevant, since a result of du Sautoy [6] implies that if  $\zeta_G(s) = \sum_{p \in P} W(p; p^{-s})$  for a rational function  $W(X; Y) = \frac{P(X; Y)}{Q(X; Y)}$ , then  $\zeta_G$  is a cyclotomic polynomial, that is,  $\zeta_G$  can be written as the product of finitely many Riemann  $\mathbb{Q}$ -functions and a Dirichlet-series of the form  $\sum_{p \in P} W(p; p^{-s})$  with  $W$  a polynomial. However, for other applications it is indeed important to study rational functions, one such example occurs in the recent work of de la Breteche and Swynerton-Dyer [3].

## 2. Proof of case 2

In this section we show that if  $W$  is not cyclotomic, then  $\sigma = 1$  is the natural boundary of the meromorphic continuation of  $D$ . For  $W$  a polynomial this was shown by du Sautoy and Grunewald [7], our proof closely follows their lines of reasoning.

The main difference between the case of a polynomial and a rational function is that for polynomials the local zeros created by different primes can never cancel, whereas for a rational function the zeros of the numerator belonging to some prime number  $p$  might coincide with zeros of the denominator belonging to some other prime  $q$ , and may therefore not contribute to the zeros or poles needed to prove that some point on the presumed boundary is a cluster point. We could exclude the possibility of cancellations by assuming some unproven hypotheses from transcendence theory, however, here we show that we can deal with this case unconditionally by proving that the amount of cancellation remains limited. We first consider the case of cancellations between the numerator and denominator coming from the same prime number.

Lemma 1. Let  $P, Q \in \mathbb{Z}[X; Y]$  be co-prime non-constant polynomials. Then there are only finitely many primes  $p$ , such that for some complex number  $s$  we have  $P(p; p^{-s}) = Q(p; p^{-s}) = 0$ .

Proof. Let  $V$  be the variety of  $\mathbb{H}P/Q$  over  $\mathbb{C}$ . Assume there are infinitely many pairs  $(p; s)$ , for which the equation  $P(p; p^{-s}) = Q(p; p^{-s}) = 0$  holds true. Then  $V$  is infinite, hence, at least one-dimensional. Since  $P$  and  $Q$  are non-constant, we

have  $V \notin C^2$ , hence,  $V$  is one-dimensional. Let  $V^0$  be a one-dimensional irreducible component, and let  $R$  be a generator of the ideal corresponding to  $V^0$ . Then  $hP; Qi \in hRi$ , that is,  $R$  divides  $P$  and  $Q$ , which implies that  $R$  is constant. But a constant polynomial cannot define a one-dimensional variety and this contradiction proves our claim.

Next we use the following graph-theoretic result, describing graphs which are rather close to trees. We call a cycle in a graph minimal if it is of length  $\geq 3$ , and not the union of two cycles of smaller length.

**Lemma 2.** Let  $G$  be a graph,  $k \geq 2$  an integer, such that every vertex has degree  $\leq 3k$ , and that there exists a symmetric relation  $\sim$  on the vertices, such that every vertex  $v$  is in relation to at most  $k$  other vertices, and every minimal cycle passing through  $v$  also passes through one of the vertices in relation to  $v$ . Then  $G$  is finite.

*Proof.* Suppose that  $G$  were infinite, and fix some vertex  $v_0$ . We call a geodesic path good if no two vertices of the path stand in relation to each other. We want to construct an infinite good path. Note that  $p_1$  and  $p_2$  are good paths of finite length, they cannot intersect in but one point, for otherwise their union would contain a cycle, and choosing one of the intersection points we would obtain a contradiction with the definition of a good path. Hence, the union of the good paths starting in  $v_0$  forms a tree. There are  $\leq 3k$  vertices connected to  $v_0$ , at most  $k$  of which are forbidden. Hence, the first layer of the tree contains at least  $2k$  points. Each of these points is connected to at least  $3k$  other points. It stands in relation to at most  $k$  of them and hence we can extend every path in at least  $2k$  ways, and of all these paths at most  $k$  stand in relation with  $v_0$ . Hence, the second layer contains at least  $4k - k$  points. Denote by  $n_i$  the number of points in the  $i$ -th layer of the tree. Then, continuing in this way, we obtain

$$n_{i+1} \geq 2kn_i - k(n_{i-1} + \dots + n_1):$$

From this and the assumption that  $k \geq 2$  it follows by induction that  $n_{i+1} \geq kn_i$ , hence, the tree and therefore the graph  $G$ , which contains the tree, is infinite.

Note the importance of symmetry: if the relation is allowed to be non-symmetric, we can get two regular trees, and identify their leaves. Then every minimal cycle passing through one point either passes through its parent node or the mirror image of the point. Thus in the absence of symmetry the result becomes wrong for arbitrarily large valency even for  $k = 2$ .

We can now prove our result on non-cancellation.

**Lemma 3.** Let  $P; Q \in \mathbb{Z}[X; Y]$  be co-prime polynomials with  $\deg_Y \leq 1$  defined as in the introduction. Let  $\epsilon > 0$  be given, and suppose that for a prime  $p_0$  sufficiently large  $P(p_0; p_0^{-s})$  has a zero on the segment  $[1 + it; 1 + it + \epsilon]$ , where  $\epsilon > 0$ . Then  $Q\left(\frac{P(p_0 p^{-s})}{P(p_0 p^{-s})}\right)$  has a zero or a pole on this segment.

*Proof.* Since the local zeros converge to the line  $\sigma = 1$ , there are only finitely many primes  $p$  for which the numerator or denominator has a zero, hence, we may assume that  $P(p; p^{-s}); Q(p; p^{-s}) \neq 0$  for  $p > p_0$ . For each prime  $p$  let  $z_1^p; \dots; z_k^p$  be the roots of the equation  $P(p; p^{-s}) = 0$  in the segment  $\sigma = 1, 0 \leq t \leq \frac{2}{\log p}$ , and let  $w_1^p; \dots; w_l^p$  be the roots of the equation  $Q(p; p^{-s}) = 0$  on this segment. Such roots need not exist but if they do then their number is bounded independently of

$p$ . The roots of the equations  $P(p; p^{-s}) = 0$  and  $Q(p; p^{-s}) = 0$  form a pattern with period  $\frac{2-i}{\log p}$ . If  $p_0$  is sufficiently large, then  $\epsilon$  becomes arbitrary small, hence, if  $p$  is not large then the equations  $P(p; p^{-s}) = 0$  and  $Q(p; p^{-s}) = 0$  do not have solutions on the line  $\sigma = \frac{1}{2} + \epsilon$ . Let  $p_1$  be the least prime for which such solutions exist. For  $p_1$  sufficiently large and  $p > p_1$ , either  $P(p; p^{-s}) = 0$  has no solution on the segment under consideration or it has at least  $\frac{\log p}{2}$  such solutions. Note that by fixing  $\epsilon$  and choosing  $p_0$  sufficiently large we can make this expression as large as we need. Further note that by choosing  $p_0$  large we can ensure, in view of Lemma 1, that  $P(p; p^{-s}) = Q(p; p^{-s}) = 0$  has no solution on the line  $\sigma = \frac{1}{2} + \epsilon$ .

We now define a bipartite graph  $G$  as follows: The vertices of the graph are all complex numbers  $z_i^p$  in one set and all complex numbers  $w_j^q$  in the other set, where  $p \leq p_0$ . Two vertices  $z_i^p, w_j^q$  are joined by an edge if there exists a complex number  $s$  with  $\sigma = \frac{1}{2} + \epsilon$ ,  $t = s + it$ , such that  $s$  is congruent to  $\frac{p}{2}$  modulo  $\frac{2-i}{\log p}$ , and congruent to  $w_j^q$  modulo  $\frac{2-i}{\log q}$ . In other words, the existence of an edge indicates that one of the zeros of  $P(p; p^{-s})$  obtained from  $z_i^p$  by periodicity cancels with one zero of  $Q(q; q^{-s})$  obtained from  $w_j^q$ . If  $\frac{P(p; p^{-s})}{Q(q; q^{-s})}$  has neither a zero nor a pole on the segment, then every zero of one of the polynomials cancels with a zero of the other polynomial, that is, every vertex has valency at least  $\frac{\log p_1}{2}$ .

We next bound the number of cycles. Suppose that  $z_{i_1}^{p_1} w_{i_2}^{p_2} \dots w_{i_{l-1}}^{p_{l-1}} z_{i_l}^{p_l}$ . Then there is a complex number  $s$  in the segment which is congruent to  $z_{i_1}^{p_1}$  modulo  $\frac{2-i}{\log p_1}$  and congruent to  $w_{i_2}^{p_2}$  modulo  $\frac{2-i}{\log p_2}$ . Going around the cycle and collecting the differences we obtain an equation of the form  $2 - i - \sum_{i=1}^l \frac{p_i}{\log p_i} = 0$ ,  $i \in \mathbb{Z}$ , which can only hold if the combined coefficients vanish for each occurring prime. However the coefficients cannot vanish if some prime occurs only once. If the cycle is minimal the same vertex cannot occur twice, hence, there is some  $j$  such that  $p_1 = p_j$ , but  $i_1 \neq i_j$ . Hence, every minimal cycle containing  $z_{i_1}^{p_1}$  must contain  $z_{i_j}^{p_j}$  or  $w_{i_j}^{p_j}$  for some  $i \neq j$ . The relation defined by  $x_i^p = x_j^q$ ,  $p = q$ ,  $x \in \mathbb{Z}$  is an equivalence relation with equivalence classes bounded by some constant  $K$ . If we choose  $p_1 > \exp(6K^{-1})$ , the assumptions of Lemma 2 are satisfied, and we conclude that  $G$  is finite.

But we already know that there is no  $p > p_0$  for which  $P(p; p^{-s}) = 0$  or  $Q(p; p^{-s}) = 0$  has a solution, that is,  $G$  is finite. This contradiction completes the proof.

Using Lemma 3 the proof now proceeds in the same fashion as in the polynomial case; for the details we refer the reader to the proof given by du Sautoy and Grunewald [7].

### 3. Development in cyclotomic factors

A rational function  $W(X; Y)$  with  $W(X; 0) = 1$  can be written as an infinite product of polynomials of the form  $(1 - X^a Y^b)$ . Here convergence is meant with respect to the topology of formal power series, that is, a product  $\prod_{i=1}^{\infty} (1 - X^{a_i} Y^{b_i})$  converges to a power series  $f$  if for each  $N$  there exists an  $i_0$ , such that for  $i_1 > i_0$  the partial product  $\prod_{i=1}^{i_1} (1 - X^{a_i} Y^{b_i})$  coincides with  $f$  for all coefficients of monomials  $X^a Y^b$  with  $a, b < N$ . The existence of such an extension is quite obvious, however, we need some explicit information on the factors that occur and we shall develop the necessary information here.

For a set  $A \subset \mathbb{R}^2$  define the convex cone  $\overline{A}$  generated by  $A$  to be the smallest convex subset containing  $A$  for all  $\lambda \geq 0$  and  $\mu \geq 1$ . A point  $a$  of  $A$  is extremal, if it is contained in the boundary of  $\overline{A}$  and there exists a tangent to  $A$  intersecting  $A$  precisely in  $a$ , or set theoretically speaking, if  $A \cap \text{fag} \notin \overline{A}$ . Note that a convex cone forms an additive semigroup as a subsemigroup of  $\mathbb{R}^2$ .

To a formal power series  $W = \sum_{n,m} a_{n,m} X^n Y^m \in \mathbb{Z}[[X; Y]]$  we associate the set  $A_W = \{f(n; m) : a_{n,m} \neq 0\} \subset \mathbb{R}^2$ . Suppose we start with a rational function  $W \in \mathbb{Z}[[X; Y]]$ , which is of the form  $\frac{P(X; Y)}{Q(X; Y)}$  with  $P(X; 0) = Q(X; 0) = 1$ . Then

$$\frac{1}{Q(X; Y)} = \sum_{n,m} \frac{X^n}{Q(X; Y)} = \sum_{n,m} b_{n,m} X^n Y^m;$$

say, where the convergence of the geometric series as a formal power series follows from the fact that every monomial in  $Q$  is divisible by  $Y$ . The set  $f(n; m) : b_{n,m} \neq 0 \subset \mathbb{R}^2$  is contained in the semigroup generated by the points corresponding to monomials in  $Q$ , but may be strictly smaller, as there could be unforeseen cancellations. Multiplying the power series by  $P(X; Y)$ , we obtain that  $A_W$  is contained within finitely many shifted copies of  $A_{Q^{-1}}$ .

Let  $(n; m)$  be an extremal point of  $A_W$ . Then we have  $W = (1 - X^n Y^m)^{a_{n,m}} W_1(X; Y)$ , where  $W_1(X; Y) = (1 - X^n Y^m)^{-a_{n,m}} W(X; Y)$ . Obviously,  $W_1(X; Y)$  is a formal power series with integer coefficients, we claim that  $A_{W_1}$  is a proper subset of  $A_W$ . In fact, the monomials of  $W_1$  are obtained by taking the monomials of  $A_W$ , multiplying them by some power of  $X^n Y^m$ , and possibly adding up the contribution of different monomials. Hence,  $A_{W_1}$  is contained in the semigroup generated by  $A_W$ . But  $(n; m)$  is not in  $A_{W_1}$ , and since  $(n; m)$  was assumed to be extremal, we obtain

$$A_{W_1} \subsetneq A_W \cap f(n; m)g \subsetneq \overline{A_W} \cap f(n; m)g \subsetneq \overline{A_W}.$$

Taking the convex cone is a hull operator, thus  $\overline{A_{W_1}}$  is a proper subset of  $\overline{A_W}$ . Since we begin and end with a subset of  $\mathbb{N}^2$ , we can repeat this procedure so that after finitely many steps the resulting power series  $W_k$  contains no non-vanishing coefficients  $a_{n,m}$  with  $n < N; m < M$ . This suffices to prove the existence of a product decomposition, in fact, if one is not interested in the occurring cyclotomic factors one could avoid power series and stay within the realm of polynomials by setting  $W_1(X; Y) = (1 + X^n Y^m)^{-a_{n,m}} W(X; Y)$  whenever  $a_{n,m}$  is negative. However, in this way we trade one operation involving power series for finitely many involving polynomials, which is better avoided for actual calculations.

While we can easily determine a super-set of  $A_{W_1}$ , in general we cannot prove that some coefficient of  $W_1$  does not vanish, that is, knowing only  $A_W$  and not the coefficients we cannot show that  $A_{W_1}$  is as large as we suspect it to be. However, it is easy to see that when eliminating one extremal point all other extremal points remain untouched. In particular, if we want to expand a polynomial  $W$  into a product of cyclotomic polynomials, at some stage we have to use every extremal point of  $A_W$ , and the coefficient attached to this point has not changed before this step, by induction it follows that the expansion as a cyclotomic product is unique.

We now assume that  $W$  is cyclotomic, while  $W$  is not. We further assume that  $W$  is a polynomial, and that the numerator of  $W$  is not divisible by a cyclotomic polynomial. We can always satisfy these assumptions by multiplying or dividing  $W$  with cyclotomic polynomials, which corresponds to the multiplying or dividing  $D$  with certain shifted  $\phi$ -functions, and does not change our problem. Our aim is

to find some information on the set  $f(n; m) : c_{n, m} \in \mathbb{Q}$ , where the coefficients  $c_{n, m}$  are defined via the expansion  $W(X; Y) = \sum (1 - X^n Y^m)^{c_{n, m}}$ .

In the first step we remove all points on the line  $\frac{n}{m} = \frac{1}{t}$ . By assumption we can do so by using finitely many cyclotomic polynomials. The resulting power series  $W_1$ . The inverse of the product of finitely many cyclotomic polynomials is a power series with poles at certain roots of unity, hence, we can express the sequence of coefficients as a polynomial in  $n$  and Ramanujan-sums  $c_d(n)$  for  $d$  dividing some integer  $q$ . Consider some point  $(n; m) \in A_W$ , and compute the coefficient attached to this point in  $W_1$ . If  $A_W$  does not contain a point  $(n^0; m^0)$ , such that  $(n - n^0; m - m^0)$  is collinear to  $(-1; 1)$ , then this coefficient is clearly 0. Otherwise we consider all points  $(n_1; m_1); \dots; (n_k; m_k)$  in  $A_W$ , which are on the parallel to  $(-1; 1)$  through  $(n; m)$ . The coefficients of  $W_1$  attached to points on  $\ell$  are linear combinations of shifted coefficients of inverse cyclotomic polynomials, hence, they can be written as some polynomial with periodic coefficients. In particular, either there are only finitely many non-vanishing coefficients, or there exists a complete arithmetic progression of non-vanishing coefficients. Hence, we find that  $A_{W_1}$  is contained within a locally finite set of lines parallel to  $(-1; 1)$ , and every line either contains only finitely many points, or a complete arithmetic progression. Suppose that every line contains only finitely many points. Then there exists some  $\epsilon > 0$ , such that  $A_{W_1}$  is contained in  $f(s; t) : s \leq -1/t + \epsilon$ , in particular,  $W_1$  is regular in  $f(z_1; z_2) : |z_1| \leq |z_2| + \epsilon$ . But  $W_1 = \frac{P}{Q W}$ , and by assumption  $P$  is not divisible by  $W$ , therefore, there exist points  $(z_1; z_2)$ , where  $W$  vanishes, but  $P$  does not, and these points are singularities of  $W_1$  satisfying  $|z_1| \leq |z_2| + \epsilon$ . Hence, there exists some line containing a complete arithmetic progression.

Let  $(x; 0) + t(-1; 1)$  be the unique line containing in finitely many elements of  $A_{W_1}$ , such that for each  $y > 0$  the line  $(y; 0) + t(-1; 1)$  contains only finitely many elements  $(n_1; m_1); \dots; (n_k; m_k)$  of  $A_{W_1}$ . Set  $\delta = \delta_x$ , that is, the distance of the right boundary of  $A_W$  from the line  $(x; 0) + t(-1; 1)$  measured horizontally, and set  $\delta_i = m_i - n_i$ , that is,  $\delta_i$  is the distance of  $(n_i; m_i)$  from the right boundary, also measured horizontally, and  $\delta = \min \delta_i > 0$ .

We now eliminate the points  $a_i$  to obtain the power series  $W_2$ . When doing so we introduce lots of new elements to the left of the line  $(x; 0) + t(-1; 1)$ , which are of no interest to us, and finitely many points on this line or to the right of this line, in fact, we can get points at most at the points of the form  $(n_i; m_i) + (n_j; m_j)$ ,  $j \geq 2N$ ,  $\delta_j > 0$ . Note that the horizontal distance from the line  $t(-1; 1)$  is additive, that is,  $A_{W_2}$  is contained in the intersection of  $A_W$  and the half-plane to the left of the line  $(x; 0) + t(-1; 1)$ , together with finitely many points between the lines  $(x; 0) + t(-1; 1)$  and  $t(-1; 1)$ , each of which has distance at least  $2\delta$  from the latter line. Repeating this procedure, we can again double this distance, and after finitely many steps this minimal distance is larger than the width of the strip, which means that we have arrived at a power series  $W_3$  such that  $A_{W_3}$  is contained in the intersection of  $A_W$  and the half-plane to the left of the line  $(x; 0) + t(-1; 1)$ . Moreover, since at each step there are only finitely many points changed on the line  $(x; 0) + t(-1; 1)$ , we see that the intersection of  $A_{W_3}$  with this line equals the intersection of  $A_{W_1}$  with this line up to finitely many inclusions or omissions. Since an infinite arithmetic progression, from which finitely many points are deleted still contains an infinite arithmetic progression, we see that  $A_{W_3}$  contains an infinite arithmetic progression.



Next we eliminate the points on  $A_W$  starting at the bottom and working upwards. When eliminating a point, we introduce (possibly infinitely many) new points, but all of them are on the left of the line  $(x;0) + t(-;1)$ . Hence, after infinitely many steps we arrive at a power series  $W_4$ , for which  $A_{W_4}$  is contained in the intersection of  $A_W$  and the open half-plane to the left of  $(x;0) + t(-;1)$ .

Fortunately, from this point on we can be less explicit. Consider the set of differences of the sets  $A_{W_i}$  from the line  $t(-;1)$ . Taking the differences is a semi-group homomorphism, hence, at each stage the set of differences is contained in the semi-group generated by the differences we started with. But since  $W$  is a polynomial, this semi-group is finitely generated, and therefore discrete. Hence, no matter how we eliminate terms, at each stage the set  $A_{W_i}$  is contained in a set of parallels to  $t(-;1)$  intersecting the real axis in a discrete set of non-positive numbers.

Collecting the cyclotomic factors used during this procedure, we have proven the following.

**Lemma 4.** Let  $W(X;Y)$  be a rational function such that  $W(X;Y)$  is a cyclotomic polynomial, but  $W$  itself is not cyclotomic. Define as above. Then there is a unique expansion  $W(X;Y) = \prod_{n,m} (1 - X^n Y^m)^{c_{n,m}}$ . The set  $C = \{f(n;m) : c_{n,m} \notin 0\}$  contains an infinite arithmetic progression with difference a multiple of  $(-;1)$ , only finitely many elements to the right of this line, and all entries are on lines parallel to  $t(-;1)$ , such that the lines intersect the real axis in a discrete set of points.

#### 4. Proof of case 3

We prove that  $\beta$  is an obstructing point. For integers  $n,m$  with  $c_{n,m} \notin 0$  the factor  $(-n+ms)^{c_{n,m}}$  creates a pole or a zero at  $\frac{n+1}{m}$ , which for  $\frac{n+1}{m} > \beta$  is to the right of the supposed boundary. Hence, if  $\beta$  is not an obstructing point, for some  $\epsilon > 0$  and all rational numbers  $\gamma \in (\beta, \beta + \epsilon)$  we would have  $\frac{n+1}{m} = \gamma$ ,  $c_{n,m} = 0$ . We now show that this is impossible by proving that there are pairs  $(n;m)$  with  $\frac{n+1}{m}$  arbitrarily close to  $\beta$ ,  $c_{n,m} \notin 0$ , such that the sum consists of a single term, and is therefore non-zero as well.

Let  $\frac{k}{\ell}$  be the slope of the rays. Let  $f(n_i;m_i)$  be a list of the starting points of the rays described in Lemma 4, where  $(n_0;m_0)$  defines the rightmost ray. Take an integer  $q$ , such that  $c_{km+n_0;q+m_0} \notin 0$  for all but finitely many natural numbers  $k$ . Let  $d$  be the greatest common divisor of  $m_0$  and  $q$ . The prime number theorem for arithmetic progressions guarantees infinitely many  $p$ , such that  $\frac{q+m_0}{d} = p$  is prime. Suppose there is a pair  $(n^0;m^0)$  belonging to another ray, such that  $c_{n^0;m^0} \notin 0$  and  $\frac{n^0+1}{m^0} = \frac{k+n_0}{\ell+m_0}$ . The point  $(n^0;m^0)$  must lie on one of the finitely many rays, hence, we can write  $n^0 = k^0 + n_1$ ,  $m^0 = \ell^0 + m_1$ . Since  $p$  is a divisor of the denominator of the right hand side, it also has to divide the denominator of the left hand side. We obtain that  $p$  divides both  $\ell^0 + m_0$  and  $\ell^0 + m_1$ . Restricting, if necessary, to an arithmetic progression, we obtain an infinite of indices such that  $\ell^0 + m_1 = t(\ell^0 + m_0)$ , where  $t \in [0;1]$  is a rational number with denominator dividing  $d$ . Hence, we obtain that the equations

$$\ell^0 + m_1 = t(\ell^0 + m_0); \quad t(k^0 + n_1) = k + n_0$$

have in finitely many solutions  $\beta_i \leq 2N$ . Two linear equations in two variables, none of which is trivial, can only have in finitely many solutions, if these equations are equivalent, that is,  $t^2 = 1$ , which implies  $t = 1$  since  $t$  is positive by definition. Hence, writing the equations as vectors, we have

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}^k = \begin{pmatrix} n_1 & n_0 \\ m_1 & m_0 \end{pmatrix};$$

that is, the vector linking  $\begin{pmatrix} n_0 \\ m_0 \end{pmatrix}$  with  $\begin{pmatrix} n_1 \\ m_1 \end{pmatrix}$  is collinear with  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}^k$ , contrary to the assumption that  $\begin{pmatrix} n_0 \\ m_0 \end{pmatrix}$  was on a ray other than that of  $\begin{pmatrix} n_1 \\ m_1 \end{pmatrix}$ . Hence, poles of  $f$ -factors accumulate at  $\infty$ . It remains to check that these poles are not cancelled by zeros of other factors. Since zeros of  $f$ -factors are never positive reals, these factors do not cause problems. Suppose that a pole of  $(ns - m)$  cancels with a zero of the local factor  $W(p; p^s)$ , that is,  $W(p; p^{(m+1)-n}) = 0$ . Since  $W$  has coefficients in  $\mathbb{Z}$ , this implies that  $p^{(m+1)-n}$  is algebraic of degree at most equal to the degree of  $W$ , hence,  $\frac{m+1}{n}$  can be reduced to a fraction with denominator at most equal to the degree of  $W$ . There are only finitely many rational numbers in the interval  $[\beta_i; \beta_i + 1]$  with bounded denominator, hence, only finitely many of the poles can be cancelled, that is,  $\infty$  is in fact an obstructing point.

For the corollary note that in cases (2)-(4)  $\infty$  is an obstructing point, that is, in these cases the first condition of the corollary holds true. In case (1) and (5), we can represent  $D$  as the product of finitely many Riemann  $f$ -functions multiplied by some function which is holomorphic in the half-plane  $\langle s \rangle$ , and has zeros only where the finitely many local factors vanish. A local factor belonging to the prime  $p$  creates a  $\frac{2-i}{\log p}$ -periodic pattern of zeros, hence, the number of zeros and poles is bounded above by the number of zeros of the finitely many  $f$ -functions, which is  $O(T \log T)$ , and the finitely many sets of periodic patterns, which create  $O(T)$  zeros. Hence,  $N(f < z >; j - z j < Tg)$  is  $O(T \log T)$ . It may happen that there are significantly less poles or zeros, if poles of one factor coincide with poles of another factor, however, we claim that under RH and the assumption of linear independence of zeros the amount of cancellation is negligible. First, if the imaginary part of zeros of  $f$  are  $\mathbb{Q}$ -linearly independent, then we cannot have  $(n_1 s - m_1) = (n_2 s - m_2) = 0$  for integers  $n_1; n_2; m_1; m_2$  with  $(n_1; m_1) \notin (n_2; m_2)$ , that is, zeros and poles of different  $f$ -factors cannot cancel. There is no cancellation among local factors, since local factors can only have zeros and never poles. Now consider cancellation among zeros of local factors and  $f$ -factors. We want to show that there are at most finitely many cancellations. Suppose otherwise. Since there are only finitely many local factors and finitely many  $f$ -factors, an infinitude of cancellation would imply that there are in finitely many cancellations among one local factor and one  $f$ -factor. The zeros of a local factor are of the form  $\beta_i + \frac{2k-i}{\log p}$ , where  $\beta_i$  is the logarithm of one of the roots of  $W(p; X) = 0$  chosen in such a way that  $0 = \beta_i < \frac{2-i}{\log p}$ . Since an algebraic equation has only finitely many roots, an infinitude of cancellations implies that for some complex number  $\alpha$  and in finitely many integers  $k$  we have  $(n(\alpha + \frac{2k-i}{\log p}) - m) = 0$ . Choose 4 different such integers  $k_1; \dots; k_4$ , and let  $\beta_1; \dots; \beta_4$  be the corresponding roots of  $W$ . Then we have  $\beta_1 - \beta_2 = \frac{2(k_1 - k_2)n}{\log p}$ ,  $\beta_3 - \beta_4 = \frac{2(k_3 - k_4)n}{\log p}$ , that is,  $(k_3 - k_4)(\beta_1 - \beta_2) = (k_1 - k_2)(\beta_3 - \beta_4)$ , which gives a linear relation among the zeros of  $W$ , contradicting our assumption. Hence, if the imaginary parts of the roots of  $W$  are  $\mathbb{Q}$ -linear independent, the number of zeros and

poles of  $D$  in some domain coincides with the sum of the numbers of zeros and poles of all factors, up to some bounded error, and our claim follows.

## 5. Examples

In this section we give examples to show that our classification is non-trivial in the sense that every case actually occurs.

Example 1. The sum  $\sum_{n=1}^{\infty} \frac{2(n)}{n^s} = \frac{(s)(s-1)}{(2s)(2s-2)}$  corresponds to the polynomial  $W(X; Y) = (1+Y)(1+XY)$ , while the sum  $\sum_{n=1}^{\infty} \frac{(n)}{n^s} = (s)(s-1)$  corresponds to the rational function  $W(X; Y) = \frac{1}{(1+Y)(1+XY)}$ .

Example 2. (a) Let  $(n)$  be the number of prime divisors of  $n$  counted with multiplicity. Then  $\sum_{n=1}^{\infty} \frac{2(n)}{n^s} = \prod_p (1 + \frac{2}{p^s} (1 - p^{-s}))$  corresponds to the rational function  $W(X; Y) = 1 + \frac{2Y}{1-Y}$  with main part  $1 + 2Y$ , which is not cyclotomic.

(b) Let  $G$  be the direct product of three copies of the Heisenberg-group,  $a'_n(G)$  the number of normal subgroups of  $G$  of index  $n$ . Then  $\sum_{n=1}^{\infty} \frac{a'_n(G)}{n^s}$  was computed by Taylor [11] and can be written as a finite product of  $\zeta$ -functions and an Euler-product of the form  $\prod_p W(p; p^{-s})$ , where  $W$  consists of 14 monomials and  $W(X; Y) = 1 - 2X^{13}Y^8$ , which is not cyclotomic.

Example 3. (a) Let  $G$  be the free nilpotent group of class two with three generators. Then  $\sum_{n=1}^{\infty} \frac{a'_n(G)}{n^s}$  can be written as a finite product of  $\zeta$ -functions and the Euler-product  $\prod_p W(p; p^{-s})$ , where

$$W(X; Y) = 1 + X^3Y^3 + X^4Y^3 + X^6Y^5 + X^7Y^5 + X^{10}Y^8;$$

We have  $W(X; Y) = 1 + X^7Y^5$ , which clearly does not divide  $W$ , hence, while  $W$  is cyclotomic,  $W$  is not. Hence,  $W$  is not case 1 or 2. Theorem 1 implies that  $7=5$  is an essential singularity of  $\sum_{n=1}^{\infty} \frac{a'_n(G)}{n^s}$ . Du Sautoy and Woodward [8] showed that in fact the line  $\Re s = 7=5$  is the natural boundary for  $\sum_{n=1}^{\infty} \frac{a'_n(G)}{n^s}$ .

(b) Now consider the product

$$f(s) = \prod_p (1 + p^{-s} + p^{1-2s})$$

Again, the polynomial  $W(X; Y) = 1 + Y + XY^2$  is not cyclotomic, while  $W$  is cyclotomic. Again, Theorem 1 implies that  $1=2$  is an obstructing point of  $f$ . However, the question whether there exists another point on the line  $\Re s = 1=2$  which is an obstructing point is essentially equivalent to the Riemann hypothesis. We have

$$f(s) = \frac{(s)(2s-1)(3s-1)}{(2s)(4s-2)} R(s) \prod_{m=1}^{\infty} \frac{((4m+1)s-2m)}{((4m+3)s-2m-1)((8m+2)s-4m)};$$

hence, if  $f$  has only finitely many zeros on the line  $\Re s = 1=2$  it, then the right hand side has only finitely many zeros in the domain  $\Re s > 1=2, \Im s > 0$ , hence,  $1=2$  is the unique obstructing point on this line. On the other hand, if  $f(s)$  has infinitely many non-real zeros on the line  $\Re s = 1=2$  it, then every point on this line is an obstructing point for  $f$  (confer [1]).

Hence, while for some polynomials the natural boundary can be determined, we do not expect any general progress in this case.

Example 4. (a) The local zeta function associated to the algebraic group  $G$  is defined as

$$Z_p(G; s) = \sum_{G_p^+} j \det(g) j_p^{-s} d$$

where  $G_p^+ = G(\mathbb{Q}_p) \setminus M_n(\mathbb{Z}_p)$ ,  $j : j_p$  denotes the  $p$ -adic valuation and  $d$  is the normalised Haar measure on  $G(\mathbb{Z}_p)$ . In particular the zeta function associated to the group  $G = \mathrm{GSp}_6$  [10] is given by

$$Z(s=3) = \sum_p (s-3)(s-5)(s-6) \left( 1 + p^{1-s} + p^{2-s} + p^{3-s} + p^{4-s} + p^{5-2s} \right);$$

The polynomial

$$W(X; Y) = 1 + (X + X^2 + X^3 + X^4)Y + X^5Y^2$$

satisfies the relation  $W(X; Y) = 1 + X^4Y$ , that is,  $W$  is cyclotomic, while  $W$  is not. Du Sautoy and Grunewald [7] showed that in the cyclotomic expansion of  $W$  there are only finitely many  $(n; m)$  with  $c_{n,m} \neq 0$  and  $\frac{n+1}{m} > 4$ , and that  $W(p; p^{-s}) = 0$  has solutions with  $< s > 4$  for infinitely many primes, hence,  $W$  is an example of type 4, and  $Z(s=3)$  has the natural boundary  $< s = 4$ .

(b) Let  $V$  be the cubic variety  $x_1x_2x_3 = x_4^3$ ,  $U$  be the open subset  $\{x_4 \neq 0\}$ ,  $H$  the usual height function. De la Breteche and Sir Swinnerton-Dyer [3] showed that  $Z(s) = \sum_{x \in U} H(x)^{-s}$  can be written as the product of finitely many  $\ell$ -functions, a function holomorphic in a half-plane strictly larger than  $< s > 3$ , and a function having an Euler-product corresponding to the rational function

$$W(X; Y) = 1 + (1 - X^3Y)(X^6Y^2 + X^5Y^3 + X^4 + X^2Y^2 + X^3Y^3 + Y^4) - X^9Y^3;$$

They showed that in the cyclotomic expansion of this function there occur only finitely many terms  $c_{n,m} X^n Y^m$  with  $c_{n,m} \neq 0$  and  $\frac{n+1}{m} > \frac{3}{4}$ , and all but finitely many local factors have a zero to the right of  $s = 3$ , hence,  $< s = 3$  is the natural boundary of  $Z(s)$ .

Example 5. Let  $J_2(n)$  be the Jacobsthal function, i.e.  $J_2(n) = \# \{f(x; y) : 1 \leq x, y \leq n; (x; y; n) = 1\}$ , and define  $g(s) = \sum_{n=1}^{\infty} \frac{(n)J_2(n)}{n^s}$ . Since  $J_2$  is multiplicative,  $g$  has an Euler-product, which can be computed to give

$$g(s) = \prod_p \left( 1 + p^{-s} - p^{2-s} \right);$$

We have

$$g(s) = \prod_p \left( 1 - p^{2-s} \right) \prod_p \left( 1 + \frac{p^{-s}}{1 - p^{2-s}} \right) = (s-2)D(s);$$

say. For  $< s > 2$  the Euler product for  $D$  converges uniformly, since

$$\sum_p \frac{p^{-s}}{1 - p^{2-s}} \leq \sum_p \frac{p}{1 - 2^2} < \infty \quad (2);$$

Hence,  $D$  is holomorphic and non-zero in  $< s > 2$ , that is, no point on the line  $< s = 2$  is an obstructing point, that is, Estermann's method cannot prove the existence of a single singularity of this function.

## 6. Comparison of our classification with the classification of du Sautoy and Woodward

In [8], du Sautoy and Woodward consider several classes of polynomials for which they can prove Conjecture 1. Since their classes do not coincide with the classes described in Theorem 1, we now describe how the two classifications compare. We will refer to the classes described in Theorem 1 as ‘cases’, while we will continue to refer to the polynomials of du Sautoy and Woodward by their original appellation of ‘type’.

Polynomials of type I are polynomials  $W$  such that  $W$  is not cyclotomic, this class coincides with polynomials in case (2).

Polynomials of type II are polynomials  $W$  such that  $W$  is cyclotomic, there are only finitely many  $c_{n,m} > 0$  with  $\frac{n+1}{m} > \frac{1}{2}$ , and for infinitely many primes we have that  $W(p; p^{-s})$  has zeros to the right of  $\frac{1}{2}$ . This class contains all polynomials in case (4), and all polynomials of type II fall under case (3) or (4), but there are polynomials in case (3) which are not of type II. For polynomials of type II they prove that the line  $\sigma = \frac{1}{2}$  is the natural boundary of meromorphic continuation of  $D$ , their result for polynomials therefore clearly supersedes the relevant parts of Theorem 1.

Polynomials of type III are polynomials  $W$  as in type II, but there are infinitely many pairs  $n; m$  with  $c_{n,m} > 0$ ,  $\frac{n+1}{m} > \frac{1}{2}$ . These polynomials fall under case (3), they show under the Riemann hypothesis that  $\sigma = \frac{1}{2}$  is a natural boundary. For such polynomials the results are incomparable, our results are unconditional, yet weaker.

Polynomials of type IV are polynomials with infinitely many pairs  $(n; m)$  satisfying  $c_{n,m} \notin 0$  and  $\frac{n+1=2}{m} > \frac{1}{2}$ , and such that with the exception of finitely many  $p$  there are no local zeros to the right of  $\sigma = \frac{1}{2}$ . For such polynomials du Sautoy and Woodward show that  $\sigma = \frac{1}{2}$  is the natural boundary, if the imaginary parts of the zeros of  $W$  are  $\mathbb{Q}$ -linearly independent. All polynomials of type IV fall under case (3), again, the results are incomparable.

Polynomials of type V are polynomials  $W$  such that  $W$  is cyclotomic, with the exception of finitely many  $p$  there are no local zeros to the right of  $\frac{1}{2}$ , and there are infinitely many pairs  $n; m$  with  $c_{n,m} \notin 0$  and  $\frac{n+1}{m} > \frac{1}{2}$ . This corresponds to case (5).

Polynomials of type VI are polynomials  $W$  such that  $W$  is cyclotomic, with the exception of finitely many  $p$  there are no local zeros to the right of  $\frac{1}{2}$ , there are infinitely many pairs  $(n; m)$  with  $c_{n,m} \notin 0$  and  $\frac{n+1}{m} > \frac{1}{2}$ , only finitely many of which satisfy  $\frac{n+1=2}{m} > \frac{1}{2}$ . These fall under case (3).

Case (1) does not occur in their classification as it is justly regarded as trivial.

## 7. Comparison with the multivariable case

The object of our study has been the Dirichlet-series  $D(s) = \sum_{n=1}^{\infty} \frac{W(n)}{n^s}$ . This will be called the  $1\frac{1}{2}$ -variable problem since the polynomial has two variables, but the Dirichlet-series depends on only one complex variable. If the coefficients of the above series have some arithmetical meaning, and this meaning translates into a statement on each monomial of  $W$ , then the Dirichlet-series  $D(s_1; s_2) = \sum_{n_1, n_2=1}^{\infty} \frac{W(n_1; n_2)}{n_1^{s_1} n_2^{s_2}}$  retains more information, and it could be fruitful to consider this function instead. Of course, the gain in information could be at the risk of the

technical difficulties introduced by considering several variables. However, here we show that the multivariable problem is actually easier than the original question of  $1\frac{1}{2}$ -variables.

Where there is no explicit reference to  $p$ , the problem of a natural boundary was completely solved by Essouabri, Lichtin and the first named author [2].

**Theorem 2.** Let  $W \in \mathbb{Q}[X_1; \dots; X_k]$  be a polynomial satisfying  $W(0; \dots; 0) = 1$ . Set  $D(s_1; \dots; s_k) = \prod_p W(p^{-s_1}; \dots; p^{-s_k})$ . Then  $D$  can be meromorphically continued to the whole complex plane if and only if  $W$  is cyclotomic. If it cannot be continued to the whole complex plane, then its maximal domain of meromorphic continuation is the intersection of a finite number of effectively computable half-spaces. The bounding hyper-plane of each of these half-spaces passes through the origin.

At first sight one may think that one can pass from the 2-dimensional by fixing  $s_1$ , however, this destroys the structure of the problem, as is demonstrated by the following.

**Example 6.** The Dirichlet-series  $D(s_1; s_2) = \prod_p (1 + (2 - p^{-s_1})p^{-s_2})$  as a function of two variables can be meromorphically continued into the set  $f(s_1; s_2) : \langle s_2 \rangle > 0; \langle s_1 + s_2 \rangle > 0$ , and the boundary of this set is the natural boundary of meromorphic continuation. If we fix  $s_1$  with  $\langle s_1 \rangle < 0$ , and view  $D$  as a function of  $s_1$ , then  $D$  can be continued to  $\mathbb{C}$  if and only if  $s_1 = 0$ . In every other case the line  $\langle s_2 \rangle = 0$  is the natural boundary.

**Proof.** The behaviour of  $D(s_1; s_2)$  follows from [2, Theorem 2]. If we fix  $s_1$ , then  $1 + (2 - p^{-s_1})p^{-s_2}$  has zeros with relatively large real part, provided that either  $\langle s_1 \rangle > 0$ , or  $\langle s_1 \rangle = 0$  and  $\langle p^{-s_1} \rangle < 0$ . In the first case we can argue as in the case that  $W$  is not cyclotomic. By the prime number theorem for short intervals we find that the number of prime numbers  $p < x$  satisfying  $\langle p^{-s_1} \rangle < 0$  is greater than  $\frac{x}{\log x}$ , and we see that we can again adapt the proof for the case  $W$  non-cyclotomic.

In other words, the natural boundary for the  $1\frac{1}{2}$ -variable problem is the same as for the 2-variable problem, with one exception, in which the  $1\frac{1}{2}$ -variable problem collapses to a 1-variable problem, and in which case the Euler-product becomes continuable beyond the 2-variable boundary.

It seems likely that this behaviour should be the prevalent one, it is less clear what precisely "this behaviour" is. One quite strong possibility is the following:

Suppose that  $D(s_1; s_2) = \prod_p W(p^{-s_1}; p^{-s_2})$  has a natural boundary at  $\langle s_1 \rangle = 0$ . Then there are only finitely many values  $s_2$ , for which the specialization  $D(\cdot; s_2)$  is meromorphically continuable beyond  $\langle s_1 \rangle = 0$ .

However, this statement is right now supported only by a general lack of examples, and the fact that example 6 looks quite natural, so we do not dare a conjecture. However we believe that some progress in this direction could be easier to obtain than directly handling Conjecture 1. In particular those cases, in which zeros of  $\zeta$  pose a serious threat for local zeros would become a lot easier since this type of cancellation can only affect a countable number of values for  $s_2$ .

## References

- [1] G. Bhowmik, J.-C. Schlage-Puchta, Natural Boundaries of Dirichlet series, *Func. Approx. Comment. Math.* XXXV II.1 (2007), 17{29.

- [2] G. Bhowmik, D. Essouabri, B. Lichtin, Meromorphic Continuation of Multivariable Euler Products, *Forum. Math.* 110 (2), (2007), 1111{1139.
- [3] R. de la Breteche, P. Swinnerton-Dyer, Fonction zêta des hauteurs associée à une certaine surface cubique, *Bull. Soc. Math. France* 135.1 (2007), 65{92.
- [4] T. Estermann, On certain functions represented by Dirichlet series, *Proc. London Math. Soc.* 27 (1928), 435{448.
- [5] G. Dahlquist, On the analytic continuation of Eulerian products, *Ark. Mat.* 1 (1952), 533{554.
- [6] M. P. F. du Sautoy, Zeta functions of groups and rings: uniformity, *Israel J. Math.* 86 (1994), 1{23.
- [7] M. du Sautoy, F. Grunewald, Zeta functions of groups: zeros and friendly ghosts, *Amer. J. Math.* 124 (2002), 1{48.
- [8] M. du Sautoy, L. Woodward, Zeta functions of groups and rings. *Lecture Notes in Mathematics*, 1925. Springer-Verlag, Berlin, 2008.
- [9] F. J. Grunewald, D. Segal, and G. C. Smith, Subgroups of finite index in nilpotent groups, *Invent. Math.* 93 (1988), 185{223.
- [10] J.-I. Igusa, Universal  $p$ -adic zeta functions and their functional equations, *Amer. J. Math.* 111 (1989), 671{716.
- [11] G. Taylor, Zeta Functions of Algebras and Resolution of Singularities, Ph.D. Thesis, University of Cambridge, 2001.

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