Input-output framework for robust stability of time-varying delay systems

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Abstract

The paper is devoted to the stability analysis of linear time varying delay. We first model the time varying delay system as an interconnected system between a known linear transformation and some operators depending explicitly on the delay. Embedding operators related to the delay into an uncertain set, stability of such system is then performed by adopting the quadratic separation approach. Having recognized that the conservatism comes from the choice of the feedback modeling and the operators definition, these first results are afterwards enhanced by using some redundant equation and scaling filter. At last, numerical examples are given to illustrate the results.

1 INTRODUCTION

Stability of linear time-delay systems has been intensively studied since several decades (see [8], [21], [10] and references therein). A such success can be explained by their applied aspect. Indeed, many processes include dead-time phenomena such as biology, chemistry, economics, as well as population dynamics [17] [18]. Processing time and propagation time in actuators and sensors generally induce also such delays, especially if some devices are physically distant. That is the challenge of networked controlled systems [4] as well as network control [22].

In the case of constant delay and unperturbed linear systems, efficient criteria based on roots location [19] allow to find the exact region of stability with respect to the value of the delay. Beside these direct methods, numerous works based either on Lyapunov functionals [10][21][6] or robustness framework (small gain theory [10], IQC [14] or quadratic separation [9]) have established interesting results to tackle the robust stability of delay systems with practical tools (like LMI). All resulting stability conditions are based on convex optimization (linear matrix inequality framework) and allow to conclude on stability intervals with respect to the delay and/or the uncertainty.

Regarding the case of time-varying delay systems, some authors have extended the upper cited results to address the stability issue of such systems. Nevertheless, the time-varying nature of the delay should be carefully handled rather than roughly adapted from methods originally developed for the constant delay case. In the Lyapunov-Krasovskii approach few studies [23][7][12][1] have proposed customized functionals able to significantly improve classical results. In the input-output approach [10][8][15][2], some terms (or operators) related to the delayed dynamics are embedded into an uncertain matrix and the method consists in ensuring the robust stability of the nominal system with respect to the uncertain domain. Hence, in one hand, the key issue relies on the choice of the interconnection modeling the delay systems (and thus the uncertain set), and in other hand, on the L_2-norm bounds which fit the uncertain set. Although the Lyapunov and the input-ouput approaches are closely related [24][9], the second one states clearly the reasons of the conservatism and how it may be reduced.

In this paper, the quadratic separation principle, belonging to the input-output framework, is considered to deal with the stability analysis of linear time-varying delay systems. First, such systems are modeled as the interconnection of a linear matrix equation with an uncertain matrix of operators. Secondly, based on previous results [2] and [15] which provide bounds on some operators, integral quadratic constraints are built. At last, a redundant equation is introduced to construct a new modeling of the delay systems. To this end, an augmented state is considered which is composed of the original state vector and its derivatives. Then defining relationship between augmented states $\dot{x}$, $\ddot{x}$, the delay h and its derivative $\dot{h}$ as a set of integral quadratic constraints allows to improve the stability criterion. Conditions are expressed in terms of linear matrix inequalities (LMI) which can be solved efficiently with semi-definite programming (SDP) solvers.

After the introduction, the paper carries on with the definition of some operators and preliminaries on quadratic separation useful to present the main result. In section 3, the prior result on robust stability is exploited to derive a stability condition for time-varying delay systems. The robust stability for the case of uncertain systems is also addressed. Numerical examples that show the effectiveness of the proposed critera is provided in section 4.

2 PRELIMINARIES

2.1 Notations and problem statement

Throughout the paper, the following notations are used. The set of L_2^n consists of all measurable functions $f : \mathbb{R}^+ \to \mathbb{C}^n$ such that the following norm $\|f\|_{L_2} = \left(\int_0^\infty (f^*(t) f(t)) \right)^{1/2} dt < \infty$. When context allows it, the superscript n of the dimension will be omitted. the set L_{2e}^n denotes the extended set of L_2^n which consists of the functions whose time truncation lies in L_2^n. For two symmetric matrices, A and B, $A > (\geq) B$ means that $A - B$ is (semi-) positive definite. A^T denotes the transpose of A. 1_n and $0_{m\times n}$ denote respectively the identity matrix of size n and null

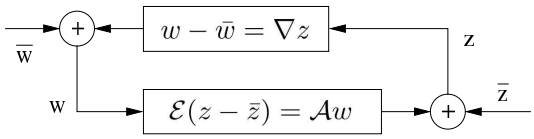

Figure 1: Feedback system

matrix of size $m \times n$. If the context allows it, the dimensions of these matrices are often omitted. $\mathrm{diag}(A,B,C)$ stands for the block diagonal matrix:

$$\mathrm{diag}(A,B,C) = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix}.$$

Let consider the following time-varying delay system:

$$\begin{cases} \dot{x}(t) = Ax(t) + A_d x(t - h(t)) & \forall t \geq 0, \\ x(t) = \phi(t) & \forall t \in [-h_{max}, 0] \end{cases} \tag{1}$$

where $x(t) \in \mathsf{R}^{\mathsf{n}}$ is the state vector, ϕ is the initial condition and A, $A_d \in \mathsf{R}^{\mathsf{n}\times\mathsf{n}}$ are constant matrices. The delay h is time-varying and the following constraints are assumed

$$h(t) \in [0, h_{\max}] \quad \text{and} \quad |\dot{h}(t)| \leq d, \tag{2}$$

where $h_{\max}$ and d are given scalar constants and may be infinite if delay independent condition and fast-varying delay condition, respectively, are looked for.

2.2 Stability analysis via quadratic separation

Coming from robust control theory, the quadratic separation provides a fruitful framework to address the stability issue of non-linear and uncertain systems [13], [20]. Recent studies [9] have shown that a such framework allows to reduce significantly the conservatism of the stability analysis of time-delay systems with constant delay. Then, in order to deal with the time-varying delay case, the quadratic separation method has been extended in [2] to handle not only the case of uncertain matrices but more generally uncertain operators. Indeed, based on the inner product and the L_{2e} space a suitable theorem is then proposed. This latter will be later used to derive stability conditions for time-varying delay systems.

Let consider the interconnection defined by Figure 1 where $\mathscr{E}$ and $\mathscr{A}$ are two, real valued, possibly non-square matrices and ∇ is a linear operator from L_{2e} to L_{2e}. For simplicity of notations, we assume in the present paper that $\mathscr{E}$ is full column rank. Assuming the well-posedness, we are interested in looking for conditions that ensure the stability of the interconnection.

Theorem 1 *The interconnected system of Figure 1 is stable if there exists a Hermitian matrix* $\Theta = \Theta^*$ *satisfying both conditions*

$$\begin{bmatrix} \mathscr{E} & -\mathscr{A} \end{bmatrix}^{\perp *} \Theta \begin{bmatrix} \mathscr{E} & -\mathscr{A} \end{bmatrix}^{\perp} > 0 \tag{3}$$

$$\forall u \in L_{2e}, \langle \begin{bmatrix} 1 \\ \nabla \end{bmatrix} u, \Theta \begin{bmatrix} 1 \\ \nabla \end{bmatrix} u \rangle \leq 0 \tag{4}$$

Proof 1 *Inspired from [20], the proof is detailed in [2].*

Basically, inequality (4) which forms an integral quadratic constraint, is built from definitions and informations on different operators which compose the matrix ∇. Then, the other one (3) provides the stability condition of the interconnection.

3 MAIN RESULTS

3.1 Defining operators

Toward modeling delay system as an interconnected system such as illustrated on Figure 1, it is required to define appropriate operators. Define the integral operator

$$\begin{aligned} \mathscr{I}: \quad & L_{2e} \to L_{2e}, \\ & x(t) \to \int_0^t x(\theta) d\theta, \end{aligned} \tag{5}$$

and the delay operator (or shift operator)

$$\begin{aligned} \mathscr{D}: \quad & L_{2e} \to L_{2e}, \\ & x(t) \to x(t-h), \end{aligned} \tag{6}$$

which constitute the fundamental elementary operators to describe a delay system. The related integral quadratic constraints are introduced in the following two lemmas. These latters will be helpful to construct inequality (4) and to derive then stability criteria for linear systems with time-varying delays in the next section.

Lemma 1 *An integral quadratic constraint for the operator* $\mathscr{I}$ *is given by the following inequality* $\forall x \in L_{2e}^n$ *and for a positive definite matrix P,*

$$\langle \begin{bmatrix} 1_n \\ \mathscr{I} 1_n \end{bmatrix} x, \begin{bmatrix} 0 & -P \\ -P & 0 \end{bmatrix} \begin{bmatrix} 1_n \\ \mathscr{I} 1_n \end{bmatrix} x \rangle < 0$$

Proof 2 *Simple calculus shows that* $\forall T > 0, \forall x \in L_{2e}^n$, *(x being truncated:* $x(t) = 0,\ \forall t > T$*)*

$$
\begin{aligned}
&\langle \begin{bmatrix} 1_n \\ \mathscr{I}1_n \end{bmatrix} x, \begin{bmatrix} 0 & -P \\ -P & 0 \end{bmatrix} \begin{bmatrix} 1_n \\ \mathscr{I}1_n \end{bmatrix} x \rangle \\
&= -2 \int_0^T x(t)^T P \int_0^t x(s) ds \\
&= -2 \int_0^T \frac{d}{dt} (\mathscr{I}x)^T P (\mathscr{I}x) dt \\
&= -(\textstyle\int_0^T x(s) ds)^T P (\int_0^T x(s) ds) < 0
\end{aligned}
$$

The second step is to derive a parameterized IQC for the operator $\mathscr{D}$:

Lemma 2 *An integral quadratic constraint for the operator* $\mathscr{D}$ *is given by the following inequality* $\forall T > 0, \forall x \in L_{2e}^n$ *and for a positive matrix Q,*

$$
\langle \begin{bmatrix} 1_n \\ \mathscr{D}1_n \end{bmatrix} x, \begin{bmatrix} -Q & 0 \\ 0 & Q(1-\dot{h}) \end{bmatrix} \begin{bmatrix} 1_n \\ \mathscr{D}1_n \end{bmatrix} x \rangle < 0 \tag{7}
$$

Proof 3 *We get that* $\forall T > 0, \forall x \in L_{2e}^n$, *(x being truncated:* $x(t) = 0,\ \forall t > T$*)*

$$
\begin{aligned}
&\langle \begin{bmatrix} 1_n \\ \mathscr{D}1_n \end{bmatrix} x, \begin{bmatrix} -Q & 0 \\ 0 & Q(1-\dot{h}) \end{bmatrix} \begin{bmatrix} 1_n \\ \mathscr{D}1_n \end{bmatrix} x \rangle \\
&= -\textstyle\int_0^{+\infty} x^T(u) Q x(u) du + \int_0^{\infty} x_d^T(t) Q x_d(t) (1-\dot{h}(t)) dt \\
&= -\textstyle\int_0^{+T} x^T(t) Q x(t) dt + \int_{-h(0)}^{T-h(T)} x^T(u) Q x(u) du \\
&= -\textstyle\int_{T-h(T)}^{T} x(u)^T Q x(u) du < 0
\end{aligned}
$$

where $x_d(t) = x(t - h(t))$.

In the next paragraph, applying the prior result exposed in Section 2.2 a rate and delay dependent stability condition for time-varying delay systems is provided.

3.2 Stability condition for time-varying delay systems

First, let us reformulate the dynamic of linear systems with time-varying delay as suggested on Figure 1 in order to apply the quadratic separation principle. System (1) can be described as the feedback

$$\underbrace{\begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}}_{w(t)} = \underbrace{\begin{bmatrix} \mathscr{I}1_n & \\ & \mathscr{D}1_n \end{bmatrix}}_{\nabla} \underbrace{\begin{bmatrix} \dot{x}(t) \\ x(t) \end{bmatrix}}_{z(t)}, \tag{8}$$

over the feedforward equation

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\mathscr{E}} \underbrace{\begin{bmatrix} \dot{x}(t) \\ x(t) \end{bmatrix}}_{z(t)} = \underbrace{\begin{bmatrix} A & A_d \\ 1 & 0 \end{bmatrix}}_{\mathscr{A}} \underbrace{\begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}}_{w(t)}.$$

This simplistic description of the system 1 gives rise, applying Theorem 1, to the well-known independent of delay (IOD) criterion [2][9]. Secondly, so as to develop delay dependent condition, an additional operator must be blended into ∇, enhancing then the time delay system description. Usually, the operator $(1-\mathscr{D})\circ\mathscr{I}$ (or in Laplace domain for the constant delay case $(1-e^{-sh})s^{-1}$), bounded by h_{max}, is added. This operator is applied to the signal $\dot{x}(t)$ and the relationship

$$\begin{aligned}(1-\mathscr{D})\circ\mathscr{I}[\dot{x}_i(t)] &= (1-\mathscr{D})\circ\mathscr{I}[\mathscr{I}^{-1}[x_i(t)]] \\ &= x_i(t)-x_i(t-h(t)),\ i=\{1,...,n\}\end{aligned}$$

should be specified in the linear equation $\mathscr{E}z(t)=\mathscr{A}w(t)$. In this paper, inspired from [15], instead of the integrator $\mathscr{I}$, a scaling filter of dimension n_ϕ of the form

$$\begin{cases} \dot{x}_\phi(t) = A_\phi x_\phi(t) + B_\phi u(t), \\ y(t) = C_\phi x_\phi(t) + D_\phi u(t), \end{cases} \tag{9}$$

is considered. The key idea is now to apply the new operator $(1-\mathscr{D})\circ\phi^{-1}$ to the filtered signal $y(t,x_i(t))$ (each component of the state vector $x(t)$ of system (1) is processed $u(t)=x_i(t),\ i=\{1,...,n\}$):

$$\begin{aligned}(1-\mathscr{D})\circ\phi^{-1}[y(t,x_i(t))] &= (1-\mathscr{D})\circ\phi^{-1}[\phi[x_i(t)]] \\ &= x_i(t)-x_i(t-h(t)),\ i=\{1,...,n\}\end{aligned}$$

The dynamical system (9) should be designed according to the following lemma.

Lemma 3 *An integral quadratic constraint for the operator $\mathscr{F}=(1-\mathscr{D})\circ\phi^{-1}$ is given by the following inequality*

$\forall x \in L_{2e}^n$ *and for a positive definite matrix R,*

$$\langle \begin{bmatrix} 1_n \\ \mathscr{F}1_n \end{bmatrix} x, \begin{bmatrix} -R & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} 1_n \\ \mathscr{F}1_n \end{bmatrix} x \rangle < 0,$$

where ϕ*, defined in (9), is a realization of any bounded rational transfert function* Φ *which satisfies*

$$\begin{cases} |\Phi(j\omega)| > 1 + \frac{1}{\sqrt{1-d}}, & \text{if } h_{max}|\omega| > 1 + \frac{1}{\sqrt{1-d}}, \\ |\Phi(j\omega)| > h_{max}|\omega|, & \text{if } h_{max}|\omega| \leq 1 + \frac{1}{\sqrt{1-d}}. \end{cases} \tag{10}$$

Proof 4 *In [15], it is shown that for all systems* ϕ *satisfying the above specifications* $\|(1-\mathscr{D})\circ\phi^{-1}\|_{L_2} \leq 1$ *holds. It means that for any v, a* L_2^n *function,*

$$\|\mathscr{F}v\|_{L_2} \leq \|v\|_{L_2}$$

$$\int_0^\infty (\mathscr{F}v(t))^T \mathscr{F}v(t) - v^T(t)v(t)dt \leq 0$$

is satisfied. Defining $v(t) = R^{1/2}x(t)$*, R being a symmetric positive definite matrix, we have*

$$\int_0^\infty (\mathscr{F}x(t))^T R \mathscr{F}x(t) - x^T(t)Rx(t)dt \leq 0.$$

Factorizing on both sides by $[x^T(t) \;\; (\mathscr{F}x(t))^T]$ *and its transposed, the IQC of the lemma is recovered.*

An example of $\Phi(s)$, proposed by [15], satisfying (10) is

$$\Phi(s) = k\frac{h_{max}^2 s^2 + ch_{max}s}{h_{max}^2 s^2 + ah_{max}s + b} \tag{11}$$

where $k = \sqrt{8/(2-d)}$, $a = \sqrt{6.5+2b}$, $b = \sqrt{50}$ and $c = \sqrt{12.5}$.

The time-varying delay system (1) is now modeled as the interconnection of

$$\underbrace{\begin{bmatrix} x(t) \\ \tilde{x}_\phi(t) \\ x(t-h(t)) \\ v_1(t) \end{bmatrix}}_{w(t)} = \nabla \underbrace{\begin{bmatrix} \dot{x}(t) \\ \dot{\tilde{x}}_\phi(t) \\ x(t) \\ \tilde{\phi}[x] \end{bmatrix}}_{z(t)} \tag{12}$$

with $\nabla = \text{diag}\left(\mathscr{I}1_{\mathsf{n}}, \mathscr{I}1_{\mathsf{n}_\phi \mathsf{n}}, \mathscr{D}1_{\mathsf{n}}, (1-\mathscr{D}) \circ \phi^{-1}1_{\mathsf{n}}\right)$ and

$$\underbrace{\begin{bmatrix} 1_{\mathsf{n}(3+\mathsf{n}_\phi)} \\ 0 \end{bmatrix}}_{\mathscr{E}} z(t) = \underbrace{\begin{bmatrix} A & 0 & A_d & 0 \\ \tilde{B}_\phi & \tilde{A}_\phi & 0 & 0 \\ 1_{\mathsf{n}} & 0 & 0 & 0 \\ \tilde{D}_\phi & \tilde{C}_\phi & 0 & 0 \\ 1_{\mathsf{n}} & 0 & -1 & -1 \end{bmatrix}}_{\mathscr{A}} w(t) \tag{13}$$

where $v_1(t) = x(t) - x(t-h(t))$,

$$\begin{array}{rclrcl} \tilde{A}_\phi & = & 1_{\mathsf{n}} \otimes A_\phi, & \tilde{B}_\phi & = & 1_{\mathsf{n}} \otimes B_\phi, \\ \tilde{C}_\phi & = & 1_{\mathsf{n}} \otimes C_\phi, & \tilde{D}_\phi & = & 1_{\mathsf{n}} \otimes D_\phi, \\ \tilde{x}_\phi & = & \begin{bmatrix} x_{\phi 1} \\ \vdots \\ x_{\phi n} \end{bmatrix}, & \tilde{\phi}[x] & = & \begin{bmatrix} \phi[x_1] \\ \vdots \\ \phi[x_n] \end{bmatrix}, \end{array} \tag{14}$$

x_i are the components of the state vector x. Note that there is one filter of dimension n_ϕ associated to each x_i. At this point, refering to the quadratic separation approach, Theorem 1 may be applied.

Theorem 2 *For given positive scalars $h_{\max}$ and d, if there exists positive definite matrices $P \in \mathbb{R}^{n(1+n_\phi)\times n(1+n_\phi)}$ and Q, $R \in \mathbb{R}^{n\times n}$, then system (1) with a time varying delay constrained by (2) is asymptotically stable if the LMI condition (3) holds with Θ (setting $\dot{h} = d$), $\mathscr{E}$ and $\mathscr{A}$ defined as (15) and (13).*

Proof 5 *First, condition (4) must be satisfied for ∇ defined as (12). Invoking all Lemmas previously defined and combining all inequalities related to each operator, it is readily seen that the separator (15) fulfills the inequality (4), where*

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{12}^* & \Theta_{22} \end{bmatrix}, \begin{array}{rl} \Theta_{11} & = \text{diag}(0_{\mathsf{n}(1+\mathsf{n}_\phi)}, -Q, -R), \\ \Theta_{12} & = \text{diag}(-P, 0_{2\mathsf{n}}), \\ \Theta_{22} & = \text{diag}(0_{\mathsf{n}(1+\mathsf{n}_\phi)}, (1-\dot{h})Q, R), \end{array} \tag{15}$$

and $P \in \mathbb{R}^{n(1+n_\phi)\times n(1+n_\phi)}$ and $Q, R \in \mathbb{R}^{n\times n}$ are positive definite matrices. It is readily seen that inequality (7) still holds for $\dot{h} = d$. Hence, regarding the separator, Θ_{22} can be chosen as $\text{diag}(0, (1-d)Q, R)$. *Then, the filter ϕ may be chosen as a realization of (11) defined by*

$$\left\{ \begin{array}{rcl} \dot{x}_{\phi i}(t) & = & \begin{bmatrix} 0 & 1 \\ -\frac{b}{h_{max}^2} & -\frac{a}{h_{max}} \end{bmatrix} x_{\phi i}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y_i(t) & = & \begin{bmatrix} -\frac{kb}{h_{max}^2} & \frac{kc-ka}{h_{max}} \end{bmatrix} x_{\phi i}(t) + ku(t) \end{array} \right. ,$$

where $x_{\phi i},\ i = \{1,...,n\}$, *represents the different states of the same filter according to the different inputs* $u(t) = \{x_1(t),...,x_n(t)\}$. *Considering Theorem 1 where the interconnection is given by (13)-(12), and (4) being proved, the LMI (3) forms the stability criterion.*

3.3 Model extension

Previous works [5] and [3], [9] have shown that redundant system modeling (for linear uncertain systems and constant delay systems, respectively) may increase the relevancy of the stability analysis. The rational behind this model extension is to provide some extra relations between the delay, its variations and the state. Using the derivative operator, an augmented state is constructed which is composed of the original state vector and its derivatives. Then defining relationship between augmented states $\dot{x}$, $\ddot{x}$, the delay h and its derivative $\dot{h}$ an enhanced stability condition is provided. Differentiating the system (1), we get:

$$\ddot{x}(t) = A\dot{x}(t) + (1-\dot{h}(t))A_d\dot{x}(t-h(t)).$$

Consider the artificially augmented system

$$\begin{cases} \dot{x}(t) = Ax(t) + A_d x(t-h(t)), \\ \ddot{x}(t) = A\dot{x}(t) + (1-\dot{h}(t))A_d\dot{x}(t-h(t)), \end{cases} \tag{16}$$

with accordingly defined initial conditions. Introducing the augmented state

$$\zeta(t) = \begin{bmatrix} \dot{x}(t) \\ x(t) \end{bmatrix},$$

and specifying the relationship between the two components of $\zeta(t)$ with the equality $[0\ \ 1]\dot{\zeta}(t) = [1\ \ 0]\zeta(t)$, we have the new descriptor augmented system

$$E\dot{\zeta}(t) = \bar{A}\zeta(t) + \bar{A}_d\zeta(t-h(t)), \tag{17}$$

where

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \bar{A} = \begin{bmatrix} A & 0 \\ 0 & A \\ 0 & 1 \end{bmatrix}, \bar{A}_d = \begin{bmatrix} A_d & 0 \\ 0 & (1-\dot{h})A_d \\ 0 & 0 \end{bmatrix}.$$

Then, the new time-varying delay system (17) can be shaped as Figure 1 with

$$\underbrace{\begin{bmatrix} \zeta(t) \\ \tilde{x}_\phi(t) \\ \zeta(t-h(t)) \\ v_2(t) \end{bmatrix}}_{w(t)} = \nabla \otimes 1_2 \underbrace{\begin{bmatrix} \dot{\zeta}(t) \\ \dot{\tilde{x}}_\phi(t) \\ \zeta(t) \\ \tilde{\phi}[\zeta] \end{bmatrix}}_{z(t)} \tag{18}$$

and

$$\underbrace{\begin{bmatrix} \mathrm{diag}\left(E, 1_{2n(2+n_\phi)}\right) \\ 0 \end{bmatrix}}_{\mathscr{E}} z(t) = \underbrace{\begin{bmatrix} \bar{A} & 0 & \bar{A}_d & 0 \\ \tilde{B}_\phi & \tilde{A}_\phi & 0 & 0 \\ 1_n & 0 & 0 & 0 \\ \tilde{D}_\phi & \tilde{C}_\phi & 0 & 0 \\ 1_n & 0 & -1 & -1 \end{bmatrix}}_{\mathscr{A}} w(t) \tag{19}$$

where $v_2(t) = \zeta(t) - \zeta(t-h(t))$, ∇ defined as (12) and redefining

$$\begin{aligned} \tilde{A}_\phi &= 1_{2n} \otimes A_\phi, \qquad \tilde{B}_\phi = 1_{2n} \otimes B_\phi, \\ \tilde{C}_\phi &= 1_{2n} \otimes C_\phi, \qquad \tilde{D}_\phi = 1_{2n} \otimes D_\phi, \\ \tilde{x}_\phi &= \begin{bmatrix} x_{\phi 1} \\ \vdots \\ x_{\phi 2n} \end{bmatrix}, \quad \tilde{\phi}[\zeta] = \begin{bmatrix} \phi[\dot{x}_1] \\ \phi[x_1] \\ \vdots \\ \phi[\dot{x}_n] \\ \phi[x_n] \end{bmatrix}. \end{aligned} \tag{20}$$

Following the same line than in the previous section, we propose:

Theorem 3 *For given positive scalars $h_{\max}$ and d, if there exists positive definite matrices $P \in \mathbb{R}^{2n(1+n_\phi)\times 2n(1+n_\phi)}$, $Q, R \in \mathbb{R}^{2n\times 2n}$ and a matrix $X \in \mathbb{R}^{2n(6+2n_\phi)\times n(7+2n_\phi)}$, then system (1) with a time varying delay constrained by (2) is asymptotically stable if the LMI condition*

$$\Theta^{[j]} + X[\mathscr{E} \ -\mathscr{A}^{[j]}] + (X[\mathscr{E} \ -\mathscr{A}^{[j]}])^T > 0$$

holds for $j = 1,2$, with $\mathscr{E}$ and $\mathscr{A}$ defined as (19). Θ is of the form of (15) with appropriate dimension. $\Theta^{[j]}$ and $\mathscr{A}^{[j]}$ are the vertices of, respectively, Θ and $\mathscr{A}$ setting $\dot{h}(t) = d_j$. The constraint (2) gives $d_1 = -d$ and $d_2 = d$.

Proof 6 *First, It follows the same idea that the one of Theorem 2. Hence, the stability of (17) will be proved if*

$$\left[\begin{array}{cc} \mathscr{E} & -\mathscr{A}(\dot{h}(t)) \end{array}\right]^{\perp *} \Theta(\dot{h}(t)) \left[\begin{array}{cc} \mathscr{E} & -\mathscr{A}(\dot{h}(t)) \end{array}\right]^{\perp} > 0$$

with $\mathscr{E}$, $\mathscr{A}$ defined as (19) and Θ is of the form of (15) with appropriate dimension. Applying the Finsler's lemma, the above inequality is equivalent to

$$\Theta(\dot{h}(t)) + X[\mathscr{E} \quad -\mathscr{A}(\dot{h}(t))] + (X[\mathscr{E} \quad -\mathscr{A}(\dot{h}(t))])^T > 0 \tag{21}$$

where X is a new decision variable of appropriate dimensions. Since, $\dot{h}(t)$ appears linearly and is bounded, invoking argument of convexity, it is sufficient to test (21) on its vertices. The inequality proposed by Theorem 3 is thus recovered. System (17) being stable, the whole state $\zeta(t)$ converges asymptotically to zero. Hence, its components $x(t)$ converge as well. The original system (1) is thus also asymptotically stable.

3.4 Robustness issue

Coming from robust control, quadratic separation provides a suitable framework to address the stability analysis of uncertain delay systems:

$$\dot{x}(t) = A(\Delta)x(t) + A_d(\Delta)x(t - h(t)) \tag{22}$$

where

$$\left[\begin{array}{cc} A(\Delta) & A_d(\Delta) \end{array}\right] = \left[\begin{array}{cc} A & A_d \end{array}\right] + B\Delta \left[\begin{array}{cc} C & C_d \end{array}\right].$$

The second term of the right hand side of the above equation describes the uncertainty characterizing system (22). The uncertain time-varying matrix Δ, belonging to $\mathbf{\Omega}$, satisfies

$$\Delta^T(t)\Delta(t) \leq 1, \ \forall \mathsf{t} \geq 0, \ \forall \Delta \in \Omega, \tag{23}$$

and models non-linear and neglected dynamics as well as parametric uncertainties. C, C_d and B are constant matrices of appropriate dimensions which structure the uncertainty. Then, according to the set of admissible uncertainties and (23), we have to find a separator U such that

$$\left\langle \left[\begin{array}{c} 1 \\ \Delta \end{array}\right] x, \underbrace{\left[\begin{array}{cc} U_1 & U_2 \\ U_2^* & U_3 \end{array}\right]}_{U} \left[\begin{array}{c} 1 \\ \Delta \end{array}\right] x \right\rangle < 0, \ \forall \Delta \in \mathbf{\Omega}. \tag{24}$$

For instance, assume Ω is a set of diagonal real valued matrices with bounded uncertainties:

$$\Omega = \left\{ \Delta = \text{diag}(\delta_1, ..., \delta_N) \mid |\delta_i| \leq \bar{\delta}_i \right\},$$

Then, inequality (24) holds with

$$U = \text{diag}\left(-\bar{\delta}_1^2 u_1, \ldots, -\bar{\delta}_N^2 u_N, u_1, \ldots, u_N\right)$$

where u_i, $i = \{1, ..., N\}$ are scalar decision variables. Eventually, we propose to analyze the robust stability of system (22) with the following Theorem.

Theorem 4 *For given positive scalars $h_{\max}$ and d, if there exists positive definite matrices $P \in \mathbb{R}^{n(1+n_\phi)\times n(1+n_\phi)}$, Q, $R \in \mathbb{R}^{n\times n}$ and matrices U_k ($k = 1,2,3$) designed according to the uncertain set Ω (i.e. such that (24) holds), then system (22) with a time varying delay constrained by (2) is robustly asymptotically stable for any uncertainty $\Delta \in \Omega$ if the LMI condition (3) holds with Θ, $\mathscr{E}$ and $\mathscr{A}$ defined as (26) and (25).*

Proof 7 *First, introducing the exogenous signals*

$$w_\Delta = \Delta z_\Delta, \text{ with } z_\Delta = Cx(t) + C_d x(t - h(t)),$$

we rewritte system (22) as the interconnection of

$$\underbrace{\begin{bmatrix} x(t) \\ \tilde{x}_\phi(t) \\ x(t-h(t)) \\ v_1(t) \\ w_\Delta(t) \end{bmatrix}}_{w(t)} = \nabla \underbrace{\begin{bmatrix} \dot{x}(t) \\ \dot{\tilde{x}}_\phi(t) \\ x(t) \\ \tilde{\phi}[x] \\ z_\Delta(t) \end{bmatrix}}_{z(t)}$$

with $\nabla = \left(\mathscr{I}1_n, \mathscr{I}1_{n_\phi n}, \mathscr{D}1_n, \mathscr{F}1_n, \Delta\right)$ *and*

$$\underbrace{\begin{bmatrix} 1_{n(3+n_\phi)+n_\Delta} \\ 0 \end{bmatrix}}_{\mathscr{E}} z(t) = \underbrace{\begin{bmatrix} A & 0 & A_d & 0 & B \\ \tilde{B}_\phi & \tilde{A}_\phi & 0 & 0 & 0 \\ 1_n & 0 & 0 & 0 & 0 \\ \tilde{D}_\phi & \tilde{C}_\phi & 0 & 0 & 0 \\ C & 0 & C_d & 0 & 0 \\ 1_n & 0 & -1 & -1 & 0 \end{bmatrix}}_{\mathscr{A}} w(t) \tag{25}$$

Table 1: The maximal allowable delays h_m for system (27)

d	0	0.1	0.2	0.5	0.8	1
Fridman et al (2002) [7]	4.472	3.604	3.033	2.008	1.364	0.999
Fridman et al (2006) [8]	1.632	1.632	1.632	1.632	1.632	1.632
Wu et al (2004) [23]	4.472	3.604	3.033	2.008	1.364	-
He et al (2007) [11]	4.472	3.605	3.039	2.043	1.492	1.345
He et al (2007) [12]	4.472	3.605	3.039	2.043	1.492	1.345
Ariba et al (2007) [1]	5.120	4.081	3.448	2.528	2.152	1.991
Kao et al (2007) [15]	6.117	4.714	3.807	2.280	1.608	1.360
Theorem 2	6.117	4.714	3.807	2.280	1.608	1.360
Theorem 3	6.117	4.794	3.995	2.682	1.957	1.602

with notations of (14). Combining every IQC related to each operators defined by lemmas and the struture of the uncertainty leading to (24), a separator of the form

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{12}^* & \Theta_{22} \end{bmatrix}, \begin{array}{ll} \Theta_{11} & = \mathrm{diag}(0_{n(1+n_\phi)}, -Q, -R, U_1), \\ \Theta_{12} & = \mathrm{diag}(-P, 0_{2n}, U_2), \\ \Theta_{22} & = \mathrm{diag}(0_{n(1+n_\phi)}, (1-d)Q, R, U_3), \end{array} \tag{26}$$

fulfills the requirement (4). Finally, condition (3) provides the robust (with respect to the uncertain set Ω) stability criterion.

4 NUMERICAL EXAMPLES

4.1 Example 1

Consider the following system,

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} x(t - h(t)). \tag{27}$$

For various d, the maximal allowable delay, h_{max}, is computed. To demonstrate the effectiveness of our criterion, results are compared against those obtained in the literature (see Table 1). On this example, compared to Lyapunov technics, robust approaches [15], Theorem 2 and 3 reduce drastically the conservatism, especially when d is close to zero *i.e.* when the delay is slowly time varying. Using the same scaling filter for bounding operators Theorem 2 recovers the results of [15], whereas Theorem 3, taking into account the derivative equation, reduces the conservatism. Indeed, the stability analysis is further improved thanks to an appropriate modeling of time-varying delay systems which brings additional informations on the system.

Table 2: The maximal allowable delays h_m for system (28)

d	0	0.1	0.5	0.8
[16]	0.241	0.234	0.188	0.110
[23]	1.149	1.106	0.924	0.760
[15]	1.416	1.302	0.974	0.829
Theorem 4	1.515	1.422	1.105	0.910

4.2 Example 2

Consider now the following time-varying and uncertain system, extracted from [16],

$$\begin{aligned}\dot{x}(t) = &\begin{bmatrix} -2+\delta_1 cos(t) & 0 \\ 0 & -1+\delta_2 sin(t) \end{bmatrix} x(t) \\ &+ \begin{bmatrix} -1+\gamma_1 cos(t) & 0 \\ -1 & -1+\gamma_2 sin(t) \end{bmatrix} x(t-h(t)).\end{aligned} \tag{28}$$

The δ_i and γ_i are uncertain but bounded parameters:

$$|\delta_1| \leq 1.6,\ |\delta_2| \leq 0.05,\ |\gamma_1| \leq 0.1,\ |\gamma_2| \leq 0.3.$$

This example can be expressed as system (22) with

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix},\ A_d = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix},$$
$$B = 1_2,\ C = \mathrm{diag}(1.6,\ 0.05),\ C_\mathrm{d} = \mathrm{diag}(0.1,\ 0.3).$$

Results are summarized in Table 2. It shows that Theorem 4 enables to find higher maximal bounds on the delay $h(t)$ than others results from the literature. The quadratic separation offers thus a suitable framework to address uncertainties. In that case, the conservatism is also related to the manner to handle uncertainties (as the design of (24)).

5 CONCLUSIONS

In this paper, the problem of the delay dependent stability analysis of a time-varying delay systems has been studied by means of quadratic separation. The delay part is embedded into an uncertain matrix of operators. Inspired from [2] and [15], tight bounds of the L_2 induced norms of operators allow to reduce the conservatism of the approach. Then using an augmented state, new modelling of time delay systems are introduced which emphasizes the relation

between $\dot{h}$ and signals $\dot{x}$ and $\ddot{x}$. The resulting criteria are then expressed in terms of a convex optimization problem with LMI constraints, allowing the use of efficient solvers. Finally, two numerical examples show that these methods reduced conservatism and improved the maximal allowable delay.

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