Very weak solutions for the Stokes equations

Chérif Amrouche, M.A. Rodriguez-Bellido

Very weak solutions for the Stokes equations.

Chérif Amrouche [a], María Ángeles Rodríguez-Bellido [b,1]

[a] *Laboratoire de Mathématiques Appliquées, CNRS UMR 5142, Université de Pau et des Pays de l'Adour, IPRA, Avenue de l'Université- 64000 Pau (France)*

[b] *Dpto. Ecuaciones Diferenciales y Análisis Numérico, Universidad de Sevilla, Aptdo. de Correos 1160 - 41080 Sevilla (Spain)*

Received *****; accepted after revision +++++

Presented by

Abstract

The concept of very weak solution introduced by Giga [9] for the Stokes equations has been intensively studied in the last years for the Navier-Stokes equations. However, a more rigorous study about the existence of traces for non regular vector fields is necessary in order to make a precise extension of the Stokes result to the Navier-Stokes case. Such study and a new and simpler proof of the existence of very weak solution for Stokes equations are made, based on density arguments and an adequate functional framework. We also obtain regularity results in fractional Sobolev spaces. All these results are obtained in the case of a bounded open set, connected of class $\mathcal{C}^{1,1}$ of $\mathbb{R}^3$ and can be extended to Laplace's equation and to other dimensions. *To cite this article: C. Amrouche, M.A. Rodríguez-Bellido, C. R. Acad. Sci. Paris, Ser. I 340 (2005).*

Résumé

Solutions très faibles pour les équations de Stokes. Le concept de solution très faible introduit par Giga [9] pour les équations de Stokes a été beaucoup étudié ces dernières années pour les équations de Navier-Stokes. Nous donnons ici une nouvelle preuve plus simple de l'existence de solution très faible pour les équations de Navier-Stokes, qui s'appuie sur des arguments de densité et un cadre fonctionnel approprié pour définir de manière plus rigoureuse les traces des champs de vecteurs peu réguliers. On obtient aussi résultats de régularité dans des espaces de Sobolev fractionnaires. Tous les résultats sont obtenus dans le cas d'un ouvert connexe de classe $\mathcal{C}^{1,1}$ de $\mathbb{R}^3$ et peuvent être étendus à l'équation de Laplace ainsi qu'aux autres dimensions. *Pour citer cet article : C. Amrouche, M.A. Rodríguez-Bellido, C. R. Acad. Sci. Paris, Ser. I 340 (2005).*

Email addresses: `cherif.amrouche@univ-pau.fr` (Chérif Amrouche), `angeles@us.es` (María Ángeles Rodríguez-Bellido).

[1] Partially supported by M.E.C. (Spain), Project MTM2006-07932, and by Junta de Andalucía, Project P06-FQM-02373.

Version française abrégée

L'objet de cette note consiste essentiellement à étudier l'existence de solutions très faibles $(\boldsymbol{u}, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)$ du problème de Stokes (S) (voir Definition 3.2). L'une des difficultés pour prouver l'existence de telles solutions consiste à donner un sens à la condition aux limites de Dirichlet. Utilisant un argument d'interpolation, cela nous permet d'en déduire l'existence de solutions appartenant à des espaces de Sobolev fractionnaires. Les principaux résultats d'existence sont donnés dans la Section 3.

1. Introduction

Let Ω be a bounded connected open set of $\mathbb{R}^3$ of class $\mathcal{C}^{1,1}$ with boundary Γ. We study the Stokes problem (S):

$$-\Delta \boldsymbol{u} + \nabla q = \boldsymbol{f} \quad \text{and} \quad \nabla \cdot \boldsymbol{u} = h \quad \text{in } \Omega, \quad \boldsymbol{u} = \boldsymbol{g} \quad \text{on } \Gamma,$$

where $\boldsymbol{u}$ denotes the velocity field and q the pressure, and both are unknown. The external force $\boldsymbol{f}$, the compressibility condition h and the boundary condition $\boldsymbol{g}$ are the data. The vector fields and matrix fields (and the corresponding spaces) are respectively denoted by boldface Roman and special Roman.

The notion of very weak solutions $(\boldsymbol{u}, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)$ for the Stokes or Navier-Stokes equations, corresponding to very irregular data, has been developed in the last years by Giga [9] (and also by Lions-Magenes [11] for the Laplace's equation, in a domain Ω of class $\mathcal{C}^\infty$), Amrouche-Girault [1] (in a domain Ω of class $\mathcal{C}^{1,1}$) and more recently by Galdi-Simader-Sohr [8], Farwig-Galdi-Sohr [7] (in a domain Ω of class $\mathcal{C}^{2,1}$, see also Schumacher [14]) and finally by Kim [10] (in a domain Ω of class $\mathcal{C}^2$ with connected boundary). In this context, the boundary condition is chosen in $\mathbf{L}^p(\Gamma)$ (see Brown-Shen [3], Conca [5], Fabes-Kenig-Verchota [6], Moussaoui [12], Shen [15], Savaré [13], or more generally in $\mathbf{W}^{-1/p,p}(\Gamma)$.

The purpose of this work is to develop a unified theory of very weak solutions for the Dirichlet problem associated to the Stokes system. One important question is to define rigorously the traces of the vector functions which are living in subspaces of $\mathbf{L}^p(\Omega)$ (see Lemma 2.3). We prove existence and uniqueness of very weak solutions $(\boldsymbol{u}, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)$ for the Stokes problem for any $1 < p < \infty$ (see Definition 3.2). Using an interpolation argument, we deduce the existence of solutions belonging to fractional Sobolev spaces $W^{s,p}(\Omega)$, with $0 \leq s \leq 2$ (see Corollary 3.6 and Theorem 3.7). Observe that the study of the Stokes problem is fundamental for the study of the Oseen and Navier-Stokes equations. The detailed proofs of the results announced in this Note are given in [2].

2. Density and trace results

We introduce the spaces: $\boldsymbol{\mathcal{D}}_\sigma(\Omega) = \{\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\Omega);\ \nabla \cdot \boldsymbol{\varphi} = 0\}$, $\quad \boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega}) = \{\boldsymbol{\psi} \in \boldsymbol{\mathcal{D}}(\overline{\Omega})^3;\ \nabla \cdot \boldsymbol{\psi} = 0\}$,

$$\mathbf{L}^p_\sigma(\Omega) = \{\boldsymbol{v} \in \mathbf{L}^p(\Omega);\ \nabla \cdot \boldsymbol{v} = 0\}, \quad \mathbf{X}_{r,p}(\Omega) = \{\boldsymbol{\varphi} \in \mathbf{W}^{1,r}_0(\Omega);\ \nabla \cdot \boldsymbol{\varphi} \in W^{1,p}_0(\Omega)\}, \quad 1 < r,\, p < \infty,$$

and we set $\mathbf{X}_{p,p}(\Omega) = \mathbf{X}_p(\Omega)$.

Lemma 2.1 *i) The space* $\boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega})$ *is dense in* $\mathbf{L}^p_\sigma(\Omega)$.

ii) The space $\boldsymbol{\mathcal{D}}(\Omega)$ *is dense in* $\mathbf{X}_{r,p}(\Omega)$ *and for all* $q \in W^{-1,p}(\Omega)$ *and* $\boldsymbol{\varphi} \in \mathbf{X}_{r',p'}(\Omega)$, *we have*

$$\langle \nabla q,\ \boldsymbol{\varphi}\rangle_{[\mathbf{X}_{r',p'}(\Omega)]' \times \mathbf{X}_{r',p'}(\Omega)} = -\langle q,\ \nabla \cdot \boldsymbol{\varphi}\rangle_{W^{-1,p}(\Omega) \times W^{1,p'}_0(\Omega)}. \tag{1}$$

It is easy then to prove the following characterization:

$$(\mathbf{X}_{r,p}(\Omega))' = \left\{\boldsymbol{f} = \nabla \cdot \mathbb{F}_0 + \nabla f_1;\ \mathbb{F}_0 \in \mathbb{L}^{r'}(\Omega),\ f_1 \in W^{-1,p'}(\Omega),\ \text{with } \mathbb{F}_0 = (\mathrm{f_{ij}})_{1 \leq \mathrm{i,j} \leq 3}\right\}. \tag{2}$$

As a consequence of Lemma 2.1 ii) and the Sobolev embeddings, we have the embeddings $\mathbf{W}^{-1,r}(\Omega) \hookrightarrow (\mathbf{X}_{r',p'}(\Omega))' \hookrightarrow \mathbf{W}^{-2,p}(\Omega)$, where the second embedding holds if $\frac{1}{r} \leq \frac{1}{p} + \frac{1}{3}$.

Giving a meaning to the trace of a very weak solution of a Stokes problem is not trivial. Remember that we are not in the classical variational framework. In this way, we need to introduce some spaces. First, we consider the space $\mathbf{Y}_{p'}(\Omega) = \{\boldsymbol{\psi} \in \mathbf{W}^{2,p'}(\Omega);\ \boldsymbol{\psi}|_\Gamma = \mathbf{0},\ (\nabla \cdot \boldsymbol{\psi})|_\Gamma = 0\}$ that can also be described (see [1]) as:

$$\mathbf{Y}_{p'}(\Omega) = \{\boldsymbol{\psi} \in \mathbf{W}^{2,p'}(\Omega);\ \boldsymbol{\psi}|_\Gamma = \mathbf{0},\ \frac{\partial \boldsymbol{\psi}}{\partial \boldsymbol{n}} \cdot \boldsymbol{n}\Big|_\Gamma = 0\}. \tag{3}$$

Note also that if $\boldsymbol{\psi} \in \mathbf{Y}_{p'}(\Omega)$, then div $\boldsymbol{\psi} \in W_0^{1,p'}(\Omega)$ and the range space of the normal derivative $\gamma_1 : \mathbf{Y}_{p'}(\Omega) \to \mathbf{W}^{1/p,p'}(\Gamma)$ is $\mathbf{Z}_{p'}(\Gamma) = \{\boldsymbol{z} \in \mathbf{W}^{1/p,p'}(\Gamma);\ \boldsymbol{z} \cdot \boldsymbol{n} = 0\}$. Secondly, we shall use the spaces:

$$\mathbf{T}_{p,r}(\Omega) = \{\boldsymbol{v} \in \mathbf{L}^p(\Omega);\ \Delta \boldsymbol{v} \in (\mathbf{X}_{r',p'}(\Omega))'\},\ \mathbf{T}_{p,r,\sigma}(\Omega) = \{\boldsymbol{v} \in \mathbf{T}_{p,r}(\Omega);\ \nabla \cdot \boldsymbol{v} = 0\},$$

endowed with the norm $\|\boldsymbol{v}\|_{\mathbf{T}_{p,r}(\Omega)} = \|\boldsymbol{v}\|_{\mathbf{L}^p(\Omega)} + \|\Delta \boldsymbol{v}\|_{[\mathbf{X}_{r',p'}(\Omega)]'}$. When $p = r$, these spaces are denoted as $\mathbf{T}_p(\Omega)$ and $\mathbf{T}_{p,\sigma}(\Omega)$, respectively.

We also introduce the space $\mathbf{H}_{p,r}(\mathrm{div};\Omega) = \{\boldsymbol{v} \in \mathbf{L}^p(\Omega);\ \nabla \cdot \boldsymbol{v} \in L^r(\Omega)\}$, which is equipped with the graph norm. The following lemma will help us to prove a trace result:

Lemma 2.2 *i) The space $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\mathbf{T}_{p,r}(\Omega)$ and in $\mathbf{T}_{p,r}(\Omega) \cap \mathbf{H}_{p,r}(\mathrm{div};\Omega)$ respectively.*

ii) The space $\boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega})$ is dense in $\mathbf{T}_{p,r,\sigma}(\Omega)$.

The following two lemma prove that the tangential trace of functions $\boldsymbol{v}$ of $\mathbf{T}_{p,r,\sigma}(\Omega)$ belongs to the dual space of $\mathbf{Z}_{p'}(\Gamma)$, which is $(\mathbf{Z}_{p'}(\Gamma))' = \{\boldsymbol{\mu} \in \mathbf{W}^{-1/p,p}(\Gamma);\ \boldsymbol{\mu} \cdot \boldsymbol{n} = 0\}$. Besides, we recall that we can decompose $\boldsymbol{v}$ into its tangential, $\boldsymbol{v}_\tau$, and normal parts, that is: $\boldsymbol{v} = \boldsymbol{v}_\tau + (\boldsymbol{v} \cdot \boldsymbol{n})\,\boldsymbol{n}$.

Lemma 2.3 *Let Ω be a bounded open set of $\mathbb{R}^3$ of class $\mathcal{C}^{1,1}$. Let $1 < p < \infty$ and $r > 1$ be such that $\frac{1}{r} \leq \frac{1}{p} + \frac{1}{3}$. The mapping $\gamma_\tau : \boldsymbol{v} \mapsto \boldsymbol{v}_\tau|_\Gamma$ on the space $\mathcal{D}(\overline{\Omega})^3$ can be extended by continuity to a linear and continuous mapping, still denoted by γ_τ, from $\mathbf{T}_{p,r}(\Omega)$ into $\mathbf{W}^{-1/p,p}(\Gamma)$, and the following Green formula holds*

$$\langle \Delta \boldsymbol{v}, \boldsymbol{\psi} \rangle_{[\mathbf{X}_{r',p'}(\Omega)]' \times \mathbf{X}_{r',p'}(\Omega)} = \int_\Omega \boldsymbol{v} \cdot \Delta \boldsymbol{\psi}\, d\boldsymbol{x} - \left\langle \boldsymbol{v}_\tau, \frac{\partial \boldsymbol{\psi}}{\partial \boldsymbol{n}} \right\rangle_{\mathbf{W}^{-1/p,p}(\Gamma) \times \mathbf{W}^{1/p,p'}(\Gamma)}, \tag{4}$$

for any $\boldsymbol{v} \in \mathbf{T}_{p,r}(\Omega)$ and $\boldsymbol{\psi} \in \mathbf{Y}_{p'}(\Omega)$.

Proof: Let be $\boldsymbol{v} \in \mathcal{D}(\overline{\Omega})^3$ and $\boldsymbol{\psi} \in \mathbf{Y}_{p'}(\Omega)$. Then (4) holds. Observe that $\mathbf{Y}_{p'}(\Omega)$ is included in $\mathbf{X}_{r',p'}(\Omega)$

Consider $\boldsymbol{\mu} \in \mathbf{W}^{1/p,p'}(\Gamma)$. Then, $\boldsymbol{\mu} = \boldsymbol{\mu}_\tau + (\boldsymbol{\mu} \cdot \boldsymbol{n})\boldsymbol{n}$. Since Ω is of class $\mathcal{C}^{1,1}$, we know that there exists $\boldsymbol{\psi} \in \mathbf{W}^{2,p'}(\Omega)$ such that $\boldsymbol{\psi} = \mathbf{0}$ and $\dfrac{\partial \boldsymbol{\psi}}{\partial \boldsymbol{n}} = \boldsymbol{\mu}_\tau$ on Γ, and verifying:

$$\|\boldsymbol{\psi}\|_{\mathbf{W}^{2,p'}(\Omega)} \leq C\,\|\boldsymbol{\mu}_\tau\|_{\mathbf{W}^{1/p,p'}(\Gamma)} \leq C\,\|\boldsymbol{\mu}\|_{\mathbf{W}^{1/p,p'}(\Gamma)}.$$

Moreover, $\boldsymbol{\psi} \in \mathbf{Y}_{p'}(\Omega)$. Therefore, we can estimate the boundary term as follows for such functions $\boldsymbol{\psi}$:

$$\left|\langle \boldsymbol{v}_\tau, \boldsymbol{\mu} \rangle_{\mathbf{W}^{-1/p,p}(\Gamma) \times \mathbf{W}^{1/p,p'}(\Gamma)}\right| \leq \|\boldsymbol{v}\|_{\mathbf{L}^p(\Omega)} \|\boldsymbol{\psi}\|_{\mathbf{W}^{2,p'}(\Omega)} + \|\Delta \boldsymbol{v}\|_{[\mathbf{X}_{r',p'}]'} \|\boldsymbol{\psi}\|_{\mathbf{X}_{r',p'}}.$$

Thus, $\|\boldsymbol{v}_\tau\|_{\mathbf{W}^{-1/p,p}(\Gamma)} \leq C\,\|\boldsymbol{v}\|_{\mathbf{T}_{p,r}(\Omega)}$. Therefore, the linear continuous mapping $\boldsymbol{v} \mapsto \boldsymbol{v}_\tau|_\Gamma$ defined on $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is continuous for the norm of $\mathbf{T}_{p,r}(\Omega)$. Since $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\mathbf{T}_{p,r}(\Omega)$, then we can extend this mapping from $\mathbf{T}_{p,r}(\Omega)$ into $\mathbf{W}^{-1/p,p}(\Gamma)$, that is, the tangential trace of functions of $\mathbf{T}_{p,r}(\Omega)$ belongs to $\mathbf{W}^{-1/p,p}(\Gamma)$ and the relation (4) holds. ■

We can also prove that $\mathcal{D}(\overline{\Omega})^3$ is dense in $\mathbf{H}_{p,r}(\mathrm{div};\Omega)$, the mapping $\gamma_{\boldsymbol{n}} : \boldsymbol{v} \mapsto \boldsymbol{v} \cdot \boldsymbol{n}|_\Gamma$ is continuous from $\mathbf{H}_{p,r}(\mathrm{div};\Omega)$ into $W^{-1/p,p}(\Gamma)$, and we have the Green formula: for any $\boldsymbol{v} \in \mathbf{H}_{p,r}(\mathrm{div};\Omega)$ and $\varphi \in W^{1,p'}(\Omega)$,

$$\int_\Omega \boldsymbol{v} \cdot \nabla \varphi\ d\boldsymbol{x} + \int_\Omega \varphi \ \mathrm{div}\ \boldsymbol{v}\ d\boldsymbol{x} = \langle \boldsymbol{v} \cdot \boldsymbol{n}, \varphi \rangle_{W^{-1/p,p}(\Gamma) \times W^{1/p,p'}(\Gamma)}.$$

3. Very weak solutions and regularity

We focus on the study of the Stokes problem (S) with the compatibility condition:

$$\int_\Omega h(\boldsymbol{x})\,d\boldsymbol{x} = \langle \boldsymbol{g}\cdot\boldsymbol{n}, 1\rangle_{W^{-1/p,p}(\Gamma)\times W^{1/p,p'}(\Gamma)}. \tag{5}$$

Basic results on weak and strong solutions of problem (S) may be summarized in the following theorem (see [1], [4]).

Theorem 3.1 *i) For every $\boldsymbol{f}\in\mathbf{W}^{-1,p}(\Omega)$, $h\in L^p(\Omega)$, $\boldsymbol{g}\in\mathbf{W}^{1-1/p,p}(\Gamma)$ satisfying the compatibility condition (5), the Stokes problem (S) has exactly one solution $\boldsymbol{u}\in\mathbf{W}^{1,p}(\Omega)$ and $q\in L^p(\Omega)/\mathbb{R}$, and there exists a constant $C>0$, depending only on p and Ω, such that:*

$$\|\boldsymbol{u}\|_{\mathbf{W}^{1,p}(\Omega)} + \|q\|_{L^p(\Omega)/\mathbb{R}} \le C\,(\|\boldsymbol{f}\|_{\mathbf{W}^{-1,p}(\Omega)} + \|h\|_{L^p(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{1-1/p,p}(\Gamma)}). \tag{6}$$

ii) Moreover, if $\boldsymbol{f}\in\mathbf{L}^p(\Omega)$, $h\in W^{1,p}(\Omega)$, $\boldsymbol{g}\in\mathbf{W}^{2-1/p,p}(\Gamma)$, then $\boldsymbol{u}\in\mathbf{W}^{2,p}(\Omega)$, $q\in W^{1,p}(\Omega)$ satisfy an analogous estimate to (6) with the corresponding norms.

We wonder about minimal necessary assumptions on $\boldsymbol{f}$, h and $\boldsymbol{g}$, in order that a very weak solution, that is, $(\boldsymbol{u},q)\in\mathbf{L}^p(\Omega)\times W^{-1,p}(\Omega)/\mathbb{R}$ exists.

We are interested here in the case of singular data satisfying the following assumptions:

$$\boldsymbol{f}\in(\mathbf{X}_{r',p'}(\Omega))',\ h\in L^r(\Omega),\ \boldsymbol{g}\in\mathbf{W}^{-1/p,p}(\Gamma),\ \text{with } \frac{1}{r}\le\frac{1}{p}+\frac{1}{3} \text{ and } r\le p. \tag{7}$$

Observe that the space $(\mathbf{X}_{r',p'}(\Omega))'$ is an intermediate space between $\mathbf{W}^{-1,r}(\Omega)$ and $\mathbf{W}^{-2,p}(\Omega)$.

Definition 3.2 (**Very weak solution for the Stokes problem)** *A pair $(\boldsymbol{u},q)\in\mathbf{L}^p(\Omega)\times W^{-1,p}(\Omega)$ is a* **very weak solution** *of (S) if the following equalities hold: For any $\boldsymbol{\varphi}\in\mathbf{Y}_{p'}(\Omega)$ and $\pi\in W^{1,p'}(\Omega)$,*

$$\begin{aligned} -\int_\Omega \boldsymbol{u}\cdot\Delta\boldsymbol{\varphi}\,d\boldsymbol{x} - \langle q,\nabla\cdot\boldsymbol{\varphi}\rangle_{W^{-1,p}(\Omega)\times W_0^{1,p'}(\Omega)} &= \langle \boldsymbol{f},\boldsymbol{\varphi}\rangle_\Omega - \langle \boldsymbol{g}_\tau, \frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{n}}\rangle_\Gamma,\\ \int_\Omega \boldsymbol{u}\cdot\nabla\pi\,d\boldsymbol{x} = -\int_\Omega h\,\pi\,d\boldsymbol{x} &+ \langle \boldsymbol{g}\cdot\boldsymbol{n},\pi\rangle_\Gamma, \end{aligned} \tag{8}$$

with $\langle\cdot,\cdot\rangle_\Omega = \langle\cdot,\cdot\rangle_{[\mathbf{X}_{r',p'}(\Omega)]'\times\mathbf{X}_{r',p'}(\Omega)}$ and $\langle\cdot,\cdot\rangle_\Gamma = \langle\cdot,\cdot\rangle_{\mathbf{W}^{-1/p,p}(\Gamma)\times\mathbf{W}^{1/p,p'}(\Gamma)}$.

Note that $W^{1,p'}(\Omega)\hookrightarrow L^{r'}(\Omega)$ and $\mathbf{Y}_{p'}(\Omega)\hookrightarrow\mathbf{X}_{r',p'}(\Omega)$ if $\frac{1}{r}\le\frac{1}{p}+\frac{1}{3}$, which means that all the brackets and integrals have a sense. We can then prove that, if $\boldsymbol{f}$, h and $\boldsymbol{g}$ satisfy (7), then $(\boldsymbol{u},q)\in\mathbf{L}^p(\Omega)\times W^{-1,p}(\Omega)$ is a very weak solution of (S) if and only if $(\boldsymbol{u},q)$ satisfies the system (S) in the sense of distributions.

Proposition 3.1 *Let $\boldsymbol{f}\in(\mathbf{X}_{p'}(\Omega))'$, $h\in L^p(\Omega)$ and $\boldsymbol{g}\in\mathbf{W}^{-1/p,p}(\Gamma)$ be given, and satisfying the compatibility condition (5). Then, the Stokes problem (S) has exactly one solution $\boldsymbol{u}\in\mathbf{T}_p(\Omega)$ and $q\in W^{-1,p}(\Omega)/\mathbb{R}$. Moreover, there exists a constant $C>0$, depending only on p and Ω, such that:*

$$\|\boldsymbol{u}\|_{\mathbf{T}_p(\Omega)} + \|q\|_{W^{-1,p}(\Omega)/\mathbb{R}} \le C\,\left(\|\boldsymbol{f}\|_{[\mathbf{X}_{p'}(\Omega)]'} + \|h\|_{L^p(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{-1/p,p}(\Gamma)}\right). \tag{9}$$

Proof: The case $\boldsymbol{f}=\mathbf{0}$ and $h=0$ is considered in [1]. Here, we generalize the result as follows:

Step 1: We suppose $\mathbf{g}\cdot\boldsymbol{n}=0$ on Γ and $\int_\Omega h(\boldsymbol{x})\,d\boldsymbol{x}=0$. It remains to consider the equivalent problem: Find $(\boldsymbol{u},q)\in\mathbf{L}^p(\Omega)\times W^{-1,p}(\Omega)/\mathbb{R}$ such that: for any $\boldsymbol{w}\in\mathbf{Y}_{p'}(\Omega)$ and any $\pi\in W^{1,p'}(\Omega)$ it holds

$$\int_\Omega \boldsymbol{u}\cdot(-\Delta\boldsymbol{w}+\nabla\pi)\,d\boldsymbol{x} - \langle q,\nabla\cdot\boldsymbol{w}\rangle_{W^{-1,p}(\Omega)\times W_0^{1,p'}(\Omega)} = \langle\boldsymbol{f},\boldsymbol{w}\rangle_{[\mathbf{X}_{p'}(\Omega)]'\times\mathbf{X}_{p'}(\Omega)} - \langle\boldsymbol{g}_\tau,\frac{\partial\boldsymbol{w}}{\partial\boldsymbol{n}}\rangle_\Gamma - \int_\Omega h\,\pi\,d\boldsymbol{x}.$$

We can prove (as in [1]) that for any pair $(\mathbf{F},\varphi)\in\mathbf{L}^{p'}(\Omega)\times(W_0^{1,p'}(\Omega)\cap L_0^{p'}(\Omega))$:

$$\left|\langle \boldsymbol{f}, \boldsymbol{w}\rangle_{[\mathbf{X}_{p'}(\Omega)]'\times \mathbf{X}_{p'}(\Omega)} - \left\langle \boldsymbol{g}_\tau, \frac{\partial \boldsymbol{w}}{\partial \boldsymbol{n}}\right\rangle_\Gamma - \int_\Omega h\,\pi\,d\boldsymbol{x}\right|$$
$$\leq C\,\left(\|\boldsymbol{f}\|_{[\mathbf{X}_{p'}(\Omega)]'} + \|\boldsymbol{g}\|_{\mathbf{W}^{-1/p,p}(\Omega)} + \|h\|_{L^p(\Omega)}\right)\,\left(\|\mathbf{F}\|_{\mathbf{L}^{p'}(\Omega)} + \|\varphi\|_{W^{1,p'}(\Omega)}\right),$$

being $(\boldsymbol{w}, \pi) \in \mathbf{Y}_{p'}(\Omega) \times W^{1,p'}(\Omega)/\mathbb{R}$ the unique solution of the Stokes (dual) problem:

$$-\Delta \boldsymbol{w} + \nabla \pi = \mathbf{F} \text{ and } \nabla\cdot \boldsymbol{w} = \varphi \text{ in } \Omega,\ \boldsymbol{w} = \mathbf{0} \text{ on } \Gamma.$$

Note that for any $k \in \mathbb{R}$, $\left|\int_\Omega h\,\pi\,d\boldsymbol{x}\right| = \left|\int_\Omega h\,(\pi + k)\,d\boldsymbol{x}\right| \leq \|h\|_{L^p(\Omega)}\|\pi\|_{L^{p'}(\Omega)/\mathbb{R}}$ and $\|\boldsymbol{w}\|_{\mathbf{W}^{2,p'}(\Omega)} + \|\pi\|_{W^{1,p'}(\Omega)/\mathbb{R}} \leq C\,\left(\|\mathbf{F}\|_{\mathbf{L}^{p'}(\Omega)} + \|\varphi\|_{W^{1,p'}(\Omega)}\right)$. From this bound, we deduce that the mapping $(\mathbf{F}, \varphi) \to \langle \boldsymbol{f}, \boldsymbol{w}\rangle_\Omega - \langle \boldsymbol{g}_\tau, \frac{\partial \boldsymbol{w}}{\partial \boldsymbol{n}}\rangle_\Gamma - \int_\Omega h\,\pi\,d\boldsymbol{x}$ defines an element of the dual space of $\mathbf{L}^{p'}(\Omega)\times(W_0^{1,p'}(\Omega)\cap L_0^{p'}(\Omega))$ with norm bounded by $C(\|\boldsymbol{f}\|_{[\mathbf{X}_{p'}(\Omega)]'} + \|h\|_{L^p(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{-1/p,p}(\Omega)})$. That means that there exists a unique $(\boldsymbol{u}, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)/\mathbb{R}$ solution of (S) satisfying the estimate (9).

Step 2: Now, we suppose that $\int_\Omega h(\boldsymbol{x})\,d\boldsymbol{x} = \langle \mathbf{g}\cdot\boldsymbol{n}, 1\rangle_\Gamma$. Define $\boldsymbol{u}_0 = \nabla\theta$ with $\theta \in W^{1,p}(\Omega)$ the solution of the Neumann problem: $\Delta\theta = h$ in Ω and $\frac{\partial\theta}{\partial \boldsymbol{n}} = \boldsymbol{g}\cdot\boldsymbol{n}$ on Γ. By Step 1, there exists a unique $(\boldsymbol{z}, q) \in \mathbf{L}^p(\Omega)\times W^{-1,p}(\Omega)/\mathbb{R}$ satisfying: $-\Delta \boldsymbol{z} + \nabla q = \boldsymbol{f} + \nabla h$ and $\nabla\cdot \boldsymbol{z} = 0$ in Ω and $\boldsymbol{z} = \boldsymbol{g} - \boldsymbol{u}_0|_\Gamma$ on Γ, where $\nabla h \in (\mathbf{X}_{p'}(\Omega))'$ and $\boldsymbol{g} - \boldsymbol{u}_0|_\Gamma$ satisfies the hypothesis of Step 1. Thus, the pair of functions $(\boldsymbol{u}, q) = (\boldsymbol{z} + \boldsymbol{u}_0, q)$ is the required solution. ■

The following result is a generalization of Proposition 4.11 in [1], where $\boldsymbol{f} = \mathbf{0}$ and $h = 0$.

Theorem 3.3 *Let $\boldsymbol{f}, h, \boldsymbol{g}$ be given satisfying (5) and (7). Then, the Stokes problem (S) has exactly one solution $(\boldsymbol{u}, q) \in \mathbf{T}_{p,r}(\Omega) \times W^{-1,p}(\Omega)/\mathbb{R}$. Moreover, there exists a constant $C > 0$, only depending on p and Ω, such that:*

$$\|\boldsymbol{u}\|_{\mathbf{T}_{p,r}(\Omega)} + \|q\|_{W^{-1,p}(\Omega)/\mathbb{R}} \leq C\,\left(\|\boldsymbol{f}\|_{[\mathbf{X}_{r',p'}(\Omega)]'} + \|h\|_{L^r(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{-1/p,p}(\Gamma)}\right) \tag{10}$$

Remark 1 i) Observe that in [8] Theorem 3, the domain was of class $\mathcal{C}^{2,1}$ (here it is of class $\mathcal{C}^{1,1}$), and the divergence term was $h \in L^p(\Omega)$ (here $h \in L^r(\Omega)$). Moreover, our solution is obtained in the space $\mathbf{T}_{p,r}(\Omega)$, which has been clearly characterized, contrary to the space $\widehat{\mathbf{W}}^{1,p}(\Omega)$ appearing in [8], which was not characterized, completely abstract and obtained as closure of $\mathbf{W}^{1,p}(\Omega)$ for the norm $\|\boldsymbol{u}\|_{\widehat{\mathbf{W}}^{1,p}(\Omega)} = \|\boldsymbol{u}\|_{\mathbf{L}^p(\Omega)} + \|A_r^{-1/2}\mathcal{P}_r\Delta\boldsymbol{u}\|_{\mathbf{L}^r(\Omega)}$, where A_r is the Stokes operator with domain equal to $\mathbf{W}^{2,p}(\Omega)\cap\mathbf{W}_0^{1,p}(\Omega)\cap\mathbf{L}^p_\sigma(\Omega)$ and $\mathcal{P}_r$ is the Helmholtz projection operator from $\mathbf{L}^r(\Omega)$ onto $\mathbf{L}^r_\sigma(\Omega)$.

ii) Existence of very weak solution $\boldsymbol{u} \in \mathbf{L}^p(\Omega)$ was proved by Kim [10] for $\boldsymbol{f} \in [\mathbf{W}_0^{1,q'}(\Omega)\cap W^{2,q'}(\Omega)]'$, for $h \in [W^{1,q'}(\Omega)]'$ and $\boldsymbol{g} \in \mathbf{W}^{-1/q,q}(\Gamma)$, but the spaces chosen for h and $\boldsymbol{f}$ are not correct either and the equivalence in Theorem 5 of [10] is not valid.

Corollary 3.4 *Let $\boldsymbol{f}, h, \boldsymbol{g}$ be given satisfying (5) and $\boldsymbol{f} = \nabla\cdot\mathbb{F}_0 + \nabla f_1$ with $\mathbb{F}_0 \in \mathbb{L}^r(\Omega)$, $f_1 \in W^{-1,p}(\Omega)$, $h \in L^r(\Omega)$, $\boldsymbol{g} \in \mathbf{W}^{1-1/r,r}(\Gamma)$. Then the solution $\boldsymbol{u}$ given by Theorem 3.3 belongs to $\mathbf{W}^{1,r}(\Omega)$. If moreover $f_1 \in L^r(\Omega)$, then q belongs to $L^r(\Omega)$. In both cases, we have analogous estimates to (10).*

Remark 2 It is clear that $\mathbf{W}^{1,r}(\Omega) \hookrightarrow \mathbf{T}_{p,r}(\Omega)$ when $\frac{1}{r} \leq \frac{1}{p} + \frac{1}{3}$, *i.e.*, $\mathbf{T}_{p,r}(\Omega)$ is an intermediate space between $\mathbf{W}^{1,r}(\Omega)$ and $\mathbf{L}^p(\Omega)$.

Corollary 3.5 *Let $h \in L^r(\Omega)$ and $\boldsymbol{g} \in \mathbf{W}^{-1/p,p}(\Gamma)$ be given, satisfying (5), with $\frac{1}{r} \leq \frac{1}{p} + \frac{1}{3}$ and $r \leq p$. Then, there exists at least one solution $\boldsymbol{u} \in \mathbf{T}_{p,r}(\Omega)$ verifying $\nabla\cdot\boldsymbol{u} = h$ in Ω and $\boldsymbol{u} = \boldsymbol{g}$ on Γ. Moreover, there exists a constant $C = C(\Omega, p, r)$ such that $\|\boldsymbol{u}\|_{\mathbf{T}_{p,r}(\Omega)} \leq C\,\left(\|h\|_{L^r(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{-1/p,p}(\Gamma)}\right)$.*

The following corollary gives Stokes solutions $(\boldsymbol{u}, q)$ in fractionary Sobolev spaces of type $\mathbf{W}^{\sigma,p}(\Omega)\times W^{\sigma-1,p}(\Omega)$, with $0 < \sigma < 2$.

Corollary 3.6 *Let s be a real number such that $0 \le s \le 1$.*

i) Let $\boldsymbol{f} = \nabla \cdot \mathbb{F}_0 + \nabla f_1$, h and $\boldsymbol{g}$ satisfying the compatibility condition (5) with $\mathbb{F}_0 \in \mathbf{W}^{s,r}(\Omega)$, $f_1 \in W^{s-1,p}(\Omega)$, $\boldsymbol{g} \in \mathbf{W}^{s-1/p,p}(\Gamma)$ and $h \in W^{s,r}(\Omega)$, with $\frac{1}{r} \le \frac{1}{p} + \frac{1}{3}$ and $r \le p$. Then, the Stokes problem (S) has exactly one solution $(\boldsymbol{u},\ q) \in \mathbf{W}^{s,p}(\Omega) \times W^{s-1,p}(\Omega)/\mathbb{R}$ satisfying the estimate

$$\|\boldsymbol{u}\|_{\mathbf{W}^{s,p}(\Omega)} + \|q\|_{W^{s-1,p}(\Omega)/\mathbb{R}} \le C\, \left(\|\mathbb{F}_0\|_{\mathbf{W}^{s,r}(\Omega)} + \|f_1\|_{W^{s-1,p}(\Omega)} + \|h\|_{W^{s,r}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{s-1/p,p}(\Gamma)}\right)$$

ii) Assume that $\boldsymbol{f} \in \mathbf{W}^{s-1,p}(\Omega)$, $\boldsymbol{g} \in \mathbf{W}^{s+1-1/p,p}(\Gamma)$ and $h \in W^{s,p}(\Omega)$, fulfill the compatibility condition (5). Then, the Stokes problem (S) has exactly one solution $(\boldsymbol{u},\ q) \in \mathbf{W}^{s+1,p}(\Omega) \times W^{s,p}(\Omega)/\mathbb{R}$ with

$$\|\boldsymbol{u}\|_{\mathbf{W}^{s+1,p}(\Omega)} + \|q\|_{W^{s,p}(\Omega)/\mathbb{R}} \le C\, \left(\|\boldsymbol{f}\|_{\mathbf{W}^{s-1,p}(\Omega)} + \|h\|_{W^{s,p}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{s+1-1/p,p}(\Gamma)}\right).$$

The following theorem provides solutions for external forces $\boldsymbol{f} \in \mathbf{W}^{s-2,p}(\Omega)$ and divergence condition $h \in W^{s-1,p}(\Omega)$ with $1/p < s < 2$. In particular, if $p = 2$, we obtain solutions in $\mathbf{H}^{1/2+\varepsilon}(\Omega) \times H^{1/2+\varepsilon}(\Omega)$, for any $0 < \varepsilon \le 3/2$.

Theorem 3.7 *Let s be a real number such that $\frac{1}{p} < s \le 2$. Let $\boldsymbol{f}, h$ and $\boldsymbol{g}$ satisfy the compatibility condition (5) with $\boldsymbol{f} \in \mathbf{W}^{s-2,p}(\Omega)$, $h \in W^{s-1,p}(\Omega)$ and $\boldsymbol{g} \in \mathbf{W}^{s-1/p,p}(\Gamma)$. Then, the Stokes problem (S) has exactly one solution $(\boldsymbol{u},\ q) \in \mathbf{W}^{s,p}(\Omega) \times W^{s-1,p}(\Omega)/\mathbb{R}$ satisfying the estimate*

$$\|\boldsymbol{u}\|_{\mathbf{W}^{s,p}(\Omega)} + \|q\|_{W^{s-1,p}(\Omega)/\mathbb{R}} \le C\, \left(\|\boldsymbol{f}\|_{\mathbf{W}^{s-2,p}(\Omega)} + \|h\|_{W^{s-1,p}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{s-1/p,p}(\Gamma)}\right). \qquad (11)$$

References

[1] C. Amrouche and V. Girault, *Decomposition of vector spaces and application to the Stokes problem in arbitrary dimension,* Czechoslovak Mathematical Journal, 44, 119 (1994), 109–140.

[2] C. Amrouche and M. A. Rodríguez-Bellido, *Stokes, Oseen and Navier-Stokes equations with singular data,* Submitted.

[3] R. M. Brown and Z. Shen, *Estimates for the Stokes operator in Lipschitz domains,* Indiana University Mathematics Journal, 44 (4) (1995), 1183-1206.

[4] L. Cattabriga, *Su un problema al contorno relativo al sistema di equazoni di Stokes,* Rend. Sem. Univ. Padova 31 (1961), 308–340.

[5] C. Conca, *Stokes equations with non-smooh data,* Rev. Math. Appl., 10 (1989), 115-122.

[6] E. B. Fabes, C. E. Kenig, G. C. Verchota, *The Dirichlet problem for the Stokes system on Lipschitz domains,* Duke Math. Jour. 57-3 (1988), 769-793.

[7] R. Farwig, G. P. Galdi, *Very weak solutions and large uniqueness classes of stationary Navier-Stokes equations in bounded domain of $\mathbb{R}^2$,* J. Diff. Equat. 227 (2006), 564-580.

[8] G. P. Galdi, C. G. Simader and H. Sohr, *A class of solutions to stationary Stokes and Navier-Stokes equations with boundary data in $W^{-1/q,q}$,* Math. Ann. , 331 (2005), 41-74.

[9] Y. Giga, *Analyticity of the semigroup generated by the Stokes operator in L_p-spaces,* Math. Z. 178 (1981), 287-329.

[10] H. Kim, *Existence and regularity of very weak solutions of the stationary Navier-Stokes equations,* Arch. Rational Mech. Anal., **193** (2009), 117-152.

[11] J. L. Lions and E. Magenes *Problèmes aux limites non homogènes et applications,* vol 1, Dunod, Paris, (1968).

[12] M. Moussaoui and A.M. Zine, *Existence and regularity results for the Stokes system with non-smooth boundary data in a polygon*, Math. Mod. Meth. Appl. Sc. vol. 8-8 (1998), 1307-1315.

[13] G. Savaré, *Regularity results for elliptic equations in Lipschitz domains*, J. Funct. Anal. 152 (1998), no. 1, 176-201.

[14] K. Schumacher, *Very weak solutions to the stationary Stokes and Stokes resolvent problem in weighted function spaces*, Ann. Univ. Ferrara Sez. VII Sci. Mat. 54 (2008), no. 1, 123-144.

[15] Z. Shen, *A note on the Dirichlet problem for the Stokes system in Lipschitz domains,* Proc. AMS 123-3 (1995), 801-811.