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# FINITE DEFICIENCY INDICES AND UNIFORM REMAINDER IN WEYL'S LAW

LUC HILLAIRET

## INTRODUCTION

Von Neumann theory classifies all self-adjoint extensions of a given symmetric operator  $A$  in terms of the so-called *deficiency indices*. Since the choice of the self-adjoint condition reflects the physics that is underlying the problem, it is natural to ask how the spectra of two different self-adjoint extensions  $A_0$  and  $A_1$  can differ.

In some cases, the deficiency indices are finite. This happens for instance in the following interesting settings from mathematical physics

- Quantum graphs (see [2, 5] and section 4.1 below),
- Pseudo-Laplacians, Šeba billiards (see [3, 6], and section 4.2 below),
- Manifolds with conical singularities (see [8, 7]),
- Hybrid manifolds (see [10]).

In any of these settings we prove the following theorem (see section 2 and the applications for a more precise version)

**Theorem 1.** *Let  $A$  be the symmetric operator associated with one of the preceding settings. There exists a constant  $C$  such that for any self-adjoint extensions  $A_0$  and  $A_1$  of  $A$  we have*

$$\forall E, |N_1(E) - N_0(E)| \leq C$$

where  $N_i$  denotes the spectral counting function of  $A_i$ .

Motivation for this result came principally from [6] and the Šeba billiard setting. In this case we can take  $A_0$  to be the standard Dirichlet Laplace operator in  $R$ , and this theorem proves that the remainder in Weyl's law is, up to a  $O(1)$  term, uniform with respect to the location of the Delta potential.

In contrast with [2, 5, 7, 10] we consider a rather crude spectral invariant. Moreover, our result relies on the min-max principle only which is less sophisticated than the analysis performed in the former references. It should be noted, however, that our result is not a straightforward byproduct of these results and should more likely be considered as a first step. From our perspective, it is quite interesting to have a general method allowing to get quite good hold on the spectral counting function before moving on to more complete spectral invariants such as heat kernel or resolvent estimates.

## 1. SETTING AND NOTATIONS

We begin by recalling some basic facts from spectral theory of self-adjoint operators as well as Von Neumann theory of self-adjoint extensions of a symmetric operator. We will use [11, 12] and [4] as references.

**1.1. Basic Spectral Theory.** We consider a Hilbert space  $\mathcal{H}$  with scalar product  $\langle \cdot, \cdot \rangle$  and associated norm  $\| \cdot \|$ .

On  $\mathcal{H}$  we consider a symmetric operator  $A$  with domain  $\text{dom}(A)$  and its adjoint  $A^*$ . The graph norm is defined on  $\text{dom}(A)$  by  $\|u\|_A^2 = \|u\|^2 + \|Au\|^2$ .

An operator is self-adjoint if  $A = A^*$ . It has compact resolvent if the injection from  $\text{dom}(A)$  into  $\mathcal{H}$  is compact.

The spectrum of a self-adjoint operator with compact resolvent consists in eigenvalues of finite multiplicities, that form a discrete set in  $\mathbb{R}$ . There exists an orthonormal basis consisting of eigenvectors.

If there exists  $C \in \mathbb{R}$  such that  $\forall u \in \text{dom}(A)$ ,  $\|Au\| \geq C\|u\|$ , the operator is called *semibounded*.

For a semibounded self-adjoint operator with compact resolvent, the spectrum can be ordered into a non-decreasing sequence  $(\lambda_n)_{n \in \mathbb{N}}$ .

The spectral counting function is then defined by

$$N(E) = \text{Card}\{\lambda_n \geq E\},$$

and by Courant-Hilbert min-max principle (see [4]) we have

$$(1.1) \quad \lambda_n = \min \left\{ \max \left\{ \frac{\langle Au, u \rangle}{\|u\|^2}, u \in F \setminus \{0\} \right\}, F \subset \mathcal{H}, F \text{ vector space s.t. } \dim F = n \right\}.$$

**1.2. Von Neumann Theory.**

This section summarizes section X.1 of [12]. We define

$$\mathcal{K}^\pm = \ker(A^* \mp \text{id})$$

$$d_\pm = \dim(\mathcal{K}^\pm).$$

We recall that we have the following decomposition of  $\text{dom}(A^*)$  (see Lemma in section X.1 of [12]).

$$\text{dom}(A^*) = \text{dom}(\bar{A}) \oplus \mathcal{K}^+ \oplus \mathcal{K}^-,$$

and that the decomposition is orthogonal with respect to the scalar product  $\langle \cdot, \cdot \rangle_{A^*}$  defined on  $\text{dom}(A^*)$  by

$$\forall u, v \in \text{dom}(A^*), \langle u, v \rangle_{A^*} := \langle u, v \rangle + \langle A^*u, A^*v \rangle.$$

We denote by  $\pi_\pm$  the orthogonal projection from  $\text{dom}(A^*)$  onto  $\mathcal{K}^\pm$  and by  $\pi_0$  the orthogonal projection onto  $\text{dom}(\bar{A})$  (so that  $\pi_0 = \text{id} - \pi_+ - \pi_-$ ).

The following theorem (theorem X.2 and corollary of [12]) provides a parameterization of all self-adjoint extensions of  $A$

**Theorem 2.** *A admits self-adjoint extensions if and only if  $d_+ = d_-$ .*

*For any self-adjoint extension  $A_{sa}$  of  $A$ , there exists a unique isometry  $U$  from  $\mathcal{K}^+$  onto  $\mathcal{K}^-$  such that  $\forall u \in \text{dom}(A_{sa}), U\pi_+(u) = \pi_-(u)$ .*

## 2. THE THEOREM

Using the notations of the preceding section we have

**Theorem 3.** *Let  $A$  be a symmetric operator with equal finite deficiency indices :*

$$d_+ = d_- = d < \infty.$$

*Suppose that there exists  $A_0$  a self-adjoint extension of  $A$  such that*

- (i)  *$A_0$  has compact resolvent,*
- (ii)  *$A_0$  is semibounded*

*Then*

- (1) *Any other self-adjoint extension also has compact resolvent and is semibounded.*
- (2) *There exist  $E_0$  such that, for any other self-adjoint extension  $A_1$  the following holds*

$$\forall E \in \mathbb{R}, |N_1(E) - N_0(E)| \leq d$$

*where  $N_i(E)$  denotes the spectral counting function of  $A_i$ .*

## 3. PROOFS

**3.1.  $A_1$  has compact resolvent.** We prove this fact by proving the stronger lemma.

**Lemma 3.1.** *If  $A_0$  has compact resolvent then the injection from  $\text{dom}(A^*)$  (equipped with  $\|\cdot\|_{A^*}$ ) into  $\mathcal{H}$  is compact.*

*Proof.* Since  $A_0$  is compact, the injection from  $(\text{dom}(A_0), \|\cdot\|_{A_0})$  into  $\mathcal{H}$  is compact. But  $\text{dom}(\bar{A})$  is closed in  $\text{dom}(A^*)$  for  $\|\cdot\|_{A^*}$ . Since  $\|\cdot\|_{A_0}$  coincide with  $\|\cdot\|_{A^*}$  on  $A_0$ , the injection from  $(\text{dom}(\bar{A}), \|\cdot\|_{A^*})$  is also compact.

Let  $(u_n)_{n \in \mathbb{N}}$  be a sequence in  $\text{dom}(A^*)$  that is  $\|\cdot\|_{A^*}$ -bounded. We can extract a subsequence from  $\pi_0 u_n$  that converges in  $\mathcal{H}$  since the injection from  $(\text{dom}(\bar{A}), \|\cdot\|_{A^*})$  into  $\mathcal{H}$  is compact. On  $\mathcal{K}^\pm$ ,  $\|\cdot\|_{A^*}$  is equivalent to  $\|\cdot\|$  and, since  $d^\pm$  are finite we can also extract convergent subsequences. This proves the Lemma.  $\square$

The Lemma says that  $A^*$  has compact resolvent and so does  $A_1$  since  $A^*$  extends  $A_1$  and the latter is closed.

**3.2.  $A_1$  is semibounded.** Let  $(\lambda_k(A_0))_{k \in \mathbb{N}}$  denote the (ordered spectrum) of  $A_0$  and consider  $n$  such that  $\lambda_n(A_0) \geq 0$ .

Consider  $F \subset \text{dom}(A_1)$  of dimension  $n + d$ . Denote by  $F_1 = F \cap \text{dom}(\bar{A}) = F \cap \ker(\text{id} - \pi_0)$ . Theorem 2 implies that  $\ker(\text{id} - \pi_0)|_{\text{dom}(A_1)}$  is of dimension  $d$ , and thus, since  $F$  is of dimension  $n + d$ ,  $F_1$  is of dimension at least  $n$ .

Moreover  $F_1 \subset \text{dom}(\bar{A}) \subset \text{dom}(A_{1,0})$ , thus, for all  $u \in F_1$  we have

$$\langle A_1 u, u \rangle = \langle A_0 u, u \rangle.$$

Since  $F_1 \subset F$  it follows that

$$\max_{u \in F, u \neq 0} \frac{\langle A_1 u, u \rangle}{\|u\|^2} \geq \max_{u \in F_1, u \neq 0} \frac{\langle A_0 u, u \rangle}{\|u\|^2}.$$

Since  $\dim F_1 \geq n$  and  $\lambda_n \geq 0$ , it follows from the min-max principle that the right-hand side is non-negative.

We thus obtain, that for all  $F \subset \text{dom}(A_1)$  of dimension  $n + d$  we have

$$\max_{u \in F, u \neq 0} \frac{\langle A_1 u, u \rangle}{\|u\|^2} \geq 0.$$

This implies that  $A_1$  has at most  $n + d - 1$  negative eigenvalues. (otherwise the subspace generated by  $n + d$  negative eigenvalues would contradict the preceding bound).

**3.3. Comparing  $N_0$  and  $N_1$ .** According to the previous section we have that  $A_1$  also is semi-bounded with compact resolvent so that we can denote by  $(\lambda_k(A_1))_{k \in \mathbb{N}}$  its ordered spectrum. We also denote by  $V_n^i$  the vector space generated by the first  $n$  eigenvectors of  $A_i$ .

For any  $n$ , set  $F = V_{n+d}^1 \cap \text{dom}(\bar{A})$ . Making the same reasoning as previously we find that  $F$  is of dimension at least  $n$  and

$$\max_{u \in V_{n+d}, u \neq 0} \left\{ \frac{\langle A_1 u, u \rangle}{\|u\|^2} \right\} \geq \max_{u \in F, u \neq 0} \left\{ \frac{\langle A_0 u, u \rangle}{\|u\|^2} \right\}.$$

By definition of  $V_{n+d}$  the left-hand-side is  $\lambda_{n+d}(A_1)$  and, using the min-max principle, the right-hand side is bounded below by  $\lambda_n(A_0)$ . Thus, we obtain,

$$\forall n, \lambda_{n+d}(A_1) \geq \lambda_n(A_0).$$

Observe that  $N_0(E)$  is characterized by

$$\lambda_{N_0(E)}(A_0) \leq E < \lambda_{N_0(E)+1}(A_0)$$

Using this and the preceding inequality we find that, for all  $E$  we have

$$E \leq \lambda_{N_0(E)+d+1}(A_1)$$

An thus, for all  $E$  we have

$$N_1(E) \leq N_0(E) + d.$$

Since  $A_0$  and  $A_1$  now play symmetric roles we also have

$$\forall n, \lambda_{n+d}(A_0) \geq \lambda_n(A_1).$$

and thus

$$N_0(E) \leq N_1(E) + d.$$

The final claim of the theorem follows.

## 4. APPLICATIONS

**4.1. Quantum graphs.** Quantum graphs are now well-studied objects from mathematical physics (see [9] for an introduction). A very rough way of defining a (finite) quantum graph is the following.

Pick  $K$  positive real numbers (the lengths), set  $\mathcal{H} = \bigoplus_{i=1}^K L^2(0, L_i)$  and  $\mathcal{D} = \bigoplus_{i=1}^K \mathcal{C}_0^\infty(0, L_i)$  and define  $A$  on  $\mathcal{D}$  by  $A(u_1 \oplus u_2 \oplus \dots \oplus u_K) = -(u_1'' \oplus u_2'' \oplus \dots \oplus u_K'')$ . This operator is symmetric. Any self-adjoint extension of  $A$  is called a quantum graph.

**Remark 4.1.** Usually quantum graphs are constructed starting from a combinatorial graph given by its edges and vertices. This combinatoric data is actually hidden in the choice of the self-adjoint condition.

One basic question is to understand to which extent the knowledge of the spectrum determines the quantum graph (i.e. the lengths and the boundary condition).

It is known that there are isospectral quantum graphs [1] and the following theorem says that, as far as counting function is concerned it is quite difficult to determine the self-adjoint condition.

**Theorem 4.** *For any quantum graph with  $K$  edges the following bound holds :*

$$\left| N(E) - \frac{\mathcal{L}}{\pi} E_+^{\frac{1}{2}} \right| \leq 3K$$

where  $\mathcal{L} := \sum_{i=1}^K L_i$  and  $E_+ := \max(E, 0)$ .

*Proof.* Fix  $K$  and the choice of the lengths. We choose one particular self-adjoint extension  $A_D$  that consists in decoupling all the edges and putting Dirichlet boundary condition on each end of each edge. The spectrum is then easily computed and we have

$$\text{spec}(A_D) = \bigcup_{i=1}^K \left\{ \frac{k^2 \pi^2}{L_i^2}, k \in \mathbb{N} \right\}.$$

In particular we have (denoting by  $N_D$  the counting function of the Dirichlet extension)

$$N_D(E) = \sum_{i=1}^K \left[ \frac{L_i}{\pi} E_+^{\frac{1}{2}} \right],$$

where  $[\cdot]$  denotes the integer part. In particular, we have

$$\left| N_D(E) - \frac{\mathcal{L}}{\pi} E_+^{\frac{1}{2}} \right| \leq K.$$

We now compute the deficiency indices. A straightforward computation yields  $d_+ = d_- = 2K$ . And thus, using the main theorem and triangular inequality we obtain that for any quantum graph

$$\left| N_D(E) - \frac{\mathcal{L}}{\pi} E_+^{\frac{1}{2}} \right| \leq 3K,$$

independently of the choice of lengths. □

**4.2. Pseudo-Laplacian with Delta potentials.** It is known that on Riemannian manifolds of dimension 2 or 3 it is possible to add so-called Delta potentials. From a spectral point of view this corresponds to choosing a finite set of points  $P$  and to consider the Riemannian Laplace operator defined on smooth functions with support in  $M \setminus P$ . (see [3] for instance)

**Remark 4.2.** This construction is also possible starting from a bounded domain in  $\mathbb{R}^2$  with, say, Dirichlet boundary condition. We obtain the so-called Šeba billiards (See [6] for instance)

There are several self-adjoint extensions and (a slight generalization of) Lemma of [3] proves that, in this setting the deficiency indices are  $d := \text{card } P$ . Following Colin de Verdière we call any such self-adjoint extension a *Pseudo-Laplacian with  $d$  Delta potentials*

Application of the theorem gives the following.

**Theorem 5.** *Let  $M$  be a closed Riemannian manifold of dimension 2 or 3. Let  $N_0$  be the counting function of the (standard) Laplace operator on  $M$ . For any pseudo-laplacian with  $d$  Delta potential the following holds*

$$|N(E) - N_0(E)| \leq d$$

It should be noted first that the bound depends only on the number of Delta potentials and not on their location, and second that the effect of adding Delta potentials is much smaller than the usual known remainder terms in Weyl's law for  $N_0(E)$ .

**4.3. Others.** There are two other settings where finite deficiency indices occur that are worth mentioning. In both case one could apply the theorem to get a Weyl's asymptotic formula independent of the choice of the self-adjoint condition up to a  $O(1)$  term. These are

- (1) Manifolds with conical singularities. The common self-adjoint extension in use corresponds to Friedrichs extension and if one changes the self-adjoint extensions at the conical points (see [8, 7]) then the counting function is affected only by some bounded correction. Observe that the deficiency index associated with the Laplace operator on the cone of opening angle  $\alpha$  is  $2\left[\frac{\alpha}{2\pi}\right] - 1$  (with  $[\cdot]$  the integer part).
- (2) The so-called *hybrid manifolds* that are studied in [10] which are obtained by, in some sense, grafting quantum graphs onto higher dimensional manifolds. Here again, one can compare the counting function of the chosen self-adjoint extension to the natural one which is obtained when all the parts are decoupled.

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