



**HAL**  
open science

# Maharam extension and spectral representation of stable processes

Emmanuel Roy

► **To cite this version:**

Emmanuel Roy. Maharam extension and spectral representation of stable processes. 2010. hal-00443893v2

**HAL Id: hal-00443893**

**<https://hal.science/hal-00443893v2>**

Preprint submitted on 26 Jan 2010 (v2), last revised 28 May 2012 (v4)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# MAHARAM EXTENSION AND SPECTRAL REPRESENTATION OF STABLE PROCESSES

EMMANUEL ROY

ABSTRACT. We give a second look at stable processes (especially stationary) by interpreting the self-similar property at the level of the Lévy measure as characteristic of a Maharam system. This allows us to derive structural results and their ergodic consequences. As a byproduct, we obtain a “stable processes” proof of Banach-Lamperti Theorem for  $\alpha < 2$ .

## 1. INTRODUCTION

In a fundamental paper [9], Rosiński revealed the hidden structure of stationary symmetric  $\alpha$ -stable ( $S\alpha S$ ) processes. Namely, he proved that, through what is called, following Hardin [5], a *minimal spectral representation*, such a process is driven by a non-singular dynamical system.

Such a result was proved to classify those processes according to their ergodic properties such as various kind of mixing. In [13], we used a different approach as we considered the whole family of stationary infinitely divisible processes without Gaussian part (called *IDp processes*). The key tool there was the Lévy measure system of the process, which was measure-preserving and not just merely non-singular. As of today, in the stable<sup>1</sup> case, it remains unclear what was the connection between the Lévy measure and the non-singular system. This is the purpose of this paper to fill the gap and go beyond both approaches.

Indeed, we will prove that Lévy measure systems of  $\alpha$ -stable processes have the form of a so-called Maharam system. This observation has some interesting consequences as it allows us to derive very quickly minimal spectral representations, to reinforce factorization results, and to refine ergodic classification.

Let us explain very loosely the mathematical features of stable distributions we will be using. Observe that stable distributions are characterized by a self-similar property which is obvious when observing the corresponding Lévy process:

If  $X_t$  is an  $\alpha$ -stable Lévy process, then  $b^{-\frac{1}{\alpha}} X_{bt}$  has the same distribution.

However, if not obvious or useful, this property is also present for any  $\alpha$ -stable object but takes another form. The common feature is to be found in the Lévy measure:

Loosely speaking, if  $\{X_t\}_{t \in S}$  is an  $\alpha$ -stable process indexed by a set  $S$ , then for any positive number  $c > 0$ , the image of the Lévy measure  $Q$  by the map  $R_c := \{x_t\}_{t \in S} \mapsto \{cx_t\}_{t \in S}$  is  $c^{-\alpha}Q$ .

---

2000 *Mathematics Subject Classification*. Primary 60G52, 60G10, 37A40; secondary 37A50.

*Key words and phrases*. Stable stationary processes, Maharam system, ergodic properties.

<sup>1</sup>we use the terminology “stable” in the traditional acceptance, which is sometimes replaced by “strictly stable” nowadays.

This property on the Lévy measure is characteristic of  $\alpha$ -stable processes and can be translated into an ergodic theoretic statement:

The measurable non-singular flow  $\{R_c\}_{c \in \mathbb{R}_+}$  is dissipative and the multiplicative coefficient  $c^{-\alpha}$  has an outstanding importance in that matter, since it reveals the structure of a Maharam's transformation. The importance is even greater when there is more invariance involved (stationary  $\alpha$ -stable processes, etc...).

The paper is organised as follows. In Section 2 we recall what a spectral representation is and in Section 3, we give the necessary background in non-singular theory. Maharam systems are introduced in Section 4 and the link with Lévy measures of stable processes is exposed in Section 5. Spectral representations are obtained in Section 6 and are refined in Section 7. We deduce from the preceding results some ergodic properties in Section 8. The continuous case is briefly explained in Section 9. The paper ends with Section 10 and a new proof of Banach-Lamperti Theorem (for  $\alpha < 2$ ).

## 2. SPECTRAL REPRESENTATION

We warn the reader that we will, most of the time, omit the implicit “ $\mu$ -a.e.” or “modulo null sets” throughout the document.

A family of functions  $\{f_t\}_{t \in T} \subset L^\alpha(\Omega, \mathcal{F}, \mu)$  where  $(\Omega, \mathcal{F}, \mu)$  is a  $\sigma$ -finite Lebesgue space is said to be a spectral representation of  $S\alpha S$  process  $\{X_t\}_{t \in T}$  if

$$\{X_t\}_{t \in T} = \left\{ \int_{\Omega} f_t(\omega) M(d\omega) \right\}_{t \in T}$$

holds in distribution,  $M$  being an independently scattered  $S\alpha S$ -random measure on  $(\Omega, \mathcal{F})$  with intensity measure  $\mu$ .

We'll say that a spectral representation is *proper* if  $\text{Supp} \{f_t, t \in T\} = \Omega$ . Of course we obtain a proper representation from a general one by removing the complement of  $\text{Supp} \{f_t, t \in T\}$ .

To express that a representation contains the strict minimum to define the process, the notion of minimality has been introduced (Hardin [5]):

A spectral representation is said to be  $\{f_t\}_{t \in T} \subset L^\alpha(\Omega, \mathcal{F}, \mu)$  *minimal* if it is proper and  $\sigma\left(\frac{f_t}{f_s} 1_{\{f_s \neq 0\}}, s, t \in T\right) = \mathcal{F}$ .

Hardin proved in [5] the existence of minimal representations for  $S\alpha S$  processes.

In the stationary case ( $T = \mathbb{R}$  or  $\mathbb{Z}$ ), Rosiński has explained the form of the spectral representation:

**Theorem 1.** (*Rosiński*) *Let  $\{f_t\}_{t \in T} \subset L^\alpha(\Omega, \mathcal{F}, \mu)$  be a minimal representation of a stationary  $S\alpha S$ -process, then there exists nonsingular flow  $\{\phi_t\}_{t \in T}$  on  $(\Omega, \mathcal{F}, \mu)$  and a cocycle  $\{a_t\}_{t \in T}$  for this flow with values in  $\{-1, 1\}$  (or in  $|z| = 1$  in the complex case) such that, for each  $t \in T$ ,*

$$f_t = a_t \left\{ \frac{d\mu \circ \phi_t}{d\mu} \right\}^{\frac{1}{\alpha}} (f_0 \circ \phi_t).$$

Those definitions readily extends to general  $\alpha$ -stable processes, replacing  $M$  by a general independently scattered  $\alpha$ -stable random measure (see [15]).

## 3. SOME TERMINOLOGY

A quadruplet  $(\Omega, \mathcal{F}, \mu, T)$  is called a *dynamical system* or shortly a *system* if  $T$  is a *non-singular automorphism* that is a bijective bi-measurable map such that  $T^*\mu \sim \mu$ . If  $T_*(\mu) = \mu$  then  $(\Omega, \mathcal{F}, \mu, T)$  is a *measure-preserving* (abr. *m.p.*) dynamical system.

A system  $(\Omega_2, \mathcal{F}_2, \mu_2, T_2)$  is said to be a *non-singular* (resp. *measure preserving*) *factor* of the system  $(\Omega_1, \mathcal{F}_1, \mu_1, T_1)$  if there exists a measurable *non-singular* (resp. *measure-preserving*) *homomorphism* between them, that is a measurable map  $\Phi$  from  $\Omega_1$  to  $\Omega_2$  such that  $\Phi T_1 = T_2 \Phi$  and  $\Phi^* \mu_1 \sim \mu_2$  (resp.  $\Phi^* \mu_1 = \mu_2$ ). If  $\Phi$  is invertible and bi-measurable it is called a *non-singular* (resp. *measure-preserving*) *isomorphism* and the system are said to be *non-singular* (resp. *measure-preserving*) *isomorphic*.

**3.1. Krieger types and factors of Von Neumannn algebras.** Consider a non-singular dynamical system  $(\Omega, \mathcal{F}, \mu, T)$ . A set  $A \in \mathcal{F}$  such that  $\mu(A) > 0$  is said to be *periodic* of period  $n$  if  $T^i A$ ,  $0 \leq i \leq n - 1$ , are disjoint and  $T^n A = A$  and *wandering* if  $T^i A$ ,  $i \in \mathbb{Z}$  are disjoint. A set is *exhaustive* if  $\cup_{k \in \mathbb{Z}} T^k A = \Omega$ . A system is *conservative* if there is no wandering set and *dissipative* if there is an exhaustive wandering set.

$(\Omega, \mathcal{F}, \mu, T)$  is said to be of *Krieger type*:

- $I_n$  if there exists an exhaustive set of period  $n$ .
- $I_\infty$  if it is dissipative.
- $\text{II}_1$  if there is no periodic set and exists an equivalent finite  $T$ -invariant measure.
- $\text{II}_\infty$  if is conservative with an equivalent infinite  $T$ -invariant continuous measure but no absolutely continuous finite  $T$ -invariant measure.
- $\text{III}$  if there is no absolutely continuous  $T$ -invariant measure.

A Von Neumann algebra  $M$  (i.e. a weakly closed self-adjoint unital subalgebra of the algebra of bounded linear operators of an Hilbert space) is said to be a factor if  $M \cap M' = \mathbb{C}1$  where  $M'$  is the commutant of  $M$ . Factors are building blocks of more general Von Neumannn algebra.

Let  $(\Omega, \mathcal{F}, \mu, T)$  be an ergodic non-singular system. Take  $\Psi \in L^\infty(\Omega, \mu)$  and consider the operators  $A_\Psi$  and  $V_j$ ,  $j \in \mathbb{Z}$ , acting on  $L^2(\Omega \times \mathbb{Z}, \mu \otimes m)$  ( $m$  is the counting measure on  $\mathbb{Z}$ ) defined by

$$(A_\Psi h)(\omega, i) := \Psi(T^i \omega) h(\omega, i), (V_j h)(\omega, i) := h(\omega, i - j)$$

The Von Neumannn algebra generated by  $A_\Psi$  and  $V_j$ ,  $j \in \mathbb{Z}$ , is a factor if and only if  $T$  is ergodic. The fundamental result here is that every factor can be realised this way.

There exists an algebraic classification of factors due to Murray-Von Neumannn and Connes using the same types as above which corresponds exactly, through the above construction, to the Krieger type of  $(\Omega, \mathcal{F}, \mu, T)$ .

## 4. MAHARAM TRANSFORMATION

**Definition 2.** A m.p. dynamical system is said to be *Maharam* if it is isomorphic to  $(\Omega \times \mathbb{R}, \mathcal{F} \otimes \mathcal{B}, \mu \otimes e^s ds, \tilde{T})$  where  $T$  is a non singular automorphism of  $(\Omega, \mathcal{F}, \mu)$  and  $\tilde{T}$  is defined by

$$\tilde{T}(\omega, s) := \left( T(\omega), s - \ln \frac{dT_*^{-1}\mu}{d\mu}(\omega) \right).$$

Observe that the dissipative flow  $\{\tau_t\}_{t \in \mathbb{R}}$  defined by  $\tau_t := (\omega, s) \mapsto (\omega, s - t)$  commutes with  $\tilde{T}$ .

Note that we have chosen the usual additive representation but we could (and eventually will !) use the following multiplicative representation of a Maharam system. Take  $0 < \alpha < 2$ , we can represent  $(\Omega \times \mathbb{R}, \mathcal{F} \otimes \mathcal{B}, \mu \otimes e^s ds, \tilde{T})$  by the system  $(\Omega \times \mathbb{R}_+^*, \mathcal{F} \otimes \mathcal{B}_+, \mu \otimes \frac{1}{s^{1+\alpha}} ds, \tilde{T}_\alpha)$  where  $\tilde{T}_\alpha$  is defined by:

$$\tilde{T}_\alpha(\omega, s) := \left( T(\omega), s \left( \frac{dT_*^{-1}\mu}{d\mu}(\omega) \right)^{\frac{1}{\alpha}} \right).$$

The isomorphism being provided by the map  $(\omega, s) \mapsto (\omega, (2 - \alpha)^{-\frac{1}{2-\alpha}} e^{(2-\alpha)s})$ .

Observe that, under this isomorphism,  $\{\tau_t\}_{t \in \mathbb{R}}$  is changed into  $\left\{ S_{\frac{t}{e^\alpha}} \right\}_{t \in \mathbb{R}_+^*}$  where  $S_t$  is the multiplication by  $t$  on the second coordinate.

In [1], the authors proved the following characterization, as a straightforward application of Krengel's representation of dissipative transformations:

**Theorem 3.** *A system  $(X, \mathcal{A}, \nu, \gamma)$  is Maharam if and only if there exists a measurable flow  $\{Z_t\}_{t \in \mathbb{R}}$  commuting with  $\gamma$  such that  $(Z_t)_* \nu = e^t \nu$ .  $\{Z_t\}_{t \in \mathbb{R}}$  corresponds to  $\{\tau_t\}_{t \in \mathbb{R}}$  under the isomorphism with the Maharam system under the additive representation.*

In the original theorem they assumed ergodicity of  $\gamma$  to prove that the resulting non-singular transformation  $T$  in the above representation was actually living on a non-atomic measure space  $(\Omega, \mathcal{F}, \mu)$ . The ergodicity assumption is therefore not necessary in the way we present this theorem.

We end this section by a very natural lemma which is part of folklore. We provide a sketch of proof.

**Lemma 4.** *Consider two Maharam systems  $(\Omega_1 \times \mathbb{R}_+^*, \mathcal{F}_1 \otimes \mathcal{B}, \mu_1 \otimes \frac{1}{s^{1+\alpha}} ds, \tilde{T}_1)$  and  $(\Omega_2 \times \mathbb{R}_+^*, \mathcal{F}_2 \otimes \mathcal{B}, \mu_2 \otimes \frac{1}{s^{1+\alpha}} ds, \tilde{T}_2)$  and denote by  $\{S_t\}_{t \in \mathbb{R}_+^*}$  and  $\{Z_t\}_{t \in \mathbb{R}_+^*}$  their respective multiplicative flows. Assume there exists a (measure-preserving) factor map (resp. isomorphism)  $\Phi$  between the two systems such that, for all  $t \in \mathbb{R}_+^*$ ,  $S_t \Phi = Z_t \Phi$ . Then  $\Phi$  induces a non-singular factor map (resp. isomorphism)  $\phi$  between  $(\Omega_1, \mathcal{F}_1, \mu_1, T_1)$  and  $(\Omega_2, \mathcal{F}_2, \mu_2, T_1)$ . If  $\phi$  is a non-singular isomorphism between  $(\Omega_1, \mathcal{F}_1, \mu_1, T_1)$  and  $(\Omega_2, \mathcal{F}_2, \mu_2, T_1)$ , then it induces a m.p. isomorphism  $\Phi$  between the two associated Maharam systems such that, for all  $t \in \mathbb{R}_+^*$ ,  $S_t \Phi = Z_t \Phi$ .*

*Proof.* We solve the isomorphism case and we skip the standard measure theoretic issues concerning null sets.

Set  $\Phi(\omega_1, t) = (\Phi_1(\omega_1, t), \Phi_2(\omega_1, t))$  and  $\Phi^{-1}(\omega_1, t) = ((\Phi^{-1})_1(\omega_1, t), (\Phi^{-1})_2(\omega_1, t))$ . The property  $\Phi \circ S_t = Z_t \circ \Phi$  for all  $t \in \mathbb{R}$  implies:

$$(\Phi_1(\omega_1, t), \Phi_2(\omega_1, t)) = \Phi(\omega_1, t) = \Phi \circ S_t(\omega_1, 1) = Z_t \circ \Phi(\omega_1, 1) = (\Phi_1(\omega_1, 1), t \Phi_2(\omega_1, 1))$$

Therefore  $\Phi_1$  (and also  $(\Phi^{-1})_1$ ) doesn't depend on its second coordinate. As

$$\begin{aligned} (\omega_1, 1) &= \Phi^{-1}\Phi(\omega_1, 1) \\ &= ((\Phi^{-1})_1(\Phi_1(\omega_1, 1), 1), (\Phi^{-1})_2(\Phi_1(\omega_1, 1), \Phi_2(\omega_1, 1))) \end{aligned}$$

and

$$\begin{aligned} (\omega_2, 1) &= \Phi\Phi^{-1}(\omega_2, 1) \\ &= (\Phi_1((\Phi^{-1})_1(\omega_2, 1), 1), \Phi_2((\Phi^{-1})_1(\omega_2, 1), (\Phi^{-1})_2(\omega_2, 1))), \end{aligned}$$

we can define  $\phi(\omega_1) := \Phi_1(\omega_1, 1)$ , it is invertible and  $\phi^{-1}(\omega_2) = (\Phi^{-1})_1(\omega_2, 1)$ .

It is easy to check that  $\phi \circ T_1 = T_2 \circ \phi$ .

Define also  $\psi(\omega_1) := \Phi_2(\omega_2, 1) > 0$ ,  $\widetilde{\mu}_2 := \phi_*\mu_1$ .

Take  $f \in L^1(\mu_2 \otimes \frac{1}{s^{1+\alpha}}ds)$ , thanks to Fubini theorem, we get

$$\begin{aligned} \int_{\Omega_2} \int_{\mathbb{R}_+^*} f(\omega_2, s) \frac{1}{s^{1+\alpha}} ds \mu_2(d\omega_2) &= \int_{\Omega_1} \left( \int_{\mathbb{R}_+^*} f \circ \Phi(\omega_1, s) \frac{1}{s^{1+\alpha}} ds \right) \mu_1(d\omega_1) \\ &= \int_{\Omega_1} \left( \int_{\mathbb{R}_+^*} f(\phi(\omega_1), s\psi(\omega_1)) \frac{1}{s^{1+\alpha}} ds \right) \mu_1(d\omega_1) = \int_{\Omega_1} \left( \int_{\mathbb{R}_+^*} f(\phi(\omega_1), s) \frac{\psi(\omega_1)^\alpha}{s^{1+\alpha}} ds \right) \mu_1(d\omega_1) \\ &= \int_{\mathbb{R}_+^*} \left( \int_{\Omega_2} f(\omega_2, s) \phi_*\widetilde{\mu}_1(d\omega_2) \right) \frac{1}{s^{1+\alpha}} ds \end{aligned}$$

We conclude that  $\phi_*\widetilde{\mu}_1 = \mu_2$  where  $\widetilde{\mu}_1 := \psi(\omega_1)^\alpha \mu_1(d\omega_1)$ . As  $\psi > 0$ ,  $\widetilde{\mu}_1 \sim \mu_1$ .

Observe that  $\psi = \left( \frac{d\mu_2}{d\phi_*\mu_1} \right)^\frac{1}{\alpha} \circ \phi$ .

Let now  $\phi$  be a non-singular isomorphism between  $(\Omega_1, \mathcal{F}_1, \mu_1, T_1)$  and  $(\Omega_2, \mathcal{F}_2, \mu_2, T_2)$

and set  $\Phi : (\omega_1, s) \mapsto \left( \phi(\omega_1), s \left( \frac{d\phi_*^{-1}\mu_2}{d\mu_1} \right)^\frac{1}{\alpha}(\omega_1) \right)$ . It is not difficult to check that  $\Phi$  is the desired m.p. isomorphism.  $\square$

*Remark 5.* From this lemma, the Maharam systems associated to  $(\Omega, \mathcal{F}, \mu_1, T)$  and  $(\Omega, \mathcal{F}, \mu_2, T)$  where  $\mu_1 \sim \mu_2$  are isomorphic (take  $\phi := \text{Id}$ ).

**4.1. Refinements of type III (see [4]).** Observe that since the flow  $\{S_t\}_{t \in \mathbb{R}}$  commutes with  $\widetilde{T}$ , it acts non-singularly on the space  $(Z, \nu)$  of ergodic components of  $\widetilde{T}$  and is called the *associated flow* of  $T$ . This flow is ergodic whenever  $T$  is ergodic. In terms of Von Neumann algebra, the associated flow corresponds exactly to the flow of weights of Connes and Takesaki [3]. The form of this flow allows to classify ergodic type III systems.

- $T$  is of type  $\text{III}_\lambda$ ,  $0 < \lambda < 1$  if the associated flow is the periodic flow  $x \mapsto x + t \pmod{-\log \lambda}$ .
- $T$  is of type  $\text{III}_0$  if the associated flow is free.
- $T$  is of type  $\text{III}_1$  if the associated flow is the trivial flow on a singleton.

In particular  $\widetilde{T}$  is ergodic if and only if  $T$  is of type  $\text{III}_1$ .

## 5. LÉVY MEASURE AS MAHARAM SYSTEM

**5.1. Lévy measure of  $\alpha$ -stable processes.** For simplicity we will only consider discrete time stationary processes.

**Definition 6.** A shift-invariant Lévy measure  $Q$  on  $(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes \mathbb{Z}})$  is the Lévy measure of an  $\alpha$ -stable stationary process if and only if it satisfies

$$(5.1) \quad (R_t)_* Q = t^{-\alpha} Q$$

for any positive  $t$ ,  $R_t$  being the multiplication by  $t$ , i.e.

$$\{x_n\}_{n \in \mathbb{Z}} \mapsto \{tx_n\}_{n \in \mathbb{Z}}.$$

**Theorem 7.** Let  $(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes \mathbb{Z}}, Q, S)$  be the Lévy measure system of an  $\alpha$ -stable stationary process. Then there exists a probability space  $(\Omega, \mathcal{F}, \mu)$ , a non singular transformation  $T$ , a function  $f \in L^\alpha(\mu)$  such that, if  $M$  denotes the map  $(\omega, t) \mapsto tf(\omega)$  then the map  $\Theta := (\omega, t) \mapsto \left\{ M \circ \tilde{T}_\alpha^n(\omega, t) \right\}_{n \in \mathbb{Z}}$  yields an isomorphism of the Maharam system  $(\Omega \times \mathbb{R}_+, \mathcal{F} \otimes \mathcal{B}_+, \mu \otimes \frac{1}{s^{1+\alpha}} ds, \tilde{T}_\alpha)$  with  $(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes \mathbb{Z}}, Q, S)$ .

*Proof.* First observe that Theorem 3 can be applied to  $(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes \mathbb{Z}}, Q, S)$  since the measurable and (obviously) dissipative flow  $\left\{ R_{e^{\frac{t}{\alpha}}} \right\}_{t \in \mathbb{R}}$  satisfies the hypothesis, thanks to Eq (5.1). Therefore, there exists an isomorphism  $\Psi$  between the Maharam system  $(\Omega \times \mathbb{R}_+, \mathcal{F} \otimes \mathcal{B}_+, \mu \otimes \frac{1}{s^{1+\alpha}} ds, \tilde{T}_\alpha)$  and  $(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes \mathbb{Z}}, Q, S)$  for an appropriate non-singular system  $(\Omega, \mathcal{F}, \mu, T)$ . Set  $f := \Psi(\omega, 1)_0$  (i.e.  $\Psi(\omega, 1)_0$  is the 0-th coordinate of the sequence  $\Psi(\omega, 1)$ ) and let us check  $f \in L^\alpha(\mu)$ . Indeed, as  $Q$  is a Lévy measure, we have:

$$\int_{\mathbb{R}^{\mathbb{Z}}} x_0^2 \wedge 1 Q(d\{x_n\}_{n \in \mathbb{Z}}) < \infty$$

but since  $\Psi$  is an isomorphism and  $\Psi(\omega, t) = \Psi \circ S_t(\omega, 1) = R_t \circ \Psi(\omega, 1) = t\Psi(\omega, 1)$ , we have:

$$\begin{aligned} \int_{\mathbb{R}^{\mathbb{Z}}} x_0^2 \wedge 1 Q(d\{x_n\}_{n \in \mathbb{Z}}) &= \int_{\Omega} \int_{\mathbb{R}_+} \Psi(\omega, t)_0^2 \wedge 1 \frac{1}{t^{\alpha+1}} dt \mu(d\omega) \\ \int_{\Omega} \int_{\mathbb{R}_+} \left( t^2 \Psi(\omega, 1)_0^2 \right) \wedge 1 \frac{1}{t^{\alpha+1}} dt \mu(d\omega) &= \left( \int_{\mathbb{R}_+} z^2 \wedge 1 \frac{1}{z^{\alpha+1}} dz \right) \int_{\Omega} |\Psi(\omega, 1)_0|^\alpha \mu(d\omega) \end{aligned}$$

after the change of variable  $z := t|\Psi(\omega, 1)_0|$ . Therefore  $\int_{\Omega} |\Psi(\omega, 1)_0|^\alpha \mu(d\omega) < \infty$ .  $\square$

In the symmetric case we can precise the theorem:

**Theorem 8.** Let  $(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes \mathbb{Z}}, Q, S)$  be the Lévy measure system of a symmetric  $\alpha$ -stable stationary process. Then there exists a probability space  $(\Omega, \mathcal{F}, \mu)$ , a non singular transformation  $T$ , a function  $f \in L^\alpha(\mu)$  and a measurable map  $\phi : \Omega \rightarrow \{-1, 1\}$  such that, if  $M$  denotes the map  $(\omega, t) \mapsto tf(\omega)$  then the map  $(\omega, t) \mapsto \left\{ M \circ \overline{T}_\alpha^n(\omega, t) \right\}_{n \in \mathbb{Z}}$  yields an isomorphism between  $(\Omega \times \mathbb{R}, \mathcal{F} \otimes \mathcal{B}, \mu \otimes \frac{1}{|s|^{1+\alpha}} ds, \overline{T}_\alpha)$  with  $(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes \mathbb{Z}}, Q, S)$ ,  $\overline{T}_\alpha$  being defined by  $(\omega, t) \mapsto \left( T\omega, \phi(\omega) t \left( \frac{dT_*^{-1}\mu}{d\mu}(\omega) \right)^{\frac{1}{\alpha}} \right)$ .

*Proof.* Start by applying Theorem 7 to the Lévy measure.

Observe that the symmetry involves the presence of a measure preserving involution  $I$ , namely  $I\{x_n\}_{n \in \mathbb{Z}} = \{-x_n\}_{n \in \mathbb{Z}}$ .  $I$  also preserves the Lévy measure of the

process. Observe also that  $I$  commutes with the shift and with the flow  $R_t$ . Therefore  $\tilde{T} := \Theta^{-1}I\Theta$  is a measure preserving automorphism of  $(\Omega \times \mathbb{R}_+^*, \mathcal{F} \otimes \mathcal{B}, \mu \otimes \frac{1}{s^{1+\alpha}} ds, \tilde{T})$  and we can apply Lemma 4 to deduce that  $\tilde{T}$  induces a non singular involution  $\phi$  on  $(\Omega, \mathcal{F}, \mu, T)$ . It is standard that such transformation admits an equivalent finite invariant measure so, up to another measure preserving isomorphism, we can assume that  $\phi$  preserves the probability measure  $\mu$ .

Using Rohklin structure theorem, the compact factor associated to the compact group  $\{\text{Id}, \phi\}$  tells us that we can represent  $(\Omega, \mathcal{F}, \mu, T)$  as  $(X \times \{-1, 1\}, \mathcal{A} \otimes \mathcal{P}\{-1, 1\}, \nu \otimes m, S_\xi)$  where  $S$  is a non-singular automorphism of  $(X, \mathcal{A}, \nu)$ ,  $m$  is the Haar measure on  $(\{-1, 1\}, \mathcal{P}\{-1, 1\})$ ,  $\xi$  a cocycle from  $X$  to  $\{-1, 1\}$  and  $S_\xi := (x, \epsilon) \mapsto (Sx, \xi(x)\epsilon)$ .

It is now clear that  $(X \times \{-1, 1\} \times \mathbb{R}_+^*, (\mathcal{A} \otimes \mathcal{P}\{-1, 1\}) \otimes \mathcal{B}, \nu \otimes m \otimes \frac{1}{s^{1+\alpha}} ds, \tilde{S}_\xi)$  is isomorphic to  $(X \times \mathbb{R}_+^*, \mathcal{A} \otimes \mathcal{B}, \nu \otimes \frac{1}{|s|^{1+\alpha}} ds, R)$  thanks to the mapping  $(x, \epsilon, s) \mapsto (x, \epsilon s)$  and  $R := (x, s) \mapsto (Sx, (\frac{dS\nu}{d\nu}(x))^{\frac{1}{\alpha}} s)$ .  $\square$

**5.2. Maharam systems as Lévy measure.** A famous theorem of Krengel [7] shows that if a system  $(\Omega, \mathcal{F}, \mu, T)$  admits no finite absolutely continuous  $T$ -invariant measure then it possesses a 2-generator, that is, there exists a measurable function  $f : \Omega \rightarrow \{0, 1\}$ ,  $\mu\{f = 1\} < \infty$ , such that  $\sigma\{f \circ T^n, n \in \mathbb{Z}\} = \mathcal{F}$ . As  $f \in L^\alpha(\mu)$  for every  $0 < \alpha < 2$ , every Maharam system based on a system without finite invariant absolutely continuous measure is isomorphic to the Lévy measure system of an  $\alpha$ -stable process for any  $\alpha$ . This is the most interesting case as it covers exactly the family of ergodic stable processes.

## 6. SPECTRAL REPRESENTATION

It is now very easy to derive spectral representation from the results of the preceding section.

**Theorem 9.** *If  $(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes \mathbb{Z}}, Q, S)$  is the Lévy measure system of an  $\alpha$ -stable stationary process, under the notation of Theorem 7,  $(\Omega, \mathcal{F}, \mu)$  together with the function  $f \in L^\alpha(\mu)$  and the non-singular automorphism  $T$  yields a spectral representation of the process. In the  $S\alpha S$  case, consider also the cocycle  $\phi$  as in Theorem 8.*

*Proof.* Build the Poisson measure over  $(\Omega \times \mathbb{R}_+^*, \mathcal{F} \otimes \mathcal{B}, \mu \otimes \frac{1}{s^{1+\alpha}} ds, \overline{T}_\alpha)$ . It is now a direct consequence of Theorem 3.12.2, page 156 in [15].  $\square$

Note that we don't have assumed any hypothesis of minimality to get this result, although it is a consequence of the following result together with Proposition 2.2 in [10].

**Theorem 10.** *Consider a proper spectral representation of the  $\alpha$ -stable stationary process of Theorem 9 given by  $(K, \mathcal{K}, \rho)$ , a non-singular automorphism  $R$  and  $g \in L^\alpha(\rho)$ . There exists a non-singular factor map  $\Phi$  between  $(K, \mathcal{K}, \rho, R)$  and  $(\Omega, \mathcal{F}, \mu, T)$ .*

*Proof.* First build the Maharam extension  $(K \times \mathbb{R}_+^*, \mathcal{K} \otimes \mathcal{B}, \rho \otimes \frac{1}{s^{1+\alpha}}, \tilde{R})$  and let  $M$  be the map  $(\kappa, t) \mapsto tg(\kappa)$ . Then  $(\kappa, t) \mapsto \left\{ M \circ \tilde{R}^n(\kappa, t) \right\}_{n \in \mathbb{Z}}$  is a factor map from  $(K \times \mathbb{R}_+^*, \mathcal{K} \otimes \mathcal{B}, \rho \otimes \frac{1}{s^{1+\alpha}}, \tilde{R})$  to the Lévy measure system associated to the



$\alpha$ -stable process. Theorem 7 and Lemma 4 imply the existence of the desired non singular factor map  $\Phi$ .  $\square$

## 7. REFINEMENTS OF THE REPRESENTATION

Factors are building blocks of Von Neumann algebra, ergodic stationary processes are building blocks of stationary processes, prime numbers are the building blocks of integers, etc. What are the building blocks of stationary infinitely divisible processes? Let's get more precise:

Given a stationary ID process  $X$ , what are the solutions to the equation (in distribution):

$$X = X_1 + X_2$$

where  $X_1$  and  $X_2$  are independent stationary ID processes. Of course, if  $Q$  is the Lévy measure of  $X$ , then taking  $X_1$  with Lévy measure  $c_1Q$  and  $X_2$  with Lévy measure  $c_2Q$  with  $c_1 + c_2 = 1$  gives a solution. If these are the only solutions, we said in [12] that  $X$  is *pure*, meaning that is impossible to reduce  $X$  to "simpler" pieces. It was then very easy to show that  $X$  is pure if and only if its Lévy measure is ergodic.

In this section, we will try to comment the above equation according to the Krieger type of the associated non-singular transformation. We won't deal with type  $\text{III}_0$  as it is very particular and remains unclear to us.

**7.1. The type  $\text{III}_1$  case, pure stable processes.** It was an open question whether there exist pure stable processes. It can now be solved thanks to the Maharam structure of the Lévy measure: an  $\alpha$ -stable process is pure if and only if the underlying non-singular system is of type  $\text{III}_1$ .

The existence of pure stable processes (guaranteed by the comments made in Section 5.2) is reassuring as it validates the specific study of stable processes. We will see in Section 8 that, "in general", a stationary stable process is weakly mixing, rigid and pure.

**7.2. The type  $\text{III}_\lambda$  case,  $0 < \lambda < 1$ .** In this section we derive the form of those  $\alpha$ -stable processes associated with an ergodic, type  $\text{III}_\lambda$  non-singular automorphism  $0 < \lambda < 1$ .

**7.2.1. Semi-stable stationary processes.** There is a weaker form of stability called semi-stability (see [16]).

**Definition 11.** A shift-invariant Lévy measure  $Q$  on  $(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes \mathbb{Z}})$  is the Lévy measure of an  $\alpha$ -semi-stable stationary process of span  $b > 0$  if and only if it satisfies

$$(R_b)_* Q = b^{-\alpha} Q$$

where  $R_b$  is the multiplication by  $b$

$$\{x_n\}_{n \in \mathbb{Z}} \mapsto \{bx_n\}_{n \in \mathbb{Z}}.$$

Of course by iterating  $R_b$ , we easily observe that  $(R_{b^n})_* Q = b^{-n\alpha} Q$  for all  $n \in \mathbb{Z}$ .

7.2.2. *Discrete Maharam extension.* Assume  $(\Omega, \mathcal{F}, \mu, T)$  is a non-singular system such that there exists  $\lambda > 0$  so that  $\frac{dT_*^{-1}\mu}{d\mu} \in \{\lambda^n, n \in \mathbb{Z}\}$   $\mu$ -almost everywhere. We can form its *discrete Maharam extension*, that is, the m.p. system  $(\Omega \otimes \mathbb{Z}, \mathcal{F} \otimes \mathcal{B}, \mu \otimes \lambda^n dn, \tilde{T})$  where  $\lambda^n dn$  stands for the measure  $\sum_{n \in \mathbb{Z}} \lambda^n \delta_n$  on  $(\mathbb{Z}, \mathcal{B})$  and  $\tilde{T}$  is defined by

$$\tilde{T}(\omega, n) = \left( T\omega, n - \log_\lambda \frac{dT_*^{-1}\mu}{d\mu}(\omega) \right).$$

As in the continuous case, thanks to Krengel's representation Theorem of dissipative transformation, we easily derive the following characterization result:

**Theorem 12.** *Let  $(X, \mathcal{A}, \nu, \gamma)$  a m.p. system. Assume there exists  $\lambda > 0$  and a dissipative transformation  $Z$  such that  $Z\gamma = \gamma Z$  and  $Z_*\nu = \lambda\nu$ . Then  $(X, \mathcal{A}, \nu, \gamma)$  has the form of (is m.p. isomorphic to) a Maharam extension of a non-singular system  $(\Omega, \mathcal{F}, \mu, T)$  so that  $\frac{dT_*^{-1}\mu}{d\mu} \in \{\lambda^n, n \in \mathbb{Z}\}$   $\mu$ -almost everywhere.*

As a direct application, the Lévy measure system of an  $\alpha$ -semi-stable stationary process with span  $b$  can be represented as a discrete Maharam extension as above, with  $\lambda = b^{-\alpha}$ .

7.2.3. *Ergodic decomposition of Maharam extension of type  $\text{III}_\lambda$  transformations.* Let  $(\Omega, \mathcal{F}, \mu, T)$  be an ergodic type  $\text{III}_\lambda$  system. Thanks to [6], up to a change of measure we can assume that  $\frac{dT_*^{-1}\mu}{d\mu} \in \{\lambda^n, n \in \mathbb{Z}\}$  and moreover  $\bigcup_{n \in \mathbb{Z}} \frac{dT_*^{-n}\mu}{d\mu} = \{\lambda^n, n \in \mathbb{Z}\}$ . Therefore, the discrete Maharam extension  $(\Omega \otimes \mathbb{Z}, \mathcal{F} \otimes \mathcal{B}, \beta\mu \otimes \lambda^n dn, \tilde{T})$ , where  $\beta := \int_0^{-\ln \lambda} e^{-s} ds$ , exists and is ergodic (thanks to a well known criteria on ergodicity of cocycles found in [17]). Now form the product system

$$\left( \Omega \otimes \mathbb{Z} \otimes [0, -\ln \lambda[, \mathcal{F} \otimes \mathcal{B} \otimes \mathcal{B}([0, -\ln \lambda]), \beta\mu \otimes \lambda^n dn \otimes \frac{e^{-s}}{\beta} ds, \tilde{T} \times \text{Id} \right).$$

The dissipative non singular flow  $S_t : (\omega, n, s) \mapsto \left( \omega, n + \left\lfloor \frac{s+t}{-\ln \lambda} \right\rfloor, s + t + \ln \lambda \left\lfloor \frac{s+t}{-\ln \lambda} \right\rfloor \right)$  satisfies  $S_t \circ \tilde{T} \times \text{Id} = \tilde{T} \times \text{Id} \circ S_t$  and  $(S_t)_* \mu \otimes \lambda^n dn \otimes e^{-s} ds = e^{-t} \mu \otimes \lambda^n dn \otimes e^{-s} ds$  and it is very easy to see that  $(\mathbb{Z} \otimes [0, -\ln \lambda[, \mathcal{B} \otimes \mathcal{B}([0, -\ln \lambda]), \lambda^n dn \otimes e^{-s} ds)$  is just a reparametrization of  $(\mathbb{R}, \mathcal{B}, e^{-s} ds)$  thanks to the mapping  $(n, s) \mapsto -n \ln \lambda + s$ .

Therefore  $(\Omega \otimes \mathbb{Z} \otimes [0, -\ln \lambda[, \mathcal{F} \otimes \mathcal{B} \otimes \mathcal{B}([0, -\ln \lambda]), \mu \otimes \lambda^n dn \otimes e^{-s} ds, \tilde{T} \times \text{Id})$  can be seen as the Maharam extension of  $(\Omega, \mathcal{F}, \mu, T)$  and then we obtain the ergodic decomposition of this system: it is the discrete Maharam extension  $(\Omega \otimes \mathbb{Z}, \mathcal{F} \otimes \mathcal{B}, \beta\mu \otimes \lambda^n dn, \tilde{T})$  randomized by the measure  $\frac{e^{-s}}{\beta} ds$  on  $[0, -\ln \lambda[$ .

7.2.4. *Application to stable processes.* Let  $(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes \mathbb{Z}}, Q, S)$  the Lévy measure system of an  $\alpha$ -stable process driven by an ergodic type  $\text{III}_\lambda$  system  $(\Omega, \mathcal{F}, \mu, T)$  and let  $f \in L^\alpha(\mu)$  be given as in Theorem 7. Let  $b > 1$  so that  $b^{-\alpha} = \lambda$ , we need to obtain a multiplicative version of the above structure adapted to our parameters. Up to a change of measure we can assume that  $\left( \frac{dT_*^{-1}\mu}{d\mu} \right)^{\frac{1}{\alpha}} \in \{b^n, n \in \mathbb{Z}\}$   $\mu$ -almost everywhere. Consider the discrete Maharam extension  $(\Omega \times G_b, \mathcal{F} \otimes \mathcal{B}, \beta\mu \otimes m_b, \tilde{T})$  (in

a multiplicative representation) where  $\beta = \int_1^b \frac{1}{s^{1+\alpha}} ds$ ,  $m_b$  is the measure  $\sum_{g \in G_b} g^{-\alpha} \delta_g$  on  $G_b := \{b^n, n \in \mathbb{Z}\}$  and  $\tilde{T} := (\omega, g) \mapsto \left( T\omega, g \left( \frac{dT_*^{-1}\mu}{d\mu}(\omega) \right)^{\frac{1}{\alpha}} \right)$ . Form the system  $(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes \mathbb{Z}}, Q^s, S)$  as a factor of  $(\Omega \times G_b, \mathcal{F} \otimes \mathcal{B}, \beta\mu \otimes m_b, \tilde{T})$  given by the mapping  $\varphi := (\omega, g) \mapsto \left\{ M \circ \tilde{T}^n(\omega, g) \right\}_{n \in \mathbb{Z}}$  where  $M(\omega, g) = gf(\omega)$  and  $Q^s = \varphi_*(\beta\mu \otimes m_b)$ .

Now, as above, we recover the Maharam extension of  $(\Omega, \mathcal{F}, \mu, T)$  by considering  $(\Omega \times G_b \times [1, b], \mathcal{F} \otimes \mathcal{B} \otimes \mathcal{B}([1, b]), \beta\mu \otimes m_b \otimes \frac{1}{\beta s^{1+\alpha}} ds, \tilde{T} \times \text{Id})$ . As the system  $(G_b \times [1, b], \mathcal{B} \otimes \mathcal{B}([1, b]), m_b \otimes \frac{1}{s^{1+\alpha}} ds)$  is isomorphic to  $(\mathbb{R}_+^*, \mathcal{B}, \frac{1}{s^{1+\alpha}} ds)$  thanks to  $(g, t) \mapsto gt$ , we obtain  $(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes \mathbb{Z}}, Q, S)$  by  $(\mathbb{R}^{\mathbb{Z}} \times [1, b], \mathcal{B}^{\otimes \mathbb{Z}} \otimes \mathcal{B}([1, b]), Q^s \otimes \frac{1}{\beta s^{1+\alpha}} ds, S \times \text{Id})$  and the map  $(\{x_n\}_{n \in \mathbb{Z}}, t) \mapsto \{tx_n\}_{n \in \mathbb{Z}}$ . At last, we can check that  $Q^s$  is a Lévy measure, indeed we know that

$$\int_{\mathbb{R}^{\mathbb{Z}}} x_0^2 \wedge 1Q(d\{x_n\}_{n \in \mathbb{Z}}) < +\infty$$

but

$$\int_{\mathbb{R}^{\mathbb{Z}}} x_0^2 \wedge 1Q(d\{x_n\}_{n \in \mathbb{Z}}) = \int_1^b \left( \int_{\mathbb{R}^{\mathbb{Z}}} (sx_0)^2 \wedge 1Q^s(d\{x_n\}_{n \in \mathbb{Z}}) \right) \frac{1}{\beta s^{1+\alpha}} ds$$

therefore, for some  $1 \leq s < b$ ,  $\int_{\mathbb{R}^{\mathbb{Z}}} (sx_0)^2 \wedge 1Q^s(d\{x_n\}_{n \in \mathbb{Z}}) < +\infty$  and this is enough to prove that  $Q^s$  is a Lévy measure.

$(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes \mathbb{Z}}, Q^s, S)$  is the Lévy measure system of an  $\alpha$ -semi-stable stationary process with span  $b$ . Heuristically, if  $X$  has Lévy measure  $Q$ ,  $X$  can be thought as the continuous sum of independent processes  $Y^t$ ,  $1 \leq t < b$  weighted by the probability measure  $\frac{1}{\beta s^{1+\alpha}} ds$  where  $\frac{1}{t} Y^t$  has Lévy measure  $Q^s$ . More formally, if  $\left( (\Omega \times G_b \times [1, b])^*, (\mathcal{F} \otimes \mathcal{B} \otimes \mathcal{B}([1, b]))^*, \left( \beta\mu \otimes m_b \otimes \frac{1}{\beta s^{1+\alpha}} ds \right)^*, (\tilde{T} \times \text{Id})_* \right)$  denotes the Poisson suspension over  $(\Omega \times G_b \times [1, b], \mathcal{F} \otimes \mathcal{B} \otimes \mathcal{B}([1, b]), \beta\mu \otimes m_b \otimes \frac{1}{\beta s^{1+\alpha}} ds, \tilde{T} \times \text{Id})$ , then, if  $I$  denotes the stochastic integral as in [8],  $X_n := I\{M_1\} \circ (\tilde{T} \times \text{Id})_*^n$  has Lévy measure  $Q$  and  $Y_n := I\{M_2\} \circ (\tilde{T} \times \text{Id})_*^n$  where  $M_1(\omega, g, s) = sgf(\omega)$  and  $M_2(\omega, g, s) = gf(\omega)$ .

We therefore observe that  $X_n$  is entirely determined by a pure  $\alpha$ -semi-stable stationary process with span  $b$ ,  $Y_n$ . It is very easy to see that  $X_n$  and  $Y_n$  share the same type of mixing.

**7.3. The type I and II cases.** This case is easy to deal with as we can assume the associated ergodic nonsingular system is actually measure preserving, that is, the Lévy measure system  $(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes \mathbb{Z}}, Q, S)$  is isomorphic to  $(\Omega \times \mathbb{R}_+^*, \mathcal{F} \otimes \mathcal{B}, \mu \otimes \frac{1}{s^{1+\alpha}} ds, \tilde{T})$  where  $T$  preserves  $\mu$  and  $\tilde{T}$  acts as  $T \times \text{Id}$ , i.e.  $\tilde{T}(\omega, t) = (T\omega, t)$ . Considering  $f \in L^\alpha(\mu)$  furnished by Theorem 7,  $(\Omega \times \mathbb{R}_+^*, \mathcal{F} \otimes \mathcal{B}, \mu \otimes \frac{1}{s^{1+\alpha}} ds, \tilde{T})$  is isomorphic to  $(\mathbb{R}^{\mathbb{Z}} \times \mathbb{R}_+^*, \mathcal{B}^{\otimes \mathbb{Z}} \otimes \mathcal{B}, Q^s \otimes \frac{1}{s^{1+\alpha}} ds, S \times \text{Id})$  through the map  $(\omega, t) \mapsto$

$(\{f \circ T^n(\omega)\}_{n \in \mathbb{Z}}, t)$  and

$$\begin{aligned} \int_{\mathbb{R}^{\mathbb{Z}}} x_0^2 \wedge 1Q(d\{x_n\}_{n \in \mathbb{Z}}) &= \int_{\Omega} \int_{\mathbb{R}_+} (tf(\omega))^2 \wedge 1 \frac{1}{t^{\alpha+1}} dt \mu(d\omega) \\ &= \int_{\mathbb{R}_+} \left( \int_{\mathbb{R}^{\mathbb{Z}}} (tx_0)^2 \wedge 1Q^s(d\{x_n\}_{n \in \mathbb{Z}}) \right) \frac{1}{t^{\alpha+1}} dt < +\infty \end{aligned}$$

For the same reason as above  $Q^s$  is a Lévy measure. We draw the same conclusions as in the preceding section taking into account that the weight is now the infinite measure  $\frac{1}{t^{\alpha+1}} dt$  on  $\mathbb{R}_+^*$  and  $Q^s$  can be any Lévy measure (of a stationary process).

## 8. ERGODIC PROPERTIES

Some ergodic properties of general IDp stationary processes have been given in terms of ergodic properties of the Lévy measure system in [13].

Let us recall the main Theorem of this paper:

**Theorem 13.** *A stationary IDp process  $(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes \mathbb{Z}}, \mathbb{P}, S)$  with Lévy measure system  $(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes \mathbb{Z}}, Q, S)$  is*

- *ergodic (and then weakly mixing) if and only if  $(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes \mathbb{Z}}, Q, S)$  has no  $S$ -invariant set of positive and finite measure*
- *mixing if and only if  $(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes \mathbb{Z}}, Q, S)$  is of zero type*
- *Bernoulli if  $(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes \mathbb{Z}}, Q, S)$  is dissipative*

For an  $\alpha$ -stable stationary processes  $(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes \mathbb{Z}}, Q, S)$ , it is more interesting to give them in terms of the associated system  $(\Omega, \mathcal{F}, \mu, T)$  related to a minimal spectral representation. This work has been undergone in the symmetric case ( $S\alpha S$ ) in a series of papers from which we extract the following main results:

**Theorem 14.** *(Samorodnitsky, [14]) A stationary  $S\alpha S$  process is ergodic (and then weakly mixing) if and only if  $(\Omega, \mathcal{F}, \mu, T)$  admits no absolutely continuous invariant probability measure.*

and (rephrased by us in term of the unitary operator associated to  $T$ , i.e.  $U_T := f \mapsto \sqrt{\frac{dT_*^{-1}\mu}{d\mu}} f \circ T$  on  $L^2(\mu)$ ):

**Theorem 15.** *(Rosiński and Samorodnitsky, [11]) A stationary  $S\alpha S$  process is mixing if and only if for all  $f \in L^2(\mu)$ ,  $\langle U_T^n f, f \rangle_{L^2(\mu)} \rightarrow 0$  as  $n$  tends to infinity (equivalently if the maximal spectral type of  $U_T$  is Rajchman).*

Our concern now is to obtain these criteria in the general stable case, using the Maharam's framework. We need first to link ergodic properties between a system and its Maharam extension. This is the purpose of the following theorem of general interest.

**Theorem 16.** *The Maharam system  $(\Omega \times \mathbb{R}_+^*, \mathcal{F} \otimes \mathcal{B}_+, \mu \otimes \frac{1}{s^{1+\alpha}} ds, \tilde{T}_\alpha)$*

- *has no  $\tilde{T}_\alpha$ -invariant set of positive and finite measure if and only if  $(\Omega, \mathcal{F}, \mu, T)$  has no  $T$ -invariant probability measure  $\nu \ll \mu$*
- *is of zero type if and only if for all  $f \in L^2(\mu)$ ,  $\langle U_T^n f, f \rangle_{L^2(\mu)} \rightarrow 0$  as  $n$  tends to infinity (equivalently if the maximal spectral type of  $U_T$  is Rajchman)*
- *is dissipative if and only if  $(\Omega, \mathcal{F}, \mu, T)$  is dissipative*

*Proof.* The first and third assertions are well known, let us prove the second.

Take  $f \in L^2(\mu)$  with full support on  $(\Omega, \mathcal{F}, \mu)$  and assume  $\langle U_T^n f, f \rangle_{L^2(\mu)} \rightarrow 0$  as  $n$  tends to infinity. Consider the Maharam extension  $(\Omega \times \mathbb{R}_+^*, \mathcal{F} \otimes \mathcal{B}_+, \mu \otimes \frac{1}{s^{1+\alpha}} ds, \tilde{T}_\alpha)$ , denotes by  $U_{\tilde{T}_\alpha}^n$  the unitary operator associated to  $\tilde{T}_\alpha$  and consider  $g \in L^2(\frac{1}{s^{1+\alpha}} ds)$ . Compute

$$\begin{aligned} \langle U_{\tilde{T}_\alpha}^n f \otimes g, f \otimes g \rangle_{L^2(\mu \otimes \frac{1}{s^{1+\alpha}} ds)} &= \int_{\Omega} \int_{\mathbb{R}_+^*} f(T^n \omega) g \left( \left( \frac{dT_*^{-n} \mu}{d\mu}(\omega) \right)^{\frac{1}{\alpha}} s \right) f(\omega) g(s) \mu(d\omega) \frac{1}{s^{1+\alpha}} ds \\ &= \int_{\Omega} f(T^n \omega) f(\omega) \int_{\mathbb{R}_+^*} g \left( \left( \frac{dT_*^{-n} \mu}{d\mu}(\omega) \right)^{\frac{1}{\alpha}} u \right) g(u) \frac{1}{s^{1+\alpha}} ds \mu(d\omega) \\ &\leq \|g\|_2^2 \int_{\Omega} \sqrt{\frac{dT_*^{-n} \mu}{d\mu}(\omega)} f(T^n \omega) f(\omega) \mu(d\omega) \\ &= \|g\|_2^2 \langle U_T^n f, f \rangle_{L^2(\mu)} \end{aligned}$$

thanks to Cauchy-Schwarz inequality. This proves that  $\langle U_{\tilde{T}_\alpha}^n f \otimes g, f \otimes g \rangle_{L^2(\mu \otimes \frac{1}{s^{1+\alpha}} ds)}$  tends to zero and this is enough to prove that  $(\Omega \times \mathbb{R}_+^*, \mathcal{F} \otimes \mathcal{B}_+, \mu \otimes \frac{1}{s^{1+\alpha}} ds, \tilde{T}_\alpha)$  is of zero type.

In the other direction:

We can assume that  $\mu$  is a probability measure. We will apply the zero type property of  $\tilde{T}$  to the functions  $1 \otimes g$  where  $g$  is a positive function in  $L^2(\frac{1}{s^{1+\alpha}} ds)$  such that  $\|g\|_p := \left( \int_{\mathbb{R}_+^*} g^p(u) \frac{1}{s^{1+\alpha}} ds \right)^{\frac{1}{p}} < \infty$  and  $\|g\|_q := \left( \int_{\mathbb{R}_+^*} g^q(u) \frac{1}{s^{1+\alpha}} ds \right)^{\frac{1}{q}} < \infty$  where  $0 < p < 1$  and  $\frac{1}{p} + \frac{1}{q} = 1$ :

$$\langle U_{\tilde{T}_\alpha}^n 1 \otimes g, 1 \otimes g \rangle_{L^2(\mu \otimes \frac{1}{s^{1+\alpha}} ds)} \rightarrow 0,$$

as  $n$  tends to infinity. But

$$\begin{aligned} \langle U_{\tilde{T}_\alpha}^n 1 \otimes g, 1 \otimes g \rangle_{L^2(\mu \otimes \frac{1}{s^{1+\alpha}} ds)} &= \int_{\Omega} \int_{\mathbb{R}_+^*} g \left( \left( \frac{dT_*^{-n} \mu}{d\mu}(\omega) \right)^{\frac{1}{\alpha}} u \right) g(u) \frac{1}{s^{1+\alpha}} ds \mu(d\omega) \\ &\geq \|g\|_p \|g\|_q \int_{\Omega} \left( \frac{dT_*^{-n} \mu}{d\mu}(\omega) \right)^{\frac{1}{p}} \mu(d\omega) \\ &\geq \|g\|_p \|g\|_q \left( \int_{\Omega} \sqrt{\frac{dT_*^{-n} \mu}{d\mu}(\omega)} \mu(d\omega) \right)^{\frac{2}{p}} \end{aligned}$$

thanks to inverse Hölder inequality and convexity. This implies that  $\int_{\Omega} \sqrt{\frac{dT_*^{-n} \mu}{d\mu}(\omega)} \mu(d\omega)$  tends to 0 as  $n$  tends to infinity. Now take any sets  $A$  and  $B$  in  $\mathcal{F}$ , then

$$\langle U_T^n 1_A, 1_B \rangle_{L^2(\mu)} = \int_{\Omega} \sqrt{\frac{dT_*^{-n} \mu}{d\mu}(\omega)} 1_A(T^n \omega) 1_B(\omega) \mu(d\omega) \leq \int_{\Omega} \sqrt{\frac{dT_*^{-n} \mu}{d\mu}(\omega)} \mu(d\omega).$$

It is classical that this is enough to prove that, for all  $f \in L^2(\mu)$ ,  $\langle U_T^n f, f \rangle_{L^2(\mu)} \rightarrow 0$  as  $n$  tends to infinity.  $\square$

We are now in position to derive the expected ergodic criteria:

**Theorem 17.** *A stationary stable process  $(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes \mathbb{Z}}, \mathbb{P}, S)$  with associated system  $(\Omega, \mathcal{F}, \mu, T)$  is*

- *ergodic (and then weakly mixing) if and only if  $(\Omega, \mathcal{F}, \mu, T)$  has no  $T$ -invariant probability measure  $\nu \ll \mu$*
- *mixing if and only if for all  $f \in L^2(\mu)$ ,  $\langle U_T^n f, f \rangle_{L^2(\mu)} \rightarrow 0$  as  $n$  tends to infinity (equivalently if the maximal spectral type of  $U_T$  is Rajchman)*
- *Bernoulli if  $(\Omega, \mathcal{F}, \mu, T)$  is dissipative*

*Proof.* This follows immediately from the combination of Theorem 7 where we showed that the Lévy measure of the process has the form of a Maharam system, and Theorem 13.  $\square$

In the forthcoming sections, we are interested in less known ergodic properties ( $K$  property and rigidity) that have been neglected in the  $\alpha$ -stable literature.

### 8.1. $K$ property.

**Definition 18.** A non-singular system  $(\Omega, \mathcal{F}, \mu, T)$  is a  $K$ -system if there exists a sub- $\sigma$ -algebra  $\mathcal{G} \subset \mathcal{F}$  such that  $T^{-1}\mathcal{G} \subset \mathcal{G}$ ,  $T^{-n}\mathcal{G} \downarrow \{\Omega, \emptyset\}$ ,  $T^n\mathcal{G} \uparrow \mathcal{F}$  and  $\frac{d\mu}{dT_*\mu}$  is  $\mathcal{G}$ -measurable.

**Theorem 19.** *If  $(\Omega, \mathcal{F}, \mu, T)$  is a  $K$ -system which is not of type  $\text{II}_1$  then its Maharam extension  $(\Omega \times \mathbb{R}_+^*, \mathcal{F} \otimes \mathcal{B}_+, \mu \otimes \frac{1}{s^{1+\alpha}} ds, \tilde{T}_\alpha)$  is remotely infinite.*

*Proof.* Let  $\mathcal{G}$  be as in Definition 18. Observe that, as  $\frac{d\mu}{dT_*\mu}$  is  $\mathcal{G}$ -measurable,  $\mathcal{G} \otimes \mathcal{B}_+$  is  $\tilde{T}_\alpha$ -invariant, that is  $\tilde{T}_\alpha^{-1}\mathcal{G} \otimes \mathcal{B}_+ \subset \mathcal{G} \otimes \mathcal{B}_+$ . Indeed, take  $g$   $\mathcal{G}$ -measurable and  $f$   $\mathcal{B}_+$ -measurable, we get:

$$g \otimes f(\tilde{T}_\alpha(\omega, s)) = \left( g(T\omega), s \left( \frac{dT_*^{-1}\mu}{d\mu}(\omega) \right)^{\frac{1}{\alpha}} \right) = \left( g(T\omega), s \left( \frac{d\mu}{dT_*\mu}(T\omega) \right)^{\frac{1}{\alpha}} \right).$$

We are going to show that  $\mathcal{P} := \bigcap_{n \in \mathbb{N}} \tilde{T}_\alpha^{-n}\mathcal{G} \otimes \mathcal{B}_+$  only contains sets of zero or infinite measure. Observe that, as  $S_t$  commutes with  $\tilde{T}_\alpha$  and preserves  $\mathcal{G} \otimes \mathcal{B}_+$  for all  $t > 0$  then,  $S_t^{-1}(\tilde{T}_\alpha^{-n}\mathcal{G} \otimes \mathcal{B}_+) \subset \tilde{T}_\alpha^{-n}\mathcal{G} \otimes \mathcal{B}_+$  and therefore  $S_t^{-1}\mathcal{P} \subset \mathcal{P}$  for all  $t > 0$ . Now consider the measurable union, say  $K$ , of  $\mathcal{P}$ -measurable sets of finite and positive measure. It is a  $\tilde{T}_\alpha$ -invariant set and a  $S_t$ -invariant set as well. Recall that the non-singular action of the flow  $S_t$  on the ergodic components of  $(\Omega \times \mathbb{R}_+^*, \mathcal{F} \otimes \mathcal{B}_+, \mu \otimes \frac{1}{s^{1+\alpha}} ds, \tilde{T}_\alpha)$  is ergodic, therefore, if  $K \neq \emptyset$  then  $K = \Omega \times \mathbb{R}_+^*$  mod.  $\nu \otimes \frac{1}{s^{1+\alpha}} ds$ .

Assume  $K = \Omega \times \mathbb{R}_+^*$ , this implies that the measure  $\mu \otimes \frac{1}{s^{1+\alpha}} ds$  is  $\sigma$ -finite on  $\mathcal{P}$  and therefore  $\mathcal{P}$  is a factor of  $(\Omega \times \mathbb{R}_+^*, \mathcal{F} \otimes \mathcal{B}_+, \mu \otimes \frac{1}{s^{1+\alpha}} ds, \tilde{T}_\alpha)$ . Now consider the quotient space  $(\Omega \times \mathbb{R}_+^*)_{/\mathcal{P}}$  that we can endow, with a slight abuse of notation with the  $\sigma$ -algebra  $\mathcal{P}$ . Let  $\rho$  be the image measure of  $\mu \otimes \frac{1}{s^{1+\alpha}} ds$  by the projection map  $\pi$ . On  $((\Omega \times \mathbb{R}_+^*)_{/\mathcal{P}}, \mathcal{P}, \rho)$   $\tilde{T}_\alpha$  and the dissipative flow  $S_t$  induce a transformation

$U$  and a dissipative flow  $Z_t$  that satisfy

$$\pi \circ \tilde{T}_\alpha = U \circ \pi, \quad \pi \circ S_t = U \circ \pi \text{ and } U \circ Z_t = Z_t \circ U$$

Of course, thanks to Theorem 3,  $\left((\Omega \times \mathbb{R}_+^*) / \mathcal{P}, \mathcal{P}, \rho, U\right)$  is a Maharam system, therefore, we can represent it as  $\left(Y \times \mathbb{R}_+^*, \mathcal{K} \otimes \mathcal{B}_+, \sigma \otimes \frac{1}{s^{1+\alpha}} ds, \tilde{L}_\alpha\right)$  for a non-singular system  $(Y, \mathcal{K}, \sigma, L)$ . Applying Lemma 4,  $\pi$  induces a non-singular factor map  $\Gamma$  from  $(\Omega, \mathcal{G}, \mu, T)$  to  $(Y, \mathcal{K}, \sigma, L)$ , which means that there exists a  $R$ -invariant  $\sigma$ -algebra  $\mathcal{Z} \subset \mathcal{G}$  such that  $\Gamma^{-1}\mathcal{K} = \mathcal{Z}$ . But we can observe, that for all  $n > 0$ , the factor  $\tilde{T}_\alpha^{-n}\mathcal{G} \otimes \mathcal{B}_+$  corresponds to a Maharam system that corresponds to the factor  $T^{-n}\mathcal{G}$  of  $(\Omega, \mathcal{G}, \mu, T)$ . Therefore, for all  $n > 0$ ,  $\mathcal{Z} \subset T^{-n}\mathcal{G}$ , i.e.  $\mathcal{Z} \subset \bigcap_{n \in \mathbb{N}} T^{-n}\mathcal{G} = \{\Omega, \emptyset\}$ . This means that  $\mathcal{K} = \{Y, \emptyset\}$ , or, in other words, that

$(Y, \mathcal{K}, \sigma, L)$  is the trivial (one point) system.  $\left(Y \times \mathbb{R}_+^*, \mathcal{K} \otimes \mathcal{B}_+, \sigma \otimes \frac{1}{s^{1+\alpha}} ds, \tilde{L}_\alpha\right)$  then possesses lots of invariant sets of positive finite measure, for example  $A := Y \times [1, 2]$ . But  $\pi^{-1}(A)$  is in turn a positive and finite measure invariant set for the system  $\left(\Omega \times \mathbb{R}_+^*, \mathcal{F} \otimes \mathcal{B}_+, \mu \otimes \frac{1}{s^{1+\alpha}} ds, \tilde{T}_\alpha\right)$ . From Theorem 16, this is not possible as, by assumption,  $(\Omega, \mathcal{F}, \mu, T)$  has no  $T$ -invariant probability measure  $\nu \ll \mu$ .

We can conclude that  $K = \emptyset$ . To prove that  $\left(\Omega \times \mathbb{R}_+^*, \mathcal{F} \otimes \mathcal{B}_+, \mu \otimes \frac{1}{s^{1+\alpha}} ds, \tilde{T}_\alpha\right)$  is remotely infinite, it remains to show that  $\bigvee_{n \in \mathbb{Z}} \tilde{T}_\alpha^{-n}\mathcal{G} \otimes \mathcal{B}_+ = \mathcal{F} \otimes \mathcal{B}_+$ . We only sketch the proof which consists into verifying that the operation of taking natural extension and Maharam extension commute:

Of course, we have  $\bigvee_{n \in \mathbb{Z}} \tilde{T}_\alpha^{-n}\mathcal{G} \otimes \mathcal{B}_+ \subset \mathcal{F} \otimes \mathcal{B}_+$ . It is not difficult to check that  $\bigvee_{n \in \mathbb{Z}} \tilde{T}_\alpha^{-n}\mathcal{G} \otimes \mathcal{B}_+$  corresponds to a Maharam system that we can represent by a  $\sigma$ -algebra  $\mathcal{H} \subset \mathcal{F}$ . But we also have  $\mathcal{G} \subset \mathcal{H}$  and as  $T^{-1}\mathcal{H} = \mathcal{H}$ , we get  $\bigvee_{n \in \mathbb{Z}} \mathcal{G} \subset \mathcal{H}$ . By assumption,  $\bigvee_{n \in \mathbb{Z}} \mathcal{G} = \mathcal{F}$  and we deduce  $\mathcal{H} = \mathcal{F}$  which implies  $\bigvee_{n \in \mathbb{Z}} \tilde{T}_\alpha^{-n}\mathcal{G} \otimes \mathcal{B}_+ = \mathcal{F} \otimes \mathcal{B}_+$ .  $\square$

As before we deduce the following result for  $\alpha$ -stable stationary processes:

**Proposition 20.** *Let  $(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes \mathbb{Z}}, \mathbb{P}, S)$  be a stationary stable process with associated system  $(\Omega, \mathcal{F}, \mu, T)$ . If  $(\Omega, \mathcal{F}, \mu, T)$  is  $K$  and not of type  $\text{II}_1$ , then  $(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes \mathbb{Z}}, \mathbb{P}, S)$  is  $K$ .*

*Proof.* This follows from Theorem 19 and from [13] where we showed that a stationary IDp process is  $K$  if its Lévy measure system is remotely infinite.  $\square$

Recall that in the probability preserving context,  $K$  is strictly stronger than mixing. In [11], to produce examples of mixing  $\alpha$ -stable stationary processes that were not based on dissipative non-singular systems, the authors considered indeed null recurrent Markov chains as base systems. These systems are well known examples of  $K$ -systems, therefore Proposition 20 shows that the associated  $\alpha$ -stable stationary processes are not just merely mixing but are indeed  $K$ .

**8.2. Rigidity.** We recall that a system  $(\Omega, \mathcal{F}, \mu, T)$  is rigid if there exists an increasing sequence  $n_k$  such that  $T^{n_k} \rightarrow \text{Id}$  in the group of non-singular automorphism on  $(\Omega, \mathcal{F}, \mu)$  (the convergence being equivalent to the weak convergence in  $L^2(\mu)$  of the associated unitary operators  $U_{T^{n_k}} : f \mapsto \sqrt{\frac{dT_*^{-n_k}\mu}{d\mu}} f \circ T^{n_k}$  to the identity)

. Observe that in the finite measure case, rigidity doesn't imply ergodicity but prevents mixing.

**Theorem 21.** *The Maharam system  $(\Omega \times \mathbb{R}_+^*, \mathcal{F} \otimes \mathcal{B}_+, \mu \otimes \frac{1}{s^{1+\alpha}} ds, \tilde{T}_\alpha)$  is rigid for the sequence  $n_k$  if and only if  $(\Omega, \mathcal{F}, \mu, T)$  is rigid for the sequence  $n_k$ .*

*Proof.* First observe that the map  $T \mapsto \tilde{T}_\alpha$  is a continuous group homomorphism from the group of non-singular automorphism of  $(\Omega, \mathcal{F}, \mu)$  to the group of measure preserving automorphism of  $(\Omega \times \mathbb{R}_+^*, \mathcal{F} \otimes \mathcal{B}_+, \mu \otimes \frac{1}{s^{1+\alpha}} ds)$ . As  $T^{n_k} \rightarrow \text{Id}$  then  $\tilde{T}_\alpha^{n_k} \rightarrow \text{Id}$  therefore  $\tilde{T}_\alpha^{n_k}$  is rigid for the sequence  $n_k$ .

Conversely, if  $\tilde{T}_\alpha$  is rigid for the same sequence, then, as

$$\left\langle U_{\tilde{T}_\alpha}^{n_k} f \otimes g, f \otimes g \right\rangle_{L^2(\mu \otimes \frac{1}{s^{1+\alpha}} ds)} \leq \|g\|_2^2 \langle U_T^{n_k} f, f \rangle_{L^2(\mu)} \leq \|g\|_2^2 \|f\|_2^2$$

and  $\left\langle U_{\tilde{T}_\alpha}^{n_k} f \otimes g, f \otimes g \right\rangle_{L^2(\mu \otimes \frac{1}{s^{1+\alpha}} ds)} \rightarrow \|g\|_2^2 \|f\|_2^2$ , we get  $\langle U_T^{n_k} f, f \rangle_{L^2(\mu)} \rightarrow \|f\|_2^2$  thus  $T$  is rigid.  $\square$

We need the following general result:

**Proposition 22.** *A stationary IDp stationary process  $(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes \mathbb{Z}}, \mathbb{P}, S)$  is rigid for the sequence  $n_k$  if and only if its Lévy measure system  $(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes \mathbb{Z}}, Q, S)$  is rigid for the sequence  $n_k$ .*

*Proof.* Consider  $X := \{X_n\}_{n \in \mathbb{Z}}$  where  $X_n := \{x_k\}_{k \in \mathbb{Z}} \mapsto x_n$  on  $\mathbb{R}^{\mathbb{Z}}$  and let  $\langle a, X \rangle$  be a finite linear combination of the coordinates.  $\exp i \langle a, X \rangle - \mathbb{E}[\exp i \langle a, X \rangle]$  is a centered square integrable vector under  $\mathbb{P}$  whose spectral measure (under  $\mathbb{P}$ ) is  $\lambda_a := |\mathbb{E}[\exp i \langle a, X \rangle]|^2 \sum_{k=1}^{\infty} \frac{1}{k!} \sigma_a^{*k}$  where  $\sigma_a$  is the spectral measure of  $\exp i \langle a, X \rangle - 1$

under  $Q$  (see [13]). Therefore  $\widehat{\sigma}_a(n_k) \rightarrow \widehat{\sigma}_a(0)$  if and only if  $\widehat{\lambda}_a(n_k) \rightarrow \widehat{\lambda}_a(0)$ . This implies that  $\exp i \langle a, X \rangle - \mathbb{E}[\exp i \langle a, X \rangle]$  is a rigid vector for  $n_k$  under  $\mathbb{P}$  if and only if  $\exp i \langle a, X \rangle - 1$  is a rigid vector for  $n_k$  under  $Q$ . Observe now that the smallest  $\sigma$ -algebra generated by vectors of the kind  $\exp i \langle a, X \rangle - \mathbb{E}[\exp i \langle a, X \rangle]$  under  $\mathbb{P}$  is  $\mathcal{B}^{\otimes \mathbb{Z}}$ , and the same is true with vectors of the kind  $\exp i \langle a, X \rangle - 1$  under  $Q$ . As in any dynamical system if there exists a rigid vector for the sequence  $n_k$  there exists a non trivial factor which is rigid for the sequence  $n_k$ , we get the announced result.  $\square$

**Proposition 23.** *A stationary stable process  $(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes \mathbb{Z}}, \mathbb{P}, S)$  with associated system  $(\Omega, \mathcal{F}, \mu, T)$  is rigid for the sequence  $n_k$  if and only if  $(\Omega, \mathcal{F}, \mu, T)$  is rigid for the sequence  $n_k$ .*

*Proof.* This is the combination of the last two results.  $\square$

*Remark 24.* Rigidity, ergodicity and type III<sub>1</sub> are proved to be generic properties among non-singular automorphisms (see [2] and [4]), therefore, loosely speaking, “in general” a stationary stable process is weakly mixing, rigid and pure.



## 9. THE CONTINUOUS CASE

Maruyama's proof of the existence of Lévy measures for continuous processes is not reliable. Therefore, we cannot start from a Lévy measure system as we did in the discrete time case to deduce the equivalent statements and for example to deduce spectral representation. However, we can observe that spectral representations indeed gives the existence of Lévy measures ! Consider a spectral representation  $\{f_t\}_{t \in \mathbb{R}}$  of a stationary process  $\{X_t\}_{t \in \mathbb{R}}$  on a space  $(\Omega, \mathcal{F}, \mu)$ . It is not difficult to check that the measure  $Q$  obtained as the image measure of  $\mu \otimes \frac{1}{s^{1+\alpha}} ds$  by the map  $(\omega, s) \mapsto \{s f_t(\omega)\}_{t \in \mathbb{R}}$ . If  $\{\theta_t\}_{t \in \mathbb{R}}$  is a non-singular measurable flow on  $(\Omega, \mathcal{F}, \mu)$ , then  $\{\tilde{\theta}_t\}_{t \in \mathbb{R}}$  is a measure-preserving measurable flow on  $(\Omega \times \mathbb{R}_+^*, \mathcal{F} \otimes \mathcal{B}, \mu \otimes \frac{1}{s^{1+\alpha}} ds)$ .

 10. A NEW PROOF OF BANACH-LAMPERTI THEOREM FOR  $\alpha < 2$ .

Banach-Lamperti theorem was central to Hardin's approach of spectral representation. This theorem gives the general form of an isometry  $U$  between  $L^\alpha(\mu_1)$  to  $L^\alpha(\mu_2)$  for  $\alpha \neq 2$ . Obtaining these representations without this theorem allows indeed to give a new proof of it if  $\alpha < 2$ , using stable processes. This theorem takes a more pleasant form if we assume that the isometry is *proper* meaning that  $\text{Supp}\{Uf, f \in L^\alpha(\mu_1)\} = X_2$  up to a null set. Of course, the general case is easily deduced.

**Theorem 25.** (*Banach-Lamperti*) *Let  $(X_1, \mathcal{A}_1, \mu_1)$  and  $(X_2, \mathcal{A}_2, \mu_2)$  be two Lebesgue spaces and  $\alpha < 2$ . If  $U$  is a proper linear isometry from  $L^\alpha(\mu_1)$  to  $L^\alpha(\mu_2)$ , then there exists a non singular transformation  $\Phi$  from  $(X_2, \mathcal{A}_2, \mu_2)$  to  $(X_1, \mathcal{A}_1, \mu_1)$  such that*

$$Uf = \left( \frac{d\mu_1}{d\Phi_*\mu_2} \circ \Phi \right)^{\frac{1}{\alpha}} f \circ \Phi.$$

*Proof.* As  $(X_1, \mathcal{A}_1, \mu_1)$  is Lebesgue, there exists a  $L^\alpha(\mu_1)$ -dense collection  $\{1_{A_n}\}_{n \in \mathbb{N}}$  of indicator functions. Consider the  $\alpha$ -stable process  $\{X_n\}_{n \in \mathbb{N}}$  defined by

$$\mathbb{E} \left[ \exp i \sum_{k=0}^n a_k X_k \right] = \exp \left( - \left\| \sum_{k=0}^n a_k 1_{A_k} \right\|_{L^\alpha(\mu_1)}^\alpha \right),$$

for all  $n \in \mathbb{N}$  and  $(a_0, \dots, a_n) \in \mathbb{R}^n$ . The fact that it is well defined is classical (see [5]).

As  $U$  is an isometry, we also get

$$\mathbb{E} \left[ \exp i \sum_{k=0}^n a_k X_k \right] = \exp \left( - \left\| \sum_{k=0}^n a_k U 1_{A_k} \right\|_{L^\alpha(\mu_2)}^\alpha \right).$$

As a consequence, we know that the maps  $\varphi_1 : (t, x_1) \mapsto (t 1_{A_0}(x_1), \dots, t 1_{A_n}(x_1), \dots)$  (resp.  $\varphi_2 : (t, x_1) \mapsto (t U 1_{A_0}(x_1), \dots, t U 1_{A_n}(x_1), \dots)$ ) sends  $(X_1 \times \mathbb{R}_+^*, \mathcal{A}_1 \otimes \mathcal{B}, \mu_1 \otimes \frac{1}{s^{1+\alpha}} ds)$  (resp.  $(X_2 \times \mathbb{R}_+^*, \mathcal{A}_2 \otimes \mathcal{B}, \mu_2 \otimes \frac{1}{s^{1+\alpha}} ds)$ ) to  $(\mathbb{R}^{\mathbb{N}}, \mathcal{B}^{\otimes \mathbb{N}}, Q)$  where  $Q$  is the Lévy measure of the process  $\{X_n\}_{n \in \mathbb{N}}$ . But  $\varphi_1$  is clearly invertible modulo null sets as  $\cup_{n \in \mathbb{N}} A_n = X_1$  (up to a null set). Therefore  $\varphi_1^{-1} \varphi_2$  maps  $(X_2 \times \mathbb{R}_+^*, \mathcal{A}_2 \otimes \mathcal{B}, \mu_2 \otimes \frac{1}{s^{1+\alpha}} ds)$  onto  $(X_1 \times \mathbb{R}_+^*, \mathcal{A}_1 \otimes \mathcal{B}, \mu_1 \otimes \frac{1}{s^{1+\alpha}} ds)$  and intertwines the flows  $S_s^2 : (t, x_2) \mapsto (ts, x_2)$  and  $S_s^1 : (t, x_1) \mapsto (ts, x_1)$ . By Lemma 4,  $\varphi_1^{-1} \varphi_2$  induces a non-singular

map  $\Phi$  from  $(X_2, \mathcal{A}_2, \mu_2)$  to  $(X_1, \mathcal{A}_1, \mu_1)$  and it is immediate to check it is the desired transformation.  $\square$

*Remark 26.* By duality, it is easy to derive the result for  $\alpha > 2$  if  $U$  is invertible.

## REFERENCES

- [1] J. Aaronson, M. Lemańczyk, and D. Volný. A cut salad of cocycles. *Fund. Math.*, 157(2-3):99–119, 1998.
- [2] O.N. Ageev and C.E. Silva. Genericity of rigid and multiply recurrent infinite measure-preserving and nonsingular transformations. *Topology Proc.*, 26(2):357–365, 2001.
- [3] A. Connes and M. Takesaki. The flow of weights on factors of type III. *Tohoku Math. Journ.*, 29:473–575, 1977.
- [4] A.I. Danilenko and C.E. Silva. Ergodic theory : Non-singular transformations. Preprint.
- [5] C.D. Hardin. On the spectral representation of symmetric stable processes. *J. Multivariate Anal.*, 12:385–401, 1982.
- [6] Y. Katznelson and B. Weiss. The classification of nonsingular actions, revisited. *Ergodic Theory Dynam. Systems*, 11:333–348, 1991.
- [7] U. Krengel. Transformations without finite invariant measure have finite strong generators. *Contributions to Ergodic Theory and Probability (Proc. Conf., Ohio State Univ., Columbus, Ohio, 1970)*, pages 133–157, 1970.
- [8] G. Maruyama. Infinitely divisible processes. *Theory Probab. Appl.*, 15(1):1–22, 1970.
- [9] J. Rosiński. On the structure of stationary stable processes. *Ann. Probab.*, 23(3):1163–1187, 1995.
- [10] J. Rosiński. Minimal integral representations of stable processes. *Probab. Math. Statist.*, 26(1):121–142, 2006.
- [11] J. Rosiński and G. Samorodnitsky. Classes of mixing stable processes. *Bernoulli*, 2(4):365–377, 1996.
- [12] E. Roy. *Mesures de Poisson, infinie divisibilité et propriétés ergodiques*. PhD thesis, Université Paris 6, 2005.
- [13] E. Roy. Ergodic properties of Poissonian ID processes. *Ann. Probab.*, 35(2):551–576, 2007.
- [14] G. Samorodnitsky. Null flows, positive flows and the structure of stationary symmetric stable processes. *Ann. Probab.*, 33(5):1782–1803, 2005.
- [15] G. Samorodnitsky and M. Taqqu. *Stable non-Gaussian random processes: stochastic models with infinite variance*. Chapman & Hall, 1994.
- [16] K.-I. Sato. *Lévy processes and infinitely divisible distributions*. Cambridge University Press, 1999.
- [17] K. Schmidt. *Cocycles on ergodic transformation groups*, volume 1 of *Macmillan Lectures in Mathematics*. Macmillan Company of India, Ltd. Dehli, 1977.

*Current address:* Laboratoire Analyse Géométrie et Applications, UMR 7539, Université Paris 13, 99 avenue J.B. Clément, F-93430 Villetaneuse, France

*E-mail address:* roy@math.univ-paris13.fr