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1. Introduction

Given (M, g), a m-dimensional Riemannian manifold, and Ω, a smooth bounded domain in M, we denote by $\lambda_1(\Omega)$ the first eigenvalue of the Laplace-Beltrami operator under 0 Dirichlet boundary condition. The critical points of the functional

$$\Omega \longmapsto \lambda_1(\Omega)\,,$$

under the volume constraint $\mathrm{Vol}(\Omega) = \alpha$ (where $\alpha \in (0, \mathrm{Vol}(M))$ is fixed) are called *extremal domains*. Smooth *extremal domains* are characterized by the property that the eigenfunctions associated to the first eigenvalue of the Laplace-Beltrami operator have constant Neumann boundary data [2]. In other words, a smooth domain is extremal if and only if there exists a positive function u_1 and a constant λ_1 such that

$$\Delta_g u_1 + \lambda_1 \, u_1 = 0\,,$$

in Ω with

$$u_1 = 0 \qquad \text{and} \qquad \nabla_n u_1 = \text{constant} \qquad \text{on} \qquad \partial\Omega\,,$$

where n denotes the inward unit normal vector to $\partial\Omega$. The theory of *extremal domains* is very reminiscent of the theory of constant mean curvature surfaces or hypersurfaces. To give some credit to this assertion, let us recall that, in the early 1970's, J. Serrin has proved that the only compact, smooth, extremal domains in Euclidean space are round balls [6], paralleling the well known result of Alexandrov asserting that round spheres are the only (embedded) compact constant mean curvature hypersurfaces in Euclidean space. More recently, F. Pacard and P. Sicbaldi have proved the existence of extremal domains close to small geodesic balls centered at critical points of the scalar curvature function [5], paralleling an earlier result of R. Ye which provides constant mean curvature topological spheres (with high mean curvature) close to small geodesic spheres centered at nondegenerate critical points of the scalar curvature function [8].

We propose the following :

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Definition 1.1. *A smooth domain $\Omega \subset \mathbb{R}^m$ is said to be an* exceptional domain *if it supports positive harmonic functions having* 0 *Dirichlet boundary data and constant (nonzero) Neumann boundary data. Any such harmonic function is called a* roof *function.*

Exceptional domains arise as limits under scaling of sequences of *extremal domains* just like minimal surfaces arise as limits under scaling of sequences of constant mean curvature surfaces. As explained above, there is a formal correspondence between *extremal domains* and constant mean curvature surfaces. In this note, we try to explain that there is also a strong analogy between *exceptional domains* and minimal surfaces. More generally, we propose the :

Definition 1.2. *A m-dimensional flat Riemannian manifold M is said to be* exceptional *if it supports positive harmonic functions having* 0 *Dirichlet boundary data and constant (nonzero) Neumann boundary data. Any such harmonic function is called a* roof *function.*

Our results raise the problem of the classification of (unbounded) smooth m-dimensional *exceptional manifolds*. In trying to address this classification problem, we provide a Weierstrass type representation formula for *exceptional flat surfaces*. When the dimension $m = 2$, we give non trivial examples of *exceptional domains* which are embedded in $\mathbb{R}^2$ and we prove a *half space* result for *exceptional domains* which are conformal to a half plane.

2. A non trivial example of *exceptional domain* in $\mathbb{R}^2$

To begin with, observe that the property of being an *exceptional domain* is preserved under the action of the group of similarities of $\mathbb{R}^m$ (generated by isometries and dilations). We now give trivial examples of *exceptional domains* in $\mathbb{R}^m$:

(i) The half space $\{x = (x_1, \ldots, x_m) \in \mathbb{R}^m \ : \ x_1 > 0\}$ is an *exceptional domain* in $\mathbb{R}^m$ since the function $u(x) = x_1$ is a positive harmonic function with 0 Dirichlet boundary data and constant Neumann boundary data.

(ii) The complement of a ball of radius 1 in $\mathbb{R}^m$ is an *exceptional domain* since, the function u defined by $u(x) := \log |x|$, when $m = 2$ and $u(x) := 1 - |x|^{2-m}$, when $m \geq 3$ is positive, harmonic and has 0 Dirichlet and constant Neumann data on the unit sphere.

(iii) The product $\Omega \times \mathbb{R}^k$ is an *exceptional domain* in $\mathbb{R}^m$ provided $\Omega \subset \mathbb{R}^{m-k}$ is an *exceptional domain* in $\mathbb{R}^{m-k}$.

In dimension $m = 2$, there exists (up to a similarity) at least another *exceptional domain*. To describe this domain, we make use of the invariance of the Laplace operator under conformal transformations. The idea is that there exists a (somehow natural) unbounded, positive harmonic function U

with 0 Dirichlet boundary condition on an infinite strip in $\mathbb{R}^2$. This function does not have constant Neumann data but we can then look for a conformal transformation h which has the property that the pull back of the harmonic function U by h has constant Neumann boundary data on the boundary of the image of the strip by h.

To proceed, it is be convenient to identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$. We claim that :

Proposition 2.1. *The domain*

$$\Omega := \left\{ w \in \mathbb{C} \, : \, |\Im \, w| < \tfrac{\pi}{2} + \cosh(\Re \, w) \right\} \, ,$$

is an exceptional domain*.*

To prove this result, we define the infinite strip

$$S := \left\{ z \in \mathbb{C} \, : \, \Im \, z \in (-\tfrac{\pi}{2}, \tfrac{\pi}{2}) \right\} \, ,$$

and the mapping

$$F(z) := z + \sinh z \, .$$

Observe that $\Omega = F(S)$. The proof of Proposition 2.1 follows from the following two results.

Lemma 2.1. *The mapping F is a conformal diffeomorphism from S into Ω.*

Proof. We can write

$$F(z) - F(z') = (z - z') \int_0^1 \left(1 + \cosh\left(tz + (1-t)z'\right)\right) \, dt \, .$$

In particular

$$\langle z - z', F(z) - F(z') \rangle = |z - z'|^2 \left(1 + \int_0^1 \Re \, \cosh\left(tz + (1-t)z'\right) dt \right) \, , \tag{2.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{C}$. Now, observe that, for all $x + i\, y \in S$, we have

$$\Re \, \cosh\left(x + i\, y\right) = \cosh x \, \cos y \geq 0 \, .$$

This, together with (2.1), implies immediately that F, restricted to S, is injective. We also have

$$|\partial_z \Lambda(z)|^2 = |1 + \cosh z|^2 = (\cosh x + \cos y)^2 \, .$$

Therefore, $\partial_z F$ does not vanish in S and this shows that F is a local diffeomorphism and the mapping F being holomorphic, it is conformal. □

We define the real valued function u on Ω by the identity

$$u(F(z)) = \Re \, \cosh z \, ,$$

for all $z \in S$. We have the :

Lemma 2.2. *The function u is harmonic and positive in Ω, vanishes and has constant Neumann boundary data on $\partial\Omega$.*

Proof. The function W defined in $\mathbb{C}$ by $W(z) := \Re \cosh z$ is harmonic. As already mentioned in the proof of the previous Lemma, $W(x + i\, y) = \cosh x \, \cos y$ and hence, the function W is both harmonic and positive in S and vanishes on ∂S. The mapping F being a conformal diffeomorphism from S to Ω, we conclude the function u is both harmonic and positive in Ω and vanishes on $\partial\Omega$. We claim that u has constant Neumann data on $\partial\Omega$. Indeed, by definition

$$u(F(z)) = \tfrac{1}{2} \left(\cosh z + \cosh \bar{z} \right).$$

Since F is holomorphic, differentiation with respect to z yields

$$2\, \partial_z u(F(z)) = \frac{\sinh z}{1 + \cosh z}.$$

Therefore

$$|\nabla u|^2 (F(z)) = \frac{\cosh x - \cos y}{\cosh x + \cos y},$$

where $z = x + i\, y$. On $\partial\Omega$, $y = \pm\pi/2$ and hence $|\nabla u| \equiv 1$. Since we already know that $u = 0$ on $\partial\Omega$, we conclude that u has constant Neumann boundary data. $\square$

The two previous Lemma complete the proof of the fact that $\Omega = F(S)$ is an *exceptional domain* in $\mathbb{R}^2$ with *roof* function given by u.

Remark 2.1. *We suspect that this example generalises to any dimension $m \geq 3$, namely that there exists a rotationally symmetric* exceptional domain *in $\mathbb{R}^m$, for all $m \geq 3$.*

3. Toward a global representation formula

Let M be a *exceptional flat surface* with smooth boundary ∂M. Let $\tilde{M}$ be its universal cover and $\partial\tilde{M}$ be the preimage of ∂M by the covering map $\tilde{M} \longrightarrow M$. In the following, we exclude the non interesting case where $\partial M = \emptyset$.

By assumption, M is a flat surface and hence $\tilde{M}$ is naturally endowed with a flat Riemannian metric g and hence with an induced complex structure which is conformal to the standard one. Also, there exists an orientation preserving isometric immersion $F : (\tilde{M}, g) \longrightarrow (\mathbb{C}, g_{\mathbb{C}})$ (where $g_{\mathbb{C}}$ is the canonical Euclidean metric on $\mathbb{C}$) which induces a smooth immersion of $\partial\tilde{M}$. Observe that F is holomorphic and that

$$\|dF\|_g = 1,$$

in $\tilde{M} \cup \partial\tilde{M}$. We define the holomorphic $(1,0)$-form

$$\Phi := dF = \partial_z F \, dz,$$

Observe that Φ does not vanish and admits a smooth extension to $\tilde{M} \cup \partial\tilde{M}$.

We let $u : M \longrightarrow \mathbb{R}^+$ be a *roof* function on M and, with slight abuse of notation, we denote also by $u : \tilde{M} \longrightarrow \mathbb{R}^+$ its lift. The *roof* function u can be normalized so that

$$\|\nabla u\|_g = 1\,, \tag{3.1}$$

on ∂M. We consider the harmonic conjugate function $v : \tilde{M} \longrightarrow \mathbb{R}$ (which is uniquely defined up to some additive constant) which is the solution of

$$\partial_z (u - i\,v) = 0 \qquad \text{(and hence} \quad \partial_{\bar{z}} (u + i\,v) = 0\,)\,. \tag{3.2}$$

And we set

$$U := u + i\,v\,.$$

Recall that U is a holomorphic function from $\tilde{M}$ into $\mathbb{C}$. The property that u takes positive values in M and vanishes on ∂M can be translated into the fact that U maps $\tilde{M}$ to

$$\mathbb{C}^+ := \{w \in \mathbb{C} \,:\, \Re w > 0\}\,,$$

and $\partial \tilde{M}$ to $i\,\mathbb{R}$. Since $\Phi \neq 0$ on $\tilde{M}$ there exists a unique holomorphic function h on $\tilde{M}$ such that

$$dU = \partial_z U\, dz = h\, \Phi\,.$$

We deduce from the fact that u vanishes on $\partial \tilde{M}$ and from (3.1) that $\nabla_n U = 1$, if n denotes the inward unit normal vector to $\partial \tilde{M}$, and hence

$$\|\partial_z U\|_g = 1 \qquad \text{on} \qquad \partial \tilde{M}\,. \tag{3.3}$$

Now, condition (3.1) translates into the fact that

$$\|\Phi\|_g = \|dF\|_g = 1 = \|dU\|_g\,,$$

on $\partial \tilde{M}$. Clearly, this is equivalent to the fact that

$$|h| = 1 \qquad \text{on} \qquad \partial \tilde{M}\,.$$

Therefore, we end up with the following data :

(i) An oriented simply connected complex surface $\tilde{M}$ with smooth boundary $\partial \tilde{M}$.

(ii) A holomorphic function U, defined on $\tilde{M}$, which takes values in $\mathbb{C}^+$ and which maps $\partial \tilde{M}$ into $i\,\mathbb{R}$.

(iii) A holomorphic function h, defined on $\tilde{M}$, such that $|h| = 1$ on $\partial \tilde{M}$ and for which the 1-form Φ defined by $\Phi := \frac{1}{h}\, dU$ does not vanish on $\tilde{M}$.

By analogy with the theory of minimal surfaces, we call these data the Weierstrass type representation formula for *exceptional flat surfaces*.

Conversely, given a set of such data, we can define the map $F : \tilde{M} \longrightarrow \mathbb{C}$ by integrating $dF = \Phi$. Thanks to (iii), this map is an immersion and its image is an immersed *exceptional flat surface* with *roof* function given by

$$u = \Re U\,.$$

In the next section, we will give some explicit examples of such constructions when $\partial\tilde{M}$ is equal to $\partial D \setminus \{\alpha_1, \ldots, \alpha_n\}$, where $\alpha_1, \ldots, \alpha_n$ is a finite collection of points on $\partial D = S^1$.

Example 3.1. *We illustrate this Weierstrass type formula by giving some (rather pathologic) example. We consider $M = \mathbb{C}^+$, the function $U(z) = z$ and*

$$F(z) = \int_0^z e^{-\sinh\zeta}\, d\zeta\,.$$

Note that $\partial_z F$ is $2i\pi$-periodic and this implies that $F(z + 2i\pi) = F(z) + C$, where the constant C is given explicitly by

$$C := i \int_0^{2\pi} e^{-i\sin s}\, ds\,.$$

Moreover we observe that, for $x > 0$,

$$F(x + iy) = F(iy) + \int_0^x e^{-\sinh(s+iy)}\, ds\,,$$

converges to $+\infty$ as $x \to +\infty$ if $y = 0$, but this quantity is bounded if $|y - \pi| < \frac{\pi}{2}$ and even admits a finite limit as $x \to +\infty$.

Hence, in addition to the regular boundary $F(i\,\mathbb{R})$ (which is a smooth periodic curve), the image of F has a singular boundary which is the set of points which are the limits $\lim_{u\to+\infty} F(x + i\,y)$, as u tends to $+\infty$, for the values of y for which this limit exists. The roof *function tends to infinity along this singular boundary.*

4. Examples of exceptional flat surfaces

Thanks to the Weierstrass type representation described in the previous section, we can give many nontrivial examples of *exceptional flat surfaces.* We keep the notations introduced in the previous section.

The construction makes use of an integer $n \in \mathbb{N} \setminus \{0\}$ and the Riemann surface $D = \{z \in \mathbb{C} \,:\, |z| < 1\}$. On D, we define the holomorphic functions

$$h(z) = z^{n-1}\,,$$

and

$$U(z) := \frac{1 + z^n}{1 - z^n}\,.$$

Then, the 1-form Φ is given by

$$\Phi(z) := \frac{2n}{(1 - z^n)^2}\, dz\,,$$

Observe that both U and Φ have singularities at the n-th roots of unity. The function F is then obtained by integrating Φ and the *roof* function u is then defined by $u = \Re U$.

(i) When $n = 1$, we can take

$$F(z) = \frac{1+z}{1-z}\,.$$

In this case, we simply have $F(D) = \mathbb{C}^+$ and we recover the fact that the half plane is an *exceptional domain.* This exceptional domain is the counterpart of the plane in the framework of minimal surfaces.

(ii) When $n = 2$, we can take

$$F(z) = \frac{2z}{1-z^2} + \log\left(\frac{z+1}{z-1}\right)\,.$$

In this case, the *exceptional flat surface* we find can be isometrically embedded in $\mathbb{C}$ and hence $F(D)$ is an *exceptional domain.* In fact, $F(D)$ corresponds (up to some similarity) to the domain Ω which has been defined in Proposition 2.1. This exceptional domain is the counterpart of the catenoid in the framework of minimal surfaces.

(iii) Finally, when $n \geq 3$ the *exceptional flat surface* we find cannot be isometrically embedded in $\mathbb{C}$ anymore. They are the counterpart, in this setting, of the minimal n-noids described in [4].

Let us analyze this example further. The function U can be written as

$$U(z) = -\frac{1}{n}\sum_{k=1}^{n} \frac{z+\alpha^k}{z-\alpha^k}\,,$$

where $\alpha := e^{i2\pi/n}$. In particular, $\Re U$ is nothing but a multiple of the sum of the Poisson kernel on the unit disc with poles at $1, \alpha, \ldots, \alpha^{n-1}$. Next, observe that

$$dU = z^{n-1}\,\frac{2n}{(1-z^n)^2}\,dz\,,$$

so that the function h is cooked up to counterbalance the zero of dU and ensure that Φ does not vanish in the unit disk, while keeping the condition $|dU|^2 = |\Phi|^2$ on ∂D.

This example can be generalized as follows : Consider n distinct points $\alpha_1, \ldots, \alpha_n \in S^1 \subset \mathbb{C}$ and $a_1, \ldots, a_n > 0$. We define

$$U(z) := -\sum_{k=1}^{n} a_k\,\frac{z+\alpha_k}{z-\alpha_k}\,.$$

It is easy to check that $\Re U$ is positive (since each function

$$z \longmapsto -\frac{z+\alpha_k}{z-\alpha_k}\,,$$

maps D to $\mathbb{C}^+$) and vanishes on $\partial D \setminus \{\alpha_1, \ldots, \alpha_n\}$. We have

$$\prod_{k=1}^{n} (z - \alpha_k)^2 \, dU = P(z) \, dz ,$$

where P is a polynomial which depends on the choice of the points $\alpha_1, \ldots, \alpha_n$ and the weights $a_1, \ldots, a_n$. Let us assume that P does not vanish on ∂D and let us denote by $z_1, \ldots, z_\ell$ the roots of P, counted with multiplicity) which belong to the unit disc. We simply define

$$h(z) := \prod_{j=1}^{\ell} \frac{z - z_j}{z \, \bar{z}_j - 1} ,$$

and the 1-form Φ by $\Phi := \frac{1}{h} \, dU$. Integration of Φ yields a $2n$ dimensional family of *exceptional flat surfaces* which are immersed in $\mathbb{C}$.

5. A global Weierstrass type representation

In this section, we show that *exceptional flat surfaces* whose immersion in $\mathbb{C}$ have finitely many *regular ends* and are locally finite coverings of $\mathbb{C}$ are precisely the examples presented in the previous section. We use the notations introduced in section §3 and we set

$$\widehat{M} := M \cup \partial M .$$

We further assume that M is *simply connected* and that $\partial M \neq \emptyset$. In particular, M has the conformal type of the unit disk D, and without loss of generality, we can assume that M is indeed equal to D and consider $\bar{D}$ as a natural compactification of M. We denote by F an orientation preserving, holomorphic, isometric immersion $F : (\widehat{M}, g) \longrightarrow (\mathbb{C}, g_{\mathbb{C}})$. Recall that

$$\|dF\|_g = 1 ,$$

on ∂M. Some natural *hypotheses* will be needed :

(H-1) *M has finitely many ends.* This means that

$$\partial M = \partial D \setminus \cup_{j=1}^{n} E_j = \cup_{j=1}^{n} I_j ,$$

where each $E_j \subset S^1$ is a closed arc and $I_j \subset S^1$ is an open arc.

(H-2) *F is proper.* This means that $F(w)$ tends to infinity as w tends to $\cup_{j=1}^{n} E_j$.

(H-3) *Each end of N is regular.* This means that the image of $I_j := (\theta_j^-, \theta_j^+)$ by F is a curve Γ_j which is asymptotically parallel to fixed directions at infinity. In other words, there exist two unit vectors τ_j^- and $\tau_j^+ \in S^1 \subset \mathbb{C}$ such that

$$\lim_{\theta \in I_j \, , \, \theta \to \theta_j^{\pm}} \frac{F(e^{i\theta})}{|F(e^{i\theta})|} = \tau_j^{\pm} .$$

Observe that this is for example the case if we assume that Γ_j have finite total curvature.

(H-4) *The mapping F is a locally finite covering.* This means that there exists $d \in \mathbb{N}^*$ such that, for any $z \in \mathbb{C}$, the cardinal of $\{\zeta \in M : F(\zeta) = z\}$ is less than or equal to d.

The main result of this section reads :

Theorem 5.1. *Assume that M be a simply connected* exceptional flat surface *and let $F : M \longrightarrow \mathbb{C}$ be an isometric immersion. Further assume that (H-1), ..., (H-4) hold and we identify M with D. Then, there exist $\mu \in \mathbb{R}$, n distinct points $\alpha_1, \dots, \alpha_n \in S^1$ and n constants $a_1, \dots, a_n > 0$ such that*

$$dF = e^{i\,\mu} \prod_{k=1}^{m} \frac{\overline{z_k}\, z - 1}{z - z_k}\, dU.$$

where $z_1, \dots, z_m \in \bar{D}$ denote the zeros (counted with multiplicity) of dU where U

$$U(z) := -\sum_{j=1}^{n} a_j\, \frac{z + \alpha_j}{z - \alpha_j}.$$

in $\bar{D}$.

The proof of the Theorem is decomposed into the following Lemmas and Propositions. We start by analyzing the ends E_j and show that they reduce to isolated points $\alpha_1, \dots, \alpha_n$. Next, we analyze the behavior of F near the points α_j and show that F does not have any essential singularity there. Then, we proceed with the analysis of the function U and show that it has the expected form. The proof of the Theorem is completed with the study of the function h.

As promised, we first analyze the sets E_j. This is the contain of the following :

Lemma 5.1. *Under the assuptions of Theorem 5.1, there exists a finite number of points $\alpha_1, \dots, \alpha_n \in \partial D = S^1$ such that $\widehat{M} = \bar{D} \setminus \{\alpha_1, \dots, \alpha_n\}$.*

Proof. We need to show that each interval E_j is reduced to a point. This essentially follows from the fact that we can prove that the capacity of E_j vanishes.

We argue by contradiction and suppose that, for some j, E_j is an arc of positive arc length. This implies that we can find some $\ell \in (0, \pi/2)$ and some arc $E \subset E_j$ of length ℓ. Our problem being invariant under the action of homographic transformation of the unit disk, without loss of generality, we can assume that E is the image of $[-\frac{\ell}{2}, \frac{\ell}{2}]$ by $s \longmapsto e^{is}$ and, reducing ℓ if this is necessary, we can also assume that the opposite arc $-E$ (which is the image of $[-\frac{\ell}{2}, \frac{\ell}{2}]$ by $s \longmapsto -e^{is}$) is contained in $S^1 \setminus \cup_{j=1}^{n} E_j$.

Recall that for any smooth function defined on (a,b) which satisfies $f(b) = 1$ and $f(a) = 0$, we have

$$1 = f(b) - f(0) = \int_a^b f'(s)\, ds \leq \left(\int_a^b (f')^2(s)\, ds \right)^{1/2} \sqrt{b-a}\,.$$

If in addition, $b - a \leq 2$, we conclude that

$$\int_a^b (f')^2(s)\, ds \geq \frac{1}{2}\,.$$

Now, assume that we are given a smooth function $f : \overline{D} \longrightarrow \mathbb{R}$ such that $f = 1$ on E and $f = 0$ on $-E$, using the previous inequality, we can write

$$\int_D \|\nabla f\|^2_{g_{\mathbb{C}}}\, dx\, dy \geq \int_{D\cap\{|x|<\sin \ell/2\}} |\partial_y f|^2\, dx\, dy \geq \int_{|x|\leq \sin(\ell/2)} \frac{1}{2}\, dx = \sin(\ell/2)\,. \tag{5.1}$$

Given $R > r > 0$ we let $\chi : \mathbb{C} \longrightarrow \mathbb{R}$ be defined by

$$\chi(z) = \begin{cases} 0 & \text{if} \quad |z| \leq r \\ \dfrac{\log \frac{|z|}{r}}{\log \frac{|z|}{R}} & \text{if} \quad r \leq |z| \leq R \\ 1 & \text{if} \quad R \leq |z|\,, \end{cases}$$

and we define $f : D \longrightarrow \mathbb{R}$ by $f := \chi \circ F$. Since F is conformal, we can write

$$\int_D \|\nabla f\|^2_{g_{\mathbb{C}}}\, dx\, dy = \int_D \|\nabla f\|^2_g\, dvol_g\,.$$

Now, using (H-4), we conclude that

$$\int_D \|\nabla f\|^2_g\, dvol_g \leq d \int_{\mathbb{C}} \|\nabla \chi\|^2_{g_{\mathbb{C}}}\, dx\, dy = d\, \frac{2\pi}{\log \frac{R}{r}}\,. \tag{5.2}$$

Fixing $r > 0$ large enough, we can ensure that f is identically equal to 0 on $-E$. Using (H-2), we see that f is identically equal to 1 on each E_j, and in particular on E. Therefore, f can be used in (5.1) which together with (5.2) yields

$$2\,\pi\, d \geq \sin(\ell/2)\, \log \frac{R}{r},$$

independently of $R > r$. Letting R tend to infinity, we get a contradiction and the proof of the result is complete. □

Therefore, we now know that

$$E_j := \{\alpha_j\}\,.$$

Without loss of generality we can assume that $\alpha_1, \ldots, \alpha_n$ are arranged counterclockwise along S^1. We agree that $\alpha_0 := \alpha_n$ and $\alpha_{n+1} := \alpha_1$ and that, for each $j = 1, \ldots, n$, the arc I_j is positively oriented and joints α_j to α_{j+1}. We now analyze the singularities of F close to α_j.

Given $j = 1, \dots, n$, we denote by $S(\alpha_j, r)$ the circle of radius $r > 0$ centered at α_j. We define

$$\gamma_j := \bar{D} \cap S(\alpha_j, r) .$$

which we assume to be oriented clockwise. The *angle* $\theta_j \in \mathbb{R}$ at α_j is defined by

$$\theta_j := - \lim_{r \to 0} \int_{\gamma_k} F^* d\theta \, ,$$

where $d\theta := \Im \frac{dz}{z}$. Observe that, thanks to (H-3), θ_j is well defined and we have

$$\tau_j^- = e^{i\theta_j} \, \tau_{j-1}^+ \, .$$

With these definition in mid, we prove the

Lemma 5.2. *Under the assumption of Theorem 5.1, the function*

$$H_j(z) := (z - \alpha_j)^{\theta_j/\pi} \, F(z),$$

is holomorphic in a neighborhood of α_j in $\bar{D} \setminus \{\alpha_j\}$ and $H_j(\alpha_j) \neq 0$.

Proof. Without loss of generality, we can assume that $\alpha_j = 1$. By right composing F with the conformal transformation $z \longmapsto \frac{1-z}{1+z}$, we can replace D by $\mathbb{C}^+$. Now, we define

$$G(z) := F(z)^{-\pi/\theta_j}$$

Observe that $G(0) = 0$ by (H-2). Moreover, (H-3) together with the definition of θ_j implies that the image by G of a neighborhood of 0 in $i\,\mathbb{R}$ is a $\mathcal{C}^1$-curve (and hence analytic). In particular, there exists some conformal transformation T such that, for some $r > 0$, the image by $T \circ G$ of $i\,(-r, r)$ is a straight line segment in $i\,\mathbb{R}$. Then, it is possible to extend $T \circ G$ into some function $\tilde{G}$ which is defined on some neighborhood of 0 in $\mathbb{C}$ by setting $\tilde{G}(z) = T(G(z))$ when $\Im z \geq 0$ and

$$\tilde{G}(z) := -\overline{T(G(-\overline{z}))} \, ,$$

when $\Im z \leq 0$. The resulting function $\tilde{G}$ in bounded in a neighborhood of 0 in $\mathbb{C}$ and holomorphic away from 0. It is well known that the singularity is then removable and hence it is holomorphic and hence $\tilde{G}$ is actually holomorphic in a neighborhood of 0. In particular, we can write

$$G(z) = z^k \, H(z) \, ,$$

close to 0 where H is a holomorphic function which does not vanish at 0. Going back to the definition of G, this implies that

$$F(z) = (z - \alpha_j)^{-k\,\theta_j/\pi} \, H_j(z)$$

where H_j is holomorphic in a neighborhood of α_j and does not vanish at α_j. But, the definition of θ_j readily implies that $k = 1$. This completes the proof of the result. $\square$

As a corollary, we conclude that

$$H(z) := F(z) \prod_{j=1}^{n} (z-\alpha_j)^{\theta_j/\pi}\,, \tag{5.3}$$

is a bounded holomorphic function in $\overline{D}$. Moreover, since F tends to infinity as z approaches α_j, this implies that $\theta_j > 0$.

We now make use of the fact that M is an *exceptional domain* and hence there is a *roof* function $u : \widehat{M} \longrightarrow [0, +\infty)$ and we can define the holomorphic function $U := u + i\, v$, where $v : \widehat{M} \longrightarrow \mathbb{R}$ is the (real valued) harmonic conjugate of u. The purpose of the next result is to show that U is precisely given by the formula used in section§4.

Lemma 5.3. *Under the assumptions of Theorem 5.1, there exist* n *constants* $a_1, \ldots, a_n > 0$ *such that*

$$U(z) = - \sum_{j=1}^{n} a_j \, \frac{z+\alpha_j}{z-\alpha_j}\,.$$

Proof. We first observe that it is possible to extend the function U to all $\mathbb{C} \setminus \{\alpha_1, \ldots, \alpha_n\}$ by defining V to be equal to U in $\overline{D} \setminus \{\alpha_1, \ldots, \alpha_n\}$ and

$$V(z) := -\overline{U(1/\overline{z})}\,,$$

when $z \in \mathbb{C} \setminus \overline{D}$. The key observation is that, since $\Re U = 0$ on $\partial D \setminus \{\alpha_1, \ldots, \alpha_n\}$, V is continuous and in fact holomorphic on $\mathbb{C} \setminus \{\alpha_1, \ldots, \alpha_n\}$. Moreover V converges to $V(\infty) := -\overline{U(0)}$ at infinity.

We proceed with the proof that the function V has no essential singularity at any α_j, this is a simple consequence of Picard's big Theorem. By definition, $\Re V$ vanishes on I_j and is positive in D. Therefore, the outward normal derivative of $\Re V$ on I_j is negative. This implies that the tangential derivative of $\Im V$ on I_j does not vanish and hence that $\Im V$ is strictly monotone on each I_j. This shows that there exists some neighborhood $\mathcal{V}$ of α_j in $\mathbb{C}$ such that any element of $i\,\mathbb{R}$ is achieved by V at most twice in $\mathcal{V}$ (that is, at most once on I_j and at most once on I_{j-1}, and certainly not in $\mathcal{V} \setminus \partial D$, since V takes values in $\mathbb{C} \setminus i\,\mathbb{R}$ away from ∂D). Picard's big Theorem [1] then implies that α_j is not an essential singularity of V. Hence α_j is either a removable singularity of V or a pole.

Since $\|\nabla u\|_g \equiv 1$ on ∂M, this forces $|\partial_z U| = |\partial_z F|$ on ∂M, and since $|\partial_z F|$ tends to $+\infty$ at α_j so does $|\partial_z U|$ and hence all α_j are poles of V.

We are now interested in the zeros of V. Since $\Re V$ takes positive values in D and negative values in $(\mathbb{C} \cup \{\infty\}) \setminus \overline{D}$, we know already that the only possible zeros of V are on ∂D. Moreover, we have already seen that, along I_j, the function $V = i\, v$ where v is strictly monotone. Furthermore since α_{j-1} and α_j are poles of V, $|V|$ must converges to $+\infty$ when one tends to α_{j-1} or α_j. Because of the continuity of v along each I_j then implies that v vanishes exactly at one point β_j on each I_j. Moreover, this zero is simple, since if it would be a zero of order $k > 1$, this would imply that the zero set of $\Re V$ near β_j contains k curves intersecting at β_j. This would then force $\Re V = \Re U$ to vanish in D, which is in contradiction with our hypothesis.

Finally, we prove that V has only simple poles. We know that V extends meromorphically to a map on $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$ with no pole nor zero at infinity. Furthermore, V has exactly n simple zeros and n poles, hence these poles must be simple. To summarize, the function V can be written as a linear combination of the constant function and functions of the form $z \longmapsto \frac{1}{z-\alpha_j}$. Without loss of generality, this amounts to say that V can be written as

$$V(z) = a - \sum_{j=1}^{n} a_j \, \frac{z+\alpha_j}{z-\alpha_j} \, ,$$

where a and the a_j are complex numbers. Using the fact that, by construction, $V(1/\overline{z}) = -\overline{V(z)}$, we conclude that $a \in i\,\mathbb{R}$ and also that $a_j \in \mathbb{R}$. Moreover, $\Re U$ being positive, this implies that the a_j are positive real numbers. This completes the proof of the result since U is defined up to the addition of some element of $i\,\mathbb{R}$. $\square$

We are now in a position to complete our analysis of the function F. Since F is an immersion $dF \neq 0$ on $\widehat{M}$. Hence there exists a unique holomorphic function h on $\widehat{M}$ such that

$$\partial_z U = h \, \partial_z F \, , \tag{5.4}$$

on $\widehat{M}$. Moreover, since $\|\nabla u\|_g \equiv 1$ on ∂M, this implies that $|h| \equiv 1$ on ∂M. In the next result, we analyze the function h, this will complete the proof of Theorem 5.1.

Lemma 5.4. *Under the assumptions of Theorem 5.1, there exists a constant $e^{i\,\mu} \in \mathbb{R}$ such that the function h defined by (5.4) has the form*

$$h(z) = e^{-i\,\mu} \prod_{k=1}^{m} \frac{z - z_k}{\overline{z_k} z - 1}.$$

where $z_1, \ldots z_m$ are the zeros of $\partial_z U$ in D counted with multiplicity.

Proof. The function h is holomorphic in D and satisfies $|h| = 1$ on $\partial D \setminus \{\alpha_1, \ldots, \alpha_n\}$. We can extend h as a holomorphic function H which defined on $(\mathbb{C} \cup \{\infty\}) \setminus \{\alpha_1, \ldots, \alpha_n\}$ by setting $H(z) := h(z)$ for all $z \in \overline{D} \setminus \{\alpha_1, \ldots, \alpha_n\}$ and

$$\overline{H(z)} := \frac{1}{h(1/\overline{z})} \, , \tag{5.5}$$

for all $z \in \mathbb{C} \setminus \overline{D}$. Clearly H is locally bounded in $\overline{D} \setminus \{\alpha_1, \ldots, \alpha_n\}$, its only singularities in $(\mathbb{C} \cup \{\infty\}) \setminus \overline{D}$ are poles which are the images by $z \longmapsto 1/\overline{z}$ of the zeros of h and hence is meromorphic outside $\{\alpha_1, \ldots, \alpha_n\}$. But, Lemma 5.2 and (5.3) imply that, near α_j, $|H|$ is bounded by a constant times

$|z-\alpha_j|^{-k_j}$ for some $k_j > 0$. Therefore, α_j is not an essential singularity of H and hence, H is meromorphic in $\mathbb{C} \cup \{\infty\}$.

Observe that $|H(z)| = 1$ on $\partial D \setminus \{\alpha_1, \ldots, \alpha_n\}$ and this implies that the points α_j are not poles of H. Therefore, the singularities α_j of H are removable. Also, we have

$$\Delta |H|^2 = 4 \partial_z \partial_{\bar z} |H|^2 = 4 |\partial_z H|^2 \geq 0 \,,$$

and since $|H| = 1$ on ∂D, the maximum principle implies that $|H| \leq 1$ in D.

Now, H being bounded in $\overline{D}$, it does not have poles in this set and this also implies that H has no zeroes in $(\mathbb{C} \cup \{\infty\}) \setminus D$ (because otherwise H would have poles in $\overline{D}$ by (5.5). Therefore, if $z_1, \ldots, z_m \in D$ denote the zeros of H (counted with multiplicity), then the poles of H are given by $1/\overline{z_1}, \ldots, 1/\overline{z_m}$ (also counted with multiplicity). It is then a simple exercise to check that H is of the form

$$H(z) = C \prod_{k=1}^{m} \frac{z - z_k}{\overline{z_k}\, z - 1} \,,$$

for some constant $C \in \mathbb{C}$. Finally, the condition that $|H(z)| = 1$ on ∂D forces $|C| = 1$. This completes the proof of the result. □

6. A Bernstein type result for 2-dimensional exceptional domains

We prove the following Bernstein type result for 2-dimensional exceptional domains :

Proposition 6.1. *Assume that Ω is a 2-dimensional* exceptional domain *which is conformal to $\mathbb{C}^+$ and let u be a* roof *function on Ω. We further assume that $\partial_x u > 0$ in Ω, then Ω is a half plane.*

Proof. Since we have assumed that Ω is conformal to $\mathbb{C}^+$, there exists a holomorphic map $\Psi : \mathbb{C}^+ \longmapsto \Omega$. We then define

$$H := (\partial_z u) \circ \Psi \,.$$

The function H is holomorphic in $\mathbb{C}^+$ and does not vanish (since we have assumed that $\partial_x u \neq 0$). Moreover, $|H| \equiv 1$ on $\partial \mathbb{C}^+$. We can write

$$H = e^{i\Theta} \,,$$

where Θ is a holomorphic function defined in $\mathbb{C}^+$ which is real valued on the imaginary axis. This means that

$$\Im \Theta = 0 \quad \text{when} \quad \Re z = 0 \,.$$

Since we have assumed that $\partial_x u > 0$, we also conclude that $\Re \Theta \in (-\pi/2, \pi/2)$.

We can extend Θ as a holomorphic function $\tilde\Theta$ in $\mathbb{C}$ as follows : for all $z \in \mathbb{C}$ such that $\Re z \geq 0$ we set

$$\tilde\Theta(z) := \Theta(z) \,,$$

while, when $\Re z < 0$, we set

$$\tilde{\Theta}(z) := \overline{\Theta(-\bar{z})}\,.$$

It is easy to check that $\tilde{\Theta}$ is a holomorphic function : in fact, the real part of Θ is extended as an even function of $\Re z$ while the imaginary part of Θ is extended as an odd function of $\Re z$. The fact that $\tilde{\Theta}$ is $\mathcal{C}^1$ is then a consequence of the fact that $\Im\,\Theta = 0$ on the imaginary axis and the fact that Θ being holomorphic, $\partial_x \Re\,\Theta = 0$ on the imaginary axis of $\mathbb{C}$.

Observe that the real part of $\tilde{\Theta}$ is a bounded harmonic function, and, as such, it has to be constant. The function $\tilde{\Theta}$ being holomorphic, we conclude that it is constant. But this implies that the gradient of u is constant and hence the level sets of u are straight lines. This implies that u only depends on one variable and hence it is a affine function. This completes the proof of the result. □

As a Corollary, we also prove the :

Corollary 6.1. *There is no exceptional domain contained in a wedge*

$$\Omega \subset \{z \in \mathbb{C} \,:\, \Re\, z \geq \kappa\, |\Im\, z|\}\,,$$

for some $\kappa > 0$.

Proof. The proof is by contradiction. If Ω were such an *exceptional domain*, there would exist on Ω a *roof* function u. One can apply the moving plane method [6], [3] to prove that $\partial_x u > 0$ and hence that $\partial\Omega$ is a graph over the y-axis. Observe that, since Ω is contained in a half plane, there is no bounded, positive, harmonic function on Ω having 0 boundary data on $\partial\Omega$ (otherwise one could use an affine function as a barrier to obtain a contradiction). Certainly, $\Omega \cup \partial\Omega$ is conformal to $\bar{D} \setminus E$ where D is the unit disc and E is a closed arc included in S^1. Necessarily, E is reduced to a point since otherwise we can construct bounded, positive, harmonic functions on E which have 0 boundary data on $S^1 \setminus E$ and these would lift to bounded, positive, harmonic function on Ω, with 0 boundary data, a contradiction. Therefore, we conclude that Ω is conformal to $\mathbb{C}^+$. The assumptions of the previous Lemma are fulfilled and hence we conclude that Ω is a half plane, which is a contradiction. □

7. Open problems

We have no non trivial example of *exceptional domain* in higher dimensions, $\mathbb{R}^m$, for $m \geq 3$ (beside the examples described in section 2). In dimension $m = 2$, it is tempting to conjecture that (up to similarity) the only *exceptional domains* which can be embedded in $\mathbb{R}^2$ are the half spaces, the complement of a ball and the example discussed in section 2.

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