



**HAL**  
open science

# Model-completion of varieties of co-Heyting algebras

Luck Darnière, Markus Junker

► **To cite this version:**

Luck Darnière, Markus Junker. Model-completion of varieties of co-Heyting algebras. 2010. hal-00445886v1

**HAL Id: hal-00445886**

**<https://hal.science/hal-00445886v1>**

Preprint submitted on 11 Jan 2010 (v1), last revised 29 May 2017 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Model completion of varieties of co-Heyting algebras

Luck Darnière  
Département de mathématiques  
Faculté des sciences  
Université d'Angers, France

Markus Junker  
Mathematisches Institut  
Abteilung für mathematische Logik  
Universität Freiburg, Deutschland

January 11, 2010

## Abstract

It is known that exactly eight varieties of Heyting algebras have a model-completion, but no concrete axiomatisation of these model-completions were known by now except for the trivial variety (reduced to the one-point algebra) and the variety of Boolean algebras. For each of the six remaining varieties we introduce two axioms and show that 1) these axioms are satisfied by all the algebras in the model-completion, and 2) all the algebras in this variety satisfying these two axioms have a certain embedding property. For four of these six varieties (those which are locally finite) this actually provides a new proof of the existence of a model-completion, this time with an explicit and finite axiomatisation.

**MSC 2000:** 06D20, 03C10

## 1 Introduction

It is known from a result of Maksimova [Mak77] that there are exactly eight varieties of Heyting algebras that have the amalgamation property (numbered  $\mathcal{H}_1$  to  $\mathcal{H}_8$ , see section 2). Only these varieties (more exactly their theories) can have a model completion<sup>1</sup> and it is known since the 1990's that this is indeed the case<sup>2</sup>. On the other hand these model-completions still remain very mysterious. In particular no algebraic (humanly readable) axiomatisation was known for any of them except  $\mathcal{H}_7$  and  $\mathcal{H}_8$ : the latter is the trivial variety reduced to the one point Heyting algebra, and the former is the variety of Boolean algebras whose model completion is well known.

---

<sup>1</sup>Basic model theoretic notions are recalled in section 2 but we may already point out a remarkable application of the existence of a model-completion for these varieties, namely that for each of the corresponding super-intuitionistic logics the second order propositional calculus IPC<sup>2</sup> is interpretable in the first order propositional calculus IPC<sup>1</sup> (in the sense of [Pit92]).

<sup>2</sup>For the varieties  $\mathcal{H}_3$  to  $\mathcal{H}_8$ , which are locally finite, the existence of a model-completion follows from the amalgamation property and [Whe76], corollary 5. For the variety  $\mathcal{H}_1$  of all Heyting algebras it is a translation in model-theoretic terms of a theorem of Pitts [Pit92] combined with [Mak77], as is explained in [GZ97]. It is also claimed in [GZ97] that the same holds for  $\mathcal{H}_2$  up to minor adaptations of [Pit92].

In this paper we partly fill this lacuna by giving new proofs for some of these results using algebraic and model-theoretic methods, guided by some geometric intuition. We first give in section 3 a complete classification of all the minimal finite extensions of a Heyting algebra  $L$ . This is done by proving that these extensions are in one-to-one correspondence with certain special triples of elements of  $L$ . Each of the remaining sections 4 to 9 is devoted to one of the varieties  $\mathcal{H}_i$ . We introduce for each of them two axioms that we call “density” and “splitting” and prove our main results:

**Theorem 1.1** *Every existentially closed model of  $\mathcal{H}_i$  satisfies the density and splitting axioms of  $\mathcal{H}_i$ .*

**Theorem 1.2** *Given a Heyting algebra  $L$  in  $\mathcal{H}_i$  and a finite substructure  $L_0$  of  $L$ , if  $L$  satisfies the density and splitting axioms of  $\mathcal{H}_i$  then every finite extension  $L_1$  of  $L_0$  admits an embedding into  $L$  which fixes  $L_0$  pointwise.*

By standard model-theoretic arguments (see fact 2.1) it follows immediately that if  $\mathcal{H}_i$  is locally finite then  $\mathcal{H}_i$  has a model-completion which is axiomatized by the density and the splitting axioms of  $\mathcal{H}_i$ . So this gives a new proof of the previously known model-completion results for  $\mathcal{H}_3$  to  $\mathcal{H}_6$ , which provides in addition a simple axiomatization of these model completions.

Unfortunately we do not fully recover the existence of a model-completion for  $\mathcal{H}_1$  and  $\mathcal{H}_2$ . However our axioms shed some new light on the algebraic structure of the existentially closed Heyting algebras in these varieties. Indeed it is noticed in [GZ97] that Heyting algebras existentially closed in  $\mathcal{H}_1$  satisfy the density axiom of  $\mathcal{H}_1$ , but nothing like the splitting property nor any condition sufficient for theorem 1.2 to hold seems to have been suspected until now.

Let us also point out an easy consequences of theorem 1.2.

**Corollary 1.3** *If  $L$  is an algebra in  $\mathcal{H}_i$  which satisfies the density and splitting axioms of  $\mathcal{H}_i$  then every finite algebra in  $\mathcal{H}_i$  embeds into  $L$ , and every algebra in  $\mathcal{H}_i$  embeds into an elementary extension of  $L$ .*

**Remark 1.4** In this paper we do not actually deal with Heyting algebras but with their duals, obtained by reversing the order. They are often called **co-Heyting algebras** in the literature. Readers familiar with Heyting algebras will certainly find annoying this reversing of the order. We apologise for this discomfort but there are good reasons for doing so. Indeed, the present work has been entirely build on a geometric intuition coming from the fundamental example<sup>3</sup> of the lattice of all subvarieties of an algebraic variety, and their counterparts in real algebraic geometry. Such lattices are co-Heyting algebras,

---

<sup>3</sup>This geometric intuition also played a role in the very beginning of the study of Heyting algebras. Indeed, co-Heyting algebras were born Brouwerian lattices in the paper of McKinsey and Tarski [MT46] which originated much of the later interest in Heyting algebras. Also the introduction of “slices” in [Hos67] seems inspired by the same geometric intuition that we use in this paper. Indeed, the dual of the co-Heyting algebra of all subvarieties of an algebraic variety  $V$  belongs to the  $d + 1$ -th slice if and only if  $V$  has dimension  $\leq d$  (see [DJ] for more on this topic).

not Heyting algebras. To see how this intuition is used in finding the proofs (and then hidden while writing the proofs) look at figure 2 in lemma 4.2.

## 2 Other notation, definitions and prerequisites

We denote by  $\mathcal{L}_{\text{lat}} = \{\mathbf{0}, \mathbf{1}, \vee, \wedge\}$  the language of distributive bounded lattice, these four symbols referring respectively to the least element, the greatest element, the join and meet operations.  $\mathcal{L}_{\text{HA}} = \mathcal{L}_{\text{lat}} \cup \{\rightarrow\}$  and  $\mathcal{L}_{\text{HA}^*} = \mathcal{L}_{\text{lat}} \cup \{-\}$  are the language of Heyting algebras and co-Heyting algebras respectively. Finite joins and meets will be denoted  $\bigvee$  and  $\bigwedge$ , with the natural convention that the join (resp. meet) of an empty family of elements is  $\mathbf{0}$  (resp.  $\mathbf{1}$ ).

The logical connectives ‘and’, ‘or’, and their iterated forms will be denoted  $\bigwedge$ ,  $\bigvee$ ,  $\bigwedge\bigwedge$  and  $\bigvee\bigvee$  respectively.

We denote by  $\mathcal{I}^\vee(L)$  the set of **join irreducible** elements of a lattice  $L$ , that is the elements  $a$  of  $L$  which can not be written as the join of any finite subset of  $L$  not containing  $a$ . Of course  $\mathbf{0}$  is never join irreducible since it is the join of the empty subset of  $L$ .

**Dualizing rules.** In order to help the reader more familiar with Heyting algebras than co-Heyting algebras, we recommend the use of the following conversion rules. For any ordered set  $L$  we denote by  $L^*$  the **dual** of  $L$ , that is the same set with the reverse order. If  $a$  is an element of  $L$  we denote by  $a^*$  the same element *seen as an element of  $L^*$* , so that we can write for any  $a, b \in L$ :

$$a \leq b \iff b^* \leq a^*$$

Indeed the star indicates that the second symbol  $\leq$  refers to the order of  $L^*$ , and the first one to the order of  $L$ . Similarly if  $L$  is a co-Heyting algebra we can write:

$$\begin{aligned} \mathbf{0}^* &= \mathbf{1} \quad \text{and} \quad \mathbf{1}^* = \mathbf{0} \\ (a \vee b)^* &= a^* \wedge b^* \quad \text{and} \quad (a \wedge b)^* = a^* \vee b^* \end{aligned}$$

The minus operation of  $\mathcal{L}_{\text{HA}^*}$  stands of course for the dual of the arrow operation of  $\mathcal{L}_{\text{HA}}$ , but beware of the order of the operands:

$$a - b = \min\{c : a \leq b \vee c\} = (b^* \rightarrow a^*)^*$$

The topological symmetric difference is defined as:

$$a \triangle b = (a - b) \vee (b - a) = (a^* \leftrightarrow b^*)^*$$

We will make extensive use of the following relation:

$$b \ll a \iff a - b = a \geq b$$

Note that  $b \ll a$  and  $b \not\ll a$  if and only if  $a = b = \mathbf{0}$ , hence  $\ll$  is a strict order on  $L \setminus \{\mathbf{0}\}$ . Note also that if  $a$  is join irreducible in  $L$  (hence non zero) then  $b \ll a$  if and only if  $b < a$ .

**The varieties of Maksimova.** We can now describe the varieties  $\mathcal{H}_1$  to  $\mathcal{H}_8$  introduced by Maksimova, more exactly the corresponding varieties  $\mathcal{V}_1$  to  $\mathcal{V}_8$  of co-Heyting algebras. Note that the intuitionistic negation  $\neg\varphi$  being defined as  $\varphi \rightarrow \perp$ , the corresponding operation in co-Heyting algebras is  $\mathbf{1} - a$ :

$$\mathbf{1} - a = (a^* \rightarrow \mathbf{1}^*)^* = (a^* \rightarrow \mathbf{0})^* = (\neg(a^*))^*$$

- $\mathcal{V}_1$  is the variety of all co-Heyting algebras.
- $\mathcal{V}_2 = \mathcal{V}_1 + [(\mathbf{1} - x) \wedge (\mathbf{1} - (\mathbf{1} - x)) = \mathbf{0}]$  is the dual of the variety of the logic of the weak excluded middle ( $\neg x \vee \neg\neg x = \mathbf{1}$ ).
- $\mathcal{V}_3 = \mathcal{V}_1 + [(((\mathbf{1} - x) \wedge x) - y) \wedge y = \mathbf{0}]$  is the dual of the second slice of Hosoi. With the terminology of [DJ],  $\mathcal{V}_3$  is the variety of co-Heyting algebras of dimension  $\leq 1$ . So a co-Heyting algebra  $L$  belongs to  $\mathcal{V}_3$  if and only if any prime filter of  $L$  which is not maximal is minimal (with respect to inclusion, among the prime filters of  $L$ ).
- $\mathcal{V}_4 = \mathcal{V}_3 + [(x - y) \wedge (y - x) \wedge (x \triangle (\mathbf{1} - y)) = \mathbf{0}]$  is the variety generated by the co-Heyting algebra  $\mathbf{L}_5$  (see figure 1).
- $\mathcal{V}_5 = \mathcal{V}_2 + [(((\mathbf{1} - x) \wedge x) - y) \wedge y = \mathbf{0}]$  is the variety generated by  $\mathbf{L}_3$  (see figure 1).
- $\mathcal{V}_6 = \mathcal{V}_1 + [(x - y) \wedge (y - x) = \mathbf{0}]$  is the variety generated by the chains.
- $\mathcal{V}_7 = \mathcal{V}_1 + [(1 - x) \wedge x = \mathbf{0}]$  is the variety of boolean algebras (which are exactly the co-Heyting algebras of dimension  $\leq 0$ ).
- $\mathcal{V}_8$  is the trivial variety  $\mathcal{V}_1 + [\mathbf{1} = \mathbf{0}]$  reduced to  $\mathbf{L}_1$  (see figure 1).

Note that the product of an empty family of co-Heyting algebras is just  $\mathbf{L}_1$ .

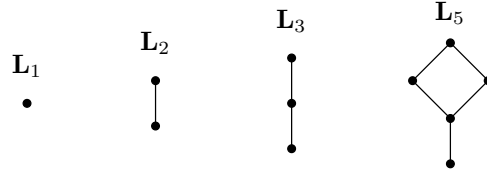


Figure 1: Four basic co-Heyting algebras

**Model-completion and super-intuitionistic logics.** For an introduction to the basic notions of first-order model-theory (language, formulas, elementary classes of structures, models and existentially closed models of a theory) we refer the reader to any introductory book, such as [Hod97] or [CK90].

Every model of a universal theory  $T$  embeds in an existentially closed model. If the class of all existentially closed models of  $T$  is elementary, then the corresponding theory  $\overline{T}$  is called the **model companion** of  $T$ . The model companion eliminates quantifiers if and only if  $T$  has the amalgamation property, in which case  $\overline{T}$  is called the **model completion** of  $T$ . By abuse of language we will speak of the model completion of a variety in place of the model completion of the theory of this variety.

It is an elementary fact that formulas in the first order intuitionistic propositional calculus (IPC<sup>1</sup>) can be considered as terms (in the usual model-theoretical sense) in the language of Heyting algebras, and formulas in the second order intuitionistic propositional calculus (IPC<sup>2</sup>) as first order formulas in the language of Heyting algebras. In particular, if a variety of Heyting algebras has a model completion then it appears, following [GZ97] that the corresponding super-intuitionistic logic has the property that IPC<sup>2</sup> is interpretable in IPC<sup>1</sup>, in the sense of Pitts [Pit92].

Finally let us recall the criterion for model completion which makes the link with theorems 1.1 and 1.2.

**Fact 2.1** *A theory  $\overline{T}$  is the model completion of a universal theory  $T$  if and only if it satisfies the two following conditions.*

1. *Every existentially closed model of  $T$  is a model of  $\overline{T}$ .*
2. *Given a model  $L$  of  $\overline{T}$ , a finitely generated substructure  $L_0$  of  $L$  and a finitely generated model  $L_1$  of  $T$  containing  $L_0$ , there is an embedding of  $L_1$  into an elementary extension of  $L$  which fixes  $L_0$  pointwise.*

**The finite model property.** A variety  $\mathcal{V}$  of co-Heyting algebras has the finite model property if any equation valid on every finite algebra in  $\mathcal{V}$  is valid on every algebra of  $\mathcal{V}$ . It is folklore (and otherwise a proof is given in [DJ] proposition 8.1) that this can be strengthened as follows:

**Fact 2.2** *A variety  $\mathcal{V}$  of co-Heyting algebra has the finite model property if for every quantifier-free  $\mathcal{L}_{\text{HA}^*}$ -formula  $\varphi(x)$  and every algebra  $L$  in  $\mathcal{V}$  such that  $L \models \exists x \varphi(x)$ , there exists a finite algebra  $L'$  in  $\mathcal{V}$  such that  $L' \models \exists x \varphi(x)$ .*

The finite model property holds obviously for every locally finite variety, but also<sup>4</sup> for  $\mathcal{V}_1$  and  $\mathcal{V}_2$ . We combine it with a bit of model-theoretic non-sense in the following lemmas.

**Lemma 2.3** *Let  $\mathcal{V}$  be a variety of co-Heyting algebras having the finite model property. Let  $\theta(x)$  and  $\phi(x, y)$  be quantifier-free  $\mathcal{L}_{\text{HA}^*}$ -formulas. Assume that for every finite co-Heyting algebra  $L_0$  and every tuple  $a$  of elements of  $L_0$  such that  $L_0 \models \theta(a)$ , there exists an extension  $L_1$  of  $L_0$  which satisfies  $\exists y \phi(a, y)$ . Then every algebra existentially closed in  $\mathcal{V}$  satisfies the following axiom:*

$$\forall x (\theta(x) \longrightarrow \exists y \phi(x, y))$$

---

<sup>4</sup>For example corollary 2.2.1 of [McK68] applies to  $\mathcal{V}_2$ , as well as to  $\mathcal{V}_1$ .

*Proof:* Let  $L$  be an existentially closed co-Heyting algebra which satisfies  $\theta(a)$  for some tuple  $a$ . Let  $\Sigma$  be its quantifier-free diagram, that is the set of all atomic and negatomic  $\mathcal{L}_{\text{HA}^*}$ -formulas (with parameters) satisfied in  $L$ . Let  $\Sigma_0$  be an arbitrary finite subset of  $\Sigma$ . The conjunction of  $\theta(a)$  and the elements of  $\Sigma_0$  is a quantifier-free formula (with parameters)  $\Delta(a, b)$ . Since  $L \models \exists x, y \Delta(x, y)$  and  $\mathcal{V}$  has the finite model property, by fact 2.2 there exists a finite co-Heyting algebra  $L_0$  and a tuple  $(a_0, b_0)$  of elements of  $L_0$  such that  $L_0 \models \Delta(a_0, b_0)$ . In particular  $L_0 \models \theta(a_0)$  hence by assumption  $L_0$  admits an extension  $L_1$  which satisfies  $\exists y \phi(a_0, y)$ . So  $L_1$  is a model of this formula and of  $\Sigma_0$  (because  $\Sigma_0$  is quantifier free and already satisfied in  $L_0$ ). We have proved that the union of  $\Sigma$  and  $\exists y \phi(a, y)$  is finitely satisfiable hence by the model-theoretic compactness theorem, it admits a model  $L'$  in which  $L$  embeds naturally. Since  $L$  is existentially closed it follows that  $L$  itself satisfies  $\exists y \phi(a, y)$ .  
■

**Lemma 2.4** *Let  $\mathcal{V}$  be a variety of co-Heyting algebras having the finite model property. Let  $\theta'(x)$  and  $\phi'(x, y)$  be  $\mathcal{L}_{\text{HA}^*}$ -formulas that are conjunctions of equations. Assume that:*

1. *There is a subclass  $\mathcal{C}$  of  $\mathcal{V}$  such that a finite co-Heyting algebra belongs to  $\mathcal{V}$  if and only if it embeds into the direct product of a finite (possibly empty) family of algebras in  $\mathcal{C}$ .*
2. *For every algebra  $L$  in  $\mathcal{C}$  and every  $a = (a_1, \dots, a_m) \in L^m$  such that  $L \models \theta'(a)$  there is an extension  $L'$  of  $L$  in  $\mathcal{V}$  and some  $b = (b_1, \dots, b_n) \in L'^n$  such that  $L' \models \phi'(a, b)$ . If moreover  $a_1 \neq \mathbf{0}$  then one can require all the  $b_i$ 's to be non zero.*

*Then every algebra existentially closed in  $\mathcal{V}$  satisfies:*

$$\forall x \left[ \left( \theta'(x) \wedge x_1 \neq \mathbf{0} \right) \rightarrow \exists y \left( \phi'(x, y) \wedge \bigwedge_{i \leq n} y_i \neq \mathbf{0} \right) \right]$$

Although somewhat tedious, this lemma will prove to be helpful for the varieties  $\mathcal{H}_2$ ,  $\mathcal{H}_4$ ,  $\mathcal{H}_5$  and  $\mathcal{H}_6$ .

*Proof:* Let  $L$  be a finite algebra in  $\mathcal{V}$  and  $a = (a_1, \dots, a_m) \in L^m$ . Assume that  $L \models \theta'(a) \wedge a_1 \neq \mathbf{0}$ .

By assumption there are  $L_1, \dots, L_r$  in  $\mathcal{C}$  such that  $L$  embeds into the direct product of the  $L_j$ 's. So each  $a_i$  can be identified with  $(a_i^1, \dots, a_i^r) \in L_1 \times \dots \times L_r$ .

For every  $j \leq r$  let  $a^j = (a_1^j, \dots, a_m^j) \in L_j^m$ . Since  $\theta'(x)$  is a conjunction of equations and  $L \models \theta'(a)$ , we have  $L_j \models \theta'(a^j)$ . Thus by assumption there is an extension  $L'_j$  of  $L_j$  and a tuple  $b^j = (b_1^j, \dots, b_n^j) \in L_j'^n$  such that  $L'_j \models \phi'(a^j, b^j)$ . Moreover if  $a_1^j \neq \mathbf{0}$  then we do require  $b_i^j \neq \mathbf{0}$  for every  $i \leq n$ .

Let  $L'$  be the direct product of the  $L'_j$ 's. For every  $i \leq n$  let  $b_i = (b_i^1, \dots, b_i^r)$  and  $b = (b_1, \dots, b_n) \in L'^n$ . The algebra  $L'$  is an extension of  $L$  in  $\mathcal{V}$ , and since  $\phi'(x, y)$  is a conjunction of equation by construction  $L' \models \phi'(a, b)$ .

Moreover  $a_1 = (a_1^1, \dots, a_1^r)$  is non zero, so there is an index  $j \leq r$  such that  $a_1^j \neq 0$ . Then by construction for every  $i \leq n$ ,  $b_i^j \neq \mathbf{0}$  hence  $b_i$  is non zero.

So we can apply lemma 2.3 to the variety  $\mathcal{V}$  with the quantifier free formulas  $\theta(x)$  and  $\phi(x, y)$  defined by:

$$\theta(x) \equiv \theta'(x) \bigwedge x_1 \neq \mathbf{0} \quad \text{and} \quad \phi(x, y) \equiv \phi'(x, y) \bigwedge \bigwedge_{i \leq n} y_i \neq \mathbf{0}$$

■

**Decreasing subsets.** For any element  $a$  and any subset  $A$  of an ordered set  $X$  we let:

$$a^\downarrow = \{b \in X : b \leq a\} \quad \text{and} \quad A^\downarrow = \bigcup_{a \in A} a^\downarrow$$

A decreasing subset of  $X$  is a subset such that  $A = A^\downarrow$ . The family  $\mathcal{L}^\downarrow(E)$  of all decreasing subsets of  $E$  are the closed sets of a topology on  $E$ , hence a co-Heyting algebra with operations:

$$A \vee B = A \cup B \quad A \wedge B = A \cap B \quad A - B = (A \setminus B)^\downarrow$$

Its completely join irreducible elements are precisely the decreasing sets  $x^\downarrow$  for  $x$  ranging over  $E$ .

It is folklore that if  $L$  is a finite co-Heyting algebra, then the map  $\iota_L : a \mapsto a^\downarrow \cap \mathcal{I}^\vee(L)$  is an  $\mathcal{L}_{\text{HA}^*}$ -isomorphism from  $L$  to the family  $\mathcal{L}^\downarrow(\mathcal{I}^\vee(L))$  of all decreasing subsets of  $\mathcal{I}^\vee(L)$ , whose inverse is the map  $A \mapsto \mathbb{W}A$ . This provides a flexible tool to construct extensions of a finite co-Heyting algebra with prescribed conditions.

**Proposition 2.5** *Let  $L$  be a finite co-Heyting algebra and  $\mathcal{I}$  an ordered set. Assume that there is an increasing map  $\pi$  from  $\mathcal{I}$  onto  $\mathcal{I}^\vee(L)$  such that for every  $\zeta \in \mathcal{I}$  and every  $x \in \mathcal{I}^\vee(L)$ :*

$$\pi(\zeta) \leq x \Rightarrow \exists \xi \in \mathcal{I}, \pi(\xi) = x \text{ and } \zeta \leq \xi$$

*Then there exists a unique  $\mathcal{L}_{\text{HA}^*}$ -embedding  $\varphi$  of  $L$  into  $\mathcal{L}^\downarrow(\mathcal{I})$  such that<sup>5</sup>  $\pi(\varphi(a)) = a^\downarrow \cap \mathcal{I}^\vee(L)$  for every  $a \in L$ .*

*Proof:* Let  $L^\downarrow$  denote  $\mathcal{L}^\downarrow(\mathcal{I}^\vee(L))$  and  $L'^\downarrow$  denote  $\mathcal{L}^\downarrow(\mathcal{I})$ . The map  $\varphi^\downarrow : L^\downarrow \rightarrow L'^\downarrow$  defined by  $\varphi^\downarrow(A) = \pi^{-1}(A)$  is clearly a morphism of bounded lattices ( $\varphi^\downarrow(A) \in L'^\downarrow$  for every  $A \in L^\downarrow$  because  $\pi$  is increasing) which is injective because  $\pi$  is surjective. Clearly  $\pi(\varphi^\downarrow(A)) = A$  for every  $A \in L^\downarrow$  so it remains to check that  $\varphi^\downarrow$  is an  $\mathcal{L}_{\text{HA}^*}$ -embedding of  $L^\downarrow$  into  $L'^\downarrow$  in order to conclude that  $\varphi : a \mapsto \varphi^\downarrow(a^\downarrow)$  is the required  $\mathcal{L}_{\text{HA}^*}$ -embedding of  $L$  into  $\mathcal{L}^\downarrow(\mathcal{I})$ .

<sup>5</sup>Note that the compositum  $\pi \circ \varphi$  is not defined. In this proposition  $\varphi(a)$  is a decreasing subset of  $\mathcal{I}$  and  $\pi(\varphi(a)) = \{\pi(\xi) : \xi \in \varphi(a)\}$ .



Let  $A, B \in L^\downarrow$  and choose any  $\zeta \in \varphi^\downarrow(A - B) = \pi^{-1}((A \setminus B)^\downarrow)$ . Then  $\pi(\zeta) \leq x$  for some  $x \in A \setminus B$ . By assumption we can find  $\xi \in \mathcal{I}^\vee(L')$  such that  $\pi(\xi) = x$  and  $\zeta \leq \xi$  so  $\zeta \in \pi^{-1}(A \setminus B)^\downarrow$ .

$$\pi^{-1}(A \setminus B)^\downarrow = (\pi^{-1}(A) \setminus \pi^{-1}(B))^\downarrow = (\varphi^\downarrow(A) \setminus \varphi^\downarrow(B))^\downarrow = \varphi^\downarrow(A) - \varphi^\downarrow(B)$$

We conclude that  $\varphi^\downarrow(A - B) \subseteq \varphi^\downarrow(A) - \varphi^\downarrow(B)$ . The reverse inclusion is immediate since  $\varphi^\downarrow(A) \subseteq \varphi^\downarrow((A - B) \cup B) = \varphi^\downarrow(A - B) \cup \varphi^\downarrow(B)$  implies that  $\varphi^\downarrow(A) - \varphi^\downarrow(B) \subseteq \varphi^\downarrow(A - B)$ . So  $\varphi^\downarrow(A - B) = \varphi^\downarrow(A) - \varphi^\downarrow(B)$ .

■

### 3 Minimal finite extensions

This section is devoted to the study of minimal finite proper extensions of a finite co-Heyting algebra  $L_0$ . We are going to show (see remark 3.3 below) that they are in one-to-one correspondence with what we call **signatures** in  $L_0$ , that is triples  $(g, H, r)$  such that  $g$  is a join irreducible element of  $L_0$ ,  $H = \{h_1, h_2\}$  is a set of one or two elements of  $L_0$  and:

- either  $r = 1$  and  $h_1 = h_2 < g$ ;
- or  $r = 2$  and  $h_1 \vee h_2$  is the unique predecessor of  $g$  (both possibilities,  $h_1 = h_2$  and  $h_1 \neq h_2$  may occur in this case).

Let  $L$  be an  $\mathcal{L}_{\text{HA}^*}$ -extension of  $L_0$  and  $x \in L$ . We introduce the following notation.

- For every  $a \in L_0$ ,  $a^- = \mathbb{W}\{b \in L_0 : b < a\}$ .
- $L_0\langle x \rangle$  denotes the  $\mathcal{L}_{\text{HA}^*}$ -substructure of  $L$  generated by  $L_0 \cup \{x\}$ .
- $g(x, L_0) = \mathbb{A}\{a \in L_0 : x \leq a\}$ .

Clearly  $a \in \mathcal{I}^\vee(L_0)$  if and only if  $a^-$  is the unique predecessor of  $a$  in  $L_0$  (otherwise  $a^- = a$ ). We say that a tuple  $(x_1, x_2)$  of elements of  $L$  is **primitive over**  $L_0$  if they are both<sup>6</sup> not in  $L_0$  and there exists  $g \in \mathcal{I}^\vee(L_0)$  such that:

1.  $g^- \wedge x_1$  and  $g^- \wedge x_2$  belong to  $L_0$ .
2. One of the following holds:
  - $x_1 = x_2$  and  $g^- \wedge x_1 \ll x_1 \ll g$ .
  - $x_1 \neq x_2$  and  $x_1 \wedge x_2 \in L_0$ ,  $g - x_1 = x_2$ ,  $g - x_2 = x_1$ .

As it will become clear after proposition 3.4 there are two different sorts of minimal extension, which correspond to the two sorts of primitive tuples above, as well as with the two sorts of signatures in  $L_0$ .

<sup>6</sup>The other conditions imply that  $x_1$  and  $x_2$  do not belong to  $L_0$ , provided they are both non zero.

**Example 3.1**  $\mathbf{L}_2 \subset \mathbf{L}_2$  is an minimal extension of the first kind, with signature  $(\mathbf{1}, \{\mathbf{0}\}, 1)$ .  $\mathbf{L}_2 \subset \mathbf{L}_2 \times \mathbf{L}_2$  and  $\mathbf{L}_3 \subset \mathbf{L}_5^*$  are minimal extensions of the second kind, with signatures  $(\mathbf{1}, \{\mathbf{0}\}, 2)$  and  $(\mathbf{1}, \{a, b\}, 2)$  respectively (where  $a, b$  are the atoms of  $\mathbf{L}_5^*$ ).

**Lemma 3.2** *Let  $(x_1, x_2) \in L^2$  be a primitive tuple over  $L_0$ , and  $g \in L_0$  be as in the definition. Then  $g = g(x_1, L_0) = g(x_2, L_0)$ .*

*Proof:* Clearly  $g(x_i, L_0) \leq g$  because  $x_i \leq g$ . On the other hand  $g(x_i, L_0) \not\leq g$  since otherwise  $g(x_i, L_0) \leq g^-$  hence *a fortiori*  $x_i \leq g^-$  and finally  $x_i = x_i \wedge g^- \in L_0$ , a contradiction.

■

So  $(g, \{g^- \wedge x_1, g^- \wedge x_2\}, \text{Card}\{x_1, x_2\})$  is a signature in  $L_0$  which is determined by  $(x_1, x_2)$  (actually by any of  $x_1, x_2$ ). We call it the **signature of the tuple**  $(x_1, x_2)$  (or simply of  $x_1$ ).

**Remark 3.3** Proposition 3.6 below implies that every minimal proper extension of a finite co-Heyting algebra  $L_0$  is generated over  $L_0$  by a primitive tuple (which is unique up to switch of the elements by proposition 3.4). Conversely it follows almost immediately from proposition 3.4 that every extension generated over  $L_0$  by a primitive tuple is minimal among the proper extensions of  $L_0$ . In section 4 we will prove that every signature in  $L_0$  is the signature of an extension generated over  $L_0$  by a primitive tuple (see remark 4.5), and this extension is unique up to isomorphism over  $L_0$  by corollary 3.5. So altogether this yields a one-to-one correspondence between the signatures in  $L_0$  and the minimal extensions of  $L_0$  up to isomorphism over  $L_0$ .

**Proposition 3.4** *Let  $L_0$  be a finite co-Heyting algebra,  $L$  an extension generated over  $L_0$  by a primitive tuple  $(x_1, x_2)$ , and let  $g = g(x_1, L_0)$ .*

*Then  $L$  is exactly the upper semi-lattice generated over  $L_0$  by  $x_1, x_2$ . It is a finite co-Heyting algebra and one of the following holds:*

1.  $x_1 = x_2$  and  $\mathcal{I}^\vee(L) = \mathcal{I}^\vee(L_0) \cup \{x_1\}$ .
2.  $x_1 \neq x_2$  and  $\mathcal{I}^\vee(L) = (\mathcal{I}^\vee(L_0) \setminus \{g\}) \cup \{x_1, x_2\}$ .

It follows that there is at most one primitive tuple which generates  $L$  over  $L_0$ . When this happens, we call the signature of this tuple simply the **signature of  $L$  in  $L_0$** . Before going any further, let us point out that conversely the signature of  $L$  in  $L_0$  *determines*  $L$ , up to isomorphism over  $L_0$ .

**Corollary 3.5** *Let  $L_1, L_2$  be two finite co-Heyting algebras both generated over a common subalgebra by a primitive tuple. If these tuples have the same signature in  $L_0$  then they are isomorphic over  $L_0$  (that is to say, there exists an isomorphism of co-Heyting algebra from  $L_1$  to  $L_2$  which fixes  $L_0$  pointwise).*

*Proof:* Assume that  $L_i$  is generated over  $L_0$  by a primitive tuple  $(x_{i,1}, x_{i,2})$  for  $i = 1, 2$  having the same signature  $(g, H, r)$  in  $L_0$ . By definition of  $H$ , changing if necessary the numbering of the  $x_{2,j}$ 's we can assume that for  $i = 1, 2$ :

$$g^- \wedge x_{1,j} = g^- \wedge x_{2,j} \quad (1)$$

By definition of  $r$ ,  $x_{1,1} \neq x_{1,2}$  if and only if  $x_{2,1} \neq x_{2,2}$ . By definition of  $g$  and by proposition 3.4 there exists a (unique) bijection  $\sigma$  from  $\mathcal{I}^\vee(L_1)$  to  $\mathcal{I}^\vee(L_2)$  which fixes  $\mathcal{I}^\vee(L_0)$  pointwise and maps each  $x_{1,j}$  to  $x_{2,j}$ . Moreover for any  $a \in \mathcal{I}^\vee(L_1)$  and any  $j = 1, 2$  we have, thanks to (1) above:

$$\begin{aligned} x_{1,j} < a &\iff a \in \mathcal{I}^\vee(L_0) \text{ and } g \leq a \\ &\iff x_{2,j} < a \end{aligned}$$

And symmetrically:

$$\begin{aligned} a < x_{1,j} &\iff a \in \mathcal{I}^\vee(L_0) \text{ and } a \leq g^- \wedge x_{1,j} \\ &\iff a \in \mathcal{I}^\vee(L_0) \text{ and } a \leq g^- \wedge x_{2,j} \\ &\iff a < x_{2,j} \end{aligned}$$

So  $\sigma$  is an order preserving bijection. Because every element in a finite distributive lattice is the join of its uniquely determined join irreducible components, it follows that  $\sigma$  uniquely extends to an isomorphism of upper-semilattices  $\varphi : L_1 \rightarrow L_2$  in the obvious way. This is actually an isomorphism of co-Heyting algebras because it is an order preserving bijection and all the  $\mathcal{L}_{\text{HA}^*}$ -structure is determined by the order.

■

We turn now to the proof of proposition 3.4.

*Proof:* Let  $L_1$  be the upper semi-lattice generated over  $L_0$  by  $x_1, x_2$ . In order to show that  $L_1 = L$  it is sufficient to check that  $L_1$  is an  $\mathcal{L}_{\text{HA}^*}$ -substructure of  $L$ . Because  $x_i \leq g$  and  $x_i \wedge g^-$  belongs to  $L_0$  by assumption,  $L_1$  is easily seen to be a sublattice of  $L$ .

Let  $a$  be any element of  $L_0$ . We first check that  $a - x_1 \in L_1$ . If  $g \leq a$  then  $(a \wedge g) - x_1 = g - x_1$  is either  $g$  or  $x_2$ . If  $g \not\leq a$  then  $a \wedge g \leq g^-$  hence  $a \wedge g \wedge x_1$  belongs to  $L_0$  hence so does  $(a \wedge g) - x_1 = (a \wedge g) - (a \wedge g \wedge x_1)$ . So in any case  $(a \wedge g) - x_1$  belongs to  $L_1$ . On the other hand  $(a - g) - x_1 = a - g$  belongs to  $L_0$  so:

$$\begin{aligned} a - x_1 &= [(a \wedge g) \vee (a - g)] - x_1 \\ &= [(a \wedge g) - x_1] \vee [(a - g) - x_1] \in L_1 \end{aligned}$$

Now we check that  $x_1 - a \in L_1$ . This is clear if  $x_1 \leq a$ . If  $x_1 \not\leq a$  then  $g \not\leq a$  hence  $x_1 \wedge a \leq g^-$ . It follows that:

$$x_1 \wedge a \leq g^- \wedge x_1 \ll x_1$$

Indeed  $g^- \wedge x_1 \ll x_1$  by assumption if  $x_1 = x_2$ , and because otherwise  $g^- \wedge x_1 < g$  hence  $g^- \wedge x_1 \ll g$  since  $g$  is join irreducible. So  $x_1 - a = x_1$  belongs to  $L_1$ .

Symmetrically  $a - x_2$  and  $x_2 - a$  belong to  $L_1$ .

Any two elements in  $L_1$  can be written  $a \vee y$  and  $a' \vee y'$  with  $a, a'$  in  $L_0$  and  $y, y'$  in  $\{\mathbf{0}, x_1, x_2\}$  hence their difference is:

$$\begin{aligned} (a \vee y) - (a' \vee y') &= [a - (a' \vee y')] \vee [y - (a' \vee y')] \\ &= [(a - a') - y'] \vee [(y - y') - a'] \end{aligned}$$

$a - a' \in L_0$  hence  $(a - a') - y' \in L_1$  by the previous computations, and similarly  $(y - y') - a' \in L_1$  because  $y - y'$  is either  $\mathbf{0}, x_1$  or  $x_2$ , indeed:

$$x_1 - x_2 = (x_1 \vee x_2) - x_2 = g - x_2 = x_1$$

It follows that  $L_1$  is an  $\mathcal{L}_{\text{HA}^*}$ -substructure of  $L$  hence  $L_1 = L$  since it contains  $L_0$  and  $x_1, x_2$ .

We turn now to the description of  $\mathcal{I}^\vee(L)$ . Since  $L_0$  is finite and  $L$  is generated by  $L_0 \cup \{x_1, x_2\}$  as an upper semi-lattice, it follows immediately that  $L$  is finite and:

$$\mathcal{I}^\vee(L) \subseteq \mathcal{I}^\vee(L_0) \cup \{x_1, x_2\} \quad (2)$$

If  $x_1 \neq x_2$  then of course  $g = x_1 \vee x_2 \notin \mathcal{I}^\vee(L)$ . Conversely if  $x_1 = x_2$  then (2) implies that  $g$  is  $\vee$ -irreducible in  $L$ , so:

$$g \in \mathcal{I}^\vee(L) \iff x_1 \neq x_2 \quad (3)$$

Assume that  $\mathcal{I}^\vee(L_0) \not\subseteq \mathcal{I}^\vee(L)$ . Let  $b \in \mathcal{I}^\vee(L_0) \setminus \mathcal{I}^\vee(L)$  and let  $y_1, \dots, y_r$  ( $r \geq 2$ ) be its  $\vee$ -irreducible components in  $L$ . By (2), each  $y_i$  either belongs to  $L_0$  or to  $\{x_1, x_2\}$ , and at least one of them does not belong to  $L_0$ . We may assume without loss of generality that  $y_1 = x_1$ . Then  $x_1 \leq b$  hence  $g \leq b$ . If  $g < b$  then  $g \ll b$  since  $b \in \mathcal{I}^\vee(L_0)$  so  $b - g = b$ , but then we have a contradiction:

$$y_1 \leq b - g \leq b - x_1 = \bigvee_{i=2}^r y_i$$

So  $b = g$ . We have proved that:

$$\mathcal{I}^\vee(L_0) \setminus \{g\} \subseteq \mathcal{I}^\vee(L) \quad (4)$$

The conclusion follows by combining (2), (3), (4) with the fact that  $\mathcal{I}^\vee(L_0) \neq \mathcal{I}^\vee(L)$  because  $L$  is a proper extension of  $L_0$ .

■

**Proposition 3.6** *Any finite proper extension  $L$  of a finite co-Heyting algebra  $L_0$  is the union of a finite tower of extensions of  $L_0$ , each of which is generated by a primitive tuple over the preceding one.*

*Proof:* It suffices to show that  $L$  contains a primitive tuple  $(x_1, x_2)$  over  $L_0$ , since then either  $L = L_0\langle x_1 \rangle$  and we are done, or one can replace  $L_0$  by  $L_0\langle x_1 \rangle$  and repeat the argument (this process must stop after finitely many steps since  $L$  is finite).

So let's take any element  $x$  minimal in  $\mathcal{T}^\vee(L) \setminus L_0$ . Observe that if  $y$  is any element of  $L$  strictly smaller than  $x$ , then all the  $\vee$ -irreducible components of  $y$  in  $L$  actually belong to  $L_0$  (by minimality of  $x$ ) so  $y \in L_0$ .

Let  $g = g(x, L_0)$ . For every  $a \in L_0$ , if  $a < g$  then  $x \not\leq a$  hence  $a \wedge x < x$ , so  $a \wedge x \in L_0$ . It follows that  $g^- \wedge x \in L_0$ . In particular  $g^- \neq g$  hence  $g \in \mathcal{T}^\vee(L_0)$ .

Moreover  $g^- \wedge x < x$  since  $x \notin L_0$ , hence  $g^- \wedge x \ll x$  because  $x$  is join irreducible in  $L$ . So in the case when  $x \ll g$  we have proved that  $(x, x)$  is primitive over  $L_0$ .

On the other hand, when  $x \not\ll g$  then  $g - (g - x) = x$ , indeed:

$$g - (g - x) = (x \vee (g - x)) - (g - x) = x - (g - x)$$

The last term is either  $\mathbf{0}$  or  $x$  due to the join irreducibility of  $x$ . But it cannot be  $\mathbf{0}$  since  $x \leq g - x$  would imply that  $g = g - x$  hence  $x \ll g$ , a contradiction. So when  $x \not\ll g$  we have proved that  $(x, g - x)$  is primitive over  $L_0$ .

■

## 4 Density and splitting in $\mathcal{V}_1$

For the variety  $\mathcal{V}_1$  of all co-Heyting algebras we introduce the following axioms D1 and S1.

**[Density D1]** For every  $a, c$  such that  $c \ll a \neq \mathbf{0}$  there exists a non zero element  $b$  such that:

$$c \ll b \ll a$$

**[Splitting S1]** For every  $a, b_1, b_2$  such that  $b_1 \vee b_2 \ll a \neq \mathbf{0}$  there exists non zero elements  $a_1$  and  $a_2$  such that:

$$\begin{aligned} a - a_2 &= a_1 \geq b_1 \\ a - a_1 &= a_2 \geq b_2 \\ a_1 \wedge a_2 &= b_1 \wedge b_2 \end{aligned}$$

Note that  $a = a_1 \vee a_2$ , so the second axioms allows to split  $a$  in two pieces  $a_1, a_2$  along  $b_1 \wedge b_2$  (so the name of “splitting”).

**Lemma 4.1** *Let  $a, c$  be two elements of a finite co-Heyting algebra  $L$ . If  $c \ll a \neq \mathbf{0}$  then there exists a finite co-Heyting algebra  $L'$  containing  $L$  and a non zero element  $b$  in  $L'$  such that:*

$$c \ll b \ll a$$

*Proof:* Let  $a_1, \dots, a_r$  be the join irreducible components of  $a$ . The idea of the proof is to add a new  $\vee$ -irreducible element  $\alpha_i$  immediately below each  $a_i$ . Let  $\mathcal{I}$  be the set  $\mathcal{I}^\vee(L)$  augmented by  $r$  new elements  $\alpha_1, \dots, \alpha_r$ . Extend the order of  $\mathcal{I}^\vee(L)$  to  $\mathcal{I}$  as follows. The  $\alpha_i$ 's are two by two incomparable, and for every  $x \in \mathcal{I}^\vee(L)$  and every  $i \leq r$ :

$$\begin{aligned} x < \alpha_i &\Leftrightarrow x < a_i \\ \alpha_i < x &\Leftrightarrow a_i \leq x \end{aligned}$$

For every  $\xi \in \mathcal{I}$  let:

$$\pi(\xi) = \begin{cases} x & \text{if } \xi = x \text{ for some } x \in L \\ a_i & \text{if } \xi = \alpha_i \text{ for some } i \leq r \end{cases}$$

This is an increasing projection of  $\mathcal{I}$  onto  $\mathcal{I}^\vee(L)$ . For every  $\zeta \in \mathcal{I}$  and every  $x \in \mathcal{I}^\vee(L)$  such that  $\pi(\zeta) \leq x$  there exists  $\xi \in \mathcal{I}$  such that  $\pi(\xi) = x$  and  $\zeta \leq \xi$ : simply take  $\xi = x$ . Thus proposition 2.5 gives an  $\mathcal{L}_{\text{HA}^*}$ -embedding  $\varphi$  of  $L$  into  $\mathcal{L}^\downarrow(\mathcal{I})$ .

Each join irreducible element  $x$  of  $L$  smaller than  $c$  is strictly smaller than some join irreducible component  $a_i$  of  $a$  because  $c \ll a$ . By construction  $x < \alpha_i < a_i$  in  $\mathcal{I}$  hence  $\varphi(x) < \alpha_i^\downarrow < \varphi(a_i)$ . These three elements of  $L'$  are join irreducible hence  $\varphi(x) \ll \alpha_i^\downarrow \ll \varphi(a_i)$ . It follows that:

$$\varphi(c) = \mathbb{W}\{\varphi(x) : x \in \mathcal{I}^\vee(L), x \leq c\} \ll \mathbb{W}_{1 \leq i \leq r} \alpha_i^\downarrow \ll \mathbb{W}_{1 \leq i \leq r} \varphi(a_i) = \varphi(a)$$

So we can take  $L' = \mathcal{L}^\downarrow(\mathcal{I})$  and  $b = \mathbb{W}_{1 \leq i \leq r} \alpha_i^\downarrow$ .

■

**Lemma 4.2** *Let  $a, b_1, b_2$  be elements of a finite co-Heyting algebra  $L$ . If  $b_1 \vee b_2 \ll a \neq \mathbf{0}$  then there exists a finite co-Heyting algebra  $L'$  containing  $L$  and non zero elements  $a_1, a_2$  in  $L'$  such that:*

$$\begin{aligned} a - a_2 &= a_1 \geq b_1 \\ a - a_1 &= a_2 \geq b_2 \\ a_1 \wedge a_2 &= b_1 \wedge b_2 \end{aligned}$$

The idea of the proof uses geometric intuition. Imagine that there exists an  $\mathcal{L}_{\text{HA}^*}$ -embedding  $\varphi$  of  $L$  into the co-Heyting algebra  $L(X)$  of all semi-algebraic closed subsets of some real semi-algebraic set  $X$ . It can be proved actually that such an embedding exists, and that moreover we can reduce to the case when  $\varphi(a)$  is equidimensional (that is its local dimension is the same at every point). So  $A = \varphi(a)$ ,  $B_1 = \varphi(b_1)$  and  $B_2 = \varphi(b_2)$  are closed semi-algebraic subsets of  $X$ . Let  $X_1 = X \setminus (B_2 \setminus B_1)$  and  $X_2 = X \setminus (B_1 \setminus B_2)$ . Glue two copies  $X'_1, X'_2$  of  $X_1$  and  $X_2$  along  $B_1 \cap B_2$ . The result  $X'$  of this glueing is a real semi-algebraic set which projects onto  $X$  in an obvious way. Figure 2 shows this construction when  $A = X$ .

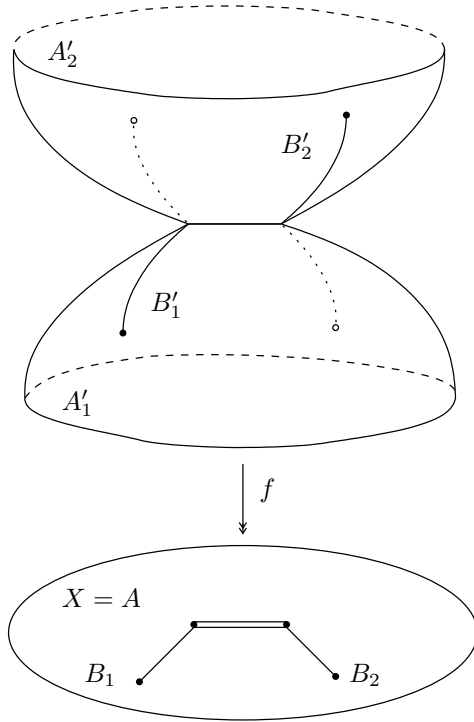


Figure 2: An example of glueing when  $A = X$ .  
The dotted lines represent cuts in the surfaces  $A'_1$  and  $A'_2$ .  
The double line represents  $B_1 \cap B_2$ .

This defines an embedding  $L(X) \hookrightarrow L(X')$  which maps any semi-algebraic subset  $Y$  of  $X$  closed in  $X$  to the preimage  $Y'$  of  $Y$  via this projection. Then  $A'$  is the union of a copy of  $A_1 = A \cap X_1$  and  $A_2 = A \cap X_2$  glued along  $B_1 \cap B_2$ . These copies  $A'_1, A'_2$  of  $A_1$  and  $A_2$  are non empty semi-algebraic subsets of  $X'$ , closed in  $X'$ , containing  $B'_1$  and  $B'_2$  respectively, such that:

$$A'_1 \cap A'_2 = B'_1 \cap B'_2$$

The additional property that  $A'_1$  (resp.  $A'_2$ ) is the topological closure in  $X'$  of  $A' \setminus A'_2$  (resp.  $A' \setminus A'_1$ ) then follows from the assumption that  $B_1 \cup B_2 \ll A$  and the fact that we reduced to the case when  $A'_1$  and  $A'_2$  are equidimensional.

*Proof:* The above geometric construction *is* a proof, provided an appropriate dictionary between real semi-algebraic sets and elements of co-Heyting algebras is given. However it would be longer to set explicitly this dictionary than to hide the geometric intuition in a shorter combinatorial proof. This is what we do now.

For each  $x \in \mathcal{I}^\vee(L)$  such that  $x \not\leq b_2$  (resp  $x \not\leq b_1$ ) let  $\xi_{x,1}$  (resp.  $\xi_{x,2}$ ) be a new symbol. For each  $x \in \mathcal{I}^\vee(L)$  such that  $x \leq b_1 \wedge b_2$  let  $\xi_{x,0}$  be a new symbol. Let  $\mathcal{I}$  be the set of all these symbols and define an order on  $\mathcal{I}$  as follows:

$$\xi_{y,j} \leq \xi_{x,i} \iff y \leq x \text{ and } \{i,j\} \neq \{1,2\}$$

The map  $\pi : \xi_{x,i} \mapsto x$  defines an increasing projection of  $\mathcal{I}$  onto  $\mathcal{I}^\vee(L)$ . For every  $\zeta \in \mathcal{I}$  and every  $x \in \mathcal{I}^\vee(L)$  such that  $\pi(\zeta) \leq x$  there exists  $\xi$  such that  $\pi(\xi) = x$  and  $\zeta \leq \xi$ . Indeed if  $\zeta = \xi_{y,j}$  for some  $j \leq 2$ , simply take  $\xi = \xi_{x,j}$ . Thus proposition 2.5 gives an  $\mathcal{L}_{\text{HA}^*}$ -embedding  $\varphi$  of  $L$  into  $\mathcal{L}^\downarrow(\mathcal{I})$ . For any  $x \in \mathcal{I}^\vee(L)$  we have:

$$\varphi(x) = \begin{cases} \xi_{x,0}^\downarrow & \text{if } x \leq b_1 \wedge b_2 \\ \xi_{x,1}^\downarrow & \text{if } x \leq b_1 \text{ and } x \not\leq b_2 \\ \xi_{x,2}^\downarrow & \text{if } x \leq b_2 \text{ and } x \not\leq b_1 \\ \xi_{x,1}^\downarrow \cup \xi_{x,2}^\downarrow & \text{otherwise.} \end{cases}$$

Let  $a_1, \dots, a_r$  be the join irreducible components of  $a$ . None of the  $a_i$ 's is smaller than  $b_1$  or  $b_2$  because by assumption  $b_1 \vee b_2 \ll a$ , so each  $\varphi(a_i) = \xi_{a_i,1}^\downarrow \cup \xi_{a_i,2}^\downarrow$ . Define:

$$\alpha_1 = \bigcup_{1 \leq i \leq r} \xi_{a_i,1}^\downarrow \quad \text{and} \quad \alpha_2 = \bigcup_{1 \leq i \leq r} \xi_{a_i,2}^\downarrow$$

By construction  $\varphi(a) - \alpha_1 = \alpha_2$  and  $\varphi(a) - \alpha_2 = \alpha_1$  and both are non empty since  $r \geq 1$  (here we use that  $a \neq \mathbf{0}$ ). Moreover, for any join irreducible element  $x$  of  $L$  such that  $x \leq b_1$ , we have  $x \leq a_j$  for some  $j \leq r$ . By definition of the order on  $\mathcal{I}$  it follows that  $\xi_{x,1} \leq \xi_{a_j,1}$  hence:

$$\varphi(x) = \xi_{x,1}^\downarrow \subseteq \xi_{a_j,1}^\downarrow \subseteq \alpha_1$$

It follows that  $\varphi(b_1) \subseteq \alpha_1$ , and symmetrically  $\varphi(b_2) \subseteq \alpha_2$ .

It remains to check that  $\alpha_1 \cap \alpha_2 = \varphi(b_1) \cap \varphi(b_2)$ . In order to do this, let  $\xi$  be any element of  $\mathcal{I}$  and  $x = \pi(\xi)$ . It is sufficient to prove that  $\xi^\downarrow \subseteq \alpha_1 \cap \alpha_2$  if and only if  $\xi^\downarrow \subseteq \varphi(b_1) \cap \varphi(b_2)$

If  $\xi^\downarrow \subseteq \varphi(b_1) \cap \varphi(b_2)$  then  $x \leq b_1 \wedge b_2$  hence  $\xi = \xi_{x,0}$  and  $x \leq a_i$  for some  $i \leq r$ . It follows that  $\xi_{x,0} \leq \xi_{a_i,1}$  so  $\xi^\downarrow \subseteq \alpha_1$ , and  $\xi_{x,0} \leq \xi_{a_i,2}$  so  $\xi^\downarrow \subseteq \alpha_2$ . With other words  $\xi^\downarrow \subseteq \alpha_1 \cap \alpha_2$ .

Conversely if  $\xi^\downarrow \subseteq \alpha_1 \cap \alpha_2$  then there exists  $i, j \leq r$  such that  $\xi^\downarrow \subseteq \xi_{a_i,1}^\downarrow$  and  $\xi^\downarrow \subseteq \xi_{a_j,2}^\downarrow$ . Thanks to the definition of the ordering on  $\mathcal{I}$  this implies that  $\xi = \xi_{x,0}$  hence  $x \leq b_1 \wedge b_2$  and so  $\xi^\downarrow \subseteq \varphi(b_1) \cap \varphi(b_2)$ .

■

**Theorem 4.3** *Every co-Heyting algebra existentially closed in  $\mathcal{V}_1$  satisfies the density axiom D1 and the splitting axiom S1.*



*Proof:* These two axioms can be written under the following form:

$$\forall x (\theta(x) \longrightarrow \exists y \phi(x, y))$$

where  $\theta(x)$  and  $\phi(x, y)$  are quantifier-free  $\mathcal{L}_{\text{HA}^*}$ -formulas. In both cases we have shown in lemmas 4.1 and 4.2 that for every finite co-Heyting algebra  $L$  and every tuple  $a$  of elements of  $L$  such that  $L \models \theta(a)$ , there exists an extension  $L'$  of  $L$  which satisfies  $\exists y \phi(a, y)$ . The result follows, by lemma 2.3.

■

Here is a partial converse of theorem 4.3.

**Theorem 4.4** *Let  $L$  be a co-Heyting algebra satisfying the density axiom D1 and the splitting axiom S1. Let  $L_0$  be a finite subalgebra of  $L$ . Let  $L_1$  be a finite co-Heyting algebra containing  $L_0$ . Then there exists an embedding of  $L_1$  into  $L$  which fixes every point of  $L_0$ .*

*Proof:* By an immediate induction based on proposition 3.6, we reduce to the case when  $L_1$  is generated over  $L_0$  by a primitive tuple. Let  $\sigma = (g, \{h_1, h_2\}, r)$  be the signature of  $L_1$  in  $L_0$ . By corollary 3.5 it is sufficient to prove that  $\sigma$  is the signature of a primitive tuple of elements  $x_1, x_2 \in L$ .

*Case 1:*  $r = 1$  so  $h_1 = h_2$ . Since  $h_1 \leq g^- \ll g$ , the splitting property S1 applied to  $g, g^-, h_1$  gives non zero elements  $y_1, y_2$  in  $L$  such that:

$$\begin{aligned} g - y_1 &= y_2 \geq g^- \\ g - y_2 &= y_1 \geq h_1 \\ y_1 \wedge y_2 &= h_1 \end{aligned}$$

We have  $y_1 - h_1 = (g - y_2) - h_1 = (g - h_1) - y_2 = g - y_2 = y_1$  hence  $h_1 \ll y_1$ . The density axiom D1 then gives  $x \in L \setminus \{0\}$  such that  $h_1 \ll x \ll y_1$ . By construction:

$$h_1 \leq g^- \wedge x \leq y_2 \wedge y_1 = h_1 \wedge h_2 \leq h_1$$

So  $g^- \wedge x = h_1 \in L_0$  and  $g^- \wedge x \ll x \ll g$ , from which it follows that  $(x, x)$  is a primitive tuple with signature  $(g, \{h_1\}, 1) = \sigma$  in  $L_0$ .

*Case 2:*  $r = 2$  so  $h_1 \vee h_2 = g^-$ . Since  $g^- \ll g$  the splitting property S1 applied to  $g, h_1, h_2$  gives non zero elements  $y_1, y_2$  in  $L$  such that:

$$\begin{aligned} g - y_1 &= y_2 \geq h_2 \\ g - y_2 &= y_1 \geq h_1 \\ y_1 \wedge y_2 &= h_1 \wedge h_2 \end{aligned}$$

We have  $h_1 \leq g^- \wedge y_1 = (h_1 \wedge y_1) \vee (h_2 \wedge y_1) = h_1 \vee (h_2 \wedge y_1)$ . On the other hand  $h_2 \wedge y_1 \leq y_2 \wedge y_1 = h_1 \wedge h_2 \leq h_1$ . Therefore  $g^- \wedge y_1 = h_1$  and symmetrically  $g^- \wedge y_2 = h_2$  so both of them belong to  $L_0$ . It follows that  $(y_1, y_2)$  is a primitive tuple with signature  $(g, \{h_1, h_2\}, 2) = \sigma$  in  $L_0$ .

■

**Remark 4.5** The above proof shows, incidentally, that any given signature in a finite co-Heyting algebra  $L_0$  is the signature of an extension of  $L_0$  generated by a primitive tuple (inside an existentially closed extension of  $L_0$ ).

**Corollary 4.6** *If  $L$  is a co-Heyting algebra satisfying the axioms D1 and S1 then any finite co-Heyting algebra embeds into  $L$ .*

*Proof:*  $\mathbf{L}_2$  is a common subalgebra of  $L$  and any co-Heyting algebra  $L_1$ . If  $L_1$  is finite, theorem 4.4 applies to  $L_0 = \mathbf{L}_2$ ,  $L_1$  and  $L$ .

■

**Corollary 4.7** *If  $L$  is a co-Heyting algebra satisfying the axioms D1 and S1,  $L_0$  a finite subalgebra of  $L$ , and  $L_1$  any extension of  $L_0$ , then  $L_1$  embeds over  $L_0$  into an elementary extension of  $L_0$  (or in  $L$  itself if  $L$  is sufficiently saturated).*

*Proof:* By standard model-theoretic argument, it suffices to show that any existential formula with parameters in  $L_0$  satisfied in  $L_1$  is satisfied in  $L$ . Let  $a$  be the list of all elements of  $L_0$  and  $\Delta(a)$  be the conjunction of the quantifier free diagram of  $L_0$ , so that a co-Heyting algebra is a model of the formula  $\Delta(a)$  if and only if  $a$  enumerates a substructure isomorphic to  $L_0$ . Let  $\exists x \theta(x, a)$  be any existential formula with parameters in  $L_0$  satisfied in  $L'$  (where  $x$  is a tuple of variables). By fact 2.2 there is a finite co-Heyting algebra  $L_1$  satisfying  $\exists x \theta(x, a) \wedge \Delta(a)$ . Since  $L_1$  models  $\Delta(a)$  it contains a subalgebra isomorphic to  $L_0$ , which we can then identify to  $L_0$ . By corollary 4.6,  $L_1$  embeds into  $L$  over  $L_0$  hence  $L$  itself models  $\exists x; \theta(x, a)$  and the conclusion follows.

■

**Remark 4.8** If  $L$  is a co-Heyting algebra satisfying the axioms D1 and S1, then every co-Heyting algebra  $L'$  embeds into an elementary extension of  $L$  by corollary 4.7 since  $\mathbf{L}_2$  is a finite common subalgebra of  $L'$  and  $L$ .

## 5 Density and splitting in $\mathcal{V}_2$

We introduce the following axioms:

[Density D2] Same as D1.

[Splitting S2] Same as S1 with the additional assumption that  $b_1 \wedge b_2 \wedge (\mathbf{1} - (b_1 - a)) = \mathbf{0}$

**Fact 5.1** *Let  $L_0$  be a finite co-Heyting algebra. Let  $x_1, \dots, x_r$  be the join irreducible components of  $\mathbf{1}$  in  $L_0$  (that is the maximal elements of  $\mathcal{I}^\vee(L_0)$ ). The following conditions are equivalent:*

1.  $L_0$  belongs to  $\mathcal{V}_2$ .
2.  $x_i \wedge x_j = \mathbf{0}$  whenever  $i \neq j$ .

3.  $L_0$  is isomorphic to a product of co-Heyting algebras  $L_1, \dots, L_r$  such  $\mathbf{1}_{L_i}$  is join irreducible.

This is folklore, but let us recall the argument.

Clearly  $\mathbf{1}_{L_0} = \mathbf{0}_{L_0}$  if and only if  $r = 0$ , in which case the whole fact is trivial. So let's assume that  $r \geq 1$ .

(1) $\Rightarrow$ (2) $\Leftarrow$ (3) is clear. (1) $\Leftarrow$ (2) is an easy computation using that  $1 - x$  is the join of all the join-irreducible components of  $\mathbf{1}$  which are not in  $x^\perp$ . (2) $\Rightarrow$ (3) is true because if we let  $y_i = \mathbb{W}_{j \neq i} x_j$  and  $L_i = L/y_i^\perp$  for every  $i \leq r$ , then it is an easy exercise to check that each  $\mathbf{1}_{L_i}$  is join irreducible and to derive from (2) that the natural map from  $L$  to the product  $L_1 \times \dots \times L_r$  is an isomorphism.

**Lemma 5.2** *Let  $L$  be a finite algebra in  $\mathcal{V}_2$  such that  $\mathbf{1}$  is join irreducible. Let  $a, c$  be any two elements of  $L$  such that  $c \ll a$ . Then there exists an extension  $L'$  of  $L$  in  $\mathcal{V}_2$  and an element  $b$  in  $L'$  such that:*

$$c \ll b \ll a$$

*If moreover  $a \neq \mathbf{0}$  then one can require that  $b \neq \mathbf{0}$ .*

*Proof:* By assumption  $\mathbf{1}$  has a unique predecessor  $x$ , thus  $L_0 = x^\perp$  has a natural structure of co-Heyting algebra.

If  $a = \mathbf{0}$  one can take  $b = \mathbf{0}$ .

If  $a = \mathbf{1}$  then  $c \leq x$ . Let  $L'$  be the co-Heyting algebra obtained by inserting one new element  $b$  between  $x$  and  $\mathbf{1}$ . Then  $a$  and  $b$  are join irreducible in  $L'$  and  $c < b < a$  hence we are done.

Otherwise  $\mathbf{0} \neq a \leq x$  thus lemma 4.2 gives an  $\mathcal{L}_{\text{HA}^*}$ -embedding  $\varphi$  of  $L_0$  into a co-Heyting algebra  $L_1$  containing a non zero element  $b$  such that  $c \ll b \ll a$ . Let  $L'$  be the co-Heyting algebra obtained by adding to  $L_1$  a new element on the top. The embedding  $\varphi$  extends uniquely to an  $\mathcal{L}_{\text{HA}^*}$ -embedding of  $L$  into  $L'$  and we are done.

■

**Lemma 5.3** *Let  $L$  be a finite algebra in  $\mathcal{V}_2$  such that  $\mathbf{1}$  is join irreducible. Let  $a, b_1, b_2$  in  $L$  be such that  $b_1 \vee b_2 \ll a$  and  $b_1 \wedge b_2 \wedge (\mathbf{1} - (\mathbf{1} - a)) = \mathbf{0}$ . Then there exists an extension  $L'$  of  $L$  in  $\mathcal{V}_2$  and elements  $a_1, a_2$  such that:*

$$\begin{aligned} a - a_2 &= a_1 \geq b_1 \\ a - a_1 &= a_2 \geq b_2 \\ a_1 \wedge a_2 &= b_1 \wedge b_2 \end{aligned}$$

*If  $a \neq \mathbf{0}$  one can require that  $a_1, a_2$  are both non zero.*

*Proof:* By assumption  $\mathbf{1}$  has a unique predecessor  $x$ , thus  $L_0 = x^\perp$  has a natural structure of co-Heyting algebra.

If  $a = \mathbf{0}$  one can take  $a_1 = a_2 = \mathbf{0}$ .

If  $a = \mathbf{1}$  then by assumption  $b_1 \wedge b_2 = \mathbf{0}$ . Let  $L'$  be an extension generated over  $L$  by a primitive tuple  $(a_1, a_2)$  with signature  $(a, \{\mathbf{0}\}, 2)$  (see remark 4.5).

By proposition 3.4  $a_1, a_2$  are exactly the two join irreducible components of  $\mathbf{1}$  in  $L'$ , and by construction  $a_1 \wedge a_2 = \mathbf{0}$  hence  $L'$  belongs to  $\mathcal{V}_2$  by fact 5.1.

Otherwise  $\mathbf{0} \neq a \leq x$  thus lemma 4.1 gives an  $\mathcal{L}_{\text{HA}^*}$ -embedding  $\varphi$  of  $L_0$  into a co-Heyting algebra  $L_1$  containing non zero elements  $a_1, a_2$  with the required properties. Let  $L'$  be the co-Heyting algebra obtained by adding to  $L_1$  a new element on the top. Clearly  $L'$  belongs to  $\mathcal{V}_2$  by fact 5.1 and the embedding  $\varphi$  extends uniquely to an  $\mathcal{L}_{\text{HA}^*}$ -embedding of  $L$  into  $L'$ , so we are done. ■

**Theorem 5.4** *Every co-Heyting algebra existentially closed in  $\mathcal{V}_2$  satisfies the density axiom D2 and the splitting axiom S2.*

*Proof:* By fact 5.1 and lemma 2.4 this follows directly from lemmas 5.2 and 5.3. ■

Here is a partial converse of theorem 5.4.

**Theorem 5.5** *Let  $L$  be an algebra in  $\mathcal{V}_2$  satisfying the density axiom D2 and the splitting axiom S2. Let  $L_0$  be a finite subalgebra of  $L$  and  $L_1$  be a finite algebra in  $\mathcal{V}_2$  containing  $L_0$ . Then there exists an embedding of  $L_1$  into  $L$  which fixes every point of  $L_0$ .*

*Proof:* By proposition 3.6 we can assume that  $L_1$  is generated over  $L_0$  by a primitive tuple  $(x_1, x_2)$ . Let  $\sigma = (g, \{h_1, h_2\}, r)$  be the signature of  $L_1$  in  $L_0$ , with  $h_i = x_i \wedge g^-$ . By corollary 3.5 it is sufficient to prove that  $\sigma$  is the signature of a primitive tuple of elements  $x_1, x_2 \in L$ .

*Case 1:*  $r = 1$  so  $h_1 = h_2$ . Since  $h_1 \leq g^- \ll g$  we have  $\mathbf{1} - g^- = \mathbf{1}$  hence obviously  $h_1 \wedge g^- \wedge (\mathbf{1} - (\mathbf{1} - g^-)) = \mathbf{0}$ . The splitting property S2 then applies to the elements  $g, g^-, h_1$  in  $L$ . Then continue like in case 1 of the proof of theorem 4.4.

*Case 2:*  $r = 2$  so  $h_1 \vee h_2 = g^-$ . If  $\mathbf{1} - g < \mathbf{1}$  then  $g$  is one of the join irreducible components of  $\mathbf{1}$  in  $L_0$ . By proposition 3.4  $x_1, x_2$  are then distinct join irreducible components of  $\mathbf{1}$  in  $L_1$ , and since  $L_1$  belongs to  $\mathcal{V}_2$  it follows that  $x_1 \wedge x_2 = \mathbf{0}$  and *a fortiori*  $h_1 \wedge h_2 = \mathbf{0}$ . On the other hand if  $\mathbf{1} - g = \mathbf{1}$  then obviously  $\mathbf{1} - (\mathbf{1} - g) = \mathbf{0}$ . So in any case we have:

$$h_1 \wedge h_2 \wedge (\mathbf{1} - (\mathbf{1} - g)) = \mathbf{0} \quad (5)$$

The splitting property S2 then applies in  $L$  to the elements  $g, h_1, h_2$ . Then continue like in case 2 of the proof of theorem 4.4. ■

**Remark 5.6** The proof shows that the minimal extension of a finite co-Heyting algebra  $L_0$  determined by a signature  $(g, \{h_1, h_2\}, r)$  belongs to  $\mathcal{V}_2$  if and only if either  $r = 1$ , or  $r = 2$  and condition (5) holds. Also the analogues of corollaries 4.6 and 4.7 hold for  $\mathcal{V}_2$  as a consequence of theorem 5.5

## 6 Density and splitting in $\mathcal{V}_3$

We introduce the following axioms:

**[Density D3]** For every  $a$  such that  $a = \mathbf{1} - (\mathbf{1} - a) \neq \mathbf{0}$  there exists a non zero element  $b$  such that  $b \ll a$ .

**[Splitting S3]** Same as S1.

A co-Heyting algebra  $L$  belongs to  $\mathcal{V}_3$  if and only if it has dimension  $\leq 1$ . If  $L$  is finite this is equivalent to say that every join irreducible element of  $L$  is either maximal or minimal (or both) in  $\mathcal{I}^\vee(L)$ .

**Lemma 6.1** *Let  $a$  be any element of a finite algebra  $L$  in  $\mathcal{V}_3$ . If  $a = \mathbf{1} - (\mathbf{1} - a) \neq \mathbf{0}$  then there exists a finite algebra  $L'$  in  $\mathcal{V}_3$  containing  $L$  and a non zero element  $b$  in  $L'$  such that  $b \ll a$ .*

*Proof:* Let  $a_1, \dots, a_r$  be the join irreducible components of  $a$  in  $L$ . The assumption that  $a = \mathbf{1} - (\mathbf{1} - a) \neq \mathbf{0}$  means that  $r \neq 0$  and all the  $a_i$ 's are join irreducible components of  $\mathbf{1}$ , that is maximal elements in  $\mathcal{I}^\vee(L_0)$ . If there exists  $i \leq r$  such that  $a_i$  is not in the same time minimal in  $L$  (that is  $a_i$  is not an atom of  $L$ ) then we can choose  $b \in \mathcal{I}^\vee(L)$  such that  $b < a_i$ . Then  $b$  is non zero and  $b \ll a_i$  because  $a_i$  is join irreducible, so *a fortiori*  $b \ll a$ . The conclusion follows, with  $L' = L$ .

It only remains to deal with the case when all the  $a_i$  are both maximal and minimal in  $\mathcal{I}^\vee(L)$ . But in this case the construction of lemma 4.1 (with  $c = \mathbf{0}$ ) gives an extension  $L'$  on  $L$  such that:

- $\mathcal{I}^\vee(L') = \mathcal{I}^\vee(L) \cup \{x_1, \dots, x_r\}$ .
- For every  $i \leq r$  and every  $x \in \mathcal{I}^\vee(L)$ ,  $x \not< x_i$  and:

$$x_i < x \iff x = a_i$$

So there are still no chain in  $\mathcal{I}^\vee(L')$  containing more than two distinct join irreducible elements, that is  $L'$  belongs to  $\mathcal{V}_2$ , and clearly:

$$\mathbf{0} \neq x_1 \vee \dots \vee x_r \ll a$$

■

**Lemma 6.2** *Let  $a, b_1, b_2$  be elements of a finite algebra  $L$  in  $\mathcal{V}_3$ . If  $b_1 \vee b_2 \ll a \neq \mathbf{0}$  then there exists a finite algebra  $L'$  in  $\mathcal{V}_3$  containing  $L$  and non zero elements  $a_1, a_2$  in  $L'$  such that:*

$$\begin{aligned} a - a_2 &= a_1 \geq b_1 \\ a - a_1 &= a_2 \geq b_2 \\ a_1 \wedge a_2 &= b_1 \wedge b_2 \end{aligned}$$

*Proof:* Same proof as for lemma 4.2. Indeed, in the extension  $L'$  of  $L$  constructed in that proof the maximal length of the chains of join irreducible elements is the same as in  $L$ . So if  $L$  belongs to  $\mathcal{V}_3$  then so does  $L'$ .

■

**Theorem 6.3** *The theory of the variety  $\mathcal{V}_3$  has a model-completion which is axiomatized by the density and splitting axioms D3 and S3.*

*Proof:* As for theorem 4.3 it immediately follows from lemmas 6.1 and 6.2, via lemma 2.3, that every algebra existentially closed in  $\mathcal{V}_3$  satisfies the axioms D3 and S3.

For the converse, by fact 2.1 it is sufficient to show that given an algebra  $L$  in  $\mathcal{V}_3$  satisfying D3 and S3, a finitely generated subalgebra  $L_0$  and a finitely generated extension  $L_1$  of  $L_0$  in  $\mathcal{V}_3$  there exists an embedding of  $L_1$  in  $L$  which fixes  $L_0$  pointwise. Since  $\mathcal{V}_3$  is locally finite,  $L_0$  and  $L_1$  are finite and by proposition 3.6 we can assume that  $L_1$  is generated by a primitive tuple  $(x_1, x_2)$ . Let  $\sigma = (g, \{h_1, h_2\}, r)$  be the signature of  $L_1$  in  $L$ , numbered so that  $h_i = x_i \wedge g^-$ . By corollary 3.5 we have to find a primitive tuple in  $L$  having signature  $\sigma$ .

*Case 1:*  $r = 1$  so  $x_1 = x_2$  and  $h_1 \ll x_1 \ll g$ . Since  $x_1, g$  are join irreducible in  $L_1$  and since  $L_1$  belongs to  $\mathcal{V}_3$ , necessarily  $g$  is a join irreducible component of  $\mathbf{1}$ ,  $x_1$  is an atom of  $L_1$ , and consequently  $h_1 = \mathbf{0}$ . The splitting axiom S3 applied to  $g, g^-, \mathbf{0}$  gives non zero elements  $y_1, y_2$  in  $L$  such that:

$$\begin{aligned} g - y_1 &= y_2 \geq g^- \\ g - y_2 &= y_1 \\ y_1 \wedge y_2 &= \mathbf{0} \end{aligned}$$

By construction  $(y_1, y_2)$  is a primitive tuple over  $L_0$  hence by lemma 3.2 and proposition 3.4 we have:

$$\mathcal{I}^\vee(L_0\langle y_1 \rangle) = (\mathcal{I}^\vee(L_0) \setminus \{g\}) \cup \{y_1, y_2\}$$

Since  $g$  was a join irreducible component of  $\mathbf{1}$  in  $L_0$ , the same then holds for  $y_1, y_2$  in  $L_0\langle y_1 \rangle$ . It follows that  $\mathbf{1} - (\mathbf{1} - y_1) = y_1$  hence the density axiom D3 gives  $x \in L \setminus \{\mathbf{0}\}$  such that  $x \ll y_1$ . *A fortiori*  $x \ll g$  and by construction  $x \wedge g^- \leq y_1 \wedge y_2 = \mathbf{0}$ . It easily follows that  $(x, x)$  is a primitive tuple with signature  $(g, \{\mathbf{0}\}, 1) = \sigma$  in  $L_0$ .

*Case 2:*  $r = 2$  so  $h_1 \vee h_2 = g^-$ . The same construction as in the case 2 of the proof of theorem 4.4 applies here and gives the conclusion.

■

## 7 Density and splitting in $\mathcal{V}_4$

We introduce the following axioms:

[Density D4] Same as D3.

[**Splitting S4**] Same as S1 with the additional assumption that  $b_1 \wedge b_2 \wedge (\mathbf{1} - a) = \mathbf{0}$ .

**Fact 7.1** *For any finite co-Heyting algebra  $L$  the following conditions are equivalent.*

1.  $L$  belongs to  $\mathcal{V}_4$ .
2.  $L$  belongs to  $\mathcal{V}_3$  (every element of  $\mathcal{I}^\vee(L)$  is either maximal or minimal) and for any three distinct join irreducible components  $x_1, x_2, x_3$  of  $\mathbf{1}$ , we have  $x_1 \wedge x_2 \wedge x_3 = \mathbf{0}$ .
3.  $L$   $\mathcal{L}_{\text{HA}^*}$ -embeds in a product of finitely many copies of  $\mathbf{L}_5$ .

This is probably well known. For lack of a reference we give here an elementary (and sketchy) proof. We can assume that  $L \neq \mathbf{L}_1$  otherwise everything is trivial.

*Proof:* (3) $\Rightarrow$ (1) is clear.

(1) $\Rightarrow$ (2) Since  $L$  belongs to  $\mathcal{V}_4$ , which is generated by  $\mathbf{L}_5$ , which belongs to  $\mathcal{V}_3$ , obviously  $L$  belongs to  $\mathcal{V}_3$ . Now assume that  $\mathbf{1}$  has at least three distinct join irreducible components  $x_1, x_2, x_3$  in  $L$ . The equation defining  $\mathcal{V}_4$  gives:

$$(x_1 - x_2) \wedge (x_2 - x_1) \wedge (x_2 \triangle (\mathbf{1} - x_1)) = \mathbf{0} \quad (6)$$

We have  $x_1 - x_2 = x_1$ ,  $x_2 - x_1 = x_2$  and  $\mathbf{1} - x_1$  is the join of all join irreducible components of  $\mathbf{1}$  except  $x_1$ . In particular it is greater than  $x_2$  and  $x_3$  so we get:

$$x_2 \triangle (\mathbf{1} - x_1) = (\mathbf{1} - x_1) - x_2 \geq x_3$$

Finally (6) becomes  $x_1 \wedge x_2 \wedge ((\mathbf{1} - x_1) - x_2) = \mathbf{0}$  hence *a fortiori*  $x_1 \wedge x_2 \wedge x_3 = \mathbf{0}$ .

(2) $\Rightarrow$ (3) We consider:

$$\mathcal{I} = \{(x_1, x_2) \in \mathcal{I}^\vee(L) \times \mathcal{I}^\vee(L) : x_1 < x_2 \text{ or } x_1 = x_2 \text{ is an atom}\}$$

$\mathcal{I}$  is ordered as follows:

$$(y_1, y_2) < (x_1, x_2) \iff y_1 = y_2 = x_1 < x_2$$

The ordered set  $\mathcal{I}$  looks like  $\mathcal{I}^\vee(L)$  except that every point of  $\mathcal{I}^\vee(L)$  strictly greater than  $r$  atoms has been “split” in  $r$  points strictly greater than only one atom. We “collapse” these  $r$  points *via* the map  $\pi$  defined for any  $\xi = (x_1, x_2) \in \mathcal{I}$  by  $\pi(\xi) = x_2$ . This defines an  $\mathcal{L}_{\text{HA}^*}$ -embedding of  $L$  into  $L' = \mathcal{L}^\downarrow(\mathcal{I})$  by means of proposition 2.5. Then (2) implies that  $\mathcal{I}$  is a finite disjoint union of copies of sets represented in figure 3. The family of all decreasing subsets of these sets are respectively isomorphic to  $\mathbf{L}_5$ ,  $\mathbf{L}_3$  and  $\mathbf{L}_2$ . Since  $\mathcal{I}^\vee(L')$  is order-isomorphic to  $\mathcal{I}$ , it follows that  $L'$  is a direct product of finitely many copies of these three algebras. Each of these copies obviously  $\mathcal{L}_{\text{HA}^*}$ -embeds into  $\mathbf{L}_5$  so we are done.  $\blacksquare$



Figure 3: The connected components of  $\mathcal{I}$

**Lemma 7.2** *Let  $a$  be any element of  $\mathbf{L}_5$  such that  $a = \mathbf{1} - (\mathbf{1} - a)$ . Then there exists an element  $b$  in  $\mathbf{L}_5$  such that  $b \ll a$ . If moreover  $a \neq \mathbf{0}$  then  $b$  can be chosen non zero.*

*Proof:* If  $a = \mathbf{0}$  one can take  $b = \mathbf{0}$ . Otherwise the assumption that  $\mathbf{1} - (\mathbf{1} - a) = a \neq \mathbf{0}$  implies that  $a$  is one of the two maximal join irreducible elements of  $L$  so we can take  $b = a^-$ .

■

**Lemma 7.3** *Let  $a, b_1, b_2$  be any elements of  $\mathbf{L}_5$  such that  $b_1 \vee b_2 \ll a$  and  $b_1 \wedge b_2 \wedge (\mathbf{1} - a) = \mathbf{0}$ . Then there exists an  $L'$  of  $L$  in  $\mathcal{V}_4$  and elements  $a_1, a_2$  in  $L'$  such that:*

$$\begin{aligned} a - a_2 &= a_1 \geq b_1 \\ a - a_1 &= a_2 \geq b_2 \\ a_1 \wedge a_2 &= b_1 \wedge b_2 \end{aligned}$$

*If moreover  $a \neq \mathbf{0}$  then  $a_1, a_2$  can be chosen both non zero.*

*Proof:* Let  $c$  denote the unique atom of  $\mathbf{L}_5$ . The first assumption on  $a, b_1, b_2$  implies that  $b_1 \vee b_2$  is either  $\mathbf{0}$  or  $c$ . In particular we can always assume that  $b_2 \leq b_1$ .

*Case 1:*  $a = \mathbf{0}$ . One can take  $a_1 = a_2 = \mathbf{0}$ .

*Case 2:*  $b_1 = b_2 = c$ . By assumption  $c \ll a$  and  $c \wedge (\mathbf{1} - a) = \mathbf{0}$  hence  $a = \mathbf{1}$ . So we can take  $L' = \mathbf{L}_5$  and for  $a_1, a_2$  the join irreducible components of  $\mathbf{1}$ .

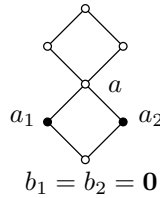


Figure 4: Case 3

*Case 3:*  $a = c$ . Then  $b_1 = b_2 = \mathbf{0}$  and we can take for  $a_1, a_2$  the atoms of the extension  $L'$  of  $\mathbf{L}_5$  shown in figure 4 (the white points are the points of  $\mathbf{L}_5$ ). Note that the four join irreducible elements of  $L'$  are either maximal or minimal, and that  $L'$  does not have three distinct maximal join irreducible elements so  $L'$  belongs to  $\mathcal{V}_4$ .



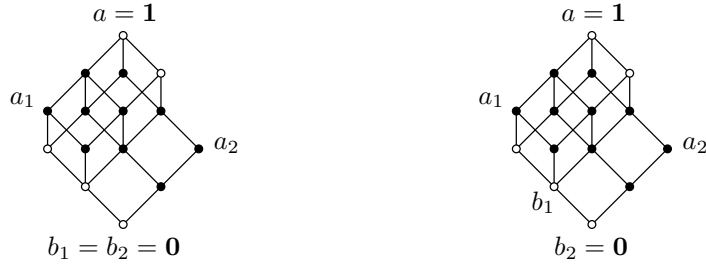


Figure 5: Cases 4 and 5

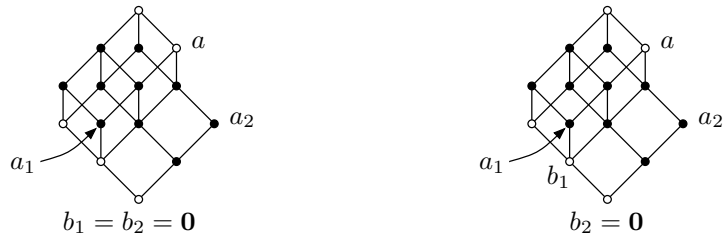


Figure 6: Cases 6 and 7

*Cases 4 to 7:* The four remaining cases when  $a > c$  are summarised in figures 5 and 6. In each case the white points represent the points of  $\mathbf{L}_5$  and one can take for  $a_1, a_2$  the points in the extension  $L'$  of  $\mathbf{L}_5$  shown in the figures. Note that  $L'$  is just  $\mathbf{L}_5 \times \mathbf{L}_3$  so it belongs to  $\mathcal{V}_4$ .

■

**Theorem 7.4** *The theory of the variety  $\mathcal{V}_4$  has a model-completion which is axiomatized by the density and splitting axioms  $D_4$  and  $S_4$ .*

*Proof:* As for theorem 6.3, the only thing which it remains to prove after lemmas 7.2 and 7.3 is that: given an algebra  $L$  in  $\mathcal{V}_4$  satisfying  $D_4$  and  $S_4$ , a finitely generated subalgebra  $L_0$  and a finitely generated extension  $L_1$  of  $L_0$  in  $\mathcal{V}_4$  generated by a primitive tuple  $(x_1, x_2)$  with signature  $\sigma = (g, \{h_1, h_2\}, r)$  in  $L_0$  (numbered so that  $h_i = x_i \wedge g^-$ ), there exists a primitive tuple in  $L$  having the same signature  $\sigma$ .

*Case 1:*  $r = 1$ . The same argument as in the case 1 in the proof of theorem 6.3 applies here (when applying  $S_4$  in place of  $S_3$  to  $g, g^-, \mathbf{0}$  the additional condition  $g^- \wedge \mathbf{0} \wedge (\mathbf{1} - g)$  is obviously satisfied).

*Case 2:*  $r = 2$  so  $h_1 \vee h_2 = g^-$ . In order to apply the splitting axiom  $S_4$  to  $g, h_1, h_2$  we have to check that  $h_1 \wedge h_2 \wedge (\mathbf{1} - g) = \mathbf{0}$ . Assume the contrary. Then  $h_1, h_2$  are non zero so  $g$  is not an atom. Since  $L_0$  belongs to  $\mathcal{V}_4 \subseteq \mathcal{V}_3$  it follows that  $g$  is maximal in  $\mathcal{I}^\vee(L_0)$  hence so are  $x_1, x_2$  in  $\mathcal{I}^\vee(L_1)$  (see proposition 3.4).

With other words  $x_1, x_2$  are two distinct join irreducible components of  $\mathcal{I}^\vee(L_1)$  and  $\mathbf{1} - g$  is the join of all the other join irreducible components of  $\mathbf{1}$  in  $L_1$ . But for any such component  $x_3$  we must have  $x_1 \wedge x_2 \wedge x_3 = \mathbf{0}$  by fact 7.1 so  $x_1 \wedge x_2 \wedge (\mathbf{1} - g) = \mathbf{0}$ . Since each  $h_i \leq x_i$  this contradicts our assumption.

So we can apply S4 to  $g, h_1, h_2$  and it gives  $y_1, y_2$  in  $L$ . Then finish like in the case 2 of the proof of theorem 4.4.

■

## 8 Density and splitting in $\mathcal{V}_5$

The density and splitting axioms for  $\mathcal{V}_5$  are respectively D5=D3 and S5=S2.

**Fact 8.1** *For a finite co-Heyting algebra  $L$  the following conditions are equivalent:*

1.  $L$  belongs to  $\mathcal{V}_5$ .
2.  $L$  belongs to  $\mathcal{V}_2$  and  $\mathcal{V}_3$ , that is every join irreducible element of  $L$  which is not an atom is a join irreducible component of  $\mathbf{1}$ , and for any two distinct join irreducible components  $x_1, x_2$  of  $\mathbf{1}$  we have  $x_1 \wedge x_2 = \mathbf{0}$ .
3.  $L$   $\mathcal{L}_{\text{HA}^*}$ -embeds in a direct product of finitely many copies of the three elements co-Heyting algebra  $\mathbf{L}_3$ .

This is probably well known, and anyway the adaptation to this context of the proof that we gave for the analogous fact 7.1 is straightforward.

**Theorem 8.2** *The theory of the variety  $\mathcal{V}_5$  has a model-completion which is axiomatized by the density and splitting axioms D5 and S5.*

*Proof:* Let  $c$  denote the unique atom of  $\mathbf{L}_3$ .

The only elements  $a$  in  $\mathbf{L}_3$  such that  $a = \mathbf{1} - (\mathbf{1} - a)$  are  $\mathbf{0}$  and  $\mathbf{1}$ . Clearly if  $a = \mathbf{0}$  then  $b = \mathbf{0}$  satisfies  $b \ll a$ , and otherwise  $b = c$  satisfies  $\mathbf{0} \neq b \ll a$ . By fact 8.1 and lemma 2.4 it follows that every algebra existentially closed in  $\mathcal{V}_5$  satisfies D5.

Now let  $a, b_1, b_2$  in  $L_3$  be such that  $b_1 \vee b_2 \ll a$  and  $b_1 \wedge b_2 \wedge (\mathbf{1} - (\mathbf{1} - a)) = \mathbf{0}$ . If  $a = \mathbf{0}$  then one can take  $a_1 = a_2 = \mathbf{0}$  as a solution for the conclusion of S5. Otherwise  $a = \mathbf{1}$  or  $a = c$  and  $b_1 = b_2 = \mathbf{0}$  thus one can take for  $a_1, a_2$  the elements of the extension  $L'$  of  $\mathbf{L}_3$  shown in figure 7 (the white points represent  $\mathbf{L}_3$ ). Note that  $L' = \mathbf{L}_2 \times \mathbf{L}_3$  belongs to  $\mathcal{V}_5$ . By fact 8.1 and lemma 2.4 again, it follows that every algebra existentially closed in  $\mathcal{V}_5$  satisfies S5.

Conversely let  $L$  in  $\mathcal{V}_5$  satisfying D5 and S5,  $L_0$  a finitely generated subalgebra and  $L_1$  a finitely generated extension of  $L_0$  in  $\mathcal{V}_5$  generated by a primitive tuple  $(x_1, x_2)$  with signature  $\sigma = (g, \{h_1, h_2\}, r)$  in  $L_0$  (numbered so that  $h_i = x_i \wedge g^-$ ). As usually it only remains to find a primitive tuple in  $L$  having the same signature  $\sigma$  in order to conclude that  $L_1$  embeds into  $L$  over  $L_0$  by corollary 3.5, hence to finish the proof by fact 2.1.

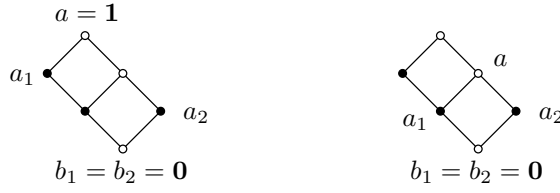


Figure 7: Solution for S5 when  $a \neq \mathbf{0}$

*Case 1:*  $r = 1$  so  $x_1 = x_2$  and  $h_1 \ll x_1 \ll g$ . Same as case 1 in the proof of theorem 6.3.

*Case 2:*  $r = 2$  so  $h_1 \vee h_2 = g^-$ . Same as case 2 in the proof of theorem 5.4 (note that  $\mathcal{V}_5$  is contained in  $\mathcal{V}_2$  when applying this proof).

■

## 9 Density and splitting in $\mathcal{V}_6$

We introduce our last axioms.

[**Density D6**] Same as D1.

[**Splitting S6**] Same as S1 with the additional assumption that  $b_1 \wedge b_2 = \mathbf{0}$ .

**Fact 9.1** *A finite co-Heyting algebra belongs to  $\mathcal{V}_6$  if and only if it embeds into a direct product of finitely many finite chains.*

This is certainly well known, and easy to check.

**Theorem 9.2** *The theory of the variety  $\mathcal{V}_6$  has a model-completion which is axiomatized by the density and splitting axioms D6 and S6.*

*Proof:* Let  $a, c$  be any elements in a finite chain  $L$  such that  $c \ll a$ . If  $a = \mathbf{0}$  then  $b = \mathbf{0}$  satisfies  $c \ll b \ll a$ . Otherwise  $c < a$  and obviously  $L$  embeds into a chain  $L'$  containing a new intermediate element  $b$  between  $a$  and  $a^-$ . Then by construction  $c \ll b \ll a$  and  $b \neq \mathbf{0}$ . By fact 9.1 and lemma 2.4 it follows that every algebra existentially closed in  $\mathcal{V}_6$  satisfies D6.

Let  $a, b_1, b_2$  be three elements in a finite chain  $L$  such that  $b_1 \vee b_2 \ll a$  and  $b_1 \wedge b_2 = \mathbf{0}$ . We may assume that  $b_2 \leq b_1$ , so by assumption  $b_2 = \mathbf{0}$ . If  $a = \mathbf{0}$  then  $a_1 = a_2 = \mathbf{0}$  satisfy the conclusion of S6. Otherwise  $b_1 < a$  and one can take for  $a_1, a_2$  the non zero points in the extension  $L'$  of  $L$  shown in figure 8 (the white points represent  $L$ ). Note that  $L' = L \times \mathbf{L}_2$  belongs to  $\mathcal{V}_2$ . By fact 9.1 and lemma 2.4 again, it follows that every algebra existentially closed in  $\mathcal{V}_6$  satisfies S6.

Conversely let  $L$  in  $\mathcal{V}_6$  be satisfying D6 and S6,  $L_0$  a finitely generated subalgebra and  $L_1$  a finitely generated extension of  $L_0$  in  $\mathcal{V}_6$  generated by a

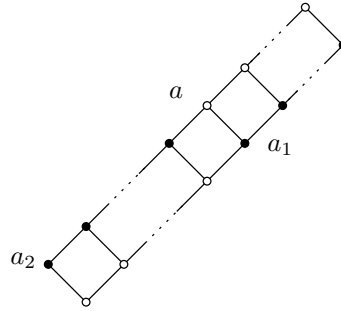


Figure 8: Solution for S6 when  $a \neq \mathbf{0}$

primitive tuple  $(x_1, x_2)$  with signature  $\sigma = (g, \{h_1, h_2\}, r)$  in  $L_0$  (numbered so that  $h_i = x_i \wedge g^-$ ). As usually it only remains to find a primitive tuple in  $L$  having the same signature  $\sigma$  in order to conclude that  $L_1$  embeds into  $L$  over  $L_0$  by corollary 3.5, hence to finish the proof by fact 2.1.

*Case 1:*  $r = 1$  so  $x_1 = x_2$  and  $h_1 \ll x_1 \ll g$ . Same as case 1 in the proof of theorem 4.4.

*Case 2:*  $r = 2$  so  $h_1 \vee h_2 = g^-$ . Since  $x_1, x_2$  are join irreducible and incomparable,  $x_1 - x_2 = x_1$  and  $x_2 - x_1 = x_2$ . By definition of  $\mathcal{V}_6$  it follows that  $x_1 \wedge x_2 = \mathbf{0}$ , hence *a fortiori*  $h_1 \wedge h_2 = \mathbf{0}$ . So the splitting axiom S6 applies to  $g, h_1, h_2$ . Then finish the proof like in case 2 of theorem 4.4.

■

## References

- [CK90] C. C. Chang and H. J. Keisler. *Model theory*, volume 73 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, third edition, 1990.
- [DJ] Luck Darnière and Markus Junker. Codimension and pseudometric in co-Heyting algebras. To appear in *Algebra Universalis*.
- [GZ97] Silvio Ghilardi and Marek Zawadowski. Model completions and r-Heyting categories. *Ann. Pure Appl. Logic*, 88(1):27–46, 1997.
- [Hod97] Wilfrid Hodges. *A shorter model theory*. Cambridge University Press, Cambridge, 1997.
- [Hos67] Tsutomu Hosoi. On intermediate logics. I. *J. Fac. Sci. Univ. Tokyo Sect. I*, 14:293–312, 1967.
- [Mak77] L. L. Maksimova. Craig’s theorem in superintuitionistic logics and amalgamable varieties. *Algebra i Logika*, 16(6):643–681, 741, 1977.

- [McK68] C. G. McKay. The decidability of certain intermediate propositional logics. *The Journal of Symbolic Logic*, 33(2):258–264, 1968.
- [MT46] J. C. C. McKinsey and Alfred Tarski. On closed elements in closure algebras. *Ann. of Math. (2)*, 47:122–162, 1946.
- [Pit92] Andrew M. Pitts. On an interpretation of second-order quantification in first-order intuitionistic propositional logic. *J. Symbolic Logic*, 57(1):33–52, 1992.
- [Whe76] William H. Wheeler. Model-companions and definability in existentially complete structures. *Israel J. Math.*, 25(3-4):305–330, 1976.