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# Weak vector and scalar potentials. Applications to Poincaré's theorem and Korn's inequality in Sobolev spaces with negative exponents.

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## Abstract

In this paper, we present several results concerning vector potentials and scalar potentials with data in Sobolev spaces with negative exponents, in a not necessarily simply-connected, three-dimensional domain. We then apply these results to Poincaré's theorem and to Korn's inequality.

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## 1 Weak versions of a classical theorem of Poincaré

In this work, (the results of which were announced in [2]),  $\Omega$  is a bounded open connected subset of  $\mathbb{R}^3$  with a Lipschitz-continuous boundary  $\Gamma$ . The notation  $X', \langle \cdot, \cdot \rangle_X$  denotes the duality pairing between a topological space  $X$  and its dual  $X'$ . The letter  $C$  denotes a constant that is not necessarily the same at its various occurrences.

We begin with a weak version of a well-known theorem of Poincaré. Here as elsewhere in this paper, “weak” means that the result to which it is attached holds as well in Sobolev spaces with negative exponents.

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**Theorem 1.1.** *Let  $\mathbf{f} \in H^{-m}(\Omega)^3$  for some integer  $m \geq 0$ . Then the following properties are equivalent:*

(i)  $H^{-m}(\Omega)^3 \langle \mathbf{f}, \boldsymbol{\varphi} \rangle_{H_0^m(\Omega)^3} = 0$  for all  $\boldsymbol{\varphi} \in V_m = \{\boldsymbol{\varphi} \in H_0^m(\Omega)^3; \operatorname{div} \boldsymbol{\varphi} = 0\}$ ,

(ii)  $H^{-m}(\Omega)^3 \langle \mathbf{f}, \boldsymbol{\varphi} \rangle_{H^m(\Omega)^3} = 0$  for all  $\boldsymbol{\varphi} \in \mathcal{V} = \{\boldsymbol{\varphi} \in \mathcal{D}(\Omega)^3; \operatorname{div} \boldsymbol{\varphi} = 0\}$

To begin with, let  $\mathbf{f} \in H^{-m}(\Omega)^3$  be such that  $\mathbf{curl} \mathbf{f} = \mathbf{0}$  in  $\Omega$ . We then use the same argument as in [8]: We know that there exist a unique  $\mathbf{u} \in H_0^m(\Omega)^3$  and a unique  $p \in H^{-m+1}(\Omega)/\mathbb{R}$  (see [5]) such that

$$\Delta^m \mathbf{u} + \mathbf{grad} p = \mathbf{f} \quad \text{and} \quad \operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega. \quad (1)$$

Hence  $\Delta^m \mathbf{curl} \mathbf{u} = \mathbf{0}$  in  $\Omega$  so that the hypoellipticity (see [10]) of the polyharmonic operator  $\Delta^m$  implies that  $\mathbf{curl} \mathbf{u} \in C^\infty(\Omega)^3$ . Since  $\operatorname{div} \mathbf{u} = 0$ , we deduce that  $\Delta \mathbf{u} = \mathbf{curl} \mathbf{curl} \mathbf{u} \in C^\infty(\Omega)^3$ . This also implies that  $\Delta^m \mathbf{u}$  belongs to  $C^\infty(\Omega)^3$  and is an irrotational vector field. By the classical Poincaré theorem, there exists  $q \in C^\infty(\Omega)^3$  such that  $\Delta^m \mathbf{u} = \mathbf{grad} q$ . Thus,  $\mathbf{f} = \mathbf{grad} (p + q)$  and, thanks to [4] proposition 2.10, the function  $p + q$  belongs to the space  $H^{-m+1}(\Omega)$ .  $\square$

We can give another proof of the implication (iv)  $\implies$  (iii) by using the following theorem:

**Theorem 1.2.** *Assume that both  $\Omega$  and  $\mathbb{R}^3 \setminus \Omega$  are simply-connected. Let  $\mathbf{u} \in H_0^m(\Omega)^3$ ,  $m \geq 0$ , be a vector field that satisfies  $\operatorname{div} \mathbf{u} = 0$  in  $\Omega$ . Then there exists a vector potential  $\boldsymbol{\psi}$  in  $H_0^{m+1}(\Omega)^3$  such that*

$$\mathbf{u} = \mathbf{curl} \boldsymbol{\psi}, \quad \operatorname{div} \Delta^{m+1} \boldsymbol{\psi} = 0 \quad \text{in } \Omega, \quad \text{and} \quad \|\boldsymbol{\psi}\|_{H^{m+1}(\Omega)^3} \leq C \|\mathbf{u}\|_{H^m(\Omega)^3}. \quad (2)$$

*Proof.* Let  $\mathbf{u} \in H_0^m(\Omega)^3$  be such that  $\operatorname{div} \mathbf{u} = 0$  in  $\Omega$  and let  $\tilde{\mathbf{u}}$  denote the extension of  $\mathbf{u}$  by  $\mathbf{0}$  in  $\mathbb{R}^3 \setminus \Omega$ . Thus  $\tilde{\mathbf{u}} \in H_0^m(\mathbb{R}^3)^3$ ,  $\operatorname{div} \tilde{\mathbf{u}} = 0$  in  $\mathbb{R}^3$ , and there exist an open ball  $B$  containing  $\bar{\Omega}$  and a vector field  $\mathbf{w} \in H_0^{m+1}(B)^3$  such that  $\tilde{\mathbf{u}} = \mathbf{curl} \mathbf{w}$ ,  $\operatorname{div} \Delta^{m+1} \mathbf{w} = 0$  in  $B$ , and

$$\|\mathbf{w}\|_{H^{m+1}(B)^3} \leq C \|\mathbf{u}\|_{H^m(B)^3}.$$

The open set  $\Omega' := B \setminus \bar{\Omega}$  is bounded, has a Lipschitz-continuous boundary and is simply-connected. Furthermore, the vector field  $\mathbf{w}' := \mathbf{w}|_{\Omega'}$  belongs to  $H^{m+1}(\Omega')^3$  and satisfies  $\mathbf{curl} \mathbf{w}' = \mathbf{0}$  in  $\Omega'$ . Therefore there exists a function  $\chi' \in H^1(\Omega')$  such that  $\mathbf{w}' = \mathbf{grad} \chi'$  in  $\Omega'$ . Hence in fact  $\chi' \in H^{m+2}(\Omega')$  and the estimate

$$\|\chi'\|_{H^{m+2}(\Omega')} \leq C \|\mathbf{w}'\|_{H^{m+1}(\Omega')^3}$$

holds. Since the function  $\chi' \in H^{m+2}(\Omega')$  can be extended to a function  $\tilde{\chi}$  in  $H^{m+2}(\mathbb{R}^3)$ , with

$$\|\tilde{\chi}\|_{H^{m+2}(\mathbb{R}^3)} \leq C \|\chi'\|_{H^{m+2}(\Omega')} \leq C \|\mathbf{w}'\|_{H^{m+1}(\Omega')^3},$$

the vector field  $\tilde{\boldsymbol{\varphi}} := \mathbf{w} - \mathbf{grad} \tilde{\chi}$  belongs to the space  $H^{m+1}(B)^3$  and satisfies  $\tilde{\boldsymbol{\varphi}}|_{\Omega'} = \mathbf{0}$ . Then the restriction  $\boldsymbol{\varphi} := \tilde{\boldsymbol{\varphi}}|_{\Omega}$  belongs to the space  $H_0^{m+1}(\Omega)^3$ , satisfies the estimate (2), and  $\mathbf{curl} \tilde{\boldsymbol{\varphi}} = \mathbf{curl} \mathbf{w} = \tilde{\mathbf{u}}$  in  $B$ . Thus  $\mathbf{u} = \mathbf{curl} \boldsymbol{\varphi}$

in  $\Omega$ . Let now  $p$  denote the unique solution in the space  $H_0^{m+2}(\Omega)$  of  $\Delta^{m+2}p = \operatorname{div} \Delta^{m+1}\boldsymbol{\varphi}$ , so that the estimate

$$\|p\|_{H^{m+2}(\Omega)} \leq C \|\boldsymbol{\varphi}\|_{H^{m+1}(\Omega)^3}$$

holds. Then the function  $\boldsymbol{\psi} = \boldsymbol{\varphi} - \mathbf{grad} p$  satisfies (2).  $\square$

We can give yet another proof of the above implication (iv)  $\implies$  (iii): Consider again the solution  $\mathbf{u} \in H_0^m(\Omega)^3$  to (1) and let  $\mathbf{v} \in H_0^{m+1}(\Omega)^3$  denote the vector potential of  $\mathbf{u}$  as given by theorem 1.2. We then have  $\Delta^m \mathbf{curl} \mathbf{u} = \mathbf{0}$ . If  $m = 2k$ , for some integer  $k \geq 1$ , then

$$\begin{aligned} {}_{H^{-m-1}(\Omega)^3} \langle \Delta^m \mathbf{curl} \mathbf{u}, \mathbf{v} \rangle_{{}_{H_0^{m+1}(\Omega)^3}} &= {}_{H^{-1}(\Omega)^3} \langle \Delta^k \mathbf{curl} \mathbf{u}, \Delta^k \mathbf{v} \rangle_{{}_{H_0^1(\Omega)^3}} \\ &= \int_{\Omega} \Delta^k \mathbf{u} \cdot \Delta^k \mathbf{curl} \mathbf{v} \, dx \\ &= \|\Delta^k \mathbf{u}\|_{L^2(\Omega)^3}^2. \end{aligned}$$

This implies that  $\Delta^k \mathbf{u} = \mathbf{0}$  in  $\Omega$  and thus  $\mathbf{u} = \mathbf{0}$  since  $\mathbf{u} \in H_0^m(\Omega)^3$ . The case  $m = 2k + 1$  follows by a similar argument.  $\square$

## 2 Scalar Potentials

Let  $\Gamma_i$ ,  $0 \leq i \leq I$ , denote the connected components of the boundary  $\Gamma$  of the domain  $\Omega$ ,  $\Gamma_0$  being the boundary of the only unbounded connected component of  $\mathbb{R}^3 \setminus \overline{\Omega}$ . We do not assume that  $\Omega$  is simply-connected, however we assume that there exist  $J$  connected and oriented surfaces  $\Sigma_j$ ,  $1 \leq j \leq J$  contained in  $\Omega$ , with the following properties: each surface  $\Sigma_j$  is an open subset of a smooth manifold, the boundary of  $\Sigma_j$  is contained in  $\Gamma$  for  $1 \leq j \leq J$ , the intersection  $\overline{\Sigma}_i \cap \overline{\Sigma}_j$  is empty for  $i \neq j$ , and finally the open set  $\Omega^\circ = \Omega \setminus \bigcup_{j=1}^J \Sigma_j$  is simply-connected and pseudo-Lipschitz in the sense of [1]. Each such surface  $\Sigma_j$  is called a cut. Finally, let  $[\cdot]_j$  denote the jump of a function over each cut  $\Sigma_j$ ,  $1 \leq j \leq J$ .

We then define the spaces

$$\begin{aligned} H(\mathbf{curl}, \Omega) &= \{\mathbf{v} \in L^2(\Omega)^3; \mathbf{curl} \mathbf{v} \in L^2(\Omega)^3\}, \\ H(\operatorname{div}, \Omega) &= \{\mathbf{v} \in L^2(\Omega)^3; \operatorname{div} \mathbf{v} \in L^2(\Omega)\}, \end{aligned}$$

each one being equipped with the graph norm, and their subspaces

$$\begin{aligned} H_0(\mathbf{curl}, \Omega) &= \{\mathbf{v} \in H(\mathbf{curl}, \Omega); \mathbf{v} \times \mathbf{n} = \mathbf{0} \quad \text{on } \Gamma\}, \\ H_0(\operatorname{div}, \Omega) &= \{\mathbf{v} \in H(\operatorname{div}, \Omega); \mathbf{v} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma\}. \end{aligned}$$

For any function  $q$  in  $H^1(\Omega^\circ)$ ,  $\mathbf{grad} q$  denotes the gradient of  $q$  in the sense of distributions in  $\mathcal{D}'(\Omega^\circ)$ . It belongs to  $L^2(\Omega^\circ)^3$  and therefore can be extended to  $L^2(\Omega)^3$ . In order to distinguish this extension from the gradient of  $q$  in  $\mathcal{D}'(\Omega)$ , we denote it by  $\widetilde{\mathbf{grad}} q$ . Finally, we remark that the space

$$K_T(\Omega) := \{\mathbf{w} \in H(\mathbf{curl}, \Omega) \cap H_0(\text{div}, \Omega); \mathbf{curl} \mathbf{w} = \mathbf{0} \text{ and } \text{div} \mathbf{w} = 0 \text{ in } \Omega\}$$

is of dimension equal to  $J$ : As shown in [1] Prop. 3.14, it is spanned by the vector fields  $\widetilde{\mathbf{grad}} q_j^T$ ,  $1 \leq j \leq J$ , where each function  $q_j^T \in H^1(\Omega^\circ)$ , which is unique up to an additive constant, satisfies

$$\begin{aligned} \Delta q_j^T &= 0 && \text{in } \Omega^\circ, \\ \partial_n q_j^T &= 0, && \text{on } \Gamma, \\ [q_j^T]_k &= \text{constant}, [\partial_n q_j^T]_k = 0, \langle \partial_n q_j^T, 1 \rangle_{\Sigma_k} = \delta_{jk} && \text{for } 1 \leq k \leq J. \end{aligned} \quad (3)$$

where  $\langle \cdot, \cdot \rangle_{\Sigma_k}$  denotes the duality pairing between the spaces  $H^{-1/2}(\Sigma_k)$  and  $H^{1/2}(\Sigma_k)$ .

**Theorem 2.1.** *Given any function  $\mathbf{f} \in L^2(\Omega)^3$  that satisfies*

$$\mathbf{curl} \mathbf{f} = \mathbf{0} \quad \text{in } \Omega \quad \text{and} \quad \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, d\mathbf{x} = 0 \quad \text{for all } \mathbf{v} \in K_T(\Omega), \quad (4)$$

*there exists a scalar potential  $\chi$  in  $H^1(\Omega)$  such that*

$$\mathbf{f} = \mathbf{grad} \chi \quad \text{in } \Omega \quad \text{and} \quad \|\chi\|_{H^1(\Omega)} \leq C \|\mathbf{f}\|_{L^2(\Omega)^3}. \quad (5)$$

*Proof.* It suffices to show that, given any vector field  $\mathbf{v} \in H_0(\text{div}, \Omega)$  such that  $\text{div} \mathbf{v} = 0$  in  $\Omega$ , there holds  $\int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, d\mathbf{x} = 0$ . Let

$$\mathbf{z} = \sum_{j=1}^J \langle \mathbf{v} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} \widetilde{\mathbf{grad}} q_j^T$$

and  $\mathbf{w} = \mathbf{v} - \mathbf{z}$ . According to [1], theorem 3.17, there exists a vector potential  $\boldsymbol{\psi} \in L^2(\Omega)^3$  that satisfies  $\mathbf{w} = \mathbf{curl} \boldsymbol{\psi}$ ,  $\text{div} \boldsymbol{\psi} = 0$  in  $\Omega$  and  $\boldsymbol{\psi} \times \mathbf{n} = \mathbf{0}$  on  $\Gamma$ . Hence

$$\int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, d\mathbf{x} = \int_{\Omega} \mathbf{f} \cdot \mathbf{curl} \boldsymbol{\psi} \, d\mathbf{x} = 0.$$

The result is then a consequence of theorem 1.1: there exists a function  $\chi \in H^1(\Omega)$  satisfying (5).  $\square$

**Remark 2.2.** (1) Any function  $\mathbf{f} \in L^2(\Omega)^3$  that satisfies  $\mathbf{curl} \mathbf{f} = \mathbf{0}$  in  $\Omega$  can be decomposed as:

$$\mathbf{f} = \mathbf{grad} \chi + \widetilde{\mathbf{grad}} p, \quad \text{with } \chi \in H^1(\Omega) \quad \text{and} \quad \widetilde{\mathbf{grad}} p \in K_T(\Omega).$$

Such a result was alluded to in [11].

(2) The second condition in (4) is trivially satisfied when  $\Omega$  is simply-connected, since  $K_T(\Omega) = \{\mathbf{0}\}$  in this case.

**Theorem 2.3.** *Given any distribution  $\mathbf{f} \in H_0(\operatorname{div}, \Omega)'$  that satisfies*

$$\mathbf{curl} \mathbf{f} = \mathbf{0} \quad \text{in } \Omega \quad \text{and} \quad {}_{H_0(\operatorname{div}, \Omega)'} \langle \mathbf{f}, \mathbf{v} \rangle_{H_0(\operatorname{div}, \Omega)} = 0 \quad \text{for all } \mathbf{v} \in K_T(\Omega), \quad (6)$$

*there exists a scalar potential  $\chi$  in  $L^2(\Omega)$  such that*

$$\mathbf{f} = \mathbf{grad} \chi \quad \text{in } \Omega \quad \text{and} \quad \|\chi\|_{L^2(\Omega)} \leq C \|\mathbf{f}\|_{H_0(\operatorname{div}, \Omega)'}. \quad (7)$$

*Proof.* Let  $\mathbf{f} \in H_0(\operatorname{div}, \Omega)'$  be such that  $\mathbf{curl} \mathbf{f} = \mathbf{0}$  in  $\Omega$ . Hence (see proposition 1 of [6]) there exist  $\boldsymbol{\psi} \in L^2(\Omega)^3$  and  $\chi_0 \in L^2(\Omega)$  such that

$$\mathbf{f} = \boldsymbol{\psi} + \mathbf{grad} \chi_0 \quad \text{in } \Omega \quad \text{and} \quad \|\boldsymbol{\psi}\|_{L^2(\Omega)^3} + \|\chi_0\|_{L^2(\Omega)} \leq C \|\mathbf{f}\|_{H_0(\operatorname{div}, \Omega)'}. \quad (8)$$

Observe that, thanks to the density of  $\mathcal{D}(\Omega)^3$  in  $H_0(\operatorname{div}, \Omega)$ ,

$${}_{H_0(\operatorname{div}, \Omega)'} \langle \mathbf{grad} \chi_0, \mathbf{v} \rangle_{H_0(\operatorname{div}, \Omega)} = 0 \quad \text{for all } \mathbf{v} \in K_T(\Omega).$$

Therefore, the function  $\boldsymbol{\psi} \in L^2(\Omega)^3$  satisfies relations (4). By theorem 2.1, there exists a function  $p \in H^1(\Omega)$  such that

$$\boldsymbol{\psi} = \mathbf{grad} p \quad \text{in } \Omega \quad \text{and} \quad \|p\|_{H^1(\Omega)} \leq C \|\boldsymbol{\psi}\|_{L^2(\Omega)^3} \leq C \|\mathbf{f}\|_{H_0(\operatorname{div}, \Omega)'}. \quad (9)$$

Hence the function  $\chi = p + \chi_0$  satisfies the announced properties.  $\square$

**Remark 2.4.** Note that this theorem is an extension of the equivalence (iii)  $\iff$  (iv) in theorem 1.1 with  $m = 1$  to the case where  $\Omega$  is not simply-connected.

More generally, let us introduce, for any integer  $m \geq 0$ , the space

$$H_0^m(\operatorname{div}, \Omega) := \{\mathbf{v} \in H_0(\operatorname{div}, \Omega); \operatorname{div} \mathbf{v} \in H_0^m(\Omega)\},$$

which coincides with  $H_0(\operatorname{div}, \Omega)$  for  $m = 0$ . Its dual space, denoted by  $H^{-m}(\operatorname{div}, \Omega)$ , can then be characterized by

$$H^{-m}(\operatorname{div}, \Omega) = \{\boldsymbol{\psi} + \mathbf{grad} \chi; \boldsymbol{\psi} \in H_0(\operatorname{div}, \Omega)', \chi \in H^{-m}(\Omega)\}.$$

One can also show that  $\mathcal{D}(\Omega)^3$  is dense in  $H_0^m(\operatorname{div}, \Omega)$  and that the following Green formula holds for any  $\chi \in H^{-m}(\operatorname{div}, \Omega)$  and  $\mathbf{v} \in H_0^m(\operatorname{div}, \Omega)$ :

$${}_{H^{-m}(\operatorname{div}, \Omega)} \langle \mathbf{grad} \chi, \mathbf{v} \rangle_{H_0^m(\operatorname{div}, \Omega)} + {}_{H^{-m}(\Omega)} \langle \chi, \operatorname{div} \mathbf{v} \rangle_{H_0^m(\Omega)} = 0. \quad (9)$$

As a consequence of theorem 2.3, it is easy to prove the following theorem, which shows that property (iv) in theorem 1.1 also holds when  $\Omega$  is not simply-connected.

**Theorem 2.5.** *For any distribution  $\mathbf{f} \in H^{-m}(\text{div}, \Omega)$  that satisfies (6), there exists a scalar potential  $\chi$  in  $H^{-m}(\Omega)$  such that*

$$\mathbf{f} = \mathbf{grad} \chi \quad \text{in } \Omega \quad \text{and} \quad \|\chi\|_{H^{-m}(\Omega)} \leq C \|\mathbf{f}\|_{H^{-m}(\text{div}, \Omega)}. \quad (10)$$

*Proof.* We give the proof when  $m = 1$ ; the general case is similar. Let  $\mathbf{f} \in H^{-1}(\text{div}, \Omega)$  satisfy (6). Then, there exist  $\boldsymbol{\psi} \in H_0(\text{div}, \Omega)'$  and  $\chi_0 \in H^{-1}(\Omega)$  such that

$$\mathbf{f} = \boldsymbol{\psi} + \mathbf{grad} \chi_0 \quad \text{in } \Omega \quad \text{and} \quad \|\boldsymbol{\psi}\|_{H_0(\text{div}, \Omega)'} + \|\chi_0\|_{H^{-1}(\Omega)} \leq C \|\mathbf{f}\|_{H^{-1}(\text{div}, \Omega)}. \quad (11)$$

Observe that, thanks to (11), we have

$${}_{H^{-1}(\text{div}, \Omega)} \langle \mathbf{grad} \chi_0, \mathbf{v} \rangle_{H_0^1(\text{div}, \Omega)} = - {}_{H^{-1}(\Omega)} \langle \chi_0, \text{div } \mathbf{v} \rangle_{H_0^1(\Omega)} = 0$$

for all  $\mathbf{v} \in K_T(\Omega)$ . By theorem 2.3, there exists a function  $p \in L^2(\Omega)$  such that  $\boldsymbol{\psi} = \mathbf{grad} p$  and the estimate (7) holds. Then the function  $\chi = \chi_0 + p$  satisfies the announced properties.  $\square$

### 3 Vector potentials in $H_0^m(\Omega)^3$

First, we recall some results concerning the existence of tangential vector potential (see [1] for proofs).

Below,  $\langle \cdot, \cdot \rangle_{\Gamma_i}$  denotes the duality pairing between the spaces  $H^{-1/2}(\Gamma_i)$  and  $H^{1/2}(\Gamma_i)$ . Given any function  $\mathbf{u} \in H(\text{div}, \Omega)$  that satisfies

$$\text{div } \mathbf{u} = 0 \quad \text{in } \Omega \quad \text{and} \quad \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Gamma_i} = 0, \quad 0 \leq i \leq I, \quad (12)$$

there exists a vector potential  $\boldsymbol{\psi}$  in  $L^2(\Omega)^3$  such that

$$\mathbf{u} = \mathbf{curl} \boldsymbol{\psi}, \quad \text{div } \boldsymbol{\psi} = 0 \quad \text{in } \Omega, \quad \text{and} \quad \boldsymbol{\psi} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma, \quad (13)$$

satisfying the estimate

$$\|\boldsymbol{\psi}\|_{L^2(\Omega)^3} \leq C \|\mathbf{u}\|_{L^2(\Omega)^3}. \quad (14)$$

Moreover, there exists a unique vector field  $\boldsymbol{\psi} \in L^2(\Omega)^3$  satisfying (13) and such that

$$\langle \boldsymbol{\psi} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0, \quad 1 \leq j \leq J, \quad (15)$$

and the estimate (14) holds. When  $\Omega$  is of class  $\mathcal{C}^{1,1}$ , then  $\boldsymbol{\psi}$  belongs to  $H^1(\Omega)^3$  and the estimate

$$\|\boldsymbol{\psi}\|_{H^1(\Omega)^3} \leq C\|\mathbf{u}\|_{L^2(\Omega)^3} \quad (16)$$

holds. If moreover  $\mathbf{u} \in H^m(\Omega)^3$  and  $\Omega$  is of class  $\mathcal{C}^{m+1,1}$ , for some integer  $m \geq 0$ , then  $\boldsymbol{\psi}$  belongs to  $H^{m+1}(\Omega)^3$  and the estimate

$$\|\boldsymbol{\psi}\|_{H^{m+1}(\Omega)^3} \leq C\|\mathbf{u}\|_{H^m(\Omega)^3} \quad (17)$$

holds. We also recall the result concerning the existence of normal vector potentials (see again [1] for proofs). For any vector field  $\mathbf{u} \in H(\text{div}, \Omega)$  that satisfies

$$\text{div } \mathbf{u} = 0 \quad \text{in } \Omega, \quad \mathbf{u} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma \quad \text{and} \quad \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0, \quad 1 \leq j \leq J, \quad (18)$$

there exists a vector potential  $\boldsymbol{\psi}$  in  $L^2(\Omega)^3$  such that

$$\mathbf{u} = \mathbf{curl } \boldsymbol{\psi}, \quad \text{div } \boldsymbol{\psi} = 0 \quad \text{in } \Omega \quad \text{and} \quad \boldsymbol{\psi} \times \mathbf{n} = \mathbf{0} \quad \text{on } \Gamma, \quad (19)$$

and the estimate

$$\|\boldsymbol{\psi}\|_{L^2(\Omega)^3} \leq C\|\mathbf{u}\|_{L^2(\Omega)^3} \quad (20)$$

holds. Moreover, there exists a unique vector field  $\boldsymbol{\psi} \in L^2(\Omega)^3$  satisfying (19) and such that

$$\langle \boldsymbol{\psi} \cdot \mathbf{n}, 1 \rangle_{\Gamma_i} = 0, \quad 0 \leq i \leq I, \quad (21)$$

and the estimate (20) holds. When  $\mathbf{u}$  is more regular, then (16) and (17) are also satisfied.

**Remark 3.1.** Let  $\mathbf{u}$  be a vector field in  $H(\text{div}, \Omega)$  that satisfies:

$$\text{div } \mathbf{u} = 0 \quad \text{in } \Omega \quad \text{and} \quad \mathbf{u} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma.$$

Using the same arguments as those of theorem 2.1, it is easy to verify that

$$\langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0, \quad 1 \leq j \leq J,$$

if and only if

$$\int_{\Omega} \mathbf{u} \cdot \mathbf{grad } q_j^T \, d\mathbf{x} = 0 \quad \text{for all } 1 \leq j \leq J.$$

Another kind of less standard but useful vector potential is given by the following theorem.

**Theorem 3.2.** *Assume that the boundary of the domain  $\Omega$  is of class  $\mathcal{C}^{1,1}$ . For any function  $\mathbf{u}$  in  $H(\text{div}, \Omega)$  satisfying (18), there exists a vector potential  $\boldsymbol{\psi}$  in  $H_0^1(\Omega)^3$ , such that*

$$\mathbf{u} = \mathbf{curl } \boldsymbol{\psi} \quad \text{and} \quad \text{div } \Delta \boldsymbol{\psi} = 0 \quad \text{in } \Omega, \quad \|\boldsymbol{\psi}\|_{H^1(\Omega)^3} \leq C\|\mathbf{u}\|_{L^2(\Omega)^3}. \quad (22)$$



*Proof.* Given any vector field  $\mathbf{u} \in H(\operatorname{div}, \Omega)$  satisfying (18), we associate the vector potential  $\boldsymbol{\psi}_0 \in H^1(\Omega)^3$  satisfying (19) and the estimate

$$\|\boldsymbol{\psi}_0\|_{H^1(\Omega)^3} \leq C\|\mathbf{u}\|_{L^2(\Omega)^3}.$$

That  $\Gamma$  is of class  $\mathcal{C}^{1,1}$  implies that the normal trace  $\boldsymbol{\psi}_0 \cdot \mathbf{n}$  belongs to  $H^{1/2}(\Gamma)$ . Hence, the fourth-order problem

$$\Delta^2 \chi = 0 \quad \text{in } \Omega, \quad \chi = 0 \quad \text{and} \quad \partial_n \chi = \boldsymbol{\psi}_0 \cdot \mathbf{n} \quad \text{on } \Gamma$$

has a unique solution  $\chi$  in  $H^2(\Omega)$  satisfying the estimate

$$\|\chi\|_{H^2(\Omega)} \leq C\|\boldsymbol{\psi}_0 \cdot \mathbf{n}\|_{H^{1/2}(\Gamma)} \leq C\|\mathbf{u}\|_{L^2(\Omega)^3}.$$

Then the vector field

$$\boldsymbol{\psi} = \boldsymbol{\psi}_0 - \mathbf{grad} \chi$$

satisfies (22). □

The vector field  $\boldsymbol{\psi}$  given by the previous theorem is unique up to vector fields belonging to the space

$$K_0^1(\Omega) := \{\mathbf{w} \in H_0^1(\Omega)^3; \mathbf{curl} \mathbf{w} = \mathbf{0} \text{ and } \operatorname{div}(\Delta \mathbf{w}) = 0 \text{ in } \Omega\}$$

(see proposition 3.4 below).

**Corollary 3.3.** *Assume that the boundary of the domain  $\Omega$  is of class  $\mathcal{C}^{m+1,1}$ , for some integer  $m \geq 0$ . For any vector field  $\mathbf{u} \in H^m(\Omega)^3$  that satisfies (18), there exists a vector potential  $\boldsymbol{\psi}$  in  $(H^{m+1}\Omega) \cap H_0^1(\Omega)^3$  satisfying*

$$\mathbf{u} = \mathbf{curl} \boldsymbol{\psi} \quad \text{and} \quad \operatorname{div} \Delta \boldsymbol{\psi} = 0 \quad \text{in } \Omega \quad \text{and} \quad \|\boldsymbol{\psi}\|_{H^{m+1}(\Omega)^3} \leq C\|\mathbf{u}\|_{H^m(\Omega)^3}.$$

*Proof.* Under the given assumptions, the vector potential  $\boldsymbol{\psi}$  given by the previous theorem belongs to  $H^{m+1}(\Omega)^3$  and its normal trace  $\boldsymbol{\psi} \cdot \mathbf{n}$  belongs to  $H^{m+1/2}(\Gamma)$ , on the one hand. On the other hand, the solution  $\chi$  to the fourth-order problem found in the previous belongs to  $H^{m+2}(\Omega)^3$ . □

We now characterize the space  $K_0^1(\Omega)$ .

**Proposition 3.4.** *Assume that the boundary of the domain  $\Omega$  is of class  $\mathcal{C}^{1,1}$ . Then the space  $K_0^1(\Omega)$  is spanned by the vector fields  $\mathbf{grad} q_i^1$ ,  $1 \leq i \leq I$ , where*

each  $q_i^1$  is the unique solution in  $H^2(\Omega)$  to the problem

$$\begin{aligned} \Delta^2 q_i^1 &= 0 && \text{in } \Omega, \\ q_i^1|_{\Gamma_0} &= 0 && \text{and } q_i^1|_{\Gamma_k} = \delta_{ik}, \quad 1 \leq k \leq I, \\ \partial_n q_i^1 &= 0 && \text{on } \Gamma, \\ \langle \partial_n \Delta q_i^1, 1 \rangle_{\Gamma_k} &= \delta_{ik} \text{ and } \langle \partial_n \Delta q_i^1, 1 \rangle_{\Gamma_0} &= -1, \quad 1 \leq k \leq I. \end{aligned} \quad (23)$$

*Proof.* First, we prove that the space  $K_0^1(\Omega)$  and the space

$$G^1 := \{\mathbf{grad} q \in H_0^1(\Omega)^3; \quad \Delta^2 q = 0 \quad \text{in } \Omega\}$$

coincide. First, it is clear that  $G^1$  is included in  $K_0^1(\Omega)$ . Second, given  $\mathbf{w} \in K_0^1(\Omega)$ , let  $\widetilde{\mathbf{w}}$  denote the extension by zero of  $\mathbf{w}$  to an open ball  $B$  containing  $\overline{\Omega}$ . Since  $\mathbf{curl} \widetilde{\mathbf{w}} = \mathbf{0}$  in  $B$ ,  $\widetilde{\mathbf{w}}$  is the gradient of a function  $q \in H^2(B)$ . Moreover,  $q = 0$  in  $B \setminus \overline{\Omega}$ , so that  $q' := q|_{\Omega}$  belongs to  $H_0^2(\Omega)$ . Since  $\mathbf{w} = \mathbf{grad} q'$ , one finds that  $\mathbf{w}$  belongs to  $G^1$ . Moreover, it is clear that the set of vector fields  $\mathbf{grad} q_i$ ,  $1 \leq i \leq I$ , where  $q_i \in H^2(\Omega)$  is the unique solution to

$$\begin{aligned} \Delta^2 q_i &= 0 \quad \text{in } \Omega, \\ q_i|_{\Gamma_0} &= 0 \quad \text{and} \quad q_i|_{\Gamma_k} = \delta_{ik}, \quad 1 \leq k \leq I, \\ \partial_n q_i &= 0 \quad \text{on } \Gamma, \end{aligned} \quad (24)$$

spans  $G^1$  ( $= K_0^1(\Omega)$ ).

One still has to check the last line of (23). Introduce now

$$M_2 := \{r \in H^2(\Omega); r|_{\Gamma_0} = 0 \text{ and } r|_{\Gamma_k} = \delta_{ik}, \quad 1 \leq k \leq I, \quad \partial_n r = 0 \text{ on } \Gamma\}.$$

For  $1 \leq i \leq I$ , the problem: find  $q_i^1$  in  $M_2$  such that

$$\forall r \in M_2, \quad \int_{\Omega} \Delta q_i^1 \Delta r \, d\mathbf{x} = -r|_{\Gamma_i}, \quad (25)$$

has a unique solution. Furthermore, the following Green's formula can be proven by a density argument, for any functions  $q$  and  $r$  in  $M_2$  with  $\Delta^2 q$  in  $L^2(\Omega)$ :

$$\int_{\Omega} (\Delta^2 q) r \, d\mathbf{x} = \int_{\Omega} \Delta q \Delta r \, d\mathbf{x} + \sum_{i=1}^I r|_{\Gamma_i} \langle \partial_n(\Delta q), 1 \rangle_{\Gamma_i}.$$

This formula implies that the solution  $q_i^1$  to (25) satisfies (23). The vector fields  $\mathbf{grad} q_i^1$ ,  $1 \leq i \leq I$ , are clearly linearly independent and they belong to  $K_0^1(\Omega)$ . Consequently, they form a basis of  $K_0^1(\Omega)$ .  $\square$

**Proposition 3.5.** *Assume that the boundary of the domain  $\Omega$  is of class  $\mathcal{C}^{1,1}$ . Given any function  $\mathbf{u}$  in  $H(\operatorname{div}, \Omega)$  satisfying (18), there exists a unique vector potential  $\boldsymbol{\psi}$  in  $H_0^1(\Omega)^3$  satisfying*

$$\mathbf{u} = \operatorname{curl} \boldsymbol{\psi}, \quad \operatorname{div} \Delta \boldsymbol{\psi} = 0 \quad \text{in } \Omega \quad \text{and} \quad \langle \partial_n(\operatorname{div} \Delta \boldsymbol{\psi}), 1 \rangle_{\Gamma_i} = 0, \quad 0 \leq i \leq I. \quad (26)$$

Moreover, the estimate (16) holds.

*Proof.* Let  $(\boldsymbol{\psi}_0 - \mathbf{grad} \chi)$  be the potential vector of  $\mathbf{u}$  given in the proof of theorem 3.2. Then the vector field

$$\boldsymbol{\psi} = \boldsymbol{\psi}_0 - \mathbf{grad} \chi + \sum_{i=1}^I \langle \partial_n(\Delta \chi), 1 \rangle_{\Gamma_i} \mathbf{grad} q_i^1$$

satisfies (26) (note that the quantities  $\langle \partial_n(\Delta \chi), 1 \rangle_{\Gamma_i}$  are well defined since  $\Delta^2 \chi = 0$ ).  $\square$

**Corollary 3.6.** *Assume that the boundary of the domain  $\Omega$  is of class  $\mathcal{C}^{m+1,1}$  for some integer  $m \geq 0$ . Given any function  $\mathbf{u}$  in  $H^m(\Omega)^3$  that satisfies (18), there exists a unique vector potential  $\boldsymbol{\psi}$  in  $(H^{m+1}\Omega) \cap H_0^1(\Omega)^3$  satisfying*

$$\mathbf{u} = \operatorname{curl} \boldsymbol{\psi}, \quad \operatorname{div} \Delta \boldsymbol{\psi} = 0 \quad \text{in } \Omega \quad \text{and} \quad \langle \partial_n(\operatorname{div} \Delta \boldsymbol{\psi}), 1 \rangle_{\Gamma_i} = 0, \quad 0 \leq i \leq I$$

and the estimate (17).

**Theorem 3.7.** *Assume that the boundary of the domain  $\Omega$  is of class  $\mathcal{C}^{2,1}$ . Given any function  $\mathbf{u}$  in  $H_0^1(\Omega)^3$  that satisfies*

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega \quad \text{and} \quad \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0, \quad 1 \leq j \leq J, \quad (27)$$

there exists a vector potential  $\boldsymbol{\psi}$  in  $H_0^2(\Omega)^3$  such that

$$\mathbf{u} = \operatorname{curl} \boldsymbol{\psi} \quad \text{and} \quad \operatorname{div} \Delta^2 \boldsymbol{\psi} = 0 \quad \text{in } \Omega \quad \text{and} \quad \|\boldsymbol{\psi}\|_{H^2(\Omega)^3} \leq C \|\mathbf{u}\|_{H^1(\Omega)^3}. \quad (28)$$

*Proof.* Given  $\mathbf{u}$  in  $H_0^1(\Omega)^3$  that satisfies (27), let  $\boldsymbol{\varphi} \in (H^2(\Omega) \cap H_0^1(\Omega))^3$  denote the vector potential given by corollary 3.6. The sixth-order problem

$$\Delta^3 \chi = 0 \quad \text{in } \Omega, \quad \chi = \frac{\partial \chi}{\partial \mathbf{n}} = 0 \quad \text{and} \quad \frac{\partial^2 \chi}{\partial \mathbf{n}^2} = \frac{\partial \boldsymbol{\varphi}}{\partial \mathbf{n}} \cdot \mathbf{n} \quad \text{on } \Gamma, \quad (29)$$

has a unique solution  $\chi \in H^3(\Omega)$  that satisfies the estimate

$$\|\chi\|_{H^3(\Omega)} \leq C \left\| \frac{\partial \boldsymbol{\varphi}}{\partial \mathbf{n}} \right\|_{H^{1/2}(\Gamma)^3} \leq C \|\boldsymbol{\varphi}\|_{H^2(\Omega)^3} \leq C \|\mathbf{u}\|_{H^1(\Omega)^3}.$$

Note that the last boundary condition in (29) can be written as

$$\left( \frac{\partial}{\partial \mathbf{n}} \mathbf{grad} \chi \right) \cdot \mathbf{n} = \frac{\partial \varphi}{\partial \mathbf{n}} \cdot \mathbf{n}.$$

For any unit tangent vector  $\boldsymbol{\tau}$  on  $\Gamma$ , we have:

$$\frac{\partial \varphi}{\partial \mathbf{n}} \cdot \boldsymbol{\tau} = \frac{\partial \varphi_i}{\partial x_j} n_j \tau_i = \frac{\partial \varphi_j}{\partial x_i} \tau_i n_j = \frac{\partial \varphi_j}{\partial \boldsymbol{\tau}} n_j = 0.$$

Also, one can show that  $(\partial_n \mathbf{grad} \chi) \cdot \boldsymbol{\tau} = 0$ , which implies that the relation  $\partial_n \mathbf{grad} \chi = \partial_n \boldsymbol{\varphi}$  holds. So, the vector field  $\boldsymbol{\psi} = \boldsymbol{\varphi} - \mathbf{grad} \chi$  belongs to  $H^2(\Omega)^3$  and satisfies (28).  $\square$

The vector field  $\boldsymbol{\psi}$  given by Theorem 3.7 is unique up to vector fields in the space

$$K_0^2(\Omega) := \{ \mathbf{w} \in H_0^2(\Omega)^3; \mathbf{curl} \mathbf{w} = \mathbf{0} \quad \text{and} \quad \text{div} \Delta^2 \mathbf{w} = 0 \text{ in } \Omega \},$$

which we now characterize.

**Proposition 3.8.** *Assume that the boundary of the domain  $\Omega$  is of class  $\mathcal{C}^{2,1}$ . Then the space  $K_0^2(\Omega)$  is spanned by the vector fields  $\mathbf{grad} q_i^2$ ,  $1 \leq i \leq I$ , where each function  $q_i^2$  is the unique solution in  $H^3(\Omega)$  to the problem*

$$\begin{aligned} \Delta^3 q_i^2 &= 0 && \text{in } \Omega, \\ q_i^2|_{\Gamma_0} &= 0 && \text{and} \quad q_i^2|_{\Gamma_k} = \delta_{ik}, \quad 1 \leq k \leq I, \\ \partial_n q_i^2 &= \partial_n^2 q_i^2 = 0 && \text{on } \Gamma, \\ \langle \partial_n(\Delta^2 q_i^2), 1 \rangle_{\Gamma_k} &= \delta_{ik} \text{ and } \langle \partial_n(\Delta^2 q_i^2), 1 \rangle_{\Gamma_0} &= -1, \quad 1 \leq k \leq I. \end{aligned} \tag{30}$$

*Proof.* First, we prove that the space  $K_0^2(\Omega)$  coincides with the space

$$G^2 := \{ \mathbf{grad} q \in H_0^2(\Omega)^3; \quad \Delta^3 q = 0 \quad \text{in } \Omega \},$$

using the same argument as in proposition 3.4. We next note that the set of vector fields  $\mathbf{grad} q_i$ ,  $1 \leq i \leq I$ , where  $q_i \in H^3(\Omega)$  is the unique solution to the problem

$$\begin{aligned} \Delta^3 q_i &= 0 && \text{in } \Omega, \\ q_i|_{\Gamma_0} &= 0 && \text{and} \quad q_i|_{\Gamma_k} = \delta_{ik}, \quad 1 \leq k \leq I, \\ \partial_n q_i &= \partial_n^2 q_i = 0 && \text{on } \Gamma, \end{aligned} \tag{31}$$

spans  $K_0^2(\Omega)$ .

Let now

$$M_3 := \{r \in H^3(\Omega); r|_{\Gamma_0} = 0, r|_{\Gamma_k} = \delta_{ik}, 1 \leq k \leq I, \partial_n r = \partial_n^2 r = 0 \text{ on } \Gamma\}.$$

For  $1 \leq i \leq I$ , the problem: find  $q_i^2$  in  $M_3$  such that

$$\forall r \in M_3, \quad \int_{\Omega} \mathbf{grad} \Delta q_i^2 \cdot \mathbf{grad} \Delta r \, d\mathbf{x} = r|_{\Gamma_i}, \quad (32)$$

has a unique solution. Furthermore, the following Green's formula can be proved by a density argument, for any functions  $q$  and  $r$  in  $M_3$  with  $\Delta^3 q$  in  $L^2(\Omega)$ :

$$\int_{\Omega} (\Delta^3 q) r \, d\mathbf{x} = - \int_{\Omega} \mathbf{grad} \Delta q \cdot \mathbf{grad} \Delta r \, d\mathbf{x} + \sum_{i=1}^I r|_{\Gamma_i} \langle \partial_n(\Delta^2 q), \cdot \rangle_{\Gamma_i}.$$

This formula shows that the solution  $q_i^2$  of (32) satisfies (30). The vector fields  $\mathbf{grad} q_i^2$ ,  $1 \leq i \leq I$ , are clearly linearly independent and they belong to  $K_0^2(\Omega)$ . Consequently, they form a basis of  $K_0^2(\Omega)$ .  $\square$

**Corollary 3.9.** *Assume that the boundary of the domain  $\Omega$  is of class  $\mathcal{C}^{2,1}$ . Given any function  $\mathbf{u}$  in  $H_0^1(\Omega)^3$  such that (27) holds, there exists a unique vector potential  $\boldsymbol{\psi}$  in  $H_0^2(\Omega)^3$  satisfying*

$$\mathbf{u} = \mathbf{curl} \boldsymbol{\psi}, \quad \text{div} \Delta^2 \boldsymbol{\psi} = 0 \text{ in } \Omega \quad \text{and} \quad \langle \partial_n(\text{div} \Delta \boldsymbol{\psi}), 1 \rangle_{\Gamma_i} = 0, \quad 0 \leq i \leq I,$$

with the corresponding estimate.

More generally, we can prove using the same arguments:

**Theorem 3.10.** *Assume that boundary of the domain  $\Omega$  is of class  $\mathcal{C}^{m+1,1}$  for some integer  $m \geq 1$ . Given any vector field  $\mathbf{u}$  in  $H_0^m(\Omega)^3$  that satisfies (27), there exists a vector potential  $\boldsymbol{\psi}$  in  $H_0^{m+1}(\Omega)^3$  such that*

$$\mathbf{u} = \mathbf{curl} \boldsymbol{\psi} \quad \text{and} \quad \text{div} \Delta^{m+1} \boldsymbol{\psi} = 0 \quad \text{in } \Omega \quad \text{and} \quad \|\boldsymbol{\psi}\|_{H^{m+1}(\Omega)^3} \leq C \|\mathbf{u}\|_{H^m(\Omega)^3}. \quad (33)$$

Moreover, there exists a unique vector potential  $\boldsymbol{\psi}$  in  $H_0^{m+1}(\Omega)^3$ , satisfying (33) and

$$\langle \partial_n \text{div} \Delta \boldsymbol{\psi}^{m+1}, 1 \rangle_{\Gamma_i} = 0, \quad 0 \leq i \leq I. \quad (34)$$

**Remark 3.11.** Similar results are found in Borchers & Sohr [7], but with different proof.

Let  $\Omega$  be a domain with a boundary of class  $\mathcal{C}^{m+1,1}$  for some integer  $m \geq 1$  and let  $\mathbf{u}$  in  $H_0^m(\Omega)^3$  be such that  $\text{div} \mathbf{u} = 0$ . If  $\Omega$  is simply-connected ( $J = 0$ ), and  $\Gamma$  is connected ( $I = 0$ ), then there exists a unique vector potential  $\boldsymbol{\psi}$  in  $H_0^{m+1}(\Omega)^3$  satisfying (33).

#### 4 Weak vector potentials

First, we note that the continuous embeddings  $H_0(\mathbf{curl}, \Omega)' \hookrightarrow H^{-1}(\Omega)^3$  and  $H_0(\operatorname{div}, \Omega)' \hookrightarrow H^{-1}(\Omega)^3$  hold. Besides, given any  $\mathbf{f} \in H^{-1}(\Omega)^3$ , we know that there exist a unique  $\mathbf{u} \in H_0^1(\Omega)^3$  and  $\chi \in L^2(\Omega)$  such that

$$\mathbf{f} = -\Delta \mathbf{u} + \mathbf{grad} \chi \quad \text{and} \quad \operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega, \quad (35)$$

and satisfying the estimate

$$\|\mathbf{u}\|_{H^1(\Omega)^3} + \|\chi\|_{L^2(\Omega)/\mathbb{R}} \leq C \|\mathbf{f}\|_{H^{-1}(\Omega)^3}.$$

Letting  $\boldsymbol{\xi} = \mathbf{curl} \mathbf{u}$ , we obtain the decomposition  $\mathbf{f} = \mathbf{curl} \boldsymbol{\xi} + \mathbf{grad} \chi$  with  $\operatorname{div} \boldsymbol{\xi} = 0$  in  $\Omega$  and  $\boldsymbol{\xi} \cdot \mathbf{n} = 0$  on  $\Gamma$ . Since  $\boldsymbol{\xi} \in L^2(\Omega)^3$  and  $\chi \in L^2(\Omega)$ , it follows that  $\mathbf{curl} \boldsymbol{\xi} \in H_0(\mathbf{curl}, \Omega)'$  and  $\mathbf{grad} \chi \in H_0(\operatorname{div}, \Omega)'$ , so that

$$H^{-1}(\Omega)^3 = H_0(\mathbf{curl}, \Omega)' + H_0(\operatorname{div}, \Omega)'. \quad (36)$$

**Proposition 4.1.** *Assume that the boundary of the domain  $\Omega$  is of class  $\mathcal{C}^{1,1}$ . Then, for any  $\mathbf{f}$  in the dual space  $H_0(\operatorname{div}, \Omega)'$ , there exist a unique  $\mathbf{u} \in (H^2(\Omega) \cap H_0^1(\Omega))^3$  and  $\chi \in L^2(\Omega)$  solution to (35) and satisfying the estimate*

$$\|\mathbf{u}\|_{H^2(\Omega)^3} + \|\chi\|_{L^2(\Omega)/\mathbb{R}} \leq C \|\mathbf{f}\|_{H_0(\operatorname{div}, \Omega)'}$$

*Proof.* Let  $\mathbf{f}$  be in the dual space of  $H_0(\operatorname{div}, \Omega)$ . We know (see proposition 1 of [6]) that there exist  $\boldsymbol{\psi} \in L^2(\Omega)^3$  and  $\chi_0 \in L^2(\Omega)$  such that

$$\mathbf{f} = \boldsymbol{\psi} + \mathbf{grad} \chi_0 \quad \text{and} \quad \|\boldsymbol{\psi}\|_{L^2(\Omega)^3} + \|\chi_0\|_{L^2(\Omega)} \leq C \|\mathbf{f}\|_{H_0(\operatorname{div}, \Omega)'}. \quad (37)$$

Thanks to the regularity of  $\Gamma$ , there exist  $\mathbf{u} \in (H^2(\Omega) \cap H_0^1(\Omega))^3$  and  $p \in H^1(\Omega)$  satisfying

$$\boldsymbol{\psi} = -\Delta \mathbf{u} + \mathbf{grad} p \quad \text{and} \quad \operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega, \quad (38)$$

with

$$\|\mathbf{u}\|_{H^2(\Omega)^3} + \|p\|_{H^1(\Omega)/\mathbb{R}} \leq C \|\boldsymbol{\psi}\|_{L^2(\Omega)^3}.$$

Then  $\mathbf{u}$  and  $\chi = p + \chi_0$  satisfy the announced properties.  $\square$

We next consider the space

$$K_N(\Omega) := \{\mathbf{w} \in H_0(\mathbf{curl}, \Omega) \cap H(\operatorname{div}, \Omega); \mathbf{curl} \mathbf{w} = \mathbf{0} \text{ and } \operatorname{div} \mathbf{w} = 0 \text{ in } \Omega\}$$

which is of dimension  $I$ . As shown in proposition 3.18 of [1], this space is spanned by the vector fields  $\mathbf{grad} q_i^N$ ,  $1 \leq i \leq N$ , where each function  $q_i^N \in$

$H^1(\Omega)$  is the unique solution to the problem

$$\begin{aligned} \Delta q_i^N &= 0 && \text{in } \Omega, \\ q_i^N &= 0 && \text{on } \Gamma_0, \quad \langle \partial_n q_i^N, 1 \rangle_{\Gamma_0} = -1, \\ q_i^N &= \text{constant} && \text{on } \Gamma_k, \quad \langle \partial_n q_i^N, 1 \rangle_{\Gamma_k} = \delta_{ik}, \text{ for } 1 \leq k \leq I. \end{aligned} \quad (39)$$

**Theorem 4.2.** *Given any distribution  $\mathbf{f}$  in the dual space  $H_0(\mathbf{curl}, \Omega)'$  that satisfies*

$$\operatorname{div} \mathbf{f} = 0 \text{ in } \Omega \text{ and } {}_{H_0(\mathbf{curl}, \Omega)'} \langle \mathbf{f}, \mathbf{v} \rangle_{H_0(\mathbf{curl}, \Omega)} = 0 \text{ for all } \mathbf{v} \in K_N(\Omega), \quad (40)$$

there exists a vector potential  $\boldsymbol{\xi}$  in  $L^2(\Omega)^3$  such that

$$\mathbf{f} = \mathbf{curl} \boldsymbol{\xi}, \quad \text{with} \quad \operatorname{div} \boldsymbol{\xi} = 0 \text{ in } \Omega \quad \text{and} \quad \boldsymbol{\xi} \cdot \mathbf{n} = 0 \text{ on } \Gamma, \quad (41)$$

and such that the following estimate holds:

$$\|\boldsymbol{\xi}\|_{L^2(\Omega)^3} \leq C \|\mathbf{f}\|_{H_0(\mathbf{curl}, \Omega)'}. \quad (42)$$

*Proof.* Let  $\mathbf{f}$  be in the dual space  $H_0(\mathbf{curl}, \Omega)'$ . According to corollary 5 of [6], there exist  $\boldsymbol{\psi} \in L^2(\Omega)^3$  and  $\boldsymbol{\xi}_0 \in L^2(\Omega)^3$  with  $\operatorname{div} \boldsymbol{\xi}_0 = 0$  in  $\Omega$  and  $\boldsymbol{\xi}_0 \cdot \mathbf{n} = 0$  on  $\Gamma$ , such that  $\mathbf{f} = \boldsymbol{\psi} + \mathbf{curl} \boldsymbol{\xi}_0$  and such that the estimate

$$\|\boldsymbol{\psi}\|_{L^2(\Omega)^3} + \|\boldsymbol{\xi}_0\|_{L^2(\Omega)^3} \leq C \|\mathbf{f}\|_{H_0(\mathbf{curl}, \Omega)'}$$

holds. Thanks to the density of  $\mathcal{D}(\Omega)^3$  in  $H_0(\mathbf{curl}, \Omega)$ , we deduce that for all  $\mathbf{v} \in K_N(\Omega)$ , we have

$${}_{H_0(\mathbf{curl}, \Omega)'} \langle \mathbf{curl} \boldsymbol{\xi}_0, \mathbf{v} \rangle_{H_0(\mathbf{curl}, \Omega)} = 0.$$

Since  $\operatorname{div} \mathbf{f} = 0$ , it follows that  $\operatorname{div} \boldsymbol{\psi} = 0$ . Then, thanks to the orthogonality relations

$${}_{H_0(\mathbf{curl}, \Omega)'} \langle \mathbf{f}, \mathbf{grad} q_i^N \rangle_{H_0(\mathbf{curl}, \Omega)} = 0 \quad \text{for all } i = 1, \dots, I,$$

the relations  $\langle \boldsymbol{\psi} \cdot \mathbf{n}, 1 \rangle_{\Gamma_i} = 0$  are satisfied for all  $i = 1, \dots, I$ . There thus exists a vector potential  $\boldsymbol{\varphi} \in L^2(\Omega)^3$  (see theorem 3.12 of [1]) such that  $\boldsymbol{\psi} = \mathbf{curl} \boldsymbol{\varphi}$ , with  $\operatorname{div} \boldsymbol{\varphi} = 0$  in  $\Omega$  and  $\boldsymbol{\varphi} \cdot \mathbf{n} = 0$  on  $\Gamma$ , and such that

$$\|\boldsymbol{\varphi}\|_{L^2(\Omega)^3} \leq C \|\boldsymbol{\psi}\|_{L^2(\Omega)^3}.$$

Hence, the vector field  $\boldsymbol{\xi} = \boldsymbol{\xi}_0 + \boldsymbol{\varphi}$  possesses the announced properties.  $\square$

**Remark 4.3.** The previous theorem has been established in [6] when  $\Gamma$  is connected, in which case  $K_N = \{\mathbf{0}\}$ .

For any integer  $m \geq 0$ , let us introduce the space

$$H_0^m(\mathbf{curl}, \Omega) := \{\mathbf{v} \in H_0(\mathbf{curl}, \Omega); \mathbf{curl} \mathbf{v} \in H_0^m(\Omega)^3\}.$$

We can easily characterize its dual space, as:

$$H^{-m}(\mathbf{curl}, \Omega) = \{\boldsymbol{\psi} + \mathbf{curl} \boldsymbol{\xi}; \boldsymbol{\psi} \in H_0(\mathbf{curl}, \Omega)', \boldsymbol{\xi} \in H^{-m}(\Omega)^3\}.$$

We can prove that  $\mathcal{D}(\Omega)^3$  is dense in  $H_0^m(\mathbf{curl}, \Omega)$  and that the following Green formula holds: for any  $\boldsymbol{\xi} \in H^{-m}(\mathbf{curl}, \Omega)$  and  $\mathbf{v} \in H_0^m(\mathbf{curl}, \Omega)$

$$H^{-m}(\mathbf{curl}, \Omega) \langle \mathbf{curl} \boldsymbol{\xi}, \mathbf{v} \rangle_{H_0^m(\mathbf{curl}, \Omega)} + H^{-m}(\Omega)^3 \langle \boldsymbol{\xi}, \mathbf{curl} \mathbf{v} \rangle_{H_0^m(\Omega)^3} = 0. \quad (43)$$

By using the decomposition (1) with  $(m+1)$  instead of  $m$ , it is easy to prove (as in Section 2) that

$$H^{-m-1}(\Omega)^3 = H^{-m}(\mathbf{curl}, \Omega) + H^{-m}(\text{div}, \Omega), \quad \text{for } m \geq 1.$$

**Theorem 4.4.** *For any distribution  $\mathbf{f}$  in the dual space  $H^{-1}(\mathbf{curl}, \Omega)$  that satisfies*

$$\text{div } \mathbf{f} = 0 \quad \text{in } \Omega \quad \text{and} \quad \langle \mathbf{f}, \mathbf{v} \rangle = 0, \quad \text{for all } \mathbf{v} \in K_N(\Omega) \quad (44)$$

*there exists a vector potential  $\boldsymbol{\xi}$  in  $H^{-1}(\Omega)^3$  such that*

$$\mathbf{f} = \mathbf{curl} \boldsymbol{\xi}, \quad \text{div } \boldsymbol{\xi} = 0 \quad \text{in } \Omega, \quad \text{and} \quad \|\boldsymbol{\xi}\|_{H^{-1}(\Omega)^3} \leq C \|\mathbf{f}\|_{H^{-1}(\mathbf{curl}, \Omega)}. \quad (45)$$

*Proof.* Given  $\mathbf{f}$  in the dual space  $H^{-1}(\mathbf{curl}, \Omega)$ , there exist  $\mathbf{f}_0 \in H_0(\mathbf{curl}, \Omega)'$  and  $\boldsymbol{\xi}_0 \in H^{-1}(\Omega)^3$  such that  $\mathbf{f} = \mathbf{f}_0 + \mathbf{curl} \boldsymbol{\xi}_0$ , and satisfying the estimate

$$\|\mathbf{f}_0\|_{H_0(\mathbf{curl}, \Omega)'} + \|\boldsymbol{\xi}_0\|_{H^{-1}(\Omega)^3} \leq C \|\mathbf{f}\|_{H^{-1}(\mathbf{curl}, \Omega)}.$$

Since  $\boldsymbol{\xi}_0 \in H^{-1}(\Omega)^3$ , there exists  $\boldsymbol{\theta}_0 \in L^2(\Omega)^3$  satisfying  $\text{div } \boldsymbol{\theta}_0 = 0$  in  $\Omega$ ,  $\boldsymbol{\theta}_0 \cdot \mathbf{n} = 0$  on  $\Gamma$ , and there exists  $\chi \in L^2(\Omega)$  such that  $\boldsymbol{\xi}_0 = \mathbf{curl} \boldsymbol{\theta}_0 + \mathbf{grad} \chi$  and

$$\|\boldsymbol{\theta}_0\|_{L^2(\Omega)^3} + \|\chi\|_{L^2(\Omega)/\mathbb{R}} \leq C \|\boldsymbol{\xi}_0\|_{H^{-1}(\Omega)^3}.$$

Since  $\mathbf{f}_0 \in H_0(\mathbf{curl}, \Omega)'$ , then  $\mathbf{f}_0 = \boldsymbol{\psi}_0 + \mathbf{curl} \boldsymbol{\varphi}_0$ , with  $\boldsymbol{\psi}_0 \in L^2(\Omega)^3$ ,  $\boldsymbol{\varphi}_0 \in L^2(\Omega)^3$ ,  $\text{div } \boldsymbol{\varphi}_0 = 0$  in  $\Omega$ ,  $\boldsymbol{\varphi}_0 \cdot \mathbf{n} = 0$  on  $\Gamma$  and

$$\|\boldsymbol{\psi}_0\|_{L^2(\Omega)^3} + \|\boldsymbol{\varphi}_0\|_{L^2(\Omega)^3} \leq C \|\mathbf{f}_0\|_{H_0(\mathbf{curl}, \Omega)'}$$

Then  $\mathbf{f} = \boldsymbol{\psi}_0 + \mathbf{curl} \boldsymbol{\varphi}_0 + \mathbf{curl} \mathbf{curl} \boldsymbol{\theta}_0 = \boldsymbol{\psi}_0 + \mathbf{curl} \boldsymbol{\mu}$ , with  $\boldsymbol{\mu} = \boldsymbol{\varphi}_0 + \mathbf{curl} \boldsymbol{\theta}_0$ ,  $\text{div } \boldsymbol{\mu} = 0$  in  $\Omega$ , and the estimate

$$\|\boldsymbol{\psi}_0\|_{L^2(\Omega)^3} + \|\boldsymbol{\mu}\|_{H^{-1}(\Omega)^3} \leq C \|\mathbf{f}\|_{H^{-1}(\mathbf{curl}, \Omega)}$$



holds.

Thanks to the density of  $\mathcal{D}(\Omega)^3$  in  $H_0^1(\mathbf{curl}, \Omega)$ , we infer that

$${}_{H^{-1}(\mathbf{curl}, \Omega)} \langle \mathbf{curl} \boldsymbol{\mu}, \mathbf{v} \rangle_{H_0^1(\mathbf{curl}, \Omega)} = 0, \quad \text{for all } \mathbf{v} \in K_N(\Omega).$$

Since  $\operatorname{div} \mathbf{f} = 0$ ,  $\operatorname{div} \boldsymbol{\psi}_0 = 0$  and therefore the condition  $\langle \boldsymbol{\psi}_0 \cdot \mathbf{n}, 1 \rangle_{\Gamma_i} = 0$  is automatically satisfied for any  $i = 0, \dots, I$ . Then by (12), there exists a vector potential  $\boldsymbol{\varphi} \in L^2(\Omega)^3$  such that

$$\boldsymbol{\psi}_0 = \mathbf{curl} \boldsymbol{\varphi}, \quad \operatorname{div} \boldsymbol{\varphi} = 0 \quad \text{in } \Omega \quad \text{and} \quad \boldsymbol{\varphi} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma,$$

and

$$\|\boldsymbol{\varphi}\|_{L^2(\Omega)^3} \leq C \|\boldsymbol{\psi}_0\|_{L^2(\Omega)^3}.$$

Hence, the vector field  $\boldsymbol{\xi} = \boldsymbol{\mu} + \boldsymbol{\varphi}$  satisfies the announced properties.  $\square$

More generally, we can prove:

**Theorem 4.5.** *Given any integer  $m \geq 0$  and any distribution  $\mathbf{f}$  in the dual space  $H^{-m}(\mathbf{curl}, \Omega)$  that satisfies (44), there exists a vector potential  $\boldsymbol{\xi}$  in  $H^{-m}(\Omega)^3$  such that*

$$\mathbf{f} = \mathbf{curl} \boldsymbol{\xi}, \quad \text{with} \quad \operatorname{div} \boldsymbol{\xi} = 0 \quad \text{in } \Omega, \quad \text{and} \quad \|\boldsymbol{\xi}\|_{H^{-m}(\Omega)^3} \leq C \|\mathbf{f}\|_{H^{-m}(\mathbf{curl}, \Omega)}.$$

## 5 Weak versions of Korn's inequality

Finally, we consider tensor fields. The next theorem generalizes theorem 3.2 of [8] and theorem 7 of [3] to Sobolev spaces with negative exponents.

In what follows, the subscript  $_s$  denotes a space of symmetric matrix fields.

**Theorem 5.1.** *Assume that  $\Omega$  is simply-connected. Given an integer  $m \geq 0$ , let  $\mathbf{e} = (e_{ij}) \in H_s^{-m}(\Omega)^{3 \times 3}$  be a symmetric matrix field that satisfies the following compatibility conditions for all  $i, j, k, l \in \{1, 2, 3\}$ :*

$$\mathcal{R}_{ijkl} := \frac{\partial^2 e_{ik}}{\partial x_l \partial x_j} + \frac{\partial^2 e_{jl}}{\partial x_k \partial x_i} - \frac{\partial^2 e_{jk}}{\partial x_l \partial x_i} - \frac{\partial^2 e_{il}}{\partial x_k \partial x_j} = 0 \quad \text{in } H^{-m-2}(\Omega). \quad (46)$$

*Then there exists a vector field  $\mathbf{v} \in H^{-m+1}(\Omega)^3$  such that  $e_{ij} = \frac{1}{2}(\partial_j v_i + \partial_i v_j)$  and  $\mathbf{v}$  is unique up to vector fields in the space  $R(\Omega) = \{\mathbf{a} + \mathbf{b} \wedge \mathbf{id}_\Omega; \mathbf{a}, \mathbf{b} \in \mathbb{R}^3\}$ .*

*Proof.* Given  $\mathbf{e} = (e_{ij}) \in H_s^{-m}(\Omega)^{3 \times 3}$ , let  $f_{ijk} := \partial_j e_{ik} - \partial_i e_{jk}$ . Then  $f_{ijk} \in H^{-m-1}(\Omega)$  and, thanks to the compatibility conditions (46), we have

$$\frac{\partial}{\partial x_l} f_{ijk} = \frac{\partial}{\partial x_k} f_{ijl}.$$

Hence the implication (iii)  $\implies$  (iv) in theorem 1.1 shows that there exist distributions  $p_{ij} \in H^{-m}(\Omega)$ , unique up to additive constants, such that  $\partial_k p_{ij} = f_{ijk}$ .

Besides, since  $\partial_k p_{ij} = -\partial_k p_{ji}$ , we can choose the distributions  $p_{ij}$  in such a way that  $p_{ij} + p_{ji} = 0$ . Noting that the distributions  $q_{ij} := e_{ij} + p_{ij}$  belong to  $H^{-m}(\Omega)$  and satisfy  $\partial_k q_{ij} = \partial_j q_{ik}$ , we again resort to theorem 1.1 to assert the existence of distributions  $v_i \in H^{-m+1}(\Omega)$ , unique up to additive constants, such that  $\partial_j v_i = q_{ij}$ .  $\square$

For any integer  $m \geq 0$ , let

$$E(\Omega) := \{\mathbf{e} \in H_s^{-m}(\Omega)^{3 \times 3}, \mathcal{R}_{ijkl}(\mathbf{e}) = 0\}$$

and

$$\dot{H}^{-m+1}(\Omega)^3 := H^{-m+1}(\Omega)^3 / R(\Omega).$$

By the previous theorem, given any  $\mathbf{e} = (e_{ij}) \in E(\Omega)$ , there exists a unique  $\dot{\mathbf{v}} = (\dot{v}_i) \in \dot{H}^{-m+1}(\Omega)^3$  such that  $e_{ij} = \frac{1}{2}(\partial_j v_i + \partial_i v_j)$ . We may thus define a linear mapping  $\mathcal{F} : E(\Omega) \rightarrow \dot{H}^{-m+1}(\Omega)^3$  by  $\mathcal{F}(\mathbf{e}) = \dot{\mathbf{v}}$ . Using the same method as in [8], we can then prove the following Korn's inequality in Sobolev spaces with negative exponents:

**Theorem 5.2.** *The linear mapping  $\mathcal{F} : E(\Omega) \rightarrow \dot{H}^{-m+1}(\Omega)^3$  is an isomorphism. Besides, there exists a constant  $C \geq 0$  such that*

$$\inf_{\mathbf{r} \in R(\Omega)} \|\mathbf{v} + \mathbf{r}\|_{H^{-m+1}(\Omega)^3} \leq C \sum_{i,j} \|\varepsilon_{ij}(\mathbf{v})\|_{H^{-m}(\Omega)} \quad \text{for all } \mathbf{v} \in H^{-m+1}(\Omega)^3,$$

and

$$\|\mathbf{v}\|_{H^{-m+1}(\Omega)^3} \leq C(\|\mathbf{v}\|_{H^{-m}(\Omega)^3} + \sum_{i,j} \|\varepsilon_{ij}(\mathbf{v})\|_{H^{-m}(\Omega)}) \quad \text{for all } \mathbf{v} \in H^{-m+1}(\Omega)^3$$

where  $\varepsilon_{ij}(\mathbf{v}) = \frac{1}{2}(\partial_j v_i + \partial_i v_j)$ .

**Remark 5.3.** Analogous techniques would likewise extend to Sobolev spaces with negative exponents the results obtained for non-simply connected domains in [9], [12] and [13].

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