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No blow-up of the 3D incompressible Euler equations

Léo Agélas *

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Abstract

One of the most challenging questions in fluid dynamics is whether the incompressible Euler equations can develop a finite-time singularity from smooth initial data. In this paper, we found that local geometry regularity of vorticity leads to a strong dynamic depletion of the nonlinear vortex stretching, thus avoiding finite-time singularity formation. Then, we prove the existence and uniqueness of global strong solutions in $C([0, +\infty[: W^{r,q}(\mathbb{R}^3)^3$ with $q > 1, r > 1 + \frac{2}{q}$, of the Euler equations as soon as the initial data $u_0 \in W^{r,q}(\mathbb{R}^3)^3$. This result gives a positive answer to the open problem about existence and smoothness of solutions of Euler equations.

1 Introduction

The Euler equations describe the motion of a fluid in $\mathbb{R}^3$. These equations have enriched many branches of mathematics, were involved in many area outside mathematical activity from weather prediction to exploding supernova (see [CONST]) and present important open physical and mathematical problems (see [CONST]). For Euler equations, non uniqueness of weak solutions has been demonstrated in [SCH], [SHN]. However, these solutions have not been obtained from a Leray solution of Navier-Stokes equations in the limit as the viscosity tends to zero (zero-viscosity limit of Navier-Stokes equations). Zero-viscosity limits of Navier-Stokes equations have only been shown to exist and to give Euler solutions in some more generalized sense known as measure-valued Euler solutions (see [DM]). Another notion of weak solutions to the Euler equations was introduced in [PLLIONS] called dissipative solutions and it was proven that they coincide with classical Euler solutions, when those exist. Recent studies indicate that the local geometric regularity of vortex lines can lead to dynamic depletion of vortex stretching [[CONST2],[CFM],[DHY],[DHY2]]. In particular, the recent results obtained in [DHY],[DHY2] show that geometric regularity of vortex lines, even in an extremely localized region containing the maximum vorticity can lead to depletion of nonlinear stretching, thus avoiding finite time singularity formation of the 3D Euler equations.

The blow-up problem for the Euler equations is a major open problem of PDE theory, of far greater physical importance than the blow-up problem for the Navier-Stokes equation, which of course is known to non-specialists because it is a Clay Millennium Problem.

It is known that if there are no singularities in the solution of the Euler equations with initial data $u_0$ on the time interval $[0,T]$, then there can be no singularities in the Navier-Stokes

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solution with the same initial data and small enough viscosity (see [CONST3]). The regularity for large enough viscosities is also known. Unfortunately, there is a gap between the two ranges of viscosities, and it is not clear how to close it.

The blow-up of smooth solutions to the Euler equations which is one of the most challenging mathematical problems in nonlinear PDE is controlled by the time integral of the maximum magnitude of the vorticity (see [BKM], [KP]). And this criterion have been refined in [CHAE] by replacing the $L^\infty$ norm of the vorticity by weaker norms close to the $L^\infty$ norm, more precisely, the blow-up of smooth solutions is only controlled by the time integral of the Besov space $B^0_{\infty,1}$ norm of two components of the vorticity.

Let us now introduce the Euler equations in $\mathbb{R}^3$ given by

$$
\begin{align*}
\begin{cases}
\frac{\partial u}{\partial t} + (u \cdot \nabla) u + \nabla p = 0, \\
\nabla \cdot u = 0,
\end{cases}
\end{align*}
$$

in which $u = u(x, t) = (u_1(x, t), u_2(x, t), u_3(x, t)) \in \mathbb{R}^3$, $p = p(x, t) \in \mathbb{R}$ denote respectively the unknown velocity field, the scalar pressure function of the fluid at the point $(x, t) \in \mathbb{R}^3 \times [0, \infty[$, with initial conditions,

$$
u(x, 0) = u_0(x) \text{ for a.e } x \in \mathbb{R}^3.
$$

In this paper, we prove that the maximum magnitude of the vorticity grows at most one exponential in time. For this, we consider the particle-trajectory mapping $X(\cdot, t) : \alpha \in \mathbb{R}^3 \mapsto X(\alpha, t) \in \mathbb{R}^3$, where $X(\alpha, t)$ is the location at time $t$ of a fluid particle initially placed at the point $\alpha$ at time $t = 0$ and such that the fluid velocity $u$ is tangent to the particle trajectory. After, we fix the time at $s > 0$ and we study the evolution in time from the integral of the initial fluid vorticity over an initial unit ball of fluid particles, $B$, to the integral of the vorticity at times $s$ over the domain $X(B, s)$, with the ball $B$ such that $X(B, s)$ contains a point where the maximum magnitude of the vorticity is reached at time $s$. The equation (E) expressing this evolution is obtained from the vorticity equation and by the fact that vorticity is stretched at time $t \in [0, s]$ by $\nabla_\alpha X(\alpha, t)$ along particle trajectories $(\omega(X(\alpha, t), t) = \nabla_\alpha X(\alpha, t) \omega_0(\alpha))$. After some manipulations, The equation (E) makes appear at the right hand side of the equality, the main obstruction in establishing the regularity of Euler solution, the stretching term which presents itself under the form $I = \int_\Omega u(y, t) f_i(y) dy$ such that $\|f_i\|_{L^\infty} \leq \|\omega_0\|_{L^\infty}$, $\int_\Omega f_i(y) dy = 0$ being the consequence that $\omega_0$ is a divergence-free vector field and $\Omega = X(B, t)$. By interpreting the term $I$ as the integral of the product between an element in $BMO_\ast(\Omega)$, the class of functions of bounded mean oscillation only on cubes inside $\Omega$ (see Section 3), $u(\cdot, t)$ and an element in $\mathcal{H}^1_1(\Omega)$, a hardy space on $\Omega$ (see Section 3), $f_i$, we are then able to use all the theory about Hardy spaces on domains that has been developed in recent years (see [JSW], [MIYA], [CKS] and [CDS]). After, using a duality result between $BMO_\ast(\Omega)$ and $\mathcal{H}^1_1(\Omega)$, we obtain a fine estimate of $I$ given by the bound $J = \|u(t)\|_{BMO_\ast(\Omega)} \|f_i\|_{\mathcal{H}^1_1(\Omega)}$. Thanks to Lemmata 3.1 and 3.2, the term $J$ is bounded by $C \|\omega(t)\|_{L^\infty} \|\omega_0\|_{L^\infty}$, where $C > 0$ is an absolute constant. Furthermore, on the left hand side of equality of (E), we have the derivative in time of the term $\int_{X(B, t)} \omega(y, t) dy$. After using an integration in time over $[0, s]$ of equation (E) and using the scaling property of Euler
equation, we infer the following inequality,
\[ \|\omega(s)\|_{L^\infty} \leq \|\omega_0\|_{L^\infty} + C \int_0^s \|\omega(t)\|_{L^\infty} \|\omega_0\|_{L^\infty} \, ds. \]

Thanks to Gronwall Lemma, we conclude that,
\[ \|\omega(s)\|_{L^\infty} \leq \|\omega_0\|_{L^\infty} \exp(C \|\omega_0\|_{L^\infty} \, s), \]
which proves that there does not exist finite time \( T > 0 \) for which \( \int_0^T \|\omega(s)\|_{L^\infty} \, ds = +\infty \).

From [BKM], we know that solution of Euler equation blow up at time \( T \) if and only if the time integral of the maximum magnitude of the vorticity diverges at time \( T \), \( \int_0^T \|\omega(t)\|_{L^\infty} \, dt = +\infty \), then we deduce that the incompressible Euler equations cannot develop a finite-time singularity from smooth initial data.

**Notations:** For any function \( \varphi \) defined on \( \mathbb{R}^3 \times [0, +\infty[ \), for all \( t \geq 0 \), we denote by \( \varphi(t) \) the function defined on \( \mathbb{R}^3 \) by \( x \mapsto \varphi(x, t) \). For any vector \( x = (x_1, x_2, x_3) \in \mathbb{R}^3 \), we denote by \( |x| \) the norm defined by \( |x| = \sqrt{\sum_{i=1}^3 x_i^2} \). For all vectors \( v = (v_i)_{1 \leq i \leq 3} \) and \( w = (w_j)_{1 \leq j \leq 3} \), we denote by \( v \otimes w \) the matrix with \( ij \)-elements given by \( (v_i w_j)_{1 \leq i, j \leq 3} \). For any bounded subset \( \Omega \subset \mathbb{R}^3 \), we denote by \( |\Omega| \) the measure of \( \Omega \) and by \( \text{diam}(\Omega) \) the diameter of \( \Omega \) defined by \( \text{diam}(\Omega) = \sup_{x,y \in \Omega} |x - y| \). For all \( \Omega \subset \mathbb{R}^3 \), we denote by \( \chi_\Omega \), the function such that \( \chi_\Omega(x) = 1, \forall x \in \Omega \) and \( \chi_\Omega(x) = 0, \forall x \notin \Omega \). For any \( x_0 \in \mathbb{R}^3 \) and \( r > 0 \), we denote by \( B(x_0, r) \), the ball of \( \mathbb{R}^3 \) of center \( x_0 \) and radius \( r \). For the sake of simplicity, for any \( q \geq 1 \), we denote by \( L^q \) (resp \( W^{r,q} \)) indifferently the Sobolev space \( L^q(\mathbb{R}^3)^3 \) (resp \( W^{r,q}(\mathbb{R}^3)^3 \)) and \( L^q(\mathbb{R}^3)^{3 \times 3} \) (resp \( W^{r,q}(\mathbb{R}^3)^{3 \times 3} \)). We denote by \( BC \) the class of bounded and continuous functions and by \( BC^m \) the class of bounded and \( m \) times continuously derivable functions. We denote by \( \text{div} \) the differential operator given by, \( \text{div} = \sum_{i=1}^3 \frac{\partial}{\partial x_i} \).

## 2 Preliminary

**Definition 2.1** If \( \Delta \) denotes the Laplace operator, the Bessel potential space \( L^p_\alpha \), \( 1 < p < \infty \), \( -\infty < \alpha < \infty \) can be defined as the space of functions (or distributions) \( f \) such that \( (I - \Delta)^{\frac{\alpha}{2}} f \) belongs to the Lebesgue space \( L^p \), normed by the corresponding Lebesgue norm. The operator \( (I - \Delta)^{\frac{\alpha}{2}} = \mathcal{G}_{-\alpha} \), which for \( \alpha > 0 \) is most easily defined by means of the Fourier transform. It corresponds, in fact, to multiplication of the Fourier transform \( \hat{f} \) of \( f \) by \( (1 + |\xi|)^{\frac{\alpha}{2}} \).

It is a theorem of A.P. Calderón that for positive integers \( \alpha \) and \( 1 < p < \infty \) the space coincides (with equivalence of norms) with the Sobolev space \( W^\alpha_\alpha \) of functions all of whose derivatives (in the distributional, or weak sense) of order at most \( \alpha \) are functions in \( L^p \).
For $\alpha > 0$ the elements of $L^p_\alpha$ are themselves $L^p$-functions, which can be represented as Bessel potentials of $L^p$-functions. In fact, the function $(1 + |\xi|^2)^{-\frac{\alpha}{2}}$ is then the Fourier transform of an integrable function, the Bessel kernel $G_\alpha(x)$. In other words, $f \in L^p_\alpha$, $1 < p < \infty$, $\alpha > 0$ if and only if there is a $g \in L^p$ such that $f(x) = \int G_\alpha(x-y)g(y)dy$, where the integral is taken over all of $\mathbb{R}^3$ with respect to the Lebesgue measure.

In what follows, we denote $W^{r,q}$ as the space $L^q_r$.

**Definition 2.2** We denote by $P$ the matrix Leray’s projection operator with components,

$$P_{i,j} = \delta_{i,j} - \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \Delta^{-1} = \delta_{i,j} - R_j R_k,$$

where $R_j$ are the Riesz transform given by $R_j = \frac{\partial}{\partial x_j} (-\Delta)^{-\frac{1}{2}} = \frac{1}{4\pi} \frac{x_j}{|x|^4}$ (see [ST]) for more details), $\Delta^{-1}$ is the inverse of Laplace operator given by $\Delta^{-1} = \frac{1}{4\pi |x|^4}$, with $*$ the convolution operator.

The operator $P$, which acts on vector-valued functions, is a projection: $P^2 = P$, annihilates gradients and maps into solenoidal (divergence-free) vectors; it is a bounded operator on (vector-valued) $L^p$, $1 < p < \infty$ and commutes with translation. We can notice that the operator $P$ can be written under the form,

$$P = I - \nabla \Delta^{-1} \text{div},$$

which yields to Helmholtz decomposition, indeed for all $v \in L^q(\mathbb{R}^3)^3$, $1 < q < \infty$,

$$v = Pv + \nabla q, \text{ with } \text{div } Pv = 0,$nabla q = \Delta^{-1} \text{div } v.$$

### 3 Hardy space and BMO space in bounded domain in $\mathbb{R}^3$

We recall the definition and some of the main properties of Hardy spaces $H^p(\mathbb{R}^3)$ introduced in [SW] (for more details on these spaces, see [FS], see also [STE2]).

**Definition 3.1** Let $0 < p < \infty$ and let $\Psi \in S(\mathbb{R}^3)$ the Schwartz class, satisfying $\int_{\mathbb{R}^3} \Psi(x) \, dx = 1$. A tempered distribution $f$ belongs to the Hardy space $H^p(\mathbb{R}^3)$ if,

$$f^*(x) = \sup_{t>0} |(\Psi_t * f)(x)| \in L^p(\mathbb{R}^3),$$

where $\Psi_t(x) = t^{-3}\Psi(t^{-1}x)$.

The (quasi)-norm of $H^p(\mathbb{R}^3)$ is defined, up to equivalence, by,

$$\|f\|_{H^p(\mathbb{R}^3)} = \|f^*\|_{L^p(\mathbb{R}^3)} \text{ for all } f \in H^p(\mathbb{R}^3).$$
In what follows, in Definition 3.1, we choose \( \Psi \in C_0^\infty(\mathbb{R}(0,1)) \) such that \( \int_{\mathbb{R}^3} \Psi(x) dx = 1 \).

Furthermore, an \( L^1 \)-function \( f \) on \( \mathbb{R}^3 \) belongs to \( \mathcal{H}^1(\mathbb{R}^3) \) if and only if its Riesz transforms \( R_j f \) all belong to \( L^1(\mathbb{R}^3) \) and
\[
\| f \|_{\mathcal{H}^1(\mathbb{R}^3)} \cong \| f \|_{L^1(\mathbb{R}^3)} + \sum_{j=1}^{3} \| R_j f \|_{L^1(\mathbb{R}^3)} \quad \text{(equivalent norms).} \tag{8}
\]

Notice that all function \( f \in \mathcal{H}^1(\mathbb{R}^3) \) satisfy
\[
\int_{\mathbb{R}^3} f(x) \, dx = 0.
\]

In [CKS], two Hardy spaces are defined on domains \( \Omega \) of \( \mathbb{R}^3 \), one which is reasonably speaking the largest, and the other which in a sense is the smallest. The largest, \( \mathcal{H}^1_r(\Omega) \), arises by restricting to \( \Omega \) arbitrary elements of \( \mathcal{H}^1(\mathbb{R}^3) \). The other, \( \mathcal{H}^1_z(\Omega) \), arises by restricting to \( \Omega \) elements of \( \mathcal{H}^1(\mathbb{R}^3) \) which are zero outside \( \Omega \). Norms on these spaces are defined as follows,
\[
\| f \|_{\mathcal{H}^1_r(\Omega)} = \inf \| F \|_{\mathcal{H}^1(\mathbb{R}^3)} \quad \text{the infimum being taken over all functions } F \in \mathcal{H}^1(\mathbb{R}^3) \text{ such that } F|\Omega = f,
\]
\[
\| f \|_{\mathcal{H}^1_z(\Omega)} = \| F \|_{\mathcal{H}^1(\mathbb{R}^3)} \quad \text{where } F \text{ is the zero extension of } f \text{ to } \mathbb{R}^3.
\]

From [CHA] (see also [AR], Theorem 1 (b2) combined with Theorem 5 (e)), the dual of \( \mathcal{H}^1_z(\Omega) \) is \( \text{BM} \Omega_r(\Omega) \), a space of locally integrable functions with
\[
\| f \|_{\text{BM} \Omega_r(\Omega)} = \sup_{Q \subset \Omega} \left( \frac{1}{|Q|} \int_Q |f(x) - f_Q| \, dx \right) < \infty, \tag{9}
\]
where \( f_Q = \frac{1}{|Q|} \int_Q f(y) \, dy \), and the supremum is taken over all cubes \( Q \) in the domain \( \Omega \).

**Lemma 3.1** Let \( \Omega \subset \mathbb{R}^3 \) a closed bounded domain and \( f \in \mathcal{H}^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3) \) such that \( \text{supp } f \subset \Omega \), then, we have,
\[
\| f \|_{\mathcal{H}^1(\mathbb{R}^3)} \leq 2|\Omega| \| f \|_{L^\infty(\mathbb{R}^3)} + \| f \|_{L^1(\mathbb{R}^3)}.
\]

**Proof.** The continuous embedding of \( \mathcal{H}^1(\mathbb{R}^3) \hookrightarrow L^1(\mathbb{R}^3) \) give us that \( f \in L^1(\mathbb{R}^3) \). Since \( \mathcal{B}(0, \epsilon) \to \{0\} \) as \( \epsilon \to 0 \), then, we can choose \( 0 < \delta < 1 \) such that,
\[
|\mathcal{B}(0, \delta) + \Omega| \leq 2|\Omega|. \tag{10}
\]

From Definition 3.1, we have,
\[
\| f \|_{\mathcal{H}^1(\mathbb{R}^3)} = \int_{\mathbb{R}^3} f^*(x) \, dx = \int_{\mathbb{R}^3} \sup_{t > 0} |(\Psi_t \ast f)(x)| \, dx \leq I + J, \tag{11}
\]
where,
\[
I = \int_{\mathbb{R}^3} \sup_{0 < t < \delta} |(\Psi_t \ast f)(x)| \, dx
\]
\[
J = \int_{\mathbb{R}^3} \sup_{t \geq \delta} |(\Psi_t \ast f)(x)| \, dx.
\]

Let us estimate the term \(I\). For all \(t > 0\), we have, \(\text{supp} \Psi_t \ast f \subset \text{supp} f + \text{supp} \Psi_t\). Since for all \(t > 0\), \(\text{supp} \Psi_t \subset \mathbb{B}(0, t)\), then, for all \(0 < t < \delta\), we have \(\text{supp} \Psi_t \subset \Omega + \mathbb{B}(0, \delta)\) and therefore \(\text{supp} \Psi_t \ast f \subset \Omega + \mathbb{B}(0, \delta)\). This implies that \(\text{supp} h \subset \Omega + \mathbb{B}(0, \delta)\), where \(h\) is the function defined by,
\[
h(x) = \sup_{0 < t < \delta} |(\Psi_t \ast f)(x)|, \quad \forall x \in \mathbb{R}^3.
\]
Moreover, for all \(0 < t < \delta\), we have \(\|\Psi_t \ast f\|_{L^\infty} \leq \|\Psi_t\|_{L^1} \|f\|_{L^\infty} = \|f\|_{L^\infty}\), which implies \(\|h\|_{L^\infty} \leq \|f\|_{L^\infty}\). Then, we obtain,
\[
I = \int_{\mathbb{R}^3} h(x) \, dx
\]
\[
= \int_{\Omega + \mathbb{B}(0, \delta)} h(x) \, dx \leq \|h\|_{L^\infty} (\Omega + \mathbb{B}(0, \delta))
\]
\[
\leq 2\|f\|_{L^\infty} |\Omega|,
\]

where we have used (10).

Let us now estimate the term \(J\). For this, we take \(N \geq 1\) and we bound the term \(J\) by two terms \(J_1\) and \(J_2\),
\[
J \leq J_1 + J_2,
\]
where,
\[
J_1 = \int_{\mathbb{R}^3} \sup_{\delta \leq t \leq N} \left| (\Psi_t \ast f)(x) \right| \, dx
\]
\[
J_2 = \int_{\mathbb{R}^3} \sup_{t > N} \left| (\Psi_t \ast f)(x) \right| \, dx.
\]

The term \(J_1\) is estimate as follows, since \(\Psi \in C_0^\infty(\mathbb{B}(0, 1))\), for all \(t > 0\), we have \(\Psi_t \in C_0^\infty(\mathbb{B}(0, t))\) and then for all \(\delta \leq t \leq N\), \(\Psi_t \in C_0^\infty(\mathbb{B}(0, N))\). Therefore, for all \(\delta \leq t \leq N\), we have \(\Psi_t \ast f \in C_0^\infty(\Omega + \mathbb{B}(0, N))\).

Moreover, by recalling \(\Psi_t = t^{-3}\Psi(t^{-1} \bullet)\), for all \(\delta \leq t \leq N\), for all \(x \in \mathbb{R}^3\), we have,
\[
\frac{\partial \Psi_t(x)}{\partial t} = -3t^{-4}\Psi(t^{-1} x) - t^{-5}(x \cdot \nabla \Psi)(t^{-1} x)
\]
and since \(\text{supp} \Psi \subset \mathbb{B}(0, 1)\), the support of the function \(x \mapsto (x \cdot \nabla \Psi)(t^{-1} x)\) is \(\mathbb{B}(0, t)\), then, we deduce for all \(\delta \leq t \leq N\),
\[
\left\| \frac{\partial \Psi_t}{\partial t} \right\|_{L^\infty} \leq 3t^{-4}\|\Psi\|_{L^\infty} + t^{-4}\|\nabla \Psi\|_{L^\infty}
\]
\[
\leq 3\delta^{-4}(\|\Psi\|_{L^\infty} + \|\nabla \Psi\|_{L^\infty})
\]
\[
= 3\delta^{-4}\|\Psi\|_{W^{1,\infty}}.
\]
Then, we obtain,

$$\sup_{\delta \leq t \leq N} \left\| \frac{\partial \Psi_t}{\partial t} \right\|_{L^\infty} \leq 3\delta^{-4} \|\Psi\|_{W^{1,\infty}},$$

which implies for all $\delta \leq t_0 < t_1 \leq N$,

$$\|\Psi_{t_1} - \Psi_{t_0}\|_{L^\infty} \leq \sup_{\delta \leq t \leq N} \left\| \frac{\partial \Psi_t}{\partial t} \right\|_{L^\infty} |t_1 - t_0| \leq 3\delta^{-4} \|\Psi\|_{W^{1,\infty}} |t_1 - t_0|.$$

Then, we deduce that for all $\delta \leq t < t' \leq N$,

$$\|\Psi_t * f - \Psi_{t'} * f\|_{L^\infty} = \| (\Psi_t - \Psi_{t'}) * f\|_{L^\infty} \leq \|\Psi_t - \Psi_{t'}\|_{L^\infty} \|f\|_{L^1} \leq 3\delta^{-4} \|\Psi\|_{W^{1,\infty}} \|f\|_{L^1} |t - t'|.$$

Let $\varepsilon > 0$ and let us introduce a subdivision of $[\delta, N]$ defined by a finite non-decreasing sequence of reals $(\tau_i)_{0 \leq i \leq n}$ such that $\tau_0 = \delta$, $\tau_n = N$ and for all $1 \leq i \leq n$, $|\tau_i - \tau_{i-1}| \leq \varepsilon$.

Since for all $\delta \leq t \leq N$, we have $\Psi_t * f \in C^0_0((\Omega + \mathbb{B}(0, N))$, then, we get,

$$J_1 = \int_{\Omega + \mathbb{B}(0, N)} \sup_{\delta \leq t \leq N} |(\Psi_t * f)(x)| \, dx.$$

For all $t \in [\delta, N]$, there exists $i_t \in [0, n] \cap \mathbb{N}$ such that $|t - \tau_{i_t}| \leq \varepsilon$, then we deduce for all $t \in [\delta, N]$ and $x \in \Omega + \mathbb{B}(0, N)$,

$$|(\Psi_t * f)(x)| \leq |(\Psi_t * f)(x) - (\Psi_{\tau_{i_t}} * f)(x)| + |(\Psi_{\tau_{i_t}} * f)(x)| \leq \|\Psi_t * f - \Psi_{\tau_{i_t}} * f\|_{L^\infty} + |(\Psi_{\tau_{i_t}} * f)(x)| \leq 3\delta^{-4} \|\Psi\|_{W^{1,\infty}} \|f\|_{L^1} |t - \tau_{i_t}| + |(\Psi_{\tau_{i_t}} * f)(x)| \leq 3\delta^{-4} \|\Psi\|_{W^{1,\infty}} \|f\|_{L^1} \varepsilon + \sup_{0 \leq i \leq n} |(\Psi_{\tau_i} * f)(x)|.$$

Therefore for all $x \in \Omega + \mathbb{B}(0, N)$,

$$\sup_{\delta \leq t \leq N} |(\Psi_t * f)(x)| \leq 3\delta^{-4} \|\Psi\|_{W^{1,\infty}} \|f\|_{L^1} \varepsilon + \sup_{0 \leq i \leq n} |(\Psi_{\tau_i} * f)(x)|.$$

Then, we obtain,

$$J_1 \leq 3\|\Omega + \mathbb{B}(0, N)\| \delta^{-4} \|\Psi\|_{W^{1,\infty}} \|f\|_{L^1} \varepsilon + \sup_{0 \leq i \leq n} \int_{\Omega + \mathbb{B}(0, N)} |(\Psi_{\tau_i} * f)(x)| \, dx.$$

Since, for all $0 \leq i \leq n$, we have,

$$\int_{\Omega + \mathbb{B}(0, N)} |(\Psi_{\tau_i} * f)(x)| \, dx \leq \int_{\mathbb{R}^3} |(\Psi_{\tau_i} * f)(x)| \, dx = \|\Psi_{\tau_i} * f\|_{L^1} \leq \|\Psi_{\tau_i}\|_{L^1} \|f\|_{L^1} = \|f\|_{L^1}.$$
Then, we deduce,

\[ J_1 \leq 3|\Omega + B(0,N)|\delta^{-4}\|\Psi\|_{W^{1,\infty}}\|f\|_{L^1} \varepsilon + \|f\|_{L^1}, \tag{15} \]

which is valid for all \( \varepsilon > 0 \), then taking the limit in (15) as \( \varepsilon \to 0 \), we get,

\[ J_1 \leq \|f\|_{L^1}. \tag{16} \]

It remains to estimate the term \( J_2 \), in fact, we have \( J_2 \to 0 \) as \( N \to \infty \). Indeed, for all \( t > N \), we have,

\[ \|\Psi_t \ast f\|_{L^\infty} \leq \|\Psi_t\|_{L^\infty}\|f\|_{L^1} \leq \frac{1}{t^3}\|\Psi\|_{L^\infty}\|f\|_{L^1}. \]

Then, we infer,

\[ \sup_{t>N}\|\Psi_t \ast f\|_{L^\infty} \leq \frac{1}{N^3}\|\Psi\|_{L^\infty}\|f\|_{L^1}. \tag{17} \]

We introduce the sequence of functions \((f_N)_{N>1}\) defined for all \( x \in \mathbb{R}^3 \) by,

\[ f_N(x) = \sup_{t>N}|(\Psi_t \ast f)(x)|. \]

Then, thanks to (17), we have for all \( x \in \mathbb{R}^3 \),

\[ f_N(x) \to 0 \text{ as } N \to \infty. \tag{18} \]

Since, for all \( x \in \mathbb{R}^3 \), \( 0 \leq f_N(x) \leq \hat{f}^+(x) \) and \( \hat{f}^+ \in L^1(\mathbb{R}^3) \) due to the fact that \( f \in \mathcal{H}^1(\mathbb{R}^3) \), therefore, thanks to Lebesgue’s dominated convergence Theorem, we deduce,

\[ \int_{\mathbb{R}^3} f_N(x) \, dx \to 0 \text{ as } N \to \infty, \]

which means,

\[ J_2 \to 0 \text{ as } N \to \infty. \tag{19} \]

Then, thanks to (11), (13), (14), (16) and (19), we deduce,

\[ \|f\|_{\mathcal{H}^1(\mathbb{R}^3)} \leq 2|\Omega|\|f\|_{L^\infty} + \|f\|_{L^1}, \]

which conclude the proof.

\[ \square \]

**Littlewood-Paley decomposition**: To prove Lemma 3.2, we need to introduce the usual dyadic unity partition of Littlewood-Paley decomposition. To this end, we take an arbitrary radial function \( \Phi \) in the Schwartz class \( \mathcal{S}(\mathbb{R}^3) \) whose Fourier transform \( \hat{\Phi} \) is such that,

\[ \text{supp} \hat{\Phi} \subset \{ \xi \in \mathbb{R}^3, |\xi| \leq 1 \} \text{ and } \hat{\Phi}(\xi) \geq \frac{1}{2} \text{ for } |\xi| \leq \frac{5}{6}. \]

We choose also a radial function \( \varphi \) in the Schwartz class \( \mathcal{S}(\mathbb{R}^3) \) whose Fourier transform \( \hat{\varphi} \) is such that,

\[ \text{supp} \hat{\varphi} \subset \{ \xi \in \mathbb{R}^3, \frac{1}{2} \leq |\xi| \leq 2 \} \text{ and } \hat{\varphi}(\xi) \geq \frac{1}{2} \text{ for } \frac{3}{5} \leq |\xi| \leq \frac{5}{3}, \]
and define $\varphi_j(x) = 2^{3j} \varphi(2^j x)$ so that $\hat{\varphi}_j(\xi) = \hat{\varphi}(2^{-j} \xi)$ for $j \in \mathbb{Z}$. We may assume,

$$\forall \xi \in \mathbb{R}^3, \hat{\Phi}(\xi) + \sum_{j \geq 0} \hat{\varphi}_j(\xi) = 1$$

For any $\xi \in \mathbb{R}^3$, we denote by,

$$\Delta_{-1} f = \mathcal{F}^{-1}(\hat{\Phi} \hat{f}) = \Phi \ast f,$$

$$\Delta_j f = \mathcal{F}^{-1}(\hat{\varphi}_j \hat{f}) = \varphi_j \ast f$$ for all $j \geq 0$.

Lemma 3.2 Let $u \in \mathbb{P}^w(r, s) \hookrightarrow BC(\mathbb{R}^3)$ for all $x \in \mathbb{R}^3$ and $y \in \mathbb{R}^3$, there exists an absolute constant $C > 0$ such that,

$$|u(x) - u(y)| \leq C \|\omega\|_{L^\infty} |x - y|,$$

where $\omega = \nabla \times u$ is the vorticity of $u$.

Proof. Thanks to the Sobolev embedding,

$$W^m, s(\mathbb{R}^3) \hookrightarrow BC(\mathbb{R}^3) \text{ for all } m > \frac{3}{s},$$

we have $u \in BC^1(\mathbb{R}^3)$. By the Biot-Savart law (see Proposition 1.3.1 in [CHE]), we have for all $x \in \mathbb{R}^3$,

$$u(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{y}{|y|^3} \times \omega(x + y) \, dy. \quad (20)$$

Denoting $K$ the kernel being in the Biot-Savart law (20), we re-write (20) as $u = K \ast \omega$.

We have also,

$$\frac{\partial u}{\partial x_j} = R_j(R \times \omega) \text{ for } j = 1, 2, 3, \quad (21)$$

where $R = (R_1, R_2, R_3)$ and $R_j$ are the Riesz transform.

Thanks to Littlewood-Paley decomposition, we have,

$$u = \sum_{j=-1}^{\infty} \Delta_j u. \quad (22)$$

Thanks to Bernstein inequalities (see Chapter 3 [LEM], see also [MEY]), there exists a constant $C > 0$ such that for all $j \in \mathbb{N}$,

$$\|\Delta_j u\|_{L^\infty} \leq C 2^{-j} \|\Delta_j \nabla u\|_{L^\infty}. \quad (23)$$

Thanks to (21) and using the same arguments as the proof of (iii) in [MIY], pages 439-440, we get that there exists a constant $C_2 > 0$ such that for all $j \in \mathbb{N}$,

$$\|\Delta_j \nabla u\|_{L^\infty} = \|\Delta_j (R \otimes (R \times \omega))\|_{L^\infty} \leq C_2 \|\Delta_j \omega\|_{L^\infty}. \quad (24)$$

Then, using (23) and (24), we obtain that there exists a constant $C_1 > 0$ such that for all $j \in \mathbb{N}$,

$$\|\Delta_j u\|_{L^\infty} \leq C_1 2^{-j} \|\Delta_j \omega\|_{L^\infty}. \quad (25)$$
We borrow some arguments used in [VIS] to obtain our proof. Let $x \in \mathbb{R}^3, y \in \mathbb{R}^3$ and $3 < p < \infty$, First case: if $|x - y| < 1$, then, thanks to (22), (24) and (25), we get that for all $N \geq -1$,

$$|u(x) - u(y)| \leq \sum_{j=-1}^{N} |\Delta_j u(x) - \Delta_j u(y)| + 2 \sum_{j=N+1}^{\infty} \|\Delta_j u\|_{L^\infty}$$

$$\leq \sum_{j=-1}^{N} \|\Delta_j \nabla u\|_{L^\infty} |x - y| + 2C_1 \sum_{j=N+1}^{\infty} 2^{-j} \|\Delta_j \omega\|_{L^\infty}$$

$$\leq \|\Delta_{-1} \nabla u\|_{L^\infty} |x - y| + C_2 \sum_{j=0}^{N} \|\Delta_j \omega\|_{L^\infty} |x - y| + 2C_1 \sum_{j=N+1}^{\infty} 2^{-j} \|\Delta_j \omega\|_{L^\infty}.$$  

Thanks to Young inequality (see Theorem IV.30 in [BRE]), there exists a real $c_\phi > 0$ depending only on $p, \Phi$ such that,

$$\|\Delta_{-1} \nabla u\|_{L^\infty} = \|\Phi \ast \nabla u\|_{L^\infty} \leq c_\phi \|\nabla u\|_{L^p}. \quad (26)$$

Thanks to Theorem 3.1.1 in [CHE], there exists a real $d_\phi > 0$ depending only on $p, \Phi$ such that,

$$\|\nabla u\|_{L^p} \leq d_\phi \|\omega\|_{L^p}. \quad (27)$$

Using (26) and (27), there exists a real $C_\phi > 0$ depending only on $p, \Phi$ such that,

$$\|\Delta_{-1} \nabla u\|_{L^\infty} \leq C_\phi \|\omega\|_{L^p}. \quad (28)$$

Let $3 < q < \infty$ and $q' > 1$ such that $\frac{1}{q} + \frac{1}{q'} = 1$, Thanks to Young inequality (see Theorem IV.30 in [BRE]), we have for $j \geq 0$,

$$\|\Delta_j \omega\|_{L^\infty} = \|\varphi_j \ast \omega\|_{L^\infty} \leq \|\varphi_j\|_{L^{q'}} \|\omega\|_{L^q} = 2^{\frac{j}{q'}} \|\varphi\|_{L^{q'}} \|\omega\|_{L^q} \quad (29)$$

Thanks to (28) and (29), we deduce,

$$|u(x) - u(y)| \leq C_\phi \|\omega\|_{L^p} |x - y| + C_2 \|\varphi\|_{L^{q'}} \|\omega\|_{L^q} |x - y| \sum_{j=0}^{N} 2^{\frac{2j}{q'}}$$

$$+ 2C_1 \|\varphi\|_{L^{q'}} \|\omega\|_{L^q} \sum_{j=N+1}^{\infty} 2^{-j(1 - \frac{2}{q'})}$$

$$= C_\phi \|\omega\|_{L^p} |x - y| + \|\varphi\|_{L^{q'}} \|\omega\|_{L^q} (C_2 |x - y| 2^{\frac{2}{q}(N+1)} + 2C_1 2^{-(1 - \frac{2}{q})(N+1)})$$

We choose $N + 1 = [-\log_2 |x - y|]$ and since $q > 3$, we deduce that there exists a constant $C_3 > 0$ such that,

$$|u(x) - u(y)| \leq C_\phi \|\omega\|_{L^p} |x - y| + C_3 \|\varphi\|_{L^{q'}} \|\omega\|_{L^q} |x - y|^{1 - \frac{2}{q}}. \quad (30)$$
Then, taking the limit as \( q \to \infty \) in (30), we obtain,

\[
|u(x) - u(y)| \leq C_p \omega \|L_p| |x - y| + C_3 \|\varphi\|_{L^1} \|\omega\|_{L^\infty} |x - y|.
\]  

(31)

Second case: if \(|x - y| \geq 1\), thanks to (22) and (25), we have,

\[
|u(x) - u(y)| \leq |\Delta_{-1} u(x) - \Delta_{-1} u(y)| + 2 \sum_{j \geq 0} \|\Delta_j u\|_{L^\infty}
\]  

\[
\leq \|\Delta_{-1} \nabla u\|_{L^\infty} |x - y| + 2 C_1 \sum_{j \geq 0} 2^{-j} \|\Delta_j \omega\|_{L^\infty}
\]  

\[
\leq \|\Delta_{-1} \nabla u\|_{L^\infty} |x - y| + 2 C_1 \sum_{j \geq 0} \|\omega\|_{L^\infty} \|\varphi_j\|_{L^1} 2^{-j}
\]  

\[
\leq C_p \omega \|L_p| |x - y| + 4 C_1 \|\varphi\|_{L^1} \|\omega\|_{L^\infty},
\]

by using (28) and the fact that \( \|\varphi_j\|_{L^1} = \|\varphi\|_{L^1} \). Since \(|x - y| \geq 1\), we infer,

\[
|u(x) - u(y)| \leq C_p \omega \|L_p| |x - y| + 4 C_1 \|\varphi\|_{L^1} \|\omega\|_{L^\infty} |x - y|.
\]  

(32)

Then, using (31) and (32), we deduce that there exists a real \( C_p > 0 \) depending only on \( p, \Phi \) and an constant \( C_4 > 0 \) such that for all \( x \in \mathbb{R}^3 \) and \( y \in \mathbb{R}^3 \), we have,

\[
|u(x) - u(y)| \leq C_p \omega \|L_p| |x - y| + C_4 \|\varphi\|_{L^1} \|\omega\|_{L^\infty} |x - y|.
\]  

(33)

Now, we use the scaling of Inequality (33), for this, we consider the function \( u_\lambda = u(\lambda \bullet) \) with \( \lambda > 0 \) a non-negative real. Since \( \nabla \cdot u = 0 \), then we have also \( \nabla \cdot u_\lambda = 0 \) and \( \omega_\lambda \) the vorticity of \( u_\lambda \) is given by \( \omega_\lambda = \lambda \omega(\lambda \bullet) \).

Let \( x \in \mathbb{R}^3 \) and \( y \in \mathbb{R}^3 \). We set \( x_\lambda = \frac{x}{\lambda} \) and \( y_\lambda = \frac{y}{\lambda} \), then using (33) with \( u_\lambda \) instead of \( u \), we obtain,

\[
|u_\lambda(x_\lambda) - u_\lambda(y_\lambda)| \leq C_p \omega_\lambda \|L_p| |x_\lambda - y_\lambda| + C_4 \|\varphi\|_{L^1} \|\omega_\lambda\|_{L^\infty} |x_\lambda - y_\lambda|.
\]  

(34)

Using the expression of \( u_\lambda, \omega_\lambda, x_\lambda \) and \( y_\lambda \), from (34), we deduce,

\[
|u(x) - u(y)| \leq C_p \lambda^{\frac{3}{2}} \|\omega\|_{L_p} |x - y| + C_4 \|\varphi\|_{L^1} \|\omega\|_{L^\infty} |x - y|.
\]  

(35)

Then, taking the limit as \( \lambda \to \infty \) in (35), we deduce,

\[
|u(x) - u(y)| \leq C_4 \|\varphi\|_{L^1} \|\omega\|_{L^\infty} |x - y|.
\]  

(36)

Since also \( \varphi \) is fixed, we conclude the proof.

\( \square \)

4 Global regularity of solution of Euler equations

Using \( \mathbb{P} \) the matrix Leray operator, Euler equations (1)-(2) can be re-written as follows,

\[
\frac{\partial u}{\partial t} + \mathbb{P}(u \cdot \nabla)u = 0,
\]  

(37)
with initial condition,
\[ u(0) = u_0. \]  
(38)

For \( u \) solution of (37)-(38), \( \omega = \nabla \times u \), the vorticity of \( u \), formally satisfies the vorticity equation,
\[ \frac{\partial \omega}{\partial t} + (u \cdot \nabla) \omega - (\omega \cdot \nabla) u = 0, \]  
(39)
with initial conditions,
\[ \omega(0) = \omega_0, \]  
(40)

where \( \omega_0 = \nabla \times u_0 \) is the vorticity of \( u_0 \).

Let us give now the proof of our Theorem.

**Theorem 4.1** Let \( u_0 \in \text{PW}^{r,q}(\mathbb{R}^3)^3 \) with \( 1 < q < \infty, \ r > 2 + \frac{3}{q} \), then there exists an unique strong solution \( u \in C([0, +\infty[, \text{PW}^{r,q}(\mathbb{R}^3))^3 \cap C^1([0, +\infty[, \text{PW}^{r-1,q}(\mathbb{R}^3))^3 \) to Euler equations (37)-(38) and moreover, there exists a constant \( C > 0 \) such that for all \( t \geq 0 \),
\[ \|\omega(t)\|_{L^\infty} \leq \|\omega_0\|_{L^\infty} \exp(C\|\omega_0\|_{L^\infty} t). \]

**Proof.** Since \( u_0 \in \text{PW}^{r,q}(\mathbb{R}^3)^3 \) and thanks to Theorem 1 in [KP2], Theorem 3.5 and 4.7 in [KP], Theorem 1 in [BB] or the results obtained in [BKM], we deduce that there exists a maximal time of existence strictly positive \( T^* > 0 \) such that there exists an unique strong solution \( u \in C([0, T^*[, \text{PW}^{r,q}(\mathbb{R}^3))^3 \) to the Euler equations (37)-(38) and if \( T^* < +\infty \), then we get,
\[ \int_0^{T^*} \|\omega(t)\|_{L^\infty} dt = +\infty. \]  
(41)

Since \( u \in C([0, T^*[, \text{PW}^{r,q}(\mathbb{R}^3))^3 \), the operator \( \mathbb{P} \) a bounded operator in \( L^q(\mathbb{R}^3) \) and thanks to Lemma X4 in [KP], from Euler equation (37), we deduce that \( u \in C^1([0, T^*[, \text{PW}^{r-1,q}(\mathbb{R}^3))^3 \).

From Helmholtz decomposition (5), we retrieve the pressure \( p \) from the velocity \( u \) with the formula,
\[ p = -\Delta^{-1}\text{div}(u \cdot \nabla)\)u). \]

Let us assume that \( T^* < +\infty \). We introduce \( T \) a non-negative real such that \( 0 < T < T^* \).

Thanks to Sobolev embedding,
\[ W^{m,q}(\mathbb{R}^3) \hookrightarrow BC(\mathbb{R}^3) \text{ for all } m > \frac{3}{q}, \]  
(42)

and since \( u \in C([0, T], W^{r,q}(\mathbb{R}^3))^3 \cap C^1([0, T], \text{PW}^{r-1,q}(\mathbb{R}^3))^3 \) with \( r > 1 + \frac{3}{q} \), we deduce that \( u \in BC^1([0, T] \times \mathbb{R}^3)^3 \).

Equation (1) represents the Eulerian description of the flow. The Lagrangian formulation of Euler equations (1) describes the flow in term of a volume preserving diffeomorphism, the time dependent map \( X : \mathbb{R}^3 \longrightarrow \mathbb{R}^3, \)
\[ \alpha \longmapsto X(\alpha, t), \quad X(\alpha, 0) = \alpha. \]
These maps represent marked fluid particle trajectories, $\alpha$ is the label of the particle, which can be seen as the location of the particle at time $t = 0$. The fact that the particle travel with velocity $u$ is expressed in the system of ordinary differential equations,

$$\frac{\partial}{\partial t}X(\alpha, t) = u(X(\alpha, t), t), \quad X(\alpha, 0) = \alpha. \quad (43)$$

The Lagrangian formulation of Euler equations and incompressibility condition given by (1) are respectively the Newton’s second law $\frac{\partial^2}{\partial t^2}X + \nabla p(X, t) = 0$ and $\det(\nabla_\alpha X) = 1$ (see Proposition 1.4 in [MB]), where $\det(\nabla_\alpha X)$ denotes the determinant of $\nabla_\alpha X$ the jacobian matrix of $X$.

The map $X$ defined by Equation (43) is well a volume preserving $C^1$-diffeomorphism from $\mathbb{R}^3$ on itself, indeed, since $u \in BC^1([0, T] \times \mathbb{R}^3)^3$, then thanks to Cauchy-Lipschitz Theorem, we deduce that the differential equation (43) admits a unique solution $X \in C^1(\mathbb{R}^3 \times [0, T])^3$ and for all $t \in [0, T]$, the map $X(t)$ is a $C^1$–diffeomorphism from $\mathbb{R}^3$ on itself (see Lemma 2.3.2 in [LERN] in particular the end of its proof, see also Theorem 2.10 combined with Theorem 2.17 in [TES]).

Thanks to Proposition 1.4 in [MB], we have for all $t \in [0, T]$,

$$\text{the mapping } X(t) : \alpha \mapsto X(\alpha, t) \text{ is volume preserving},$$

$$\text{and } \det(\nabla_\alpha X(\alpha, t)) = 1. \quad (44)$$

Gathering these results, we obtain the desired result, for any $t \in [0, T]$, the mapping

$$X(t) \text{ is a volume preserving } C^1 \text{ – diffeomorphism from } \mathbb{R}^3 \text{ on itself.} \quad (45)$$

Let us give an estimate about the length of any segment transformed during the time by the application $X$.

Let $e_0 = (1, 1, 1)$, from (43), we have for all $t \in [0, T]$ and $\alpha \in \mathbb{R}^3$,

$$\nabla_\alpha X(\alpha, t) = e_0 + \int_0^t \nabla u(X(\alpha, \tau), \tau) \nabla_\alpha X(\alpha, \tau) \, d\tau. \quad (46)$$

Then, we deduce from (46), for all $t \in [0, T]$,

$$|\nabla_\alpha X(\alpha, t)| \leq \sqrt{3} + \int_0^t \|\nabla u(\tau)\|_{L^\infty} |\nabla_\alpha X(\alpha, \tau)| \, d\tau.$$

Then, thanks to Gronwall Lemma, we obtain for all $t \in [0, T]$ and $\alpha \in \mathbb{R}^3$,

$$|\nabla_\alpha X(\alpha, t)| \leq \sqrt{3} \exp \left( \int_0^t \|\nabla u(\tau)\|_{L^\infty} \, d\tau \right).$$

Therefore, we get for all $t \in [0, T]$,

$$\|\nabla_\alpha X(t)\|_{L^\infty} \leq \sqrt{3} \exp \left( \int_0^t \|\nabla u(\tau)\|_{L^\infty} \, d\tau \right). \quad (47)$$

Let us show now that for all $t \in [0, T]$, the maximum of the magnitude of the vorticity $\omega(t)$ is reached for some points of $\mathbb{R}^3$. 

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Let us fix \( s \in \mathbb{R} \), we have for all \( \alpha \)

\[
\omega(x, t) \rightarrow 0 \quad \text{as} \quad |x| \rightarrow +\infty.
\]

For all \( t \in [0, T] \), the application \( x \mapsto |\omega(x, t)| \) is a continuous function from \( \mathbb{R}^3 \) to \([0, +\infty[\) and since \( |\omega(x, t)| \rightarrow 0 \) as \( |x| \rightarrow +\infty \), therefore we deduce that the function \( x \mapsto |\omega(x, t)| \) reaches its maximum value for some \( x(t) \in \mathbb{R}^3 \).

Then, we have for all \( t \in [0, T] \),

\[
|\omega(x(t), t)| = \|\omega(t)\|_{L^\infty(\mathbb{R}^3)}.
\]

For any \( s \in [0, T] \), due to (45), there exists an unique \( \alpha(s) \in \mathbb{R}^3 \) such that the flow \( X \) starting from \( \alpha(s) \) (which means \( X(\alpha(s), 0) = \alpha(s) \)) pass through the particle \( x(s) \) at time \( s \), that means \( X(\alpha(s), s) = x(s) \).

It is well known that for 3D Euler flows (see Proposition 1.8 in [MB] or Proposition page 24 in [CM]), we have for all \( t \in [0, T] \) and \( \alpha \in \mathbb{R}^3 \),

\[
\omega(X(\alpha, t), t) = \nabla_\alpha X(\alpha, t) \omega_0(\alpha).
\]

Let us fix \( s \in [0, T] \) and \( x(s) \) such that (48) holds. Let \( R \) a non-negative real such that \( 0 < R \leq 1 \) and let \( B(\alpha(s), R) \) be the ball of center \( \alpha(s) \) with radius \( R \).

For any \( t \in [0, T] \), we introduce the subset of \( \mathbb{R}^3 \), \( V_{R,s}(t) = X(B(\alpha(s), R), t) \), thanks to (44), we have the conservation of volume,

\[
|V_{R,s}(t)| = |B(\alpha(s), R)|.
\]

For all \( 0 \leq t \leq s \), by using the change of variables \( y = X(\alpha, t) \) with \( \alpha \in B(\alpha(s), R) \) and since \( \det(\nabla_\alpha X(\alpha, t)) = 1 \), we deduce,

\[
\int_{V_{R,s}(t)} \omega(y, t) \, dy = \int_{B(\alpha(s), R)} \omega(X(\alpha, t), t) \, d\alpha.
\]

Therefore, thanks to (51), (43) and (39) we deduce that for all \( 0 \leq t \leq s \),

\[
\frac{d}{dt} \int_{V_{R,s}(t)} \omega(y, t) \, dy = \int_{B(\alpha(s), R)} \frac{d}{dt} \omega(X(\alpha, t), t) \, d\alpha
\]

\[
= \int_{B(\alpha(s), R)} ((\omega \cdot \nabla)u)(X(\alpha, t), t) \, d\alpha.
\]

Since \( (\omega \cdot \nabla)u = \nabla u \cdot \omega \) and thanks to (49), we deduce,

\[
((\omega \cdot \nabla)u)(X(\alpha, t), t) = \nabla u(X(\alpha, t), t) \cdot \nabla X(\alpha, t) \omega_0(\alpha)
\]

\[
= \nabla(u(X(\alpha, t), t)) \omega_0(\alpha).
\]

Then, using (52), (53) and an integration by parts, we obtain for all \( 0 \leq t \leq s \),

\[
\frac{d}{dt} \int_{V_{R,s}(t)} \omega(y, t) \, dy = \int_{B(\alpha(s), R)} \nabla(u(X(\alpha, t), t)) \omega_0(\alpha) \, d\alpha
\]

\[
= - \int_{B(\alpha(s), R)} (u(X(\alpha, t), t)) \cdot \nabla \omega_0(\alpha) \, d\alpha
\]

\[
+ \int_{\partial B(\alpha(s), R)} u(X(\alpha, t), t) \omega_0(\alpha) \cdot n(\alpha) \, d\gamma(\alpha),
\]
where \( \mathbf{n}(\alpha) \) is the unit normal outward to \( B(\alpha(s), R) \) in \( \alpha \in \partial B(\alpha(s), R) \).

We can notice that the unit normal outward to any ball of center \( \alpha(s) \) in a point \( \alpha \) on its boundary, can be expressed as follows,

\[
\mathbf{n}(\alpha) = \frac{\alpha - \alpha(s)}{|\alpha - \alpha(s)|},
\]

(55)

For \( \alpha = \alpha(s) \), we set,

\[
\mathbf{n}(\alpha) = (1, 0, 0).
\]

(56)

In what follows, \( \mathbf{n} \) is considered as a function from \( \mathbb{R}^3 \) to \( \mathbb{R}^3 \) defined by (55) and (56).

Since \( \nabla \cdot \omega_0 = 0 \), from (54), we obtain for all \( 0 \leq t \leq s \),

\[
\frac{d}{dt} \int_{V_{R,s}(t)} \omega(y, t) \, dy = \int_{\partial B(\alpha(s), R)} u(X(\alpha, t), t) \, \omega_0(\alpha) \cdot \mathbf{n}(\alpha) \, d\gamma(\alpha).
\]

(57)

We integrate Equation (57) over \( t \in [0, s] \) and we get,

\[
\int_{V_{R,s}(s)} \omega(y, s) \, dy = \int_{V_{R,s}(0)} \omega(y, 0) \, dy + \int_0^s \int_{\partial B(\alpha(s), R)} u(X(\alpha, t), t) \, \omega_0(\alpha) \cdot \mathbf{n}(\alpha) \, d\gamma(\alpha) \, dt.
\]

(58)

We notice that \( V_{R,s}(0) = B(\alpha(s), R) \) and \( \omega(y, 0) = \omega_0(y) \) for all \( y \in \mathbb{R}^3 \), then we obtain,

\[
\int_{V_{R,s}(s)} \omega(y, s) \, dy = \int_{B(\alpha(s), R)} \omega_0(y) \, dy + \int_0^s \int_{\partial B(\alpha(s), R)} u(X(\alpha, t), t) \, \omega_0(\alpha) \cdot \mathbf{n}(\alpha) \, d\gamma(\alpha) \, dt.
\]

(59)

We integrate Inequality (59) over \( R \in [0, 1] \) and we obtain,

\[
\int_0^1 \int_{V_{R,s}(s)} \omega(y, s) \, dy \, dR = \int_0^1 \int_{B(\alpha(s), R)} \omega_0(y) \, dy \, dR
\]

\[
+ \int_0^s \int_0^1 \int_{\partial B(\alpha(s), R)} u(X(\alpha, t), t) \, \omega_0(\alpha) \cdot \mathbf{n}(\alpha) \, d\gamma(\alpha) \, dR \, dt.
\]

(60)

For all \( t \in [0, s] \), we have,

\[
\int_{\partial B(\alpha(s), R)} u(X(\alpha, t), t) \, \omega_0(\alpha) \cdot \mathbf{n}(\alpha) \, d\gamma(\alpha) = \frac{\partial}{\partial R} \left( \int_{B(\alpha(s), R)} u(X(\alpha, t), t) \, \omega_0(\alpha) \cdot \mathbf{n}(\alpha) \, d\alpha \right).
\]

(61)

Then, we deduce,

\[
\int_0^1 \int_{\partial B(\alpha(s), R)} u(X(\alpha, t), t) \, \omega_0(\alpha) \cdot \mathbf{n}(\alpha) \, d\gamma(\alpha) \, dR = \int_{B(\alpha(s), 1)} u(X(\alpha, t), t) \, \omega_0(\alpha) \cdot \mathbf{n}(\alpha) \, d\alpha.
\]

(62)

Owing to (62), from (60), for all \( t \in [0, s] \), we obtain,

\[
\int_0^1 \int_{V_{R,s}(s)} \omega(y, s) \, dy \, dR = \int_0^1 \int_{B(\alpha(s), R)} \omega_0(y) \, dy \, dR
\]

\[
+ \int_0^s \int_0^1 \int_{B(\alpha(s), 1)} u(X(\alpha, t), t) \, \omega_0(\alpha) \cdot \mathbf{n}(\alpha) \, d\gamma(\alpha) \, dR \, dt.
\]

(63)
Therefore, from (63), we deduce,
\[
\left| \int_0^1 \int_{V_{R,s}(s)} \omega(y, s) \, dy \, dR \right| \leq \|\omega_0\|_{L^\infty} \int_0^1 |B(\alpha(s), R)| \, dR \\
+ \int_0^s \int_{B(\alpha(s), 1)} u(X(\alpha, t), t) \, \omega_0(\alpha) \cdot \mathbf{n}(\alpha) \, d\gamma(\alpha) \, dt.
\]

On one hand, we have,
\[
\left| \int_0^1 \int_{V_{R,s}(s)} \omega(y, s) \, dy \, dR \right|
= \int_0^1 \int_{V_{R,s}(s)} \omega(X(\alpha(s), s), s) \, dy \, dR + \int_0^1 \int_{V_{R,s}(s)} \omega(y, s) - \omega(X(\alpha(s), s), s) \, dy \, dR
\geq \int_0^1 \int_{V_{R,s}(s)} \omega(y, s) \, dy \, dR - \int_0^1 \int_{V_{R,s}(s)} |\omega(y, s) - \omega(X(\alpha(s), s), s)| \, dy \, dR
\geq |\omega(X(\alpha(s), s), s)| \int_0^1 |V_{R,s}(s)| \, dR \geq |\omega(X(\alpha(s), s), s)| \int_0^1 \int_{V_{R,s}(s)} |\omega(y, s) - \omega(X(\alpha(s), s), s)| \, dy \, dR.
\]

On the other hand, we have for all \( y \in V_{R,s}(s) \),
\[
|\omega(X(\alpha(s), s), s) - \omega(y, s)| \leq \|\nabla \omega(s)\|_{L^\infty} \text{diam}(V_{R,s}(s))
\]
and also \(|\omega(X(\alpha(s), s), s)| = |\omega(x(s), s)| = \|\omega(s)\|_{L^\infty}\), then from (65), we deduce,
\[
\left| \int_0^1 \int_{V_{R,s}(s)} \omega(y, s) \, dy \, dR \right| \geq \|\omega(s)\|_{L^\infty} \int_0^1 |V_{R,s}(s)| \, dR \\
- \|\nabla \omega(s)\|_{L^\infty} \int_0^1 \text{diam}(V_{R,s}(s)) |V_{R,s}(s)| \, dR.
\]

From (64) and thanks to (66), we deduce,
\[
\|\omega(s)\|_{L^\infty} \int_0^1 |V_{R,s}(s)| \, dR \leq \|\omega_0\|_{L^\infty} \int_0^1 |B(\alpha(s), R)| \, dR \\
+ \|\nabla \omega(s)\|_{L^\infty} \int_0^1 \text{diam}(V_{R,s}(s)) |V_{R,s}(s)| \, dR \\
+ \int_0^s \int_{B(\alpha(s), 1)} u(X(\alpha, t), t) \, \omega_0(\alpha) \cdot \mathbf{n}(\alpha) \, d\gamma(\alpha) \, dt.
\]

Thanks to (50), we have \(|V_{R,s}(s)| = |B(\alpha(s), R)| = \frac{4\pi}{3} R^3\) and since \(\int_0^1 \frac{4\pi}{3} R^3 \, dR = \frac{\pi}{3}\), then, from (67), we obtain,
\[
\frac{\pi}{3} \|\omega(s)\|_{L^\infty} \leq \frac{\pi}{3} \|\omega_0\|_{L^\infty} + \|\nabla \omega(s)\|_{L^\infty} \int_0^1 \text{diam}(V_{R,s}(s)) \frac{4\pi}{3} R^3 \, dR \\
+ \int_0^s \int_{B(\alpha(s), 1)} u(X(\alpha, t), t) \, \omega_0(\alpha) \cdot \mathbf{n}(\alpha) \, d\gamma(\alpha) \, dt.
\]
Let us give now an estimate of \( \text{diam}(V_{R,s}(s)) \). We recall \( V_{R,s}(s) = X(B(\alpha(s), R), s) \), then we can write \( \text{diam}(V_{R,s}(s)) \) as follows,

\[
\text{diam}(V_{R,s}(s)) = \sup_{\alpha, \beta \in B(\alpha(s), R)} |X(\alpha, s) - X(\beta, s)|.
\]

(69)

Thanks to (47), we deduce,

\[
\text{diam}(V_{R,s}(s)) \leq 2\sqrt{3}R \exp \left( \int_0^s \| \nabla u(\tau) \|_{L^\infty} \, d\tau \right).
\]

(70)

Therefore, we deduce,

\[
\int_0^1 \text{diam}(V_{R,s}(s)) \frac{4\pi}{3} R^3 \, dR \leq \frac{8\pi}{\sqrt{3}} \exp \left( \int_0^s \| \nabla u(\tau) \|_{L^\infty} \, d\tau \right) \int_0^1 R^4 \, dR = \frac{8\pi}{5\sqrt{3}} \exp \left( \int_0^s \| \nabla u(\tau) \|_{L^\infty} \, d\tau \right).
\]

(71)

Thanks to (71), from (68), we obtain,

\[
\frac{\pi}{3} \| \omega(s) \|_{L^\infty} \leq \frac{\pi}{3} \| \omega_0 \|_{L^\infty} + \frac{8\pi}{\sqrt{3}} \exp \left( \int_0^s \| \nabla u(\tau) \|_{L^\infty} \, d\tau \right) \| \nabla \omega(s) \|_{L^\infty}
\]

\[
+ \int_0^s \left| \int_{B(\alpha(s), 1)} u(\alpha, t), t \omega_0(\alpha) \cdot n(\alpha) \, da \right| \, dt.
\]

(72)

For all \( t \in [0, T] \), we introduce the set \( V_s(t) = X(B(\alpha(s), 1), t) \). Then, thanks to (44), we have,

\[
|V_s(t)| = |B(\alpha(s), 1)| = \frac{4\pi}{3}.
\]

(73)

We introduce the inverse \( A(y, t) = X^{-1}(y, t) \) called the back to label map which satisfies \( A(X(\alpha, t), t) = \alpha \) and \( X(A(y, t), t) = y \).

Thanks to (44), for all \( t \in [0, s] \), using the change of variable \( y = X(\alpha, t) \) with \( \alpha \in B(\alpha(s), 1) \), we deduce,

\[
\int_{V_s(t)} u(y, t) \omega_0(A(y, t)) \cdot n(A(y, t)) \, dy = \int_{B(\alpha(s), 1)} u(\alpha, t), t \omega_0(\alpha) \cdot n(\alpha) \, da.
\]

(74)

Then, from (72), using (74), we deduce,

\[
\frac{\pi}{3} \| \omega(s) \|_{L^\infty} \leq \frac{\pi}{3} \| \omega_0 \|_{L^\infty} + \frac{8\pi}{\sqrt{3}} \exp \left( \int_0^s \| \nabla u(\tau) \|_{L^\infty} \, d\tau \right) \| \nabla \omega(s) \|_{L^\infty}
\]

\[
+ \int_0^s \left| \int_{V_s(t)} u(y, t) \omega_0(A(y, t)) \cdot n(A(y, t)) \, dy \right| \, dt.
\]

(75)

Let us prove now, that there exists a constant \( C > 0 \) such that for all \( t \in [0, s] \)

\[
\left| \int_{V_s(t)} u(y, t) \omega_0(A(y, t)) \cdot n(A(y, t)) \, dy \right| \leq C \| \omega(t) \|_{L^\infty} \| \omega_0 \|_{L^\infty}.
\]

For all \( t \in [0, s] \), we introduce the function \( f_t \) defined from \( \mathbb{R}^3 \) to \( \mathbb{R} \) by,
indeed,

\[ f_t(y) = \omega_0(A(y, t)) \cdot n(A(y, t)) \chi_{V_s(t)}(y) \quad \text{for all } y \in \mathbb{R}^3. \quad (76) \]

Notice that,

\[ \int_{\mathbb{R}^3} f_t(y) \, dy = 0, \quad (77) \]

indeed,

\[ \int_{\mathbb{R}^3} f_t(y) \, dy = \int_{V_s(t)} \omega_0(A(y, t)) \cdot n(A(y, t)) \, dy \]

\[ = \int_{B(\alpha(s), 1)} \omega_0(\alpha) \cdot n(\alpha) \, d\alpha, \]

by using the change of variable \( y = X(\alpha, t) \) with \( \alpha \in B(\alpha(s), 1) \). Since, we have,

\[ \int_{B(\alpha(s), 1)} \omega_0(\alpha) \cdot n(\alpha) \, d\alpha = \int_0^1 \frac{\partial}{\partial r} \left( \int_{B(\alpha(s), r)} \omega_0(\alpha) \cdot n(\alpha) \, d\alpha \right) \, dr \]

\[ = \int_0^1 \left( \int_{\partial B(\alpha(s), r)} \omega_0(\alpha) \cdot n(\alpha) \, d\gamma(\alpha) \right) \, dr. \]

On one hand, for all \( r > 0 \), we have,

\[ \int_{\partial B(\alpha(s), r)} \omega_0(\alpha) \cdot n(\alpha) \, d\gamma(\alpha) = \int_{B(\alpha(s), r)} \nabla \cdot \omega_0(\alpha) \, d\alpha. \]

On the other, we have \( \nabla \cdot \omega_0 = 0 \), then, we deduce,

\[ \int_{B(\alpha(s), 1)} \omega_0(\alpha) \cdot n(\alpha) \, d\alpha = 0, \]

and therefore, we get \( \int_{\mathbb{R}^3} f_t(y) \, dy = 0 \). From (76), we get \( \|f_t\|_{L^\infty} \leq \|\omega_0\|_{L^\infty} \), moreover since \( \text{supp} \, f_t \subset V_s(t) \) and thanks to (73), we get for all \( q_0 > 1 \), \( \|f_t\|_{L^{q_0}} \leq \|\omega_0\|_{L^\infty} \left( \frac{4\pi}{3} \right)^{\frac{1}{q_0}} \). Therefore for all \( q_0 > 1 \), \( f_t \in L^{q_0}(\mathbb{R}^3) \) and \( \int_{\mathbb{R}^3} f_t(y) \, dy = 0 \), then, we deduce that \( f_t \in \mathcal{H}^1(\mathbb{R}^3) \) (see [STE2], chapter III, §1, 1.2.4).

For any \( t \in [0, s] \), since the space \( BMO_r(V_s(t)) \) is the dual of \( \mathcal{H}^1_s(V_s(t)) \) (see Section 3), then we obtain,

\[ \left| \int_{V_s(t)} u(y, t) \omega_0(A(y, t)) \cdot n(A(y, t)) \, dy \right| = \left| \int_{V_s(t)} u(y, t) f_t(y) \, dy \right| \]

\[ \leq \|u(t)\|_{BMO_r(V_s(t))} \|f_t\|_{\mathcal{H}^1_s(V_s(t))} \]

\[ = \|u(t)\|_{BMO_r(V_s(t))} \|f_t\|_{\mathcal{H}^1(\mathbb{R}^3)}. \]

Due to the equivalence of norms in Hardy space \( \mathcal{H}^1(\mathbb{R}^3) \) and thanks to Lemma 3.1, there exists an absolute constant \( g > 0 \) such that,

\[ \|f_t\|_{\mathcal{H}^1(\mathbb{R}^3)} \leq g \left( \|f_t\|_{L^1} + 2|V_s(t)| \|f_t\|_{L^\infty} \right) \]

\[ \leq 3g|V_s(t)| \|f_t\|_{L^\infty}. \quad (79) \]
From (76), we obtain,
\[ \| f_t \|_{H^1(\mathbb{R}^3)} \leq 3q |V_s(t)| \| \omega_0 \|_{L^\infty}. \]

Thanks to (73), we deduce,
\[ \| f_t \|_{H^1(\mathbb{R}^3)} \leq 4\pi \| \omega_0 \|_{L^\infty}. \] (80)

From (9), we have,
\[ \| u(t) \|_{BMO(V_s(t))} = \sup_{Q \subset V_s(t)} \left( \frac{1}{|Q|} \int_Q |u(x, t) - u_Q(t)| \, dx \right), \] (81)

where \( u_Q(t) = \frac{1}{|Q|} \int_Q u(y, t) \, dy \), and the supremum is taken over all cubes \( Q \) in the domain \( V_s(t) \) whose sides are parallel to the coordinate axes. For all \( x \in Q \subset V_s(t) \), using Lemma 3.2, we get,
\[ |u(x, t) - u_Q(t)| \leq \frac{1}{|Q|} \int_Q |u(x, t) - u(y, t)| \, dy \leq \| \omega(t) \|_{L^\infty} \text{diam} Q. \]

Then, from (81), we deduce,
\[ \| u(t) \|_{BMO(V_s(t))} \leq \| \omega(t) \|_{L^\infty} \sup_{Q \subset V_s(t)} \text{diam} Q. \] (82)

Since, for all \( Q \) cube of \( \mathbb{R}^3 \), the measure of \( Q \) is given by,
\[ |Q| = \left( \frac{\text{diam}(Q)}{\sqrt{3}} \right)^3. \]

For all \( Q \subset V_s(t) \), cube of \( \mathbb{R}^3 \), we have, \( |Q| \leq |V_s(t)| \), and we deduce,
\[ \text{diam} Q \leq \sqrt{3} |V_s(t)|^{\frac{1}{3}}. \] (83)

Then, using (82), (83) and (73), we deduce,
\[ \| u(t) \|_{BMO(V_s(t))} \leq \sqrt{3} \left( \frac{4\pi}{3} \right)^{\frac{1}{3}} \| \omega(t) \|_{L^\infty}. \] (84)

Then, using (80) and (84), from (78), we deduce that there exists an absolute constant \( C > 0 \) such that,
\[ \left| \int_{V_s(t)} u(y, t) \omega_0(A(y, t)) \cdot n(A(y, t)) \, dy \right| \leq C \| \omega(t) \|_{L^\infty} \| \omega_0 \|_{L^\infty}. \] (85)

Using (85), from (75), we deduce that there exists an absolute constant \( C_1 > 0 \) such that,
\[ \| \omega(s) \|_{L^\infty} \leq \| \omega_0 \|_{L^\infty} + C_1 \exp \left( \int_0^s \| \nabla u(\tau) \|_{L^\infty} \, d\tau \right) \| \nabla \omega(s) \|_{L^\infty} \]
\[ + C_1 \int_0^s \| \omega(t) \|_{L^\infty} \| \omega_0 \|_{L^\infty} \, dt, \] (86)
which is valid for all \( s \in [0, T] \) and therefore valid for all \( s \in [0, T^*] \).

Now, we use the scaling property of Euler equations (37) and the vorticity equations (39), in other words, for all \( \lambda > 0 \) and \( \beta > 0 \), the couple of vector-valued functions \((u^{\lambda, \beta}, \omega^{\lambda, \beta})\) defined for all \((z, \sigma)\) on \(\mathbb{R}^3 \times \left[0, \frac{T^*}{\lambda \beta}\right)\) by:

\[
\begin{align*}
  u^{\lambda, \beta}(z, \sigma) &= \lambda u(\beta z, \lambda \beta \sigma) \\
  \omega^{\lambda, \beta}(z, \sigma) &= \lambda \beta \omega(\beta z, \lambda \beta \sigma)
\end{align*}
\]

(87)

are respectively solution of Euler equations (37) and vorticity equations (39) for the initial data \(\lambda u_0(\beta \cdot)\) and \(\lambda \beta \omega(\beta \cdot)\).

Then, Inequality (86) holds for the couple \((u^{\lambda, \beta}, \omega^{\lambda, \beta})\) and we get for all \(\sigma \in \left[0, \frac{T^*}{\lambda \beta}\right[\)

\[
\begin{align*}
  \|\omega^{\lambda, \beta}(\sigma)\|_{L^\infty} &\leq \|\omega_0\|_{L^\infty} + C_1 \|\nabla u^{\lambda, \beta}(\sigma)\|_{L^\infty} \exp \left(\int_0^\sigma \|\nabla u^{\lambda, \beta}(\theta)\|_{L^\infty} \, d\theta\right) \\
  &\quad + C_1 \|\omega_0\|_{L^\infty} \int_0^\sigma \|\omega^{\lambda, \beta}(\theta)\|_{L^\infty} \, d\theta.
\end{align*}
\]

(88)

For all \(s \in [0, T^*]\), setting \(\sigma = \frac{s}{\lambda \beta}\) in (88) and using (87), we deduce,

\[
\begin{align*}
  \|\omega(s)\|_{L^\infty} &\leq \|\omega_0\|_{L^\infty} + C_1 \|\nabla \omega(s)\|_{L^\infty} \exp \left(\int_0^s \|\nabla u(\tau)\|_{L^\infty} \, d\tau\right) \\
  &\quad + C_1 \|\omega_0\|_{L^\infty} \int_0^s \|\omega(\tau)\|_{L^\infty} \, d\tau,
\end{align*}
\]

(89)

then taking the limit as \(\beta \to 0\) in (89), we obtain for all \(s \in [0, T^*]\),

\[
\|\omega(s)\|_{L^\infty} \leq \|\omega_0\|_{L^\infty} + C_1 \|\omega_0\|_{L^\infty} \int_0^s \|\omega(t)\|_{L^\infty} \, dt.
\]

Then, thanks to Gronwall Lemma, we deduce for all \(s \in [0, T^*]\),

\[
\|\omega(s)\|_{L^\infty} \leq \|\omega_0\|_{L^\infty} \exp(C_1 \|\omega_0\|_{L^\infty} s).
\]

(90)

Then, we have \(\int_0^{T^*} \|\omega(s)\|_{L^\infty} \, ds \leq \frac{1}{C_1} \exp(C_1 \|\omega_0\|_{L^\infty} T^*) < +\infty\), which leads to a contradiction with (41), therefore \(T^* = +\infty\) and Inequality (90) holds for all \(s \geq 0\), which conclude the proof. \(\square\)

**Remark 4.1** Theorem 4.1 have been established for all \(1 < q < \infty\) and \(r > 2 + \frac{2}{q}\), in fact, the result holds also for any \(r > 1 + \frac{3}{q}\), indeed since \(C_0^\infty(\mathbb{R}^3)\) is dense in \(W^{r,q}(\mathbb{R}^3)\) and thanks to Theorem 3.6 in [KP], then Theorem 4.1 follows for any \(r > 1 + \frac{2}{q}\).

**References**

[AR] P. Auscher and E. Russ : Hardy spaces and divergence operators on strongly Lipschitz domains of \(\mathbb{R}^n\), arxiv :math/0201301v1, 2002


