HOMOGENEOUS KÄHLER AND HAMILTONIAN MANIFOLDS

BRUCE GILLIGAN, CHRISTIAN MIEBACH, AND KARL OELJEKLAUS

Dedicated to Alan T. Huckleberry

Abstract. We consider actions of reductive complex Lie groups $G = K^{\mathbb{C}}$ on Kähler manifolds X such that the K–action is Hamiltonian and prove then that the closures of the G–orbits are complex-analytic in X. This is used to characterize reductive homogeneous Kähler manifolds in terms of their isotropy subgroups. Moreover we show that such manifolds admit K–moment maps if and only if their isotropy groups are algebraic.

1. Introduction

A reductive complex Lie group G is a complex Lie group admitting a compact real form K, i. e. $G = K^{\mathbb{C}}$. Equivalently a finite covering of G is of the form $S \times Z = S \times (\mathbb{C}^*)^k$, where S is a semisimple complex Lie group. It is well known that every complex reductive Lie group admits a unique structure as a linear algebraic group. Holomorphic or algebraic actions of reductive Lie groups appear frequently in complex and algebraic geometry and interesting connections arise between the structure of the orbits of such groups and the isotropy subgroups of the orbits.

A result of this type was proved independently by Matshushima [19] and Onishchik [20]. They consider G a complex reductive Lie group and H a closed complex subgroup of G and show that G/H is Stein if and only if H is a reductive subgroup of G. In [1] Barth and Otte prove that the holomorphic separability of the homogeneous space G/H implies H is an algebraic subgroup of the reductive group G.

In the case of semisimple actions, it is known that Kähler is equivalent to algebraic in the sense that S/H is Kähler if and only if H is an algebraic subgroup of the complex semisimple Lie group S, see [2] and [3]. The simple example of an elliptic curve $\mathbb{C}^*/\mathbb{Z}$ shows that this result does not hold in the reductive case. Instead homogeneous Kähler manifolds $X = G/H$ with $G = S \times (\mathbb{C}^*)^k$ reductive are characterized by the two conditions $S \cap H$ is algebraic and $SH \subset G$ is closed, as we shall prove. If, in addition to the existence of a Kähler form, there exists a K–moment map on X, then X is called a *Hamiltonian* G–manifold. Huckleberry has conjectured that the isotropy groups in a Hamiltonian G–manifold are algebraic. In the present paper we prove that this is indeed the case.

The moment map plays a decisive role in our proof which depends in an essential way on the work of Heinzner-Migliorini-Polito [11]. In the third section of their paper

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they investigate the closure of certain orbits and prove the following: suppose T is an algebraic torus acting holomorphically on a complex space X such that the semistable quotient $\pi\colon X \to X/\!\!/T$ exists. Let A be a subanalytic set in X such that $\pi|_A\colon A \to X/\!\!/T$ is proper. Then $T \cdot A$ is subanalytic in X. We use the moment map in order to ensure the existence of the semistable T–quotient locally. This is sufficient to show that the G–orbits are locally subanalytic and hence locally closed in the Hamiltonian G–manifold X. Moreover, we deduce from this fact that the closure of any G–orbit is complex-analytic in X. This generalizes previous work of [22] and [8] to non-compact Kähler manifolds.

Our work was partially motivated by [17], where Margulis constructed discrete subgroups Γ of $\mathrm{SL}(2,\mathbb{R}) \ltimes \mathbb{R}^3$ which are free groups generated by two elements. These groups Γ can be divided into two non-empty classes depending on whether the induced action of Γ on $\mathbb{R}^3$ is properly discontinuous or not. The associated homogeneous complex manifolds $(\mathrm{SL}(2,\mathbb{C}) \ltimes \mathbb{C}^3)/\Gamma$ are not Kähler in the "non properly discontinuous" case by Corollary 3.6. It seems to be a difficult problem to decide the Kähler question for these quotients in the "properly discontinuous" case.

The paper is organized as follows. In section 2, the definitions of K–moment maps and Hamiltonian actions are recalled. Furthermore two lemmata are proved for later use. The main result of section 3 is the analyticity of orbit closures. Since for $G = S$ semisimple there always is a moment map, any semisimple Lie group action on a Kähler manifold has locally closed orbits.

In section 4 we prove that the reductive homogeneous manifold $X = G/H$ is Hamiltonian if and only if H is an algebraic subgroup of G and use this to give a new proof of the main results in [3] and [2]. Finally, in the last section this result is used in order to prove our characterization of those closed complex subgroups $H \subset G$ such that $X = G/H$ admits a Kähler form.

2. Hamiltonian G–manifolds

Let $G = K^{\mathbb{C}}$ be a complex reductive Lie group with maximal compact subgroup K. Let X be a complex manifold endowed with a holomorphic G–action.

We denote the Lie algebra of K by $\mathfrak{k}$. The group K acts via the coadjoint representation on the dual $\mathfrak{k}^*$. In the following equivariance of a map with values in $\mathfrak{k}^*$ is always meant with respect to the coadjoint action. If $\xi \in \mathfrak{k}$, we write ξ_X for the holomorphic vector field on X whose flow is given by $(t,x) \mapsto \exp(t\xi) \cdot x$. If ω is a K–invariant Kähler form on X, then the contracted form $\iota_{\xi_X}\omega$ is closed for every $\xi \in \mathfrak{k}$. By definition, a K–equivariant smooth map $\mu\colon X \to \mathfrak{k}^*$ is a moment map for the K–action on X if for each $\xi \in \mathfrak{k}$ the smooth function $\mu^\xi \in \mathcal{C}^\infty(X)$, $\mu^\xi(x) := \mu(x)\xi$, verifies $d\mu^\xi = \iota_{\xi_X}\omega$. The K–action on X is called *Hamiltonian* if an equivariant moment map $\mu\colon X \to \mathfrak{k}^*$ exists. Note that, if μ is a moment map and if $\lambda \in \mathfrak{k}^*$ is a K–fixed point, then $\mu + \lambda$ is another moment map on X.

Definition 2.1. We say that X is a *Hamiltonian G–manifold* if X admits a K–invariant Kähler form such that the K–action on X is Hamiltonian with equivariant moment map $\mu\colon X \to \mathfrak{k}^*$.

Remark. If G is semisimple, then every Kähler manifold X on which G acts holomorphically is a Hamiltonian G–manifold which can be seen as follows. Let dk be the

normalized Haar measure of K. If ω is any Kähler form on X, then $\widehat{\omega} := \int_K k^*\omega dk$ is a K–invariant Kähler form on X. Since K is semisimple, there exists a unique equivariant moment map $\mu\colon X \to \mathfrak{k}^*$ by Theorem 26.1 in [9].

In this paper we will often use the following.

Lemma 2.2. *Let X be a Hamiltonian G–manifold and let $\widetilde{G}$ be a complex reductive subgroup of G. Then every $\widetilde{G}$–stable complex submanifold $\widetilde{X}$ of X is a Hamiltonian $\widetilde{G}$–manifold.*

Proof. We may assume without loss of generality that $\widetilde{G} = \widetilde{K}^{\mathbb{C}}$ for some compact subgroup $\widetilde{K} \subset K$. Composing the moment map $\mu\colon X \to \mathfrak{k}^*$ with the orthogonal projection onto $\widetilde{\mathfrak{k}}^*$ we obtain a moment map for the $\widetilde{K}$–action on X. Restricting this map to the Kähler manifold $\widetilde{X}$ we see that the $\widetilde{K}$–action on $\widetilde{X}$ is Hamiltonian. □

Example. Let $G \to \mathrm{GL}(V)$ be a holomorphic representation of the complex reductive group G on a finite dimensional complex vector space V. Then each G–stable complex submanifold of V or of $\mathbb{P}(V)$ is a Hamiltonian G–manifold. In particular, if H is an algebraic subgroup of G, then the homogeneous space G/H is a quasi-projective variety (see e. g. Theorem 5.1 in [7]) and hence a Hamiltonian G–manifold.

For later use we note the following

Lemma 2.3. *Let (X, ω) be a Hamiltonian G-manifold with $\mu : X \to \mathfrak{k}^*$ its moment map and let $p\colon \widetilde{X} \to X$ be a topological covering. If the G–action lifts to $\widetilde{X}$, then $(\widetilde{X}, p^*\omega)$ is a Hamiltonian G–manifold with moment map $p^*\mu$.*

Proof. We equip $\widetilde{X}$ with the unique complex structure such that p is locally biholomorphic. If the G–action lifts to $\widetilde{X}$, then G acts holomorphically on $\widetilde{X}$ and p is G–equivariant. Consequently, $p^*\omega$ is a K–invariant Kähler form on $\widetilde{X}$.

For $\xi \in \mathfrak{k}$ let $\xi_{\widetilde{X}}$ and ξ_X be the induced vector fields on $\widetilde{X}$ and X, respectively. Since p is equivariant, we have $p_*\xi_{\widetilde{X}} = \xi_X$. Hence, we obtain

$$d(p^*\mu)^\xi = dp^*\mu^\xi = p^*d\mu^\xi = p^*\iota_{\xi_X}\omega = \iota_{\xi_{\widetilde{X}}}p^*\omega,$$

which shows that $p^*\mu$ is an equivariant moment map for the K–action on $\widetilde{X}$. □

3. Local closedness of G–orbits

Let X be a Hamiltonian G–manifold where $G = K^{\mathbb{C}}$ is a complex reductive group. Suppose that X is G–connected, i. e. that X/G is connected. In particular, X has only finitely many connected components since this is true for G. We want to show that the topological closure of every G–orbit is complex-analytic in X.

We fix a maximal torus T_0 in K. Then $T := T_0^{\mathbb{C}}$ is a maximal algebraic torus in G and the moment map $\mu\colon X \to \mathfrak{k}^*$ induces by restriction a moment map $\mu_T\colon X \to \mathfrak{t}_0^*$ for the T_0–action on X. Since $\mathfrak{t}_0$ is Abelian, for every $\lambda \in \mathfrak{t}_0^*$ the shifted map $\mu_T + \lambda$ is again a moment map for T_0. Consequently, every $x \in X$ lies in the zero fiber of some moment map for the T_0–action on X which has the following consequences (see [10]).

Theorem 3.1. *Let G be a complex reductive group and X be a G–connected Hamiltonian G–manifold.*

(1) *Every isotropy group T_x is complex reductive and hence the connected component of the identity $(T_x)^0$ is a subtorus of T.*
(2) *For every $x \in X$ there exists a complex submanifold S of X which contains x such that the map $T \times_{T_x} S \to T \cdot S$, $[t, y] \mapsto t \cdot y$, is biholomorphic onto its open image.*
(3) *For $\lambda \in \mathfrak{t}_0^*$ we define $\mathcal{S}_\lambda := \{x \in X;\ \overline{T \cdot x} \cap \mu_T^{-1}(\lambda) \neq \emptyset\}$. Then $\mathcal{S}_\lambda$ is a T–stable open subset of X such that the semistable quotient (see [11]) $\mathcal{S}_\lambda \to \mathcal{S}_\lambda /\!\!/ T$ exists. Moreover, the inclusion $\mu_T^{-1}(\lambda) \hookrightarrow \mathcal{S}_\lambda$ induces a homeomorphism $\mu_T^{-1}(\lambda)/T_0 \cong \mathcal{S}_\lambda /\!\!/ T$.*

Remark. Properties (1) and (2) imply that if the T–action on X is known to be almost free, then it is locally proper.

For the following we have to review the definition of subanalytic sets. For more details we refer the reader to [4] and to [12].

Let M be a real analytic manifold. A subset $A \subset M$ is called *semianalytic* if every point in M has an open neighborhood Ω such that $A \cap \Omega = \bigcup_{k=1}^{r} \bigcap_{l=1}^{s} A_{kl}$, where every A_{kl} is either of the form $\{f_{kl} = 0\}$ or $\{f_{kl} > 0\}$ for $f_{kl} \in \mathcal{C}^\omega(\Omega)$. A subset $A \subset M$ is called *subanalytic* if every element of M admits an open neighborhood Ω such that $A \cap \Omega$ is the image of a semianalytic set under a proper real analytic map. We note that finite intersections and finite unions as well as topological closures of subanalytic sets are subanalytic. Finally we call a set $A \subset M$ *locally subanalytic* if there are open sets $U_1, \ldots, U_k \subset M$ such that $A \subset U_1 \cup \cdots \cup U_k$ and such that $A \cap U_j$ is subanalytic in U_j for every j. For later use we cite the following theorem of Hironaka ([12]).

Theorem 3.2. *Let $\Phi\colon M \to N$ be a real analytic map between real analytic manifolds and let $A \subset M$ be subanalytic. If $\Phi|_{\overline{A}}\colon \overline{A} \to N$ is proper, then $\Phi(A)$ is subanalytic in N.*

It is shown in [11] that, if the semistable quotient $X \to X /\!\!/ T$ exists globally, then the semistable quotient $X \to X /\!\!/ G$ exists. The first step in the proof of this theorem consists in showing that the existence of $X /\!\!/ T$ implies that the G–orbits are subanalytic and thus locally closed in X. In our situation the semistable quotient of X with respect to T exists only locally (in the sense of Theorem 3.1(3)). As we will see this implies that the G–orbits in X are locally subanalytic which is sufficient for them to be locally closed.

The following lemma is the essential ingredient in the proof of this statement.

Lemma 3.3. *Let $A \subset X$ be a compact subanalytic set. Then $T \cdot A$ is locally subanalytic in X.*

Proof. Since A is compact, we have $A \subset \bigcup_{k=1}^{n} \mathcal{S}_{\lambda_k}$. For every $k = 1, \ldots, n$ let U_k be an open subanalytic subset of $\mathcal{S}_{\lambda_k}$ such that $\overline{U}_k \subset \mathcal{S}_{\lambda_k}$ is compact and such that $A \subset \bigcup_{k=1}^{n} U_k$. Consequently, for every k the intersection $A \cap \overline{U}_k$ is a compact subanalytic subset of $\mathcal{S}_{\lambda_k}$. Since for each k the semistable quotient $\mathcal{S}_{\lambda_k} \to \mathcal{S}_{\lambda_k} /\!\!/ T$ exists, we conclude from the proposition in Section 3 of [11] that $T \cdot (A \cap \overline{U}_k) = (T \cdot A) \cap (T \cdot \overline{U}_k)$ is subanalytic in $\mathcal{S}_{\lambda_k}$. It follows that for every k the intersection $(T \cdot A) \cap (T \cdot U_k)$ is subanalytic in the open set $T \cdot U_k \subset X$. Since we have $T \cdot A = T \cdot \big(\bigcup_{k=1}^{n} A \cap U_k\big) = \bigcup_{k=1}^{n} \big((T \cdot A) \cap (T \cdot U_k)\big)$, we conclude that $T \cdot A$ is locally subanalytic in X. $\square$

Lemma 3.4. *Let $A \subset X$ be (locally) subanalytic. Then $K \cdot A$ is (locally) subanalytic in X.*

Proof. Since K is compact, the real analytic map $\Phi\colon K \times X \to X$, $(k, x) \mapsto k \cdot x$, is proper: For every compact subset $C \subset X$ the inverse image $\Phi^{-1}(C)$ is closed and contained in $K \times (K \cdot C)$, hence compact. We conclude that the restriction of Φ to $K \times \overline{A}$ is proper. Therefore Hironaka's theorem 3.2, [12] implies that $\Phi(K \times A) = K \cdot A$ is subanalytic.

If A is locally subanalytic, then A is covered by relatively compact subanalytic open sets U such that $A \cap \overline{U}$ is subanalytic. Then it follows as above that $K \cdot (A \cap \overline{U})$ is subanalytic, and consequently $K \cdot A$ is locally subanalytic. □

Now we are in a position to prove the main result of this section.

Theorem 3.5. *Suppose X is a G–connected Hamiltonian G–manifold, where G is a complex reductive group. Then*

(1) every G–orbit is locally subanalytic and in particular locally closed in X,
(2) the boundary of every G–orbit contains only G–orbits of strictly smaller dimension, and
(3) the closure of every G–orbit is complex-analytic in X.

Proof. For every $x \in X$ the orbit $K \cdot x$ is a compact real analytic submanifold of X. By Lemma 3.3 the set $T \cdot (K \cdot x)$ is locally subanalytic in X. Thus Lemma 3.4 implies that $K \cdot \big(T \cdot (K \cdot x)\big)$ is locally subanalytic as well. Because of $G = KTK$ every G–orbit is locally subanalytic.

In order to see that the G–orbits are locally closed, we take $U_1 \cup \cdots \cup U_k$ to be an open covering of $G \cdot x$ such that for every j the intersection $(G \cdot x) \cap U_j$ is subanalytic in U_j. Since the boundary of a subanalytic set is again subanalytic and of strictly smaller dimension, we see that $(G \cdot x) \cap U_j$ contains an interior point of its closure in U_j. Moving this point with the G–action it follows that $(G \cdot x) \cap U_j$ is open in its closure in U_j. Consequently, $G \cdot x$ is locally closed.

For the second claim it is sufficient to note that the dimension of an orbit $G \cdot x$ can be checked in the intersection with an open set U such that $(G \cdot x) \cap U$ is subanalytic in U. More precisely, let $x, y \in X$ such that $G \cdot y \subset \overline{G \cdot x}$. Since $\{x, y\}$ is compact subanalytic, there are finitely many open sets $U_1, \ldots, U_k$ such that $(G \cdot x) \cup (G \cdot y) \subset U_1 \cup \cdots \cup U_k$ and such that $\big((G \cdot x) \cup (G \cdot y)\big) \cap U_j$ is subanalytic in U_j for every j. Suppose $y \in U_1$. Then $(G \cdot y) \cap U_1$ lies in the closure of $(G \cdot x) \cap U_1$ in U_1. After possibly shrinking U_1 we may assume that $(G \cdot y) \cap U_1$ is subanalytic in U_1 which implies that $(G \cdot x) \cap U_1$ is also subanalytic in U_1. Hence, we obtain $\dim G \cdot y = \dim (G \cdot y) \cap U_1 < \dim (G \cdot x) \cap U_1 = \dim G \cdot x$ as was to be shown.

Finally let $x_0 \in X$ and $E := \{x \in X;\ \dim G \cdot x < \dim G \cdot x_0\}$. The set E is complex-analytic and its complement $\Omega := X \setminus E$ is G–invariant. Since the boundary of $G \cdot x_0$ contains only orbits of strictly smaller dimension by the previous claim, the orbit $G \cdot x_0$ is closed in Ω and therefore a complex submanifold of Ω. We will show that $\overline{G \cdot x_0}$ is complex-analytic in X by applying Bishop's theorem ([5]). For this we must check that every point $x \in E$ has an open neighborhood $U \subset X$ such that $U \cap (G \cdot x_0)$ has finite volume with respect to some hermitian metric on X. Without loss of generality we may assume that $x \in \overline{G \cdot x_0} \cap E$ holds. According to what we have already shown we

find an open neighborhood $U \subset X$ of x such that $U \cap \overline{G \cdot x_0}$ is subanalytic in U. After possibly shrinking U we may assume that U is biholomorphic to the unit ball in $\mathbb{C}^n$. It is known (see the remark following Proposition 1.4 in [16]) that the $2k$–dimensional Hausdorff volume (where $k := \dim_{\mathbb{C}} G \cdot x_0$) of $U \cap (G \cdot x_0)$ is finite. Since $U \cap (G \cdot x_0)$ is also an immersed submanifold of U, the $2k$–dimensional Hausdorff volume coincides with the geometric volume associated with the standard hermitian metric on $\mathbb{C}^n$ (see page 48 in [21]). This observation allows us to deduce from Bishop's theorem that $\overline{G \cdot x_0}$ is complex-analytic in X. □

Remark. We restate the following fact which is shown in the third part of the proof and might be of independent interest: Let $E \subset \mathbb{B}_n$ be a complex-analytic subset. Suppose that $A \subset \mathbb{B}_n \setminus E$ is complex-analytic and that $A \subset \mathbb{B}_n$ is locally subanalytic and an injectively immersed complex submanifold. Then the topological closure of A in $\mathbb{B}_n$ is complex-analytic.

Remark. In [22] holomorphic actions of complex reductive groups G on compact Kähler manifolds X are considered. Under the additional assumption that the G–action on X is projective it is shown that for every $x \in X$ the closure $\overline{G \cdot x}$ is complex-analytic in X. Sommese's notion of projectivity of a G–action on X is equivalent to the fact that G acts trivially on the Albanese torus $\mathrm{Alb}(X)$. Hence, by Proposition 1 on page 269 in [13], G acts projectively on X if and only if X is a Hamiltonian G–manifold.

In [8] some properties of algebraic group actions are extended to the more general class $\mathcal{C}$ of compact complex spaces that are the meromorphic images of compact Kähler spaces and it is shown that the orbit closures are complex analytic in this setting.

From the remark after Definition 2.1 we obtain the following

Corollary 3.6. *Let $G = S$ be a semisimple complex Lie group acting holomorphically on the Kähler manifold X. Then the S–orbits are locally closed in X.*

4. Homogeneous Hamiltonian G–manifolds

Let $G = K^{\mathbb{C}}$ be a connected complex reductive group and let H be a closed complex subgroup of G. Suppose that the homogeneous space $X = G/H$ admits a K–invariant Kähler form ω. We want to show that the existence of a K–equivariant moment map $\mu\colon X \to \mathfrak{k}^*$ implies that H is an algebraic subgroup of G.

Example. If G is Abelian, i. e. if $G = (\mathbb{C}^*)^k$, then the fact that G/H is Kähler does not imply that H is algebraic as the example of an elliptic curve $\mathbb{C}^*/\mathbb{Z}$ shows. However, if G/H is a Hamiltonian G–manifold, then by Theorem 3.1(1) the group H is complex reductive and hence algebraic.

Example. Suppose that $X = G/H$ is a Hamiltonian G–manifold with moment map $\mu\colon X \to \mathfrak{k}^*$. If $\mu^{-1}(0) \neq \emptyset$, then the semistable quotient $X/\!\!/G$ exists (and is a point) and thus $X = G/H$ is Stein by [11]. In this case H is a reductive complex subgroup of G and hence is algebraic, see [19] and [20].

We will need the following technical result.

Lemma 4.1. *Let Γ be a discrete subgroup of G normalizing H such that $X = (G/H)/\Gamma$ is a Hamiltonian G–manifold with moment map μ. Suppose that Γ acts by holomorphic*

transformations on a complex manifold Y and that Y admits a Γ–invariant Kähler form ω_Y. Recall that the twisted product $(G/H) \times_\Gamma Y$ is by definition the quotient of $(G/H) \times Y$ by the diagonal Γ–action $\gamma \cdot (gH, y) := (\gamma \cdot gH, \gamma \cdot y)$. Then $(G/H) \times_\Gamma Y$ is a Hamiltonian G–manifold with moment map $\widehat{\mu}\colon (G/H) \times_\Gamma Y \to \mathfrak{k}^*$, $\widehat{\mu}[gH, y] := \mu(gH\Gamma)$.

Proof. Let $p\colon G/H \to (G/H)/\Gamma$ be the quotient map and let ω be a K–invariant Kähler form on $(G/H)/\Gamma$. Then $p^*\omega$ is a K– and Γ–invariant Kähler form on G/H and thus $p^*\omega + \omega_Y$ is a Γ–invariant Kähler form on $(G/H) \times Y$. Hence, we see that $\widehat{Y} := (G/H) \times_\Gamma Y$ is Kähler.

The map $\widehat{\mu}$ is well-defined and K–equivariant. Let $\xi_{\widehat{Y}}$ be the vector field on $\widehat{Y}$ induced by $\xi \in \mathfrak{k}$. Let $U \subset (G/H)/\Gamma$ be an open set such that the bundle $q\colon \widehat{Y} = (G/H) \times_\Gamma Y \to (G/H)/\Gamma$ is trivial over U, i. e. such that $q^{-1}(U) \cong U \times Y$. For every $[gH, y] \in U$ the vector $\xi_{\widehat{Y}}[gH, y]$ corresponds to $\big(\xi_{(G/H)/\Gamma}\big(p(gH)\big), 0\big) \in T_{p(gH)}(G/H)/\Gamma \oplus T_yY$. Moreover, we have $d\widehat{\mu}^\xi = d\mu^\xi$ in this trivialization. By construction of the Kähler form on $\widehat{Y}$ we conclude that $\widehat{\mu}$ is a moment map for the K–action on $\widehat{Y} = (G/H) \times_\Gamma Y$. □

We will first prove the algebraicity of H under the assumption that H is a discrete subgroup of G. In this case we write Γ instead of H.

Proposition 4.2. *Let G be a connected complex reductive group and let Γ be a discrete subgroup of G such that $X = G/\Gamma$ is a Hamiltonian G–manifold. Then Γ is finite.*

Proof. Let us briefly recall the Jordan decomposition of elements in the affine algebraic group $G = K^{\mathbb{C}}$ (see Chapter I.4 in [7]). Suppose that G is a subgroup of $\mathrm{GL}(N, \mathbb{C})$. An element $\gamma \in G$ is called semisimple if the matrix representing γ is diagonalizable, and unipotent if the matrix $\gamma - I_N$ is nilpotent. It can be shown that these notions do not depend on the chosen embedding $G \hookrightarrow \mathrm{GL}(N, \mathbb{C})$. Moreover, every element $\gamma \in G$ has a (unique) Jordan decomposition $\gamma = \gamma_{\mathsf{s}}\gamma_{\mathsf{u}} = \gamma_{\mathsf{u}}\gamma_{\mathsf{s}}$ in G, where γ_{s} is semisimple and γ_{u} is unipotent.

Suppose there is an element $\gamma \in \Gamma$ with $\gamma_{\mathsf{u}} \neq e$. Then there exists a nilpotent element $\xi \in \mathfrak{g}$ with $\gamma_{\mathsf{u}} = \exp(\xi)$. Since the group $\exp(\mathbb{C}\xi)$ is closed in G, the same holds for the cyclic group $\langle\gamma\rangle := \{\gamma^m;\ m \in \mathbb{Z}\} \cong \mathbb{Z}$. Lemma 2.3 implies then that $G/\langle\gamma\rangle$ is a Hamiltonian G–manifold. The group $\langle\gamma\rangle$ acts on $\mathbb{C}^*$ by $\gamma^m \cdot z := e^{im}z$. Applying Lemma 4.1 we conclude that the twisted product $G \times_{\langle\gamma\rangle} \mathbb{C}^*$ is a Hamiltonian G–manifold. Since the G–orbits in this twisted product intersect $\mathbb{C}^*$ in $\langle\gamma\rangle$–orbits and since these orbits are dense in the S^1–orbits in $\mathbb{C}^*$, we arrive at a contradiction to Theorem 3.5. Consequently, every $\gamma \in \Gamma$ must be semisimple.

If $\gamma = \gamma_{\mathsf{s}}$, then the Zariski closure of the cyclic group generated by γ is either finite or a complex torus $T \cong (\mathbb{C}^*)^l$ for some $l \geq 1$. Assume that the latter holds. Then $T/(\Gamma \cap T)$ is a Hamiltonian T–manifold by Lemma 2.2, and consequently $\Gamma \cap T$ must be finite. Since $\langle\gamma\rangle$ is contained in $\Gamma \cap T$, it follows that T is finite, a contradiction. Thus every element of Γ is semisimple and generates a finite group. According to Lemma 2.1 in [1] the group Γ is finite. □

Remark. If the group G is semisimple, then every holomorphic G–manifold which admits a Kähler form is Hamiltonian. Hence, we have given a new proof for Theorem 3.1 in [3].

Now we return to the general case that H is any closed complex subgroup of G such that $X = G/H$ is a Hamiltonian G–manifold. The following theorem the proof of which can be found in [1] gives a necessary and sufficient condition for H to be algebraic.

Theorem 4.3. *For $h \in H$ let $\mathcal{A}(h)$ denote the Zariski closure of the cyclic group generated by h in G. The group H is algebraic if and only if $\mathcal{A}(h)$ is contained in H for every $h \in H$.*

Using this result we now prepare the proof of our main theorem in this section.

Let $h \in H$. In order to have better control over the group $\mathcal{A}(h)$ we follow closely an idea which is described on page 107 in [1]. For this let $h = h_{\mathsf{s}}h_{\mathsf{u}}$ be the Jordan decomposition of h in G. As we already noted above, if h is semisimple, then $\mathcal{A}(h)$ is either finite or isomorphic to $(\mathbb{C}^*)^l$. In the first case we have $\mathcal{A}(h) \subset H$. In the second case, $X = G/H$ is a Hamiltonian $\mathcal{A}(h)$–manifold which implies that the orbit $\mathcal{A}(h) \cdot eH \cong \mathcal{A}(h)/\big(\mathcal{A}(h) \cap H\big)$ is Hamiltonian. Hence, $\mathcal{A}(h) \cap H$ is algebraic which yields $\mathcal{A}(h) \subset H$.

If h is unipotent, then there exists a simple three dimensional closed complex subgroup S of G containing h (see [14]). Again $X = G/H$ is a Hamiltonian S–manifold. Hence the orbit $S \cdot eH \cong S/(S \cap H)$ is Hamiltonian and in particular Kähler. We have to show that $S \cap H$ is algebraic in S. Then we have $\mathcal{A}(h) \subset S \cap H \subset H$, as was to be shown. Algebraicity of $S \cap H$ will be a consequence of the following lemma for which we give here a direct proof.

Lemma 4.4. *Let H be a closed complex subgroup of $S = \mathrm{SL}(2, \mathbb{C})$. If S/H is Kähler, then H is algebraic.*

Proof. Since every Lie subalgebra of $\mathfrak{s} = \mathfrak{sl}(2, \mathbb{C})$ is conjugate to $\{0\}$, to $\mathbb{C}\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, to $\mathbb{C}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, to a Borel subalgebra $\mathfrak{b}$, or to $\mathfrak{s}$, we conclude that the identity component H^0 is automatically algebraic. Therefore it suffices to show that H has only finitely many connected components since then H is the finite union of translates of H^0 which is algebraic.

For $H^0 = S$ this is trivial. Since the normalizer of a Borel subgroup B of S coincides with B, we see that $H^0 = B$ implies $H = B$, hence that H is algebraic in this case. If H^0 is a maximal algebraic torus in S, then its normalizer in S has two connected components, thus H has at most two connected components as well.

Suppose that H^0 is unipotent. Then its normalizer is a Borel subgroup. If H has infinitely many connected components, we find an element $h \in H \setminus H^0$ which generates a closed infinite subgroup Γ of S. Then $S/(\Gamma H^0)$ is Kähler (for it covers S/H), and we conclude from Lemma 4.1 that $(S/H^0) \times_\Gamma \mathbb{C}^*$ is a Hamiltonian S–manifold where Γ acts on $\mathbb{C}^*$ by $\gamma^m \cdot z := e^{im}z$. As above this contradicts Theorem 3.5.

Since the case that H^0 is trivial, i. e. that H is discrete, has already been treated, the proof is finished. □

Now suppose that $h = h_{\mathsf{s}}h_{\mathsf{u}}$ with $h_{\mathsf{s}} \neq e$ and $h_{\mathsf{u}} \neq e$. In this case there is a simple three dimensional closed complex subgroup S of the centralizer of $\mathcal{A}(h_{\mathsf{s}})$ which contains $\mathcal{A}(h_{\mathsf{u}})$. Then $\mathcal{A}(h) \subset S\mathcal{A}(h_{\mathsf{s}})$ and a finite covering of $S\mathcal{A}(h_{\mathsf{s}})$ is isomorphic to $\mathrm{SL}(2, \mathbb{C}) \times (\mathbb{C}^*)^l$. (We may suppose that $\mathcal{A}(h_{\mathsf{s}})$ has positive dimension, if not, we are essentially in the previous case.) Moreover, there is a closed complex subgroup $\widetilde{H}$ of

$\widetilde{G} = \mathrm{SL}(2,\mathbb{C}) \times (\mathbb{C}^*)^l$ containing the element $h = \left(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), (e^{a_1}, \ldots, e^{a_l})\right)$ such that $\widetilde{G}/\widetilde{H}$ is Hamiltonian. We must show that $\mathcal{A}(h) = \left(\begin{smallmatrix} 1 & \mathbb{C} \\ 0 & 1 \end{smallmatrix}\right) \times (\mathbb{C}^*)^l$ is contained in $\widetilde{H}$.

In order to simplify the notation we will continue to write G and H instead of $\widetilde{G}$ and $\widetilde{H}$. The following observation is central to our argument.

Lemma 4.5. *We may assume without loss of generality that* $H \cap (\mathbb{C}^*)^l = \{e\}$.

Proof. For this note that the action of $(\mathbb{C}^*)^l$ on G/H is Hamiltonian. This implies that $H \cap (\mathbb{C}^*)^l$ is a central subtorus T of G. Consequently, $G/H \cong (G/T)/(H/T)$. If H/T is algebraic in G/T, then H is algebraic in G. □

Let p_1 and p_2 denote the projections of $G = \mathrm{SL}(2,\mathbb{C}) \times (\mathbb{C}^*)^l$ onto $\mathrm{SL}(2,\mathbb{C})$ and $(\mathbb{C}^*)^l$, respectively.

Lemma 4.6. *The map* $p_1 \colon G \to \mathrm{SL}(2,\mathbb{C})$ *maps the group* H *isomorphically onto a closed complex subgroup of* $\mathrm{SL}(2,\mathbb{C})$.

Proof. We show first that $p_1(H)$ is closed in $\mathrm{SL}(2,\mathbb{C})$. For this note that G/H is a Hamiltonian $(\mathbb{C}^*)^l$–manifold. By Theorem 3.5 all $(\mathbb{C}^*)^l$–orbits are locally closed in G/H. Since $(\mathbb{C}^*)^l$ is the center of G, we have $(\mathbb{C}^*)^l \cdot (gH) = g \cdot \big((\mathbb{C}^*)^l \cdot eH\big)$. Hence all $(\mathbb{C}^*)^l$–orbits have the same dimension. This implies that all $(\mathbb{C}^*)^l$–orbits are closed in G/H. Consequently, $(\mathbb{C}^*)^l H$ is closed in G which shows that $p_1(H) = \mathrm{SL}(2,\mathbb{C}) \cap (\mathbb{C}^*)^l H$ is closed.

Since the restriction of p_1 to the closed subgroup H of G is a surjective holomorphic homomorphism onto $p_1(H)$ with kernel $H \cap (\mathbb{C}^*)^l = \{e\}$, the claim follows. □

If $p_1(H) = \mathrm{SL}(2,\mathbb{C})$, then $p_2 \colon H \cong \mathrm{SL}(2,\mathbb{C}) \to (\mathbb{C}^*)^l$ must be trivial. But this contradicts the fact that $(e^{a_1}, \ldots, e^{a_l})$ is contained in $p_2(H)$. Therefore $p_1(H)$ must be a proper closed subgroup of $\mathrm{SL}(2,\mathbb{C})$ which contains the element $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. In particular, we conclude that H^0 is solvable.

There are essentially three possibilities. The image $p_1(H)$ is a Borel subgroup of $\mathrm{SL}(2,\mathbb{C})$ (which implies that H is a connected two-dimensional non-Abelian subgroup of G), or $p_1(H)^0 = \left(\begin{smallmatrix} 1 & \mathbb{C} \\ 0 & 1 \end{smallmatrix}\right)$, or $p_1(H)$ is discrete containing $\left(\begin{smallmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{smallmatrix}\right)$. If $p_1(H)$ is discrete, then H is discrete. We have already shown that H is finite in this case, hence algebraic.

Remark. Suppose that $p_1(H)$ is the Borel subgroup of upper triangular matrices in $\mathrm{SL}(2,\mathbb{C})$. The map $p_2|_H \colon H \to p_2(H)$ is a surjective homomorphism with kernel $H' = H \cap \mathrm{SL}(2,\mathbb{C})$. Thus we have $H \cap \mathrm{SL}(2,\mathbb{C}) = \left(\begin{smallmatrix} 1 & \mathbb{C} \\ 0 & 1 \end{smallmatrix}\right)$.

Suppose that $p_1(H)$ is one-dimensional or a Borel subgroup. We know that $p_2(H)$ is a closed complex subgroup of $(\mathbb{C}^*)^l$ containing $(e^{a_1}, \ldots, e^{a_l})$. Since $\dim p_2(H) = 1$, we conclude that $p_2(H)^0 = \big\{(e^{ta_1}, \ldots, e^{ta_l});\ t \in \mathbb{C}\big\}$. Lemma 2.3 implies that if G/H is Hamiltonian, then the same holds for G/H^0. The possibilities for H^0 are $H^0 = \left\{\left(\left(\begin{smallmatrix} e^{ta_0} & s \\ 0 & e^{-ta_0} \end{smallmatrix}\right), (e^{ta_1}, \ldots, e^{ta_l})\right);\ t, s \in \mathbb{C}\right\}$ (if $p_1(H)$ is a Borel subgroup) or $H^0 = \left\{\left(\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right), (e^{ta_1}, \ldots, e^{ta_l})\right);\ t \in \mathbb{C}\right\}$. We have to show that in both cases G/H^0 is not Hamiltonian.

Let us first consider the case that

$$H = \left\{\left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, (e^{ta_1}, \ldots, e^{ta_l})\right);\ t \in \mathbb{C}\right\}.$$

Let $T = (\mathbb{C}^*)^{l-1} \times \{1\} \subset (\mathbb{C}^*)^l$ and $\widetilde{G} := \mathrm{SL}(2,\mathbb{C}) \times T$. Then we have $T \cap p_2(H) =: \Gamma \cong \mathbb{Z}$ and $(\mathbb{C}^*)^l / p_2(H) \cong T/\Gamma$. Moreover, G/H is a Hamiltonian $\widetilde{G}$–manifold and we have $G/H \cong \widetilde{G}/\widetilde{H}$, where $\widetilde{H} = \widetilde{G} \cap H \cong \mathbb{Z}$. This contradicts our result in the discrete case. Hence, G/H cannot be Hamiltonian.

Finally, suppose that

$$H = \left\{ \left(\begin{pmatrix} e^{ta_0} & s \\ 0 & e^{-ta_0} \end{pmatrix}, (e^{ta_1}, \ldots, e^{ta_l}) \right) ;\ t, s \in \mathbb{C} \right\}.$$

Again we consider $T = (\mathbb{C}^*)^{l-1} \times \{1\}$ and $\widetilde{G} = \mathrm{SL}(2,\mathbb{C}) \times T$. We have $\widetilde{H} = \widetilde{G} \cap H = \Gamma \ltimes \left(\begin{smallmatrix} 1 & \mathbb{C} \\ 0 & 1 \end{smallmatrix}\right)$ where $\Gamma \cong \mathbb{Z}$. As in the discrete case we let Γ act on $\mathbb{C}^*$ by $\gamma^m \cdot z := e^{im} z$ and consider the twisted product $(\widetilde{G}/\widetilde{H}^0) \times_\Gamma \mathbb{C}^*$. If G/H is Hamiltonian, then the same holds for $\widetilde{G}/\widetilde{H}$ and thus for $\widetilde{G}/\widetilde{H}^0$. Then $(\widetilde{G}/\widetilde{H}^0) \times_\Gamma \mathbb{C}^*$ is Hamiltonian by Lemma 4.1. Since the Γ–orbits in $\mathbb{C}^*$ are not locally closed, this contradicts Theorem 3.5 and we conclude that G/H is not Hamiltonian.

Summarizing our discussion in this section we proved the following.

Theorem 4.7. *Let G be a connected complex reductive group and let H be a closed complex subgroup. If $X = G/H$ is a Hamiltonian G–manifold, then H is an algebraic subgroup of G.*

In particular we obtain the following result which was originally proved in [3] and [2].

Corollary 4.8. *Let S be a connected semisimple complex Lie group and let H be a closed complex subgroup of S. If S/H admits a Kähler form, then H is an algebraic subgroup of S.*

5. Homogeneous Kähler manifolds

Let $G = K^{\mathbb{C}}$ be a connected complex reductive Lie group. In this section we characterize those closed complex subgroups H of G for which $X = G/H$ admits a Kähler form.

According to [7], Corollary I.2.3, the commutator group $S := G'$ is a connected algebraic subgroup of G and, since G is reductive, S is semisimple. Let $Z := \mathcal{Z}(G)^0 \cong (\mathbb{C}^*)^k$. Then $G = SZ$ and $S \cap Z$ is finite.

Theorem 5.1. *Let G be a reductive complex Lie group and $H \subset G$ a closed complex subgroup. Then the manifold $X = G/H$ admits a Kähler form if and only if $S \cap H \subset S$ is algebraic and SH is closed in G.*

Proof. After replacing G by a finite cover we may assume that $G = S \times Z$. Suppose first that G/H is Kähler. Then $S/(S \cap H)$ is also Kähler and hence $S \cap H \subset S$ is algebraic by Corollary 4.8. By Theorem 3.5 all S–orbits in G/H are open in their closures and their boundaries only contain orbits of strictly smaller dimension. In view of the reductive group structure of G the S–orbits in X all have the same dimension. This implies that every S–orbit in G/H is closed. Consequently, SH is closed in G.

Now suppose that $S \cap H \subset S$ is algebraic and that SH is closed in G. Although it is not used in the proof we remark that we may assume that H is solvable, since otherwise one can divide by the (ineffective) semisimple factor of H. Consider the fibration

$$X = G/H \to G/SH.$$

The base G/SH is an Abelian complex Lie group and the fiber is $SH/H = S/(S\cap H)$. There is a subgroup $(\mathbb{C}^*)^l \cong Z_1 \subset Z \cong (\mathbb{C}^*)^k$ such that $G_1 := S \times Z_1 \subset G$ acts transitively on X and $Z_1 \cap SH$ is discrete. With $H_1 := H \cap G_1$ we have that $X = G_1/H_1$, that SH_1 is closed in G_1 and that the base of the fibration

$$X = G_1/H_1 \to G_1/SH_1 = Z_1/(Z_1 \cap SH_1)$$

is a discrete quotient of Z_1. Furthermore $S \cap H = S \cap H_1$ is an algebraic subgroup of S.

Let $\Gamma_1 := Z_1 \cap SH_1$ and $\Gamma_2 \subset Z_1$ be a discrete subgroup such that $\Gamma_1 \cap \Gamma_2 = \{e\}$ and such that $\Gamma := \Gamma_1 + \Gamma_2$ is a discrete cocompact subgroup of Z_1. Since H_1' is contained in $S \cap H_1$, we can define the closed complex subgroup $H_2 \subset G_1$ to be the group generated by H_1 and $\Gamma_2(S \cap H_1)$. One still has that $H_2 \cap S = H_1 \cap S = H \cap S$ is algebraic in S and that SH_2 is closed in G_1. Hence $X = G_1/H_1 \to G_1/H_2$ is a covering map and one sees that in order to finish the proof it is sufficient to show that the base G_1/H_2 admits a Kähler form.

So we may drop the indices and have to prove that a reductive quotient $X = G/H$ of $G = S \times (\mathbb{C}^*)^k = S \times Z$ by a closed complex solvable subgroup H with $S \cap H \subset S$ algebraic, $SH \subset G$ closed and G/SH a compact torus, is Kähler.

Let $\mathrm{p}_1 \colon G \to S$ be the projection onto S. Note that the algebraic Zariski closure $\overline{H}$ of H in G is the product $\widehat{H} \times (\mathbb{C}^*)^k$, where $\widehat{H}$ is the Zariski closure of the projection $\mathrm{p}_1(H)$ of H in S. The commutator group H' of H is also the commutator group of $\overline{H}$ and of $\widehat{H}$ and is contained in $H \cap S$. Therefore one gets a natural algebraic right action of $\overline{H}$ on the homogeneous manifold $Y := S/(S \cap H)$ given by

$$(\star) \qquad \overline{h}\big(s(S \cap H)\big) := s\big(\mathrm{p}_1(\overline{h})\big)^{-1}(S \cap H).$$

As a consequence we can equivariantly compactify the $S\overline{H}$–manifold Y to an almost homogeneous projective $S\overline{H}$–manifold $\overline{Y}$, see [18], Proposition 3.1, and [15], Proposition 3.9.1. Since $X = G/H$ is realizable as a quotient of the manifold $S/(S\cap H)\times(\mathbb{C}^*)^k$ by the natural action of $H/(S\cap H)$, where the S–factor of the action is given by $(\star)$, we see that X is an open set in a holomorphic fiber bundle $\overline{X}$ with a compact torus as base and the simply connected projective manifold $\overline{Y}$ as fiber. Finally we apply Blanchard's theorem, see [6], p. 192, to get a Kähler form on $\overline{X}$ and then, by restriction, on X also. The theorem is proved. □

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Department of Mathematics and Statistics, University of Regina, Regina, Canada S4S 0A2
E-mail address: gilligan@math.uregina.ca

LATP-UMR(CNRS) 6632, CMI-Université d'Aix-Marseille I, 39, rue Joliot-Curie, F-13453 Marseille Cedex 13, France
E-mail address: miebach@cmi.univ-mrs.fr

LATP-UMR(CNRS) 6632, CMI-Université d'Aix-Marseille I, 39, rue Joliot-Curie, F-13453 Marseille Cedex 13, France
E-mail address: karloelj@cmi.univ-mrs.fr