Semi-classical behaviour of Schrödinger's dynamics : revivals of wave packets on hyperbolic trajectory

Olivier Lablée

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Abstract

The aim of this paper is to study the semi-classical behaviour of Schrödinger's dynamics for an one-dimensional quantum Hamiltonian with a classical hyperbolic trajectory. As in the regular case (elliptic trajectory), we prove, that for an initial wave packets localized in energy, the dynamics follows the classical motion during short time. This classical motion is periodic and the period T_{hyp} is order of $|\ln h|$. And, for large time, a new period T_{rev} for the quantum dynamics appears : the initial wave packets form again at $t = T_{rev}$. Moreover for the time $t = \frac{p}{q} T_{rev}$ a fractionnal revivals phenomenon of the initial wave packets appears : there is a formation of a finite number of clones of the original wave packet.

Schrödinger's dynamics, revivals of wave packets, semi-classical analysis, hyperbolic trajectory, Schrödinger operator with double wells potential.

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1 Introduction

1.1 Context and motivation

For P_h a pseudo-differential operator (here $h > 0$ is the semi-classical parameter), and for ψ_0 an inital state the quantum dynamics is governed by the famous Schrödinger equation :

$$ih\frac{\partial \psi(t)}{\partial t} = P_h \psi(t).$$

In this paper, we present a detailed study, in the semi-classical regime $h \to 0$, of the behaviour of Schrödinger's dynamics for an one-dimensional quantum Hamiltonian

$$P_h \, : \, D\left(P_h\right) \subset L^2\left(\mathbb{R}\right) \to L^2\left(\mathbb{R}\right)$$

with a classical hyperbolic trajectory : the principal symbol $p \in \mathcal{C}^{\infty}(\mathbb{R}^2, \mathbb{R})$ of P_h has a hyperbolic non-degenerate singularity.

Dynamics in the regular case and for elliptic non-degenerate singularity have been the subject of many research in physics **[Av-Pe], [LAS], [Robi1], [Robi2], [BKP]**, **[Bl-Ko]** and, more recently in mathematics **[Co-Ro]**, **[Rob], [Pau1]**, **[Pau2], [Lab2]**. The strategy to understand the long times behaviour of dynamics is to use the spectrum of the operator P_h. In the regular case, the spectrum of P_h is given by the famous Bohr-Sommerfeld rules (see for example **[He-Ro], [Ch-VuN]**, **[Col8]**) : in first approximation, the spectrum of P_h in a compact set is a sequence of real numbers with a gap of size h. The classical trajectories are periodic and supported on elliptic curves.

In the case of hyperbolic singularity we have a non-periodic trajectory supported on a "height" figure (see figure 2). The spectrum near this singularity is more complicated than in the regular case. Y. Colin de Verdière and B. Parisse give an implicit singular Bohr-Sommerfeld rules for hyperbolic singularity (**[Co-Pa1]**, **[Co-Pa2]** and **[Co-Pa3]**). This quantization formula is too implicit for using it directly in our motivation. In **[Lab3]** we have an explicit

description of the spectrum for an one-dimesional pesudo-differential operator near a hyperbolic non-degenerate singularity.

1.2 Results

With above description, we propose a study of quantum dynamics for large times ($\gg |\ln h|$) . We prove that for a localized initial state, at the begining the dynamics is periodic with a period equal to $T_{hyp} = C\,|\ln h|$ (see corollary 5.12). This period T_{hyp} corresponds to the classical Hamiltonian flow period.

Next for large time scale, a new period T_{rev} of the quantum dynamics appears : this is the revivals phenomenon (like in regular case **[Co-Ro]**, **[Rob]**, **[Pau1]**, **[Pau2]**, **[Lab2]**). For $t = T_{hyp}$ the packet relocalize in the form of a quantum revival.

We have also the phenomenon of fractional revivals of initial wave packets for time $t = \frac{p}{q} T_{rev}$, with $\frac{p}{q} \in \mathbb{Q}$: there is a formation of a finite number of clones of the original wave packet ψ_0 with a constant amplitude (see theorems 6.18, 6.19 & 6.20) and differing in the phase plane from the initial wave packet by fractions $\frac{p}{q} T_{hyp}$ (see theorem 6.15).

1.3 Paper organization

The paper is organized as follows. In section 2 we give some preliminaries about the strategy for analyse the dynamics of a quantum Hamiltonian. In this section we define a simple way to understand the evolution of $t \mapsto \psi(t)$ by the autocorrelation function :

$$\mathbf{c}(t) := |\langle \psi(t), \psi_0 \rangle_{\mathcal{H}}| .$$

In section 3 we describe the hyperbolic singularities mathematical context; we also recall the principal theorem of **[Lab3]**. This theorem provides the spectrum of the operator P_h near the singularity. Section 4 is devoted to define an initial wave packets ψ_0 localized in energy. In part 5 we prove that the quantum dynamics follows the classical motion during short time (see corollary 5.12). This classical motion is periodic and the period T_{hyp} is order of $|\ln h|$. In the last part (part 6) we detail the analysis of revivals phenomenon, see theorem 6.7 for full-revival theorem and see theorem 6.15 for fractionnal-revivals phenomenon.

2 Quantum dynamics and autocorrelation function

2.1 The quantum dynamics

For a quantum Hamiltonian $P_h : D(P_h) \subset \mathcal{H} \to \mathcal{H}$, $\mathcal{H}$ is a Hilbert space, the Schrödinger dynamics is governed by the Schrödinger equation :

$$ih\frac{\partial \psi(t)}{\partial t} = P_h \psi(t).$$

With the functional calculus, we can reformulate this equation with the unitary group $U(t) = \left\{ e^{-i\frac{t}{h}P_h} \right\}_{t\in\mathbb{R}}$. Indeed, for a initial state $\psi_0 \in \mathcal{H}$, the evolution

given by :

$$\psi(t) = U(t)\psi_0 \in \mathcal{H}.$$

2.2 Return and autocorrelation function

We now introduce a simple tool to understand the behaviour of the vector $\psi(t)$: a quantum analog or the Poincaré return function.

Definition. The quantum return functions of the operator P_h and for an initial state ψ_0 is defined by :

$$\mathbf{r}(t) := \langle \psi(t), \psi_0 \rangle_{\mathcal{H}};$$

and the autocorrelation function is defined by :

$$\mathbf{c}(t) := |\mathbf{r}(t)| = |\langle \psi(t), \psi_0 \rangle_{\mathcal{H}}|.$$

The previous function measures the return on initial state. This function is the overlap of the time dependent quantum state $\psi(t)$ with the initial state ψ_0. Since the initial state ψ_0 is normalized, the autocorrelation function takes values in the compact set $[0, 1]$. Then, if we have an orthonormal basis of eigenvectors $(e_n)_{n\in\mathbb{N}}$:

$$P_h e_n = \lambda_n(h) e_n$$

with

$$\lambda_1(h) \leq \lambda_2(h) \leq \cdots \leq \lambda_n(h) \to +\infty;$$

we get, for all integer n

$$\left(e^{-i\frac{t}{h}P_h}\right) e_n = \left(e^{-i\frac{t}{h}\lambda_n(h)}\right) e_n.$$

So for a initial vector $\psi_0 \in D(P_h) \subset \mathcal{H}$, let us denote by $(c_n)_{n\in\mathbb{N}} = (c_n(h))_{n\in\mathbb{N}}$ the sequence of $\ell^2(\mathbb{N})$ given $(c_n)_n = \pi(\psi_0)$, where π is the projector (unitary operator) :

$$\pi : \begin{cases} \mathcal{H} \to \ell^2(\mathbb{N}) \\ \\ \psi \mapsto < \psi, e_n >_{\mathcal{H}} . \end{cases}$$

Then, for all $t \geq 0$ we have

$$\psi(t) = U(t)\psi_0 = \left(e^{-i\frac{t}{h}P_h}\right) \left(\sum_{n\in\mathbb{N}} c_n e_n\right)$$

$$= \sum_{n\in\mathbb{N}} c_n e^{-i\frac{t}{h}\lambda_n(h)} e_n.$$

So, for all $t \geq 0$ we obtain

$$\mathbf{r}(t) = \sum_{n\in\mathbb{N}} |c_n|^2 e^{-i\frac{t}{h}\lambda_n(h)};\ \mathbf{c}(t) = \left|\sum_{n\in\mathbb{N}} |c_n|^2 e^{-i\frac{t}{h}\lambda_n(h)}\right|.$$

2.3 Strategy for study the autocorrelation function

The strategy, performed by the physicists (**[Av-Pe], [LAS], [Robi1]**, **[Robi2], [BKP]**, **[Bl-Ko]**) is the following :

1. We define a initial vector $\psi_0 = \sum_{n\in\mathbb{N}} c_n e_n$ localized in the following sense : the sequence $(c_n)_{n\in\mathbb{N}}$ is localized close to a quantum number n_0 (depends on h and a energy level $E \in \mathbb{R}$).

2. Next, the idea is to expand by a Taylor formula's the eigenvalues $\lambda_n(h)$ around the energy level E :

$$\lambda_n(h) = \lambda_{n_0}(h) + \lambda'_{n_0}(h)\,(n-n_0) + \frac{\lambda''_{n_0}(h)}{2}\,(n-n_0)^2 + \frac{\lambda^{(3)}_{n_0}(h)}{6}\,(n-n_0)^3 + \cdots$$

   (here $\lambda_{n_0}(h)$ is the closest eigenvalue to E), hence we get for all $t \geq 0$

$$\mathbf{c}(t) = \left| \sum_{n\in\mathbb{N}} |c_n|^2 \, e^{-it\left[\frac{\lambda'_{n_0}(h)}{h}(n-n_0) + \frac{\lambda''_{n_0}(h)}{2h}(n-n_0)^2 + \frac{\lambda^{(3)}_{n_0}(h)}{6h}(n-n_0)^3 + \cdots\right]} \right|.$$

3. And, for small values of t, the first approximation of the autocorrelation function $\mathbf{c}(t)$ is the function

$$\mathbf{c}_1(t) := \left| \sum_{n\in\mathbb{N}} |c_n|^2 \, e^{-it\frac{\lambda'_{n_0}(h)}{h}(n-n_0)} \right|;$$

   and for larger values of t, the order 2-approximation is

$$\mathbf{c}_2(t) := \left| \sum_{n\in\mathbb{N}} |c_n|^2 \, e^{-it\left[\frac{\lambda'_{n_0}(h)}{h}(n-n_0) + \frac{\lambda''_{n_0}(h)}{2h}(n-n_0)^2\right]} \right|.$$

In section 5 we study the function $t \mapsto \mathbf{c}_1(t)$ and in section 6 we study $t \mapsto \mathbf{c}_2(t)$.

3 The context of hyperbolic singularity

3.1 Link between spectrum and geometry : semi-classical analysis

For explain the philosophy of semi-classical analysis start by an example : for a real number $E > 0$; the equation

$$-\frac{h^2}{2}\Delta_g \varphi = E\varphi$$

(where Δ_g denotes the Laplace-Beltrami operator on a Riemaniann manifold (M, g)) admits the eigenvectors φ_k as solution if

$$-\frac{h^2}{2}\lambda_k = E.$$

Hence if $h \to 0^+$ then $\lambda_k \to +\infty$. So there exists a correspondence between the semi-classical limit ($h \to 0^+$) and large eigenvalues.

The asympotic's of large eigenvalues for the Laplace-Beltrami operator Δ_g on a Riemaniann manifold (M, g), or more generally for a pseudo-differential operator P_h, is linked to a symplectic geometry : the phase space geometry. This is the same phenomenon between quantum mechanics (spectrum, operator algebra) and classical mechanics (length of periodic geodesics, symplectic geometry). More precisely, for a pseudo-differential operator P_h on $L^2(M)$ with a principal symbol $p \in \mathcal{C}^\infty(T^\star M)$, there exist a link between the geometry of the foliation $\left(p^{-1}(\lambda)\right)_{\lambda \in \mathbb{R}}$ and the spectrum of the operator P_h. Indeed, we have the famous result :

$$(P_h - \lambda I_d)\, u_h = O(h^\infty)$$

then

$$MS(u_h) \subset p^{-1}(\lambda);$$

where $MS(u_h) \subset T^\star M$ denote the microsupport of the function $u_h \in L^2(M)$.

3.2 Hyperbolic singularity

In dimension one, a point $(x_0, \xi_0) \in T^\star M$ is a non-degenerate hyperbolic singularity of the symbol function $p \in \mathcal{C}^\infty(T^\star M)$ if and only if :

1. $dp(x_0, \xi_0) = 0$;
2. the eigenvalues of the Hessian matrix $\nabla^2 p(x_0, \xi_0)$ are pairwise distinct;
3. if, in some local symplectic coordinates (x, ξ) the algebra spanned by $\nabla^2 p(x_0, \xi_0)$ has a basis of the form $q = x\xi$.

Remark 3.1. There exists analogue definition for completely integrable systems, see for example the book of San Vũ Ngọc **[VuN]**.

The canonical example in dimension 1 is the Schrödinger operator with double wells potential :

$$P_h = -\frac{h^2}{2}\Delta + V;$$

we assume $V \in \mathcal{C}^\infty(\mathbb{R})$, for all $x \in \mathbb{R}$, $V(x) \geq C$ and $\lim\limits_{|x| \to \infty} V(x) = +\infty$. Here the principal symbol of P_h is the function

$$p(x, \xi) = \frac{\xi^2}{2} + V(x).$$

With the previous hypotheses on the potential V, the operator P_h is self-adjoint, the spectrum is a sequence of real numbers $(\lambda_n(h))_{n \geq 0}$ and the eigenvectors $(e_n)_{n \geq 0}$ be an orthonormal basis of the Hilbert space $L^2(\mathbb{R})$ (for example see the survey **[Lab1]**). We also suppose that the potential V admits exactly one local non-degenerate maximum. Without loss generality, we may suppose

$$V(0) = 0;\ V'(0) = 0;\ V''(0) < 0.$$

Then the foliation associated to the principal symbol $p(x, \xi) = \xi^2/2 + V(x)$ admits a singular fiber $\Lambda_0 = p^{-1}(0)$.

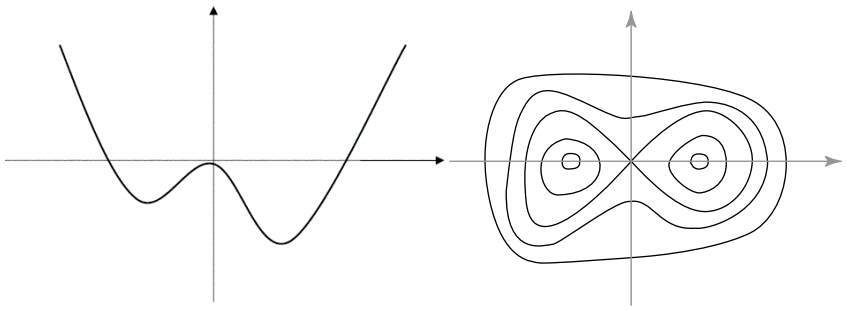

Fig. 1. & 2. On figure 1 (left) the potential function $V(x)$. On figure 2 (right) the associated foliation in the phase plane.

3.3 Spectrum near the singularity

In **[Co-Pa1]**, **[Co-Pa2]** and **[Co-Pa3],** Y. Colin de Verdière and B. Parisse gives an implicit singular Bohr-Sommerfeld rules. An explicit description is given in **[Lab3]**. Record here this description. We take the presentation from the paper **[Lab3]** and refer to this paper for more details.

3.3.1 Some notations (see [Lab3])

In **[Co-Pa1]**, **[Co-Pa2]**, **[Co-Pa3]** and **[Lab3]** we have the smooth function $E \mapsto \varepsilon(E)$ defined for $|E| \leq \delta$, where δ is a real constant and not depends to h; (this function is linked to a normal form **[CLP], [Co-Pa1]**) defined by

$$\varepsilon(E) = \sum_{j=0}^{+\infty} \varepsilon_j(E) h^j$$

We have (see **[Co-Pa1]**, **[Co-Pa2]**, **[Co-Pa3]** and **[Lab3]**) the equality $\varepsilon_0(0) = 0$ and $\varepsilon_0'(0) = 1/\sqrt{-V''(0)}$. Hence, if we use the Taylor formula on the smooth function ε_0 ; for all $E \in [-\delta, \delta]$ we get

$$\varepsilon(E) = \frac{E}{\sqrt{-V''(0)}} + O(E^2) + \sum_{j=1}^{+\infty} \varepsilon_j(E) h^j.$$

So, for $\lambda \in [-1, 1]$; and h small enough (for have $[-h, h] \subset [-\delta, \delta]$) we get

$$\varepsilon(\lambda h) = \frac{\lambda h}{\sqrt{-V''(0)}} + O(h^2) + \sum_{j=1}^{+\infty} \varepsilon_j(\lambda h) h^j.$$

In the papers **[Co-Pa1]**, **[Co-Pa2]**, **[Co-Pa3]** and **[Lab3]** we also use two smooth functions S^+ and S^- :

$$S^{+/-}(E) = \sum_{j=0}^{+\infty} S_j^{+/-}(E) h^j;$$

where the functions are $E \to S_j^{+/-}(E)$ are $\mathcal{C}^\infty$ smooth. This functions are usually called singular actions. Let us also denote $\theta_{+/-}(E) := S^{+/-}(E)/h$. This functions have a holonomy interpretation. In **[Lab3]** we consider the functions $E \mapsto F_h(E)$ and $E \mapsto G_h(E)$ defined on the compact set $[-\delta, \delta]$ by :

$$F_h(E) := -\frac{\theta_+(E) + \theta_-(E)}{2} + \frac{\pi}{2} + \frac{\varepsilon(E)}{h}\ln(h) + \arg\left(\Gamma\left(\frac{1}{2} + i\frac{\varepsilon(E)}{h}\right)\right);$$

Γ is the Gamma function, and by

$$G_h(E) := \frac{\theta_+(E) - \theta_-(E)}{2}.$$

On the compact set $[-1, 1]$, let us consider the functions $\lambda \mapsto f_h(\lambda)$ and $\lambda \mapsto g_h(\lambda)$ defined by :

$$f_h(\lambda) := F_h(\lambda h);\ g_h(\lambda) := G_h(\lambda h).$$

For finish, let us consider the functions $\lambda \mapsto \mathcal{Y}_h(\lambda)$ and $\lambda \mapsto \mathcal{Z}_h(\lambda)$ defined on the compact set $[-1, 1]$ by

$$\mathcal{Y}_h(\lambda) := f_h(\lambda) - \arccos\left(\frac{\cos(g_h(\lambda))}{\sqrt{1 + \exp(2\pi\varepsilon(\lambda h)/h)}}\right);$$

$$\mathcal{Z}_h(\lambda) := f_h(\lambda) + \arccos\left(\frac{\cos(g_h(\lambda))}{\sqrt{1 + \exp(2\pi\varepsilon(\lambda h)/h)}}\right).$$

We have the following result (see **[Lab3]**) :

Proposition 3.2. *For h small enough, the function $\mathcal{Y}_h$ (resp. $\mathcal{Z}_h$) is a bijection from $[-1, 1]$ onto $\mathcal{Y}_h([-1, 1])$ (resp. onto $\mathcal{Z}_h([-1, 1])$). Moreover on the compact set $[-1, 1]$ we have*

$$\mathcal{Y}_h'(\lambda) = \frac{\ln(h)}{\sqrt{-V''(0)}} + O(1).$$

Similary for the function $\mathcal{Z}_h$.

Since the functions $\mathcal{Y}_h$ et $\mathcal{Z}_h$ are bijectives, we can consider

$$\mathcal{A}_h := \mathcal{Y}_h^{-1} :\ \mathcal{Y}_h([-1, 1]) \to [-1, 1];$$

$$\mathcal{B}_h := \mathcal{Z}_h^{-1} :\ \mathcal{Z}_h([-1, 1]) \to [-1, 1].$$

Notation 3.3. Let us denotes :

$$I_h := \{k \in \mathbb{Z},\ 2\pi k \in \mathcal{Y}_h([-1, 1])\} = \frac{\mathcal{Y}_h([-1, 1])}{2\pi} \cap \mathbb{Z};$$

$$J_h := \{\ell \in \mathbb{Z},\ 2\pi\ell \in \mathcal{Z}_h([-1, 1])\} = \frac{\mathcal{Z}_h([-1, 1])}{2\pi} \cap \mathbb{Z}.$$

3.3.2 The main theorem

The main theorem of the paper **[Lab3]** is the the following :

Theorem 3.4. [Lab3]. The semi-classical spectrum of P_h in the compact set $\left[-\sqrt{h}, \sqrt{h}\right]$ is the disjoint union

$$(\alpha_k(h))_{k\in \mathbf{I}_h} \bigsqcup (\beta_\ell(h))_{\ell\in \mathbf{J}_h}$$

of two families $(\alpha_k(h))_k$ and $(\beta_\ell(h))_\ell$ such that $\alpha_k(h) := h\mathcal{A}_h(2\pi k) \in \mathbb{R}$, $\beta_\ell(h) := h\mathcal{B}_h(2\pi\ell) \in \mathbb{R}$. The functions $\mathcal{A}_h$ and $\mathcal{B}_h$ are $\mathcal{C}^\infty$ smooth. The families $(\alpha_k(h))_k$, $(\beta_\ell(h))_\ell$ are strictly non-increasing and :

$$\beta_{k+1}(h) < \alpha_k(h) < \beta_k(h) < \alpha_{k-1}(h).$$

Moreover, the spectral gap is of order $O(h/\left|\ln(h)\right|)$; e.g : there exists $C, C' > 0$ such that :

$$\frac{Ch}{|\ln(h)|} \leq |\alpha_{k+1}(h) - \alpha_k(h)| \leq \frac{C'h}{|\ln(h)|}, \frac{Ch}{|\ln(h)|} \leq |\beta_{k+1}(h) - \beta_k(h)| \leq \frac{C'h}{|\ln(h)|}.$$

Corollary 3.5. *The number of eigenvalues in the compact set* $\left[-\sqrt{h}, \sqrt{h}\right]$ *is of order* $|\ln(h)|/\sqrt{h}$.

In this paper, for technical reason, we just use the shape of the spectrum in the compact set $[-h, h] \subset \left[-\sqrt{h}, \sqrt{h}\right]$. More precisely we choose an initial wave packet ψ_0 localized in a compact set of size h. Indeed, the estimates in lemma 5.1 and 6.1 are very difficult to enable in the compact set $\left[-\sqrt{h}, \sqrt{h}\right]$.

Notation 3.6. Let us denote by Θ_h the set of index for eigenvalues into the compact set $[-h, h]$; e.g

$$\Theta_h := \{n \in \mathbb{N},\ \lambda_n(h) \in [-h, h]\}.$$

4 Some preliminaries

In this section, we define partials autocorrelations functions, next we define an initial vector ψ_0. And for finish, we introduce the set $\Delta \subset \mathbb{N}$, this set is useful for making estimates in sections 5 and 6.

4.1 Partials autocorrelations functions

Now, let be a initial vector $\psi_0 = \sum_{n\in\mathbb{N}} c_n e_n$, then we have for all $t \geq 0$

$$\mathbf{r}(t) = \sum_{n\in\Theta_h} |c_n|^2 e^{-i\frac{t}{h}\lambda_n(h)} + \sum_{n\in\mathbb{N}-\Theta_h} |c_n|^2 e^{-i\frac{t}{h}\lambda_n(h)}.$$

After, we let us consider an initial vector localized near the singularity in a set of size h; hence the second series

$$\sum_{n\in\mathbb{N}-\Theta_h} |c_n|^2 e^{-i\frac{t}{h}\lambda_n(h)}$$

will be equal to $O\left|\frac{1}{\ln(h)^\infty}\right|$. Next, the idea is to use the families $(\alpha_k(h))_k$ and $(\beta_\ell(h))_\ell$ from the theorem above : as the eigenvalues of P_h in the compact set $[-h,h]$ are distinct; for all index $k \in \mathbf{I}_h$ and $\ell \in \mathbf{J}_h$ there exists a unique pair $(\sigma_1(k),\sigma_2(\ell)) \in \Theta_h^2$ such that :

$$\begin{cases} h\mathcal{A}_h(2\pi k) = \lambda_{\sigma_1(k)}(h) \\ \\ h\mathcal{B}_h(2\pi \ell) = \lambda_{\sigma_2(\ell)}(h). \end{cases}$$

So, we can consider the applications :

$$\sigma_1 \, : \begin{cases} \mathbf{I}_h \to \Theta_h \\ \\ k \mapsto \sigma_1(k) \end{cases} \quad \text{and} \quad \sigma_2 \, : \begin{cases} \mathbf{J}_h \to \Theta_h \\ \\ \ell \mapsto \sigma_2(\ell). \end{cases}$$

Clearly, this applications are injectives; hence $\mathbf{I}_h$ is isomorphic to $\sigma_1(\mathbf{I}_h)$ (similary for $\mathbf{J}_h$ and $\sigma_2(\mathbf{J}_h)$). Moreover we have

$$\Theta_h = \sigma_1(\mathbf{I}_h) \bigsqcup \sigma_2(\mathbf{J}_h).$$

So, we get the equality :

$$\begin{aligned} \mathbf{r}(t) &= \sum_{k\in\sigma_1(\mathbf{I}_h)} |c_k|^2 \, e^{-i\frac{t}{h}\lambda_k(h)} + \sum_{\ell\in\sigma_2(\mathbf{J}_h)} |c_\ell|^2 \, e^{-i\frac{t}{h}\lambda_\ell(h)} \\ &\qquad + \sum_{n\in\mathbb{N}-\Theta_h} |c_n|^2 \, e^{-i\frac{t}{h}\lambda_n(h)} \\ &= \sum_{k\in\mathbf{I}_h} \left|c_{\sigma_1(k)}\right|^2 e^{-it\mathcal{A}_h(2\pi k)} + \sum_{\ell\in\mathbf{J}_h} \left|c_{\sigma_2(\ell)}\right|^2 e^{-it\mathcal{B}_h(2\pi \ell)} \\ &\qquad + \sum_{n\in\mathbb{N}-\Theta_h} |c_n|^2 \, e^{-i\frac{t}{h}\lambda_n(h)}. \end{aligned}$$

Now, our goal here is to study the series $t \mapsto \mathbf{a}$ et $t \mapsto \mathbf{b}(t)$:

Definition 4.1. Let us consider partial autocorrelation functions :

$$\mathbf{a}(t) := \sum_{k\in\mathbf{I}_h} \left|c_{\sigma_1(k)}\right|^2 e^{-it\mathcal{A}_h(2\pi k)}, \; \mathbf{b}(t) := \sum_{\ell\in\mathbf{J}_h} \left|c_{\sigma_2(\ell)}\right|^2 e^{-it\mathcal{B}_h(2\pi \ell)}.$$

4.2 Choice of initial state

4.2.1 Prologue

Let us define an initial vector $\psi_0 = \sum_{n\in\mathbb{N}} c_n e_n$ localized near the real number hE (where $E \in [-1,1]$).

Definition 4.2. Let us consider the quantum integers $n_0 = n_0(h,E)$ and $m_0 = m_0(h,E)$ defined by

$$n_0 := \arg\min_{n\in\mathbf{I}_h} |h\mathcal{A}_h(2\pi n) - hE| \quad \text{and} \quad m_0 := \arg\min_{m\in\mathbf{I}_h} |h\mathcal{B}_h(2\pi m) - hE| \, .$$

Remark 4.3. Without loss of generality, we may suppose n_0 and m_0 are unique.

The integer n_0 (resp. m_0) is the index of the eigenvalues from the family $(\alpha_k(h))_k$ (resp. $(\beta_\ell(h))_\ell$) the closest to the real number hE. As the spectral gap is of order $O(h/|\ln(h)|)$ there exists $C > 0$ such that :

$$|h\mathcal{A}_h(2\pi n_0) - hE| \leq \frac{Ch}{|\ln h|},\ |h\mathcal{B}_h(2\pi m_0) - hE| \leq \frac{Ch}{|\ln h|}.$$

So, we get :

Lemma 4.4. *As $h \to 0$, we have*

$$n_0 \sim \frac{N}{h} \text{ and } m_0 \sim \frac{M}{h}$$

where N and M non-null real numbers.

Proof. We give the proof for the integer n_0. As

$$|\mathcal{A}_h(2\pi n_0) - E| \leq \frac{C}{|\ln h|};$$

e.g.

$$\mathcal{A}_h(2\pi n_0) = E + O\left(\frac{1}{|\ln h|}\right),$$

we get

$$2\pi n_0 = \mathcal{Y}_h\left(E + O\left(\frac{1}{|\ln h|}\right)\right)$$

$$= f_h\left(E + O\left(\frac{1}{|\ln h|}\right)\right) - \arccos\left(\frac{\cos\left(g_h\left(E + O\left(\frac{1}{|\ln h|}\right)\right)\right)}{\sqrt{1 + \exp\left(2\pi\varepsilon\left(hE + O\left(\frac{h}{|\ln h|}\right)\right)/h\right)}}\right).$$

By definition :

$$f_h\left(E + O\left(\frac{1}{|\ln h|}\right)\right) = F_h\left(hE + O\left(\frac{h}{|\ln h|}\right)\right)$$

$$= -\frac{\theta_+\left(hE + O\left(h/|\ln(h)|\right)\right) + \theta_-\left(hE + O\left(h/|\ln(h)|\right)\right)}{2} + \frac{\pi}{2}$$

$$+\frac{\varepsilon\left(hE + O\left(h/|\ln(h)|\right)\right)}{h}\ln(h) + \arg\left(\Gamma\left(\frac{1}{2} + i\frac{\varepsilon\left(hE + O\left(h/|\ln(h)|\right)\right)}{h}\right)\right).$$

Hence, if we multiply $2\pi n_0$ by h we obtain

$$2\pi n_0 h = -h\frac{\theta_+\left(hE + O\left(h/|\ln(h)|\right)\right) + \theta_-\left(hE + O\left(h/|\ln(h)|\right)\right)}{2}$$

$$+\frac{\pi}{2}h + \varepsilon\left(hE + O\left(h/|\ln(h)|\right)\right)\ln(h) + h\arg\left(\Gamma\left(\frac{1}{2} + i\frac{\varepsilon\left(hE + O\left(h/|\ln(h)|\right)\right)}{h}\right)\right)$$

$$-h\arccos\left(\frac{\cos\left(g_h\left(E + O\left(\frac{1}{|\ln h|}\right)\right)\right)}{\sqrt{1 + \exp\left(2\pi\varepsilon\left(hE + O\left(\frac{h}{|\ln h|}\right)\right)/h\right)}}\right).$$

Let us evaluate this five terms. As the function

$$E \mapsto -\frac{\theta_+ \left(hE + O\left(h/\left|\ln(h)\right|\right)\right) + \theta_- \left(hE + O\left(h/\left|\ln(h)\right|\right)\right)}{2}$$

admit a asymptotic expansion in power of h from -1 to $+\infty$, we have

$$-h\frac{\theta_+ \left(hE + O\left(h/\left|\ln(h)\right|\right)\right) + \theta_- \left(hE + O\left(h/\left|\ln(h)\right|\right)\right)}{2} = O(1);$$

more precisely, for $h \to 0$

$$-h\frac{\theta_+ \left(hE + O\left(h/\left|\ln(h)\right|\right)\right) + \theta_- \left(hE + O\left(h/\left|\ln(h)\right|\right)\right)}{2} \to -\frac{1}{2}\left(S_{-1}^{+}(0) + S_{-1}^{-}(0)\right) \neq 0.$$

Next :

$$\varepsilon\left(hE + O\left(h/\left|\ln(h)\right|\right)\right)\ln(h) = \left[\frac{hE + O\left(h/\left|\ln(h)\right|\right)}{\sqrt{-V''(0)}} + O(h^2)\right]\ln(h) \to 0.$$

For finish, as

$$\arg\left(\Gamma\left(\frac{1}{2} + i\frac{\varepsilon\left(hE + O\left(h/\left|\ln(h)\right|\right)\right)}{h}\right)\right) = O(1)$$

and

$$\arccos\left(\frac{\cos\left(g_h\left(E + O\left(\frac{1}{\left|\ln h\right|}\right)\right)\right)}{\sqrt{1 + \exp\left(2\pi\varepsilon\left(hE + O\left(\frac{h}{\left|\ln h\right|}\right)\right)/h\right)}}\right) = O(1);$$

we get, for $h \to 0$:

$$h\arg\left(\Gamma\left(\frac{1}{2} + i\frac{\varepsilon\left(hE + O\left(h/\left|\ln(h)\right|\right)\right)}{h}\right)\right) \to 0$$

and

$$h\arccos\left(\frac{\cos\left(g_h\left(E + O\left(\frac{1}{\left|\ln h\right|}\right)\right)\right)}{\sqrt{1 + \exp\left(2\pi\varepsilon\left(hE + O\left(\frac{h}{\left|\ln h\right|}\right)\right)/h\right)}}\right) \to 0.$$

Hence,

$$\lim_{h\to 0} 2\pi n_0 h = -\frac{1}{2}\left(S_{-1}^{+}(0) + S_{-1}^{-}(0)\right);$$

so we prove the lemma. □

4.2.2 First definition (non definitive)

For technical reason, let us introduce the following sequence :

Definition 4.5. Let us consider :

$$\mu_{n,0} := \begin{cases} h\mathcal{A}_h(2\pi n_0) \ \ \textbf{if } n \in \sigma_1(\mathbf{I}_h) \\ h\mathcal{B}_h(2\pi m_0) \ \ \textbf{if } n \in \sigma_2(\mathbf{J}_h). \\ 0 \ \ \textbf{if } n \in \mathbb{N} - \Theta_h. \end{cases}$$

For $n \in \Theta_h = \sigma_1(\mathbf{I}_h) \bigsqcup \sigma_2(\mathbf{J}_h)$, the sequence $(\mu_{n,0})_n$ take only two values :

- the closest eigenvalue from the family $(\alpha_k(h))_k$ to the real number hE;
- or the closest eigenvalue from the family $(\beta_\ell(h))_\ell$ to the real number hE.

Now, we can give the first (non definitive) definition of our initial state.

Definition 4.6. Let us consider the sequence $(c_n(h))_{n\in\mathbb{Z}}$ defined by :

$$c_n := c_n(h) = K_h \chi\left(\frac{\mu_n - \mu_{n,0}}{h^{\alpha'}} |\ln h|^{\gamma'}\right) \chi_0\left(\frac{\mu_n - \mu_{n,0}}{2h}\right),\ n \in \mathbb{Z}$$

where $\chi \in \mathcal{S}(\mathbb{R})$, non null, non-negative and even, α' and γ' are two reals numbers; $\chi_0 \in \mathcal{D}(\mathbb{R})$ such that $\chi_0 \equiv 1$ on the set $]-1,1[$ and $\text{supp}(\chi_0) \subset [-1,1]$. We also denote

$$K_h := \left\| \chi\left(\frac{\mu_n - \mu_{n,0}}{h^{\alpha'}} |\ln h|^{\gamma'}\right) \chi_0\left(\frac{\mu_n - \mu_{n,0}}{2h}\right) \right\|_{\ell^2(\mathbb{N})}.$$

Let us detail this choice :

1. The term $\chi\left(\frac{\mu_n - \mu_{n,0}}{h^{\alpha'}} |\ln h|^{\gamma'}\right)$ localizes around the energy level hE (for technical reason we localize around the closest eigenvalues to E_h.
2. Constants α' et γ' are coefficients for dilate the function χ.
3. The function χ_0 is a cut-off for eigenvalues out of the compact set $[-h,h]$.

4.2.3 Choice of parameter α' and γ'

The only way to have a localization for initial state larger than the spectral gap, and in the same time, to have a localization in the compact set $[-h,h]$; e.g to have in the same time :

$$\frac{h}{|\ln h|} \ll \frac{h^{\alpha'}}{|\ln h|^{\gamma'}} \leq h;$$

is to take

$$\alpha' = 1;\ 0 \leq \gamma' < 1.$$

Remark 4.7. The larger choice for $h^{\alpha'} / |\ln h|^{\gamma'} = h / |\ln h|^{\gamma'}$ is to take $\gamma' = 0$.

Under this assumptions, we can forget the function χ_0 and we get :

4.2.4 Second definition

Definition 4.8. Let us consider the sequence $(c_n(h))_{n\in\mathbb{Z}}$ defined by :

$$c_n := K_h \chi\left(\frac{\mu_n - \mu_{n,0}}{h} |\ln h|^{\gamma'}\right),\ n \in \mathbb{Z}$$

where $\chi \in \mathcal{S}(\mathbb{R})$, non null, non-negative and even; $0 \leq \gamma' < 1$; and

$$K_h := \left\| \chi\left(\frac{\mu_n - \mu_{n,0}}{h} |\ln h|^{\gamma'}\right) \right\|_{\ell^2(\mathbb{N})}.$$

Now, in the partial autocorrelations functions $\mathbf{a}(t)$ and $\mathbf{b}(t)$ appears the sequences $\left(c_{\sigma_1(k)}\right)_k$ and $\left(c_{\sigma_2(\ell)}\right)_\ell$; for simplicity let us consider the following sequences :

Notation 4.9. Let us denote

$$a_n := c_{\sigma_1(n)} = K_h \chi\left(\left(\mathcal{A}_h(2\pi n) - \mathcal{A}_h(2\pi n_0)\right)|\ln h|^{\gamma'}\right);$$

$$b_m := c_{\sigma_2(m)} = K_h \chi\left(\left(\mathcal{B}_h(2\pi m) - \mathcal{B}_h(2\pi m_0)\right)|\ln h|^{\gamma'}\right).$$

4.2.5 Last definition

By Lagrange's theorem there exists a real number $\zeta = \zeta(n, h, E) \in \mathcal{Y}_h([-1, 1])$ such that

$$\mathcal{A}_h(2\pi n) - \mathcal{A}_h(2\pi n_0) = \mathcal{A}_h'(\zeta) 2\pi (n - n_0);$$

since (see proposition 3.2)

$$\mathcal{A}_h'(\zeta) = \frac{2\pi}{\frac{|\ln(h)|}{\sqrt{-V''(0)}} + O(1)};$$

we get

$$\mathcal{A}_h(2\pi n) - \mathcal{A}_h(2\pi n_0) = \frac{2\pi (n - n_0)\sqrt{-V''(0)}}{|\ln h|}\left(1 + O\left|\frac{1}{|\ln h|}\right|\right).$$

Definition 4.10. Let us consider the sequence $(a_n)_{n\in\mathbb{Z}} = (a_n(h))_{n\in\mathbb{Z}}$ defined by :

$$a_n := K_h \chi\left(\frac{n - n_0}{|\ln h|^{1-\gamma'}}\right),\ n \in \mathbb{Z}$$

where $\chi \in \mathcal{S}(\mathbb{R})$, non null, non-negative and even; $0 \leq \gamma' < 1$; and

$$K_h := \left\| \chi\left(\frac{n - n_0}{|\ln h|^{1-\gamma'}}\right)\right\|_{\ell^2(\mathbb{N})}.$$

So, clearly the sequence $(a_n)_n \in \ell^2(\mathbb{Z})$. Now, let us evaluate the constant of normalization K_h. Start by the :

Lemma 4.11. *For a function $\varphi \in \mathcal{S}(\mathbb{R})$ and $\varepsilon \in]0, 1]$; we have :*

$$\sum_{\ell\in\mathbb{Z},\, |\ell|\geq 1} \left|\varphi\left(\frac{\ell}{\varepsilon}\right)\right| = O(\varepsilon^\infty).$$

Proof. The starting point is the following remark : for all function $\psi \in \mathcal{S}(\mathbb{R})$ and for all $\varepsilon \in]0, 1]$ we have

$$\sum_{\ell\in\mathbb{Z},\, |\ell|\geq 1} \left|\psi\left(\frac{\ell}{\varepsilon}\right)\right| = O(1).$$

Indeed, for all $\ell \in \mathbb{Z}$, $|\ell| \geq 1$

$$\left|\psi\left(\frac{\ell}{\varepsilon}\right)\right| \leq \frac{M}{1+\left(\frac{\ell}{\varepsilon}\right)^2} = \frac{\varepsilon^2 M}{\varepsilon^2+\ell^2}$$

$$\leq \frac{M}{\ell^2}$$

and since

$$\sum_{\ell\in\mathbb{Z},\,|\ell|\geq 1} \frac{M}{\ell^2} < +\infty.$$

we conclude.

Next, we apply this to the function $\psi(x) := x^{2N}\varphi(x)$, where $N \in \mathbb{N}$; and we get for all $N \geq 1$:

$$\sum_{\ell\in\mathbb{Z},\,|\ell|\geq 1} \left|\varphi\left(\frac{\ell}{\varepsilon}\right)\right| \leq \varepsilon^{2N} \sum_{\ell\in\mathbb{Z}} \left|\psi\left(\frac{\ell}{\varepsilon}\right)\right| = O(\varepsilon^{2N}).$$

So we prove the lemma. □

Theorem 4.12. We have

$$K_h = \frac{1}{\sqrt{\mathfrak{F}\left(\chi^2\right)(0)}\,|\ln h|^{\frac{1-\gamma'}{2}}} + O\left|\frac{1}{\ln(h)^\infty}\right|;$$

so

$$\|a_n\|_{\ell^2(\mathbb{N})} = 1 + O\left|\frac{1}{\ln(h)^\infty}\right|.$$

Proof. By Poisson formula and the lemma above we get the equality

$$\sum_{n\in\mathbb{Z}} \chi^2\left(\frac{n-n_0}{|\ln h|^{1-\gamma'}}\right) = |\ln h|^{1-\gamma'} \sum_{\ell\in\mathbb{Z}} \mathfrak{F}\left(\chi^2\right)\left(-\ell\,|\ln h|^{1-\gamma'}\right)$$

$$= |\ln h|^{1-\gamma'}\left[\mathfrak{F}\left(\chi^2\right)(0) + O\left|\frac{1}{\ln(h)^\infty}\right|\right]$$

Now, start with the equality

$$\sum_{n\in\mathbb{N}} \chi^2\left(\frac{n-n_0}{|\ln h|^{1-\gamma'}}\right) = \sum_{n\in\mathbb{Z}} \chi^2\left(\frac{n-n_0}{|\ln h|^{1-\gamma'}}\right) - \sum_{n=-\infty}^{-1} \chi^2\left(\frac{n-n_0}{|\ln h|^{1-\gamma'}}\right)$$

$$= |\ln h|^{1-\gamma'}\,\mathfrak{F}\left(\chi^2\right)(0) + O\left|\frac{1}{\ln(h)^\infty}\right| - \sum_{n=-\infty}^{-1} \chi^2\left(\frac{n-n_0}{|\ln h|^{1-\gamma'}}\right).$$

Since the function $\mathfrak{F}\left(\chi^2\right)$ is even :

$$\sum_{n=-\infty}^{-1} \chi^2\left(\frac{n-n_0}{|\ln h|^{1-\gamma'}}\right) = \sum_{n=1}^{+\infty} \chi^2\left(\frac{n+n_0}{|\ln h|^{1-\gamma'}}\right) \leq B_k\,|\ln h|^{(1-\gamma')k} \sum_{n=1}^{+\infty} \frac{1}{|n+n_0|^k}.$$

And,

$$B_k \left|\ln h\right|^{(1-\gamma')k} \sum_{n=1}^{+\infty} \frac{1}{\left|n+n_0\right|^k} \leq B_k \left|\ln h\right|^{(1-\gamma')k} \int_0^{\infty} \frac{du}{(u+n_0)^k}$$

$$= \left|\ln h\right|^{(1-\gamma')(k-1)} \frac{B_k}{(k-1)} \frac{1}{n_0^{k-1}}.$$

Now, since for $h \to 0$ we have $n_0 \sim \frac{N}{h}$, we get, for $h \to 0$

$$\left|\ln h\right|^{(1-\gamma')k} \frac{B_k}{(k-1)} \frac{1}{n_0^{k-1}} \sim \frac{B_k}{N^{k-1}(k-1)} h^{k-1} \left|\ln h\right|^{(1-\gamma')k}$$

consequently

$$\sum_{n=-\infty}^{-1} \chi^2 \left(\frac{n-n_0}{\left|\ln h\right|^{1-\gamma'}} \right) = O\left| \frac{1}{\ln(h)^{\infty}} \right|.$$

So, we proove that

$$\left\| \chi \left(\frac{n-n_0}{\left|\ln h\right|^{1-\gamma'}} \right) \right\|_{\ell^2(\mathbb{N})}^2 = \left|\ln h\right|^{1-\gamma'} \mathfrak{F}\left(\chi^2\right)(0) + O\left| \frac{1}{\ln(h)^{\infty}} \right|;$$

hence $K_h = \frac{1}{\sqrt{\mathfrak{F}(\chi^2)(0)} \left|\ln h\right|^{\frac{1-\gamma'}{2}}} + O\left| \frac{1}{\ln(h)^{\infty}} \right|$. For finish, we have

$$\left\|a_n\right\|_{\ell^2(\mathbb{N})}^2 = K_h^2 \left|\ln h\right|^{1-\gamma'} \left[\mathfrak{F}\left(\chi^2\right)(0) + O\left| \frac{1}{\ln(h)^{\infty}} \right| \right] = 1 + O\left| \frac{1}{\ln(h)^{\infty}} \right|.$$

□

4.3 The set Δ

Definition 4.13. Let us define the sets of integers $\Delta = \Delta(h, E)$ and $\Gamma = \Gamma(h, E)$ by :

$$\Delta := \left\{ n \in \mathbb{N},\ \left|n - n_0\right| \leq \left|\ln(h)\right|^{\gamma} \right\} \subset \mathbb{N}$$

and

$$\Gamma := \mathbb{N} - \Delta$$

where γ is a real number such that $\gamma < 1$ and $\gamma + \gamma' > 1$.

Remark 4.14. Since $\gamma < 1$, we have $\left|\ln(h)\right|^{\gamma} < \left|\ln(h)\right|$, hence for $h \to 0$ we have

$$\mathrm{Card}\left(\mathbf{I}_h\right) \sim \left|\ln(h)\right|.$$

So, for h small enough we obtain $\Delta \subset \mathbf{I}_h$. On the other hand, since $\gamma + \gamma' > 1$ we have $\left|\ln(h)\right|^{1-\gamma'} \ll \left|\ln(h)\right|^{\gamma}$; this mean that the set Δ is larger than the localization of initial state.

Lemma 4.15. *We have*

$$\sum_{n \in \Gamma} \left|a_n\right|^2 = O\left| \frac{1}{\ln(h)^{\infty}} \right|.$$

Proof. The starting point is the following inequality :

$$\sum_{n\in\Gamma} |a_n|^2 \leq \sum_{n\in\mathbb{Z},\, |n-n_0|>h^{\delta-1}} |a_n|^2.$$

Next, with the same argument as in the lemma 4.11, we show that for all integer $N \geq 1$

$$\sum_{n\in\mathbb{Z}} \left(\frac{n-n_0}{h^{\delta'-1}}\right)^{2N} |a_n|^2 = O(1).$$

Without loss generality, we may suppose that $n_0 = 0$. Next we write :

$$\sum_{n\in\mathbb{Z},\, |n|>h^{\delta-1}} |a_n|^2 = h^{2N(\delta'-1)} \sum_{n\in\mathbb{Z},\, |n|>h^{\delta-1}} |a_n|^2 \left(\frac{n}{h^{\delta'-1}}\right)^{2N} \frac{1}{n^{2N}}$$

$$\leq \frac{h^{2N(\delta'-1)}}{h^{2N(\delta-1)}} \sum_{n\in\mathbb{Z}} |a_n|^2 \left(\frac{n}{h^{\delta'-1}}\right)^{2N} = O\left(h^{2N(\delta'-\delta)}\right).$$

Since $\delta' > \delta$, we obtain $\sum_{n\in\Gamma} |a_n|^2 = O(h^\infty)$. □

5 Order 1 approximation : hyperbolic period

5.1 Introduction

The aim of this section is to study the series :

$$\mathbf{a} \,:\, t \mapsto \sum_{n\in\mathbb{N}} |a_n|^2 \, e^{-it\mathcal{A}_h(2\pi n)}.$$

Unfortunately this function is too difficult to understand. So, in this section we use an approximation of $t \mapsto \mathbf{a}(t)$, the tricks is explained in section 2.3. Here, by a Taylor's formula we get :

$$\mathbf{a}(t) = \sum_{n\in\mathbb{N}} |a_n|^2 \, e^{-it\left(\mathcal{A}_h(2\pi n_0) + \mathcal{A}_h'(2\pi n_0)2\pi(n-n_0) + \mathcal{A}_h''(\zeta)2\pi^2(n-n_0)^2\right)}$$

where $\zeta = \zeta(n,h,E) \in \mathcal{Y}_h\left([-1,1]\right)$.

Next, we need the following lemma.

Lemma 5.1. *Uniformly, on the compact set* $[-1,1]$ *we have :*

$$\lambda \mapsto \left(\mathcal{A}_h'' \circ \mathcal{Y}_h\right)(\lambda) = O\left|\frac{1}{\ln(h)^3}\right|.$$

Proof. Derivatives formulas gives for all $x \in \mathcal{Y}_h\left([-1,1]\right)$, the equality

$$\mathcal{A}_h''(x) = \frac{-\left(\mathcal{Y}_h'' \circ \mathcal{A}_h\right)(x)}{\left(\mathcal{Y}_h' \circ \mathcal{A}_h\right)^3(x)};$$

hence, for $\lambda \in [-1,1]$

$$\left(\mathcal{A}_h'' \circ \mathcal{Y}_h\right)(\lambda) = \frac{-\mathcal{Y}_h''(\lambda)}{\left(\mathcal{Y}_h'\right)^3(\lambda)}.$$

First, for all $\lambda \in [-1,1]$

$$\mathcal{Y}_h''(\lambda) = -h^2 \frac{\theta_+''(\lambda h) + \theta_-''(\lambda h)}{2} + h\varepsilon''(\lambda h)\ln(h)$$

$$+\frac{\partial^2}{\partial\lambda^2}\left[\arg\left(\Gamma\left(\frac{1}{2}+i\frac{\varepsilon(\lambda h)}{h}\right)\right)\right] - \frac{\partial^2}{\partial\lambda^2}\left[\arccos\left(\frac{\cos\left(g_h(\lambda)\right)}{\sqrt{1+\exp\left(2\pi\varepsilon(\lambda h)/h\right)}}\right)\right].$$

And, for all $\lambda \in [-1,1]$ we have

$$-h^2\frac{\theta_+''(\lambda h) + \theta_-'(\lambda h)}{2} = O(h)$$

and

$$h\varepsilon''(\lambda h)\ln(h) = O(h\ln(h)).$$

Next, we study the term $\frac{\partial^2}{\partial\lambda^2}\left[\arg\left(\Gamma\left(\frac{1}{2}+i\frac{\varepsilon(\lambda h)}{h}\right)\right)\right]$: for all $\lambda \in [-1,1]$ we have :

$$\frac{\partial}{\partial\lambda}\left[\arg\left(\Gamma\left(\frac{1}{2}+i\frac{\varepsilon(\lambda h)}{h}\right)\right)\right] = \frac{\partial}{\partial\lambda}\left[\mathrm{Im}\left(\ln\left(\Gamma\left(\frac{1}{2}+i\frac{\varepsilon(\lambda h)}{h}\right)\right)\right)\right]$$

$$= \mathrm{Im}\left[\frac{\partial}{\partial\lambda}\left(\ln\left(\Gamma\left(\frac{1}{2}+i\frac{\varepsilon(\lambda h)}{h}\right)\right)\right)\right] = \mathrm{Im}\left[\frac{\Gamma'\left(\frac{1}{2}+i\frac{\varepsilon(\lambda h)}{h}\right)i\varepsilon'(\lambda h)}{\Gamma\left(\frac{1}{2}+i\frac{\varepsilon(\lambda h)}{h}\right)}\right]$$

$$= \varepsilon'(\lambda h)\mathrm{Re}\left[\frac{\Gamma'\left(\frac{1}{2}+i\frac{\varepsilon(\lambda h)}{h}\right)}{\Gamma\left(\frac{1}{2}+i\frac{\varepsilon(\lambda h)}{h}\right)}\right] = \varepsilon'(\lambda h)\mathrm{Re}\left(\Psi\left(\frac{1}{2}+i\frac{\varepsilon(\lambda h)}{h}\right)\right);$$

hence, for all $\lambda \in [-1,1]$

$$\frac{\partial^2}{\partial\lambda^2}\left[\arg\left(\Gamma\left(\frac{1}{2}+i\frac{\varepsilon(\lambda h)}{h}\right)\right)\right]$$

$$= \frac{\partial}{\partial\lambda}\left[\varepsilon'(\lambda h)\mathrm{Re}\left(\Psi\left(\frac{1}{2}+i\frac{\varepsilon(\lambda h)}{h}\right)\right)\right]$$

$$= h\varepsilon''(\lambda h)\mathrm{Re}\left(\Psi\left(\frac{1}{2}+i\frac{\varepsilon(\lambda h)}{h}\right)\right) - \left(\varepsilon'(\lambda h)\right)^2\mathrm{Im}\left(\Psi^{(1)}\left(\frac{1}{2}+i\frac{\varepsilon(\lambda h)}{h}\right)\right),$$

wher $\Psi^{(1)}$ is the first-derivative of di-Gamma function ($\Psi(z) := \frac{\Gamma'(z)}{\Gamma(z)}$, see **[Ab-St]**). Hence the function

$$\lambda \mapsto \frac{\partial}{\partial\lambda}\left[\varepsilon'(\lambda h)\mathrm{Re}\left(\Psi\left(\frac{1}{2}+i\frac{\varepsilon(\lambda h)}{h}\right)\right)\right]$$

is equal to $O(1)$ on the compact set $[-1,1]$. For finish, we estimate the term :

$$\lambda \mapsto \frac{\partial^2}{\partial\lambda^2}\left[\arccos\left(\frac{\cos\left(g_h(\lambda)\right)}{\sqrt{1+\exp\left(2\pi\varepsilon(\lambda h)/h\right)}}\right)\right];$$

for all $\lambda \in [-1,1]$ we obtain

$$\frac{\partial^2}{\partial\lambda^2}\left[\arccos\left(\frac{\cos\left(g_h(\lambda)\right)}{\sqrt{1+\exp\left(2\pi\varepsilon(\lambda h)/h\right)}}\right)\right] = \frac{N_h(\lambda)}{D_h(\lambda)}$$

where $N_h(\lambda)$ is a polynomial (with coefficients does not depends on h) in the variables :

$$\left(\cos\left(g_h(\lambda)\right), \sin\left(g_h(\lambda)\right), \frac{\partial}{\partial\lambda}\left(g_h(\lambda)\right), \frac{\partial^2}{\partial\lambda^2}\left(g_h(\lambda)\right)\right.$$

$$\left.\frac{\partial}{\partial\lambda}\left(\varepsilon(\lambda h)/h\right), \frac{\partial^2}{\partial\lambda^2}\left(\varepsilon(\lambda h)/h\right), \exp\left(2\pi\varepsilon(\lambda h)/h\right)\right).$$

Hence $\lambda \mapsto N_h(\lambda) = O(1)$ on the compact set $[-1,1]$. Next we have

$$D_h(\lambda) = \left(1+\exp\left(2\pi\varepsilon(\lambda h)/h\right)\right)^2\left(1+\exp\left(2\pi\varepsilon(\lambda h)/h\right)-\cos^2\left(g_h(\lambda)\right)\right)^{\frac{3}{2}};$$

since the function $\lambda \mapsto \exp\left(2\pi\varepsilon(\lambda h)/h\right) > 0$ is equal to $O(1)$ on the compact set $[-1,1]$, we see easily that on the compact set $[-1,1]$ we have

$$\lambda \mapsto \frac{1}{D_h(\lambda)} = O(1).$$

Next, on the set $[-1,1]$

$$\lambda \mapsto \frac{\partial^2}{\partial\lambda^2}\left[\arccos\left(\frac{\cos\left(g_h(\lambda)\right)}{\sqrt{1+\exp\left(2\pi\varepsilon(\lambda h)/h\right)}}\right)\right] = O(1).$$

Then on the compact set $[-1,1]$ we obtain that $\lambda \mapsto \mathcal{Y}_h''(\lambda) = O(1)$. For finish, since, uniformly on the compact set $[-1,1]$ we have the equality

$$\mathcal{Y}_h'(\lambda) = \frac{\ln(h)}{\sqrt{-V''(0)}} + O(1);$$

we deduce that $\lambda \mapsto \left(\mathcal{A}_h'' \circ \mathcal{Y}_h\right)(\lambda) = O\left|\frac{1}{\ln(h)^3}\right|$ on $[-1,1]$. □

5.2 Definition of a first order approximation time scale

We have the following approximation result :

Proposition 5.2. *Let α a real number such that $\alpha < 3-2\gamma$. Then, uniformly for all $t \in \left[0, |\ln(h)|^{\alpha}\right]$ we have :*

$$\mathbf{a}(t) = \sum_{n\in\mathbb{N}} |a_n|^2 e^{-it\left(\mathcal{A}_h(2\pi n_0)+\mathcal{A}_h'(2\pi n_0)2\pi(n-n_0)\right)} + O\left(|\ln(h)|^{\alpha+2\gamma-3}\right).$$

Proof. Let us introduce the difference $\varepsilon(t) := \varepsilon(t,h)$ defined by :

$$\varepsilon(t) := \left|\sum_{n\in\mathbb{N}} |a_n|^2 e^{-it\mathcal{A}_h(2\pi n)} - \sum_{n\in\mathbb{N}} |a_n|^2 e^{-it\left(\mathcal{A}_h(2\pi n_0)+\mathcal{A}_h'(2\pi n_0)2\pi(n-n_0)\right)}\right|.$$

Taylor-Lagrange's formula give the existence of a real number $\zeta = \zeta(n,h,E) \in \mathcal{Y}_h([-1,1])$ such that

$$\mathcal{A}_h(2\pi n) = \mathcal{A}_h(2\pi n_0) + \mathcal{A}_h'(2\pi n_0)2\pi(n-n_0) + \mathcal{A}_h''(\zeta)2\pi^2(n-n_0)^2;$$

hence for all $t \geq 0$ we get

$$\varepsilon(t) = \left| \sum_{n\in\mathbb{N}} |a_n|^2 e^{-it\left(\mathcal{A}_h(2\pi n_0)+\mathcal{A}_h'(2\pi n_0)2\pi(n-n_0)+\mathcal{A}_h''(\zeta)2\pi^2(n-n_0)^2\right)} \right.$$

$$\left. - \sum_{n\in\mathbb{N}} |a_n|^2 e^{-it\left(\mathcal{A}_h(2\pi n_0)+\mathcal{A}_h'(2\pi n_0)2\pi(n-n_0)\right)} \right|$$

$$= \left| \sum_{n\in\mathbb{N}} |a_n|^2 e^{-it\left(\mathcal{A}_h(2\pi n_0)+\mathcal{A}_h'(2\pi n_0)2\pi(n-n_0)\right)} \left[e^{-it\mathcal{A}_h''(\zeta)2\pi^2(n-n_0)^2} - 1 \right] \right|.$$

With the sets Γ,Δ and by triangular inequality, we obtain for all $t \geq 0$

$$\varepsilon(t) \leq \left| \sum_{n\in\Delta} |a_n|^2 e^{-it\left(\mathcal{A}_h(2\pi n_0)+\mathcal{A}_h'(2\pi n_0)2\pi(n-n_0)\right)} \left[e^{-it\mathcal{A}_h''(\zeta)2\pi^2(n-n_0)^2} - 1 \right] \right|$$

$$+ \left| \sum_{n\in\Gamma} |a_n|^2 e^{-it\left(\mathcal{A}_h(2\pi n_0)+\mathcal{A}_h'(2\pi n_0)2\pi(n-n_0)\right)} \left[e^{-it\mathcal{A}_h''(\zeta)2\pi^2(n-n_0)^2} - 1 \right] \right|. \qquad (1)$$

First, look at the term the right part of the inequality above, for all $t \geq 0$ we have

$$\left| \sum_{n\in\Gamma} |a_n|^2 e^{-it\left(\mathcal{A}_h(2\pi n_0)+\mathcal{A}_h'(2\pi n_0)2\pi(n-n_0)\right)} \left[e^{-it\mathcal{A}_h''(\zeta)2\pi^2(n-n_0)^2} - 1 \right] \right|$$

$$\leq 2 \sum_{n\in\Gamma} |a_n|^2 = O\left| \frac{1}{\ln(h)^\infty} \right|$$

by the lemma 4.15.

Now look at the left term of the inequality above. With the lemma 5.1: for all integer $n \in \Delta$ and for all real number $t \in \left[0, |\ln(h)|^\alpha\right]$ we have

$$t\mathcal{A}_h''(\zeta)2\pi^2(n-n_0)^2 \leq M |\ln(h)|^{\alpha+2\gamma-3}$$

where $M > 0$. Consequently, for all integer $n \in \Delta$ and for all $t \in \left[0, |\ln(h)|^\alpha\right]$ we have

$$e^{-it\mathcal{A}_h''(\zeta)2\pi^2(n-n_0)^2} - 1 = O\left(|\ln(h)|^{\alpha+2\gamma-3} \right);$$

hence for all $t \in \left[0, |\ln(h)|^\alpha\right]$

$$\left| \sum_{n\in\Delta} |a_n|^2 e^{-it\left(\mathcal{A}_h(2\pi n_0)+\mathcal{A}_h'(2\pi n_0)2\pi(n-n_0)\right)} \left[e^{-it\mathcal{A}_h''(\zeta)2\pi^2(n-n_0)^2} - 1 \right] \right|$$

$$\leq O\left(|\ln(h)|^{\alpha+2\gamma-3} \right) \sum_{n\in\Delta} |a_n|^2$$

$$\leq O\left(|\ln(h)|^{\alpha+2\gamma-3}\right)\sum_{n\in\mathbb{N}}|a_n|^2 = O\left(|\ln(h)|^{\alpha+2\gamma-3}\right).$$

So, for all $t\in\left[0,|\ln(h)|^{\alpha}\right]$ we get finally

$$\varepsilon(t) = O\left(|\ln(h)|^{\alpha+2\gamma-3}\right).$$

□

We conclude that, the principal term of the partial autocorrelation $\mathbf{a}(t)$ in the time-scale $\left[0,|\ln(h)|^{\alpha}\right]$ is:

Definition 5.3. The principal term of the partial autocorrelation $\mathbf{a}(t)$ is defined by :

$$\mathbf{a_1}\ : t\mapsto \sum_{n\in\mathbb{N}}|a_n|^2\, e^{-it\left(\mathcal{A}_h(2\pi n_0)+\mathcal{A}'_h(2\pi n_0)2\pi(n-n_0)\right)}.$$

We also define the function $\widetilde{\mathbf{a_1}}$ by

$$\widetilde{\mathbf{a_1}}\ : t\mapsto \sum_{n\in\mathbb{N}}|a_n|^2\, e^{-it\mathcal{A}'_h(2\pi n_0)2\pi(n-n_0)}.$$

Hence

$$\mathbf{a_1}(t) = e^{-it\mathcal{A}_h(2\pi n_0)}\widetilde{\mathbf{a_1}}(t).$$

Now, we study this serie in details.

5.3 Periodicity of the principal term

Clearly, the function $t\mapsto|\mathbf{a_1}(t)|$ is $1/\mathcal{A}'_h(2\pi n_0)$-periodic, so the approximation $\mathbf{a_1}$ defines an important characteristic time scale :

Definition 5.4. Let us define the hyperbolic period $T_{hyp} = T_{hyp}(h,E)$ by

$$T_{hyp} := \frac{1}{\mathcal{A}'_h(2\pi n_0)}.$$

Note that, since $\mathcal{A}'_h(\zeta) = 2\pi/\left(\frac{|\ln(h)|}{\sqrt{-V''(0)}}+O(1)\right)$; hence for $h\to 0$ we have $T_{hyp}\sim K\,|\ln(h)|$; where $K>0$ (not depends on h).

5.4 Geometric interpretation of the period

The period T_{hyp} correspond to the period of the classical flow. The term $|\ln(h)|$ is the signature of the hyperbolic singularity : indeed, the term $|\ln(h)|$ correspond to the period of the classical flow with a initial point on the disc $D(O,\sqrt{h}) := \left\{(x,\xi)\in\mathbb{R}^2,\ \sqrt{x^2+\xi^2}=\sqrt{h}\right\}$.

Theorem 5.5. [Lab3]. Let us consider the point $m_h = \left(\sqrt{h},0\right)\in T^*\mathbb{R}$. Then the Hamiltonian's flow associated to the function p and with initial point m_h is periodic and the period τ_h verify, for $h\to 0$, the following equivalent :

$$\tau_h \sim \frac{|\ln(h)|}{K}$$

where the constant $K\neq 0$ and not depend on h.

Remark 5.6. See also the paper **[DB-Ro]** for a similar result.

5.5 Comparison between hyperbolic period and the time scale $\left[0, |\ln(h)|^{\alpha}\right]$

Since $\gamma < 1$, we have $1 < 3 - 2\gamma$; hence there exist a real number α such that $\alpha \in]1, 3 - 2\gamma[$. Consequently, we get

$$|\ln(h)| < |\ln(h)|^{\alpha} < |\ln(h)|^{3-2\gamma}.$$

So, we can make a "good choice" for α : for h small enough we have :

$$\left[0, T_{hyp}\right] \subset \left[0, |\ln(h)|^{\alpha}\right].$$

5.6 Behaviour of autocorrelation function on a hyperbolic period

Now, let us study in details the function $\mathbf{a_1}(t)$ on the period $\left[0, T_{hyp}\right]$. Start by a technical proposition :

Proposition 5.7. *For all $t \geq 0$, we have the equality :*

$$\sum_{n\in\mathbb{Z}} |a_n|^2 e^{-it2\pi(n-n_0)\frac{1}{T_{hyp}}} = \frac{1}{\mathfrak{F}(\chi^2)(0)} \sum_{\ell\in\mathbb{Z}} \mathfrak{F}\left(\chi^2\right)\left(-|\ln h|^{1-\gamma'}\left(\ell + \frac{t}{T_{hyp}}\right)\right).$$

Proof. Let us consider the function Ω_t defined by

$$\Omega_t : \begin{cases} \mathbb{R} \to \mathbb{C} \\ \\ x \mapsto |a_x|^2 e^{-it2\pi(x-n_0)\frac{1}{T_{hyp}}} \end{cases}$$

where $t \in \mathbb{R}$ is a parameter, and let us recall a_x is defined by

$$a_x = \frac{1}{\sqrt{\mathfrak{F}(\chi^2)(0)}\,|\ln h|^{\frac{1-\gamma'}{2}}} \chi\left(\frac{x - n_0}{|\ln h|^{1-\gamma'}}\right);$$

thus

$$\sum_{n\in\mathbb{Z}} |a_n|^2 e^{-it2\pi(n-n_0)\frac{1}{T_{hyp}}} = \sum_{n\in\mathbb{Z}} \Omega_t(n).$$

So clearly, the function $\Omega_t \in \mathcal{S}(\mathbb{R})$, then the Fourier transform $\mathfrak{F}(\Omega_t)$ is equal, for all $\zeta \in \mathbb{R}$, to

$$\mathfrak{F}(\Omega_t)(\zeta) = \frac{1}{\mathfrak{F}(\chi^2)(0)\,|\ln h|^{1-\gamma'}} e^{it2\pi n_0 \frac{1}{T_{hyp}}} \mathfrak{F}\left(\chi^2\left(\frac{x - n_0}{|\ln h|^{1-\gamma'}}\right) e^{-it2\pi x \frac{1}{T_{hyp}}}\right)(\zeta).$$

$$= \frac{1}{\mathfrak{F}(\chi^2)(0)} e^{-2i\pi n_0 \zeta} \mathfrak{F}\left(\chi^2\right)\left(-|\ln h|^{1-\gamma'}\left(\zeta + \frac{t}{T_{hyp}}\right)\right).$$

With the Poisson formula we get

$$\sum_{n\in\mathbb{Z}} \Omega_t(n) = \sum_{\ell\in\mathbb{Z}} \mathfrak{F}(\Omega_t)(\ell)$$

$$= \frac{1}{\mathfrak{F}\left(\chi^2\right)(0)} \sum_{\ell \in \mathbb{Z}} \mathfrak{F}\left(\chi^2\right)\left(-\left|\ln h\right|^{1-\gamma'}\left(\ell + \frac{t}{T_{hyp}}\right)\right).$$

□

So, now, our goal is to study the behaviour of the serie :

$$t \mapsto \frac{1}{\mathfrak{F}\left(\chi^2\right)(0)} \sum_{\ell \in \mathbb{Z}} \mathfrak{F}\left(\chi^2\right)\left(-\left|\ln h\right|^{1-\gamma'}\left(\ell + \frac{t}{T_{hyp}}\right)\right).$$

Since the function $\mathfrak{F}\left(\chi^2\right) \in \mathcal{S}(\mathbb{R})$, it's clear that the only index $\ell \in \mathbb{Z}$ such that $\left(\ell + \frac{t}{T_{hyp}}\right)$ is close to zero are important. More precisely :

Definition 5.8. For all $t \in \mathbb{R}$, let us define $\ell(t) = \ell(t, h, E)$ the closest integer to the real number $-t/T_{hyp}$; e.g :

$$\ell(t) + \frac{t}{T_{hyp}} = d\left(t, T_{hyp}\mathbb{Z}\right);$$

where $d(.,.)$ denote the Euclidiean distance on $\mathbb{R}$.

Remark 5.9. Without loss of generality, we may suppose $\ell(t)$ is unique. On the other hand, for all integer $\ell \in \mathbb{Z}$ such that $\ell \neq \ell(t)$ we get :

$$\left|\ell + \frac{t}{T_{hyp}}\right| \geq \frac{1}{2}.$$

Lemma 5.10. *For a function $\varphi \in \mathcal{S}(\mathbb{R})$ and $\varepsilon \in]0,1]$ then, uniformly for $u \in \mathbb{R}$ we have :*

$$\sum_{\ell \in \mathbb{Z},\, |\ell+u| \geq \frac{1}{2}} \left|\varphi\left(\frac{\ell+u}{\varepsilon}\right)\right| = O(\varepsilon^{\infty}).$$

Proof. We see easily that, uniformly for $u \in \mathbb{R}$ we have

$$\sum_{\ell \in \mathbb{Z},\, |\ell+u| \geq \frac{1}{2}} \left|\varphi\left(\frac{\ell+u}{\varepsilon}\right)\right| = O(1).$$

Indeed, since $\varepsilon \in]0,1]$, without loss of generality, we may suppose $u \in [-1,1]$. And since for all $u \in [-1,1]$ we have

$$\left|\varphi\left(\frac{\ell+u}{\varepsilon}\right)\right| \leq \frac{M}{1 + \frac{(\ell+u)^2}{\varepsilon^2}} = \frac{\varepsilon^2 M}{\varepsilon^2 + (\ell+u)^2}$$

$$\leq \frac{M}{(\ell+u)^2} = \frac{M}{\left|\ell^2 - 2\left|\ell\right|\right|};$$

and we conclude.

Next,

$$\sum_{\ell \in \mathbb{Z},\, |\ell+u| \geq \frac{1}{2}} \left|\varphi\left(\frac{\ell+u}{\varepsilon}\right)\right| = \sum_{\ell \in \mathbb{Z},\, |\ell+u| \geq \frac{1}{2}} \left(\frac{\ell+u}{\varepsilon}\right)^{2N} \left|\varphi\left(\frac{\ell+u}{\varepsilon}\right)\right| \frac{\varepsilon^{2N}}{(\ell+u)^{2N}}$$

$$\leq \varepsilon^{2N} 4^N \sum_{\ell \in \mathbb{Z}} \left(\frac{\ell + u}{\varepsilon} \right)^{2N} \left| \varphi \left(\frac{\ell + u}{\varepsilon} \right) \right|.$$

To conclude the proof, we apply that to the function $\psi(x) := x^{2N} \varphi(x)$. □

Theorem 5.11. Uniformly for $t \geq 0$ we have :

$$\sum_{n \in \mathbb{N}} |a_n|^2 e^{\frac{-it2\pi(n-n_0)}{T_{hyp}}} = \frac{1}{\mathfrak{F}(\chi^2)(0)} \mathfrak{F}\left(\chi^2\right) \left(- |\ln h|^{1-\gamma'} d\left(t, T_{hyp}\mathbb{Z}\right)\right) + O\left| \frac{1}{\ln(h)^\infty} \right|.$$

Proof. Since the proposition 5.7 and with the lemma above we get for all $t \geq 0$

$$\sum_{n \in \mathbb{Z}} |a_n|^2 e^{-it2\pi(n-n_0)\frac{t}{T_{hyp}}} = \frac{1}{\mathfrak{F}(\chi^2)(0)} \sum_{\ell \in \mathbb{Z}} \mathfrak{F}\left(\chi^2\right) \left(- |\ln h|^{1-\gamma'} \left(\ell + \frac{t}{T_{hyp}} \right)\right)$$

$$= \frac{1}{\mathfrak{F}(\chi^2)(0)} \mathfrak{F}\left(\chi^2\right) \left(- |\ln h|^{1-\gamma'} d\left(t, T_{hyp}\mathbb{Z}\right)\right) + O\left| \frac{1}{\ln(h)^\infty} \right|.$$

Next, for all $t \geq 0$

$$\sum_{n \in \mathbb{N}} |a_n|^2 e^{-it2\pi(n-n_0)\frac{1}{T_{hyp}}}$$

$$= \sum_{n \in \mathbb{Z}} |a_n|^2 e^{-it2\pi(n-n_0)\frac{1}{T_{hyp}}} - \sum_{n=-\infty}^{-1} |a_n|^2 e^{-it2\pi(n-n_0)\frac{1}{T_{hyp}}};$$

$$= \frac{1}{\mathfrak{F}(\chi^2)(0)} \sum_{\ell \in \mathbb{Z}} \mathfrak{F}\left(\chi^2\right) \left(- |\ln h|^{1-\gamma'} \left(\ell + \frac{t}{T_{hyp}} \right)\right)$$

$$- \sum_{n=-\infty}^{-1} |a_n|^2 e^{-it2\pi(n-n_0)\frac{1}{T_{hyp}}}.$$

Hence, by triangular inequality, we have for all $t \geq 0$

$$\left| \sum_{n \in \mathbb{N}} |a_n|^2 e^{-it2\pi(n-n_0)\frac{1}{T_{hyp}}} - \frac{1}{\mathfrak{F}(\chi^2)(0)} \mathfrak{F}\left(\chi^2\right) \left(- |\ln h|^{1-\gamma'} d\left(t, T_{hyp}\mathbb{Z}\right)\right) \right|$$

$$\leq \sum_{\ell \neq \ell(t)} \left| \frac{1}{\mathfrak{F}(\chi^2)(0)} \mathfrak{F}\left(\chi^2\right) \left(- |\ln h|^{1-\gamma'} \left(\ell + \frac{t}{T_{hyp}} \right)\right) \right| + \sum_{n=1}^{+\infty} |a_{-n}|^2.$$

Since, for all $t \geq 0$, we have

$$\frac{1}{\mathfrak{F}(\chi^2)(0)} \sum_{\ell \neq \ell(t)} \mathfrak{F}\left(\chi^2\right) \left(- |\ln h|^{1-\gamma'} \left(\ell + \frac{t}{T_{hyp}} \right)\right) = O\left| \frac{1}{\ln(h)^\infty} \right|;$$

and

$$\sum_{n=1}^{+\infty} |a_{-n}|^2 = O\left| \frac{1}{\ln(h)^\infty} \right|.$$

So we prove the theorem. □

Now, we can formulate the main result of the section 5 :

Corollary 5.12. *We have :*
(i) *for t such that $t \in T_{hyp}\mathbb{Z}$ we get*

$$\sum_{n\in\mathbb{N}} |a_n|^2 e^{-it\frac{2\pi(n-n_0)}{T_{hyp}}} = 1.$$

(ii) *For all $\varepsilon > 0$ such that $\varepsilon < 1-\gamma'$, and for t such that $\left|d\left(t, T_{hyp}\mathbb{Z}\right)\right| > |\ln h|^{\gamma'-1+\varepsilon}$; we get*

$$\sum_{n\in\mathbb{N}} |a_n|^2 e^{-it\frac{2\pi(n-n_0)}{T_{hyp}}} = O\left|\frac{1}{\ln(h)^{\infty}}\right|.$$

Proof. The first point is clear. For the second : if

$$\left|-|\ln h|^{1-\gamma'}\left(\ell(t)+t\mathcal{A}_h'(2\pi n_0)\right)\right| > |\ln h|^{\varepsilon}$$

thus, we have :

$$\left|\mathfrak{F}\left(\chi^2\right)\left(-|\ln h|^{1-\gamma'}\left(\ell(t)+t\mathcal{A}_h'(2\pi n_0)\right)\right)\right| \leq \frac{B_k}{\left(1+|\ln h|^{\varepsilon}\right)^k}$$

$$\leq B_k |\ln h|^{-\varepsilon k}.$$

for all integer $k \geq 1$; so we prove the second point. □

6 Second order approximation : revival period

6.1 Introduction

In this this section, we use a second order approximation of the function $t \mapsto \mathbf{a}(t)$, indeed by a Taylor's formula we have for all $t \geq 0$

$$\mathbf{a}(t) = \sum_{n\in\mathbb{N}} |a_n|^2 e^{-it\left(\mathcal{A}_h(2\pi n_0)+\mathcal{A}_h'(2\pi n_0)2\pi(n-n_0)+\mathcal{A}_h''(2\pi n_0)2\pi^2(n-n_0)^2+\mathcal{A}_h^{(3)}(\zeta)\frac{(2\pi)^3}{6}(n-n_0)^3\right)}$$

where $\zeta = \zeta(n,h,E)$.

We need the :

Lemma 6.1. *Uniformly, on the compact set $[-1,1]$ we have*

$$\lambda \mapsto \left(\mathcal{A}_h^{(3)} \circ \mathcal{Y}_h\right)(\lambda) = O\left|\frac{1}{\ln(h)^4}\right|.$$

Proof. With the derivatives formulas, we have for all $x \in \mathcal{Y}_h([-1,1])$

$$\mathcal{A}_h^{(3)}(x) = \frac{-\left(\mathcal{Y}_h^{(3)} \circ \mathcal{A}_h\right)(x)}{\left(\mathcal{Y}_h' \circ \mathcal{A}_h\right)^4(x)} + \frac{3\left(\mathcal{Y}_h'' \circ \mathcal{A}_h\right)^2(x)}{\left(\mathcal{Y}_h' \circ \mathcal{A}_h\right)^5(x)};$$

hence, for all $\lambda \in [-1,1]$ we get

$$\left(\mathcal{A}_h^{(3)} \circ \mathcal{Y}_h\right)(\lambda) = \frac{-\mathcal{Y}_h^{(3)}(\lambda)}{\left(\mathcal{Y}_h'\right)^4(\lambda)} + \frac{3\left(\mathcal{Y}_h''\right)^2(\lambda)}{\left(\mathcal{Y}_h'\right)^5(\lambda)}.$$

First, let us estimate the function $\lambda \mapsto \mathcal{Y}_h^{(3)}(\lambda)$. For all $\lambda \in [-1,1]$ we have

$$\mathcal{Y}_h^{(3)}(\lambda) = -h^3\frac{\theta_+^{(3)}(\lambda h) + \theta_-^{(3)}(\lambda h)}{2} + h^2\varepsilon^{(3)}(\lambda h)\ln(h)$$

$$+\frac{\partial^3}{\partial\lambda^3}\left[\arg\left(\Gamma\left(\frac{1}{2} + i\frac{\varepsilon(\lambda h)}{h}\right)\right)\right] - \frac{\partial^3}{\partial\lambda^3}\left[\arccos\left(\frac{\cos\left(g_h(\lambda)\right)}{\sqrt{1+\exp\left(2\pi\varepsilon(\lambda h)/h\right)}}\right)\right].$$

Uniformly, on the compact set $[-1,1]$ we have

$$\lambda \mapsto -h^3\frac{\theta_+^{(3)}(\lambda h) + \theta_-^{(3)}(\lambda h)}{2} = O(h^2)$$

and

$$\lambda \mapsto h^2\varepsilon^{(3)}(\lambda h)\ln(h) = O(h^2\ln(h)).$$

Next, let us estimate the function $\lambda \mapsto \frac{\partial^3}{\partial\lambda^3}\left[\arg\left(\Gamma\left(\frac{1}{2} + i\frac{\varepsilon(\lambda h)}{h}\right)\right)\right]$: for all $\lambda \in [-1,1]$

$$\frac{\partial^3}{\partial\lambda^3}\left[\arg\left(\Gamma\left(\frac{1}{2} + i\frac{\varepsilon(\lambda h)}{h}\right)\right)\right]$$

$$= \frac{\partial}{\partial\lambda}\left[h\varepsilon''(\lambda h)\mathrm{Re}\left(\Psi\left(\frac{1}{2} + i\frac{\varepsilon(\lambda h)}{h}\right)\right) - \left(\varepsilon'(\lambda h)\right)^2\mathrm{Im}\left(\Psi^{(1)}\left(\frac{1}{2} + i\frac{\varepsilon(\lambda h)}{h}\right)\right)\right]$$

$$= h^2\varepsilon^{(3)}(\lambda h)\mathrm{Re}\left(\Psi\left(\frac{1}{2} + i\frac{\varepsilon(\lambda h)}{h}\right)\right) + h\varepsilon''(\lambda h)\frac{\partial}{\partial\lambda}\left[\mathrm{Re}\left(\Psi\left(\frac{1}{2} + i\frac{\varepsilon(\lambda h)}{h}\right)\right)\right]$$

$$-2h\varepsilon'(\lambda h)\varepsilon''(\lambda h)\mathrm{Im}\left(\Psi^{(1)}\left(\frac{1}{2} + i\frac{\varepsilon(\lambda h)}{h}\right)\right) - \left(\varepsilon'(\lambda h)\right)^2\frac{\partial}{\partial\lambda}\left[\mathrm{Im}\left(\Psi^{(1)}\left(\frac{1}{2} + i\frac{\varepsilon(\lambda h)}{h}\right)\right)\right]$$

$$= h^2\varepsilon^{(3)}(\lambda h)\mathrm{Re}\left(\Psi\left(\frac{1}{2} + i\frac{\varepsilon(\lambda h)}{h}\right)\right) - 3h\varepsilon''(\lambda h)\varepsilon'(\lambda h)\mathrm{Im}\left(\Psi^{(1)}\left(\frac{1}{2} + i\frac{\varepsilon(\lambda h)}{h}\right)\right)$$

$$+\left(\varepsilon'(\lambda h)\right)^3\mathrm{Re}\left(\Psi^{(2)}\left(\frac{1}{2} + i\frac{\varepsilon(\lambda h)}{h}\right)\right);$$

thus, the function $\lambda \mapsto \frac{\partial^3}{\partial\lambda^3}\left[\arg\left(\Gamma\left(\frac{1}{2} + i\frac{\varepsilon(\lambda h)}{h}\right)\right)\right]$ is equal to a $O(1)$ on the compact set [-1,1]. Now, for finish, let us estimate the function

$$\lambda \mapsto \frac{\partial^3}{\partial\lambda^3}\left[\arccos\left(\frac{\cos\left(g_h(\lambda)\right)}{\sqrt{1+\exp\left(2\pi\varepsilon(\lambda h)/h\right)}}\right)\right];$$

with the notations introduced in the proof of lemma 5.1, for all $\lambda \in [-1,1]$ we have

$$\frac{\partial^3}{\partial\lambda^3}\left[\arccos\left(\frac{\cos\left(g_h(\lambda)\right)}{\sqrt{1+\exp\left(2\pi\varepsilon(\lambda h)/h\right)}}\right)\right] = \frac{\partial}{\partial\lambda}\left(\frac{N_h}{D_h}\right)$$

$$= \frac{N_h'(\lambda)D_h(\lambda) - N_h(\lambda)D_h'(\lambda)}{D_h^2(\lambda)};$$

we have seen into the proof of lemma 5.1, that, on the compact set $[-1,1]$

$$\lambda \mapsto \frac{1}{D_h^2(\lambda)} = O(1).$$

By the same argument as into the prof of lemma 5.1, we show that $\lambda \mapsto N_h'(\lambda)D_h(\lambda) - N_h(\lambda)D_h'(\lambda)$ is to equal to a $O(1)$ on the compact set $[-1,1]$. Consequently for all $\lambda \in [-1,1]$ the function

$$\lambda \mapsto \frac{\partial^3}{\partial \lambda^3}\left[\arccos\left(\frac{\cos\left(g_h(\lambda)\right)}{\sqrt{1+\exp\left(2\pi\varepsilon(\lambda h)/h\right)}}\right)\right]$$

is to equal to a $O(1)$ on the compact set $[-1,1]$. Thus, on $[-1,1]$ we have $\lambda \mapsto \mathcal{Y}_h^{(3)}(\lambda) = O(1)$.

Since, we have, uniformly, on the compact set $[-1,1]$ the equality

$$\mathcal{Y}_h'(\lambda) = \frac{\ln(h)}{\sqrt{-V''(0)}} + O(1),$$

we deduce, that the function

$$\lambda \mapsto \frac{-\mathcal{Y}_h^{(3)}(\lambda)}{\left(\mathcal{Y}_h'\right)^4(\lambda)} = O\left|\frac{1}{\ln(h)^4}\right|$$

on the compact set $[-1,1]$.

In the proof of lemma 5.1 we have seen, that, on the compact set $[-1,1]$:

$$\lambda \mapsto \mathcal{Y}_h''(\lambda) = O\left|\frac{1}{\ln(h)^3}\right|;$$

hence, on the compact set $[-1,1]$, we get

$$\lambda \mapsto \frac{\left(\mathcal{Y}_h''\right)^2(\lambda)}{\left(\mathcal{Y}_h'\right)^5(\lambda)} = O\left|\frac{1}{\ln(h)^5}\right|;$$

thus

$$\lambda \mapsto \left(\mathcal{A}_h^{(3)} \circ \mathcal{Y}_h\right)(\lambda) = \frac{-\mathcal{Y}_h^{(3)}(\lambda)}{\left(\mathcal{Y}_h'\right)^4(\lambda)} + \frac{3\left(\mathcal{Y}_h''\right)^2(\lambda)}{\left(\mathcal{Y}_h'\right)^5(\lambda)}$$

is equal to a $O\left|\frac{1}{\ln(h)^4}\right|$ on the set $[-1,1]$. □

Notation 6.2. Let us denote by $Q_2(X) = Q_2(h, n_0, X)$ the polynomial of $\mathbb{R}_2[X]$ defined by :

$$Q_2(X) := \mathcal{A}_h(2\pi n_0) + \mathcal{A}_h'(2\pi n_0)2\pi(X - n_0) + \mathcal{A}_h''(2\pi n_0)2\pi^2(X - n_0)^2.$$

6.2 Definition of a new time scale

Proposition 6.3. *Let β be a real number such that $\beta < 4 - 3\gamma$. Then, uniformly for all $t \in \left[0, |\ln(h)|^{\beta}\right]$ we have :*

$$\mathbf{a}(t) = \sum_{n\in\mathbb{N}} |a_n|^2 e^{-itQ_2(n)} + O\left(|\ln(h)|^{\beta+3\gamma-4}\right).$$

Proof. Let us introduce the difference $\varepsilon(t) := \varepsilon(t,h)$ defined by :

$$\varepsilon(t) := \left|\mathbf{a}(t) - \sum_{n\in\mathbb{N}} |a_n|^2 e^{-itQ_2(n)}\right|.$$

Taylor-Lagrange's formula gives the existence of a real number $\zeta = \zeta(n,h,E) \in \mathcal{Y}_h([-1,1])$ such that :

$$\mathcal{A}_h(2\pi n) = \mathcal{A}_h(2\pi n_0) + \mathcal{A}_h'(2\pi n_0)2\pi(n-n_0)$$

$$+\mathcal{A}_h''(2\pi n_0)2\pi^2(n-n_0)^2 + \mathcal{A}_h^{(3)}(\zeta)\frac{(2\pi)^3}{6}(n-n_0)^3;$$

hence, for all $t \geq 0$ we get

$$\varepsilon(t) = \left|\sum_{n\in\mathbb{N}} |a_n|^2 e^{-it\left(Q_2(n)+\mathcal{A}_h^{(3)}(\zeta)\frac{(2\pi)^3}{6}(n-n_0)^3\right)} - \sum_{n\in\mathbb{N}} |a_n|^2 e^{-itQ_2(n)}\right|$$

$$= \left|\sum_{n\in\mathbb{N}} |a_n|^2 e^{-itQ_2(n)}\left[e^{-it\mathcal{A}_h^{(3)}(\zeta)\frac{(2\pi)^3}{6}(n-n_0)^3} - 1\right]\right|.$$

With the sets Γ, Δ and by triangular inequality, we obtain for all $t \geq 0$

$$\varepsilon(t) \leq \left|\sum_{n\in\Delta} |a_n|^2 e^{-itQ_2(n)}\left[e^{-it\mathcal{A}_h^{(3)}(\zeta)\frac{(2\pi)^3}{6}(n-n_0)^3} - 1\right]\right|$$

$$+\left|\sum_{n\in\Gamma} |a_n|^2 e^{-itQ_2(n)}\left[e^{-it\mathcal{A}_h^{(3)}(\zeta)\frac{(2\pi)^3}{6}(n-n_0)^3} - 1\right]\right|.$$

Since :

$$\left|\sum_{n\in\Gamma} |a_n|^2 e^{-itQ_2(n)}\left[e^{-it\mathcal{A}_h^{(3)}(\zeta)\frac{(2\pi)^3}{6}(n-n_0)^3} - 1\right]\right| \leq 2\sum_{n\in\Gamma} |a_n|^2 = O\left|\frac{1}{\ln(h)^{\infty}}\right|$$

by the lemma 4.15.

On the other hand, with the lemma 6.1: for all integer $n \in \Delta$ and for all real number $t \in \left[0, |\ln(h)|^{\beta}\right]$

$$\left|t\mathcal{A}_h^{(3)}(\zeta)\frac{(2\pi)^3}{6}(n-n_0)^3\right| \leq M|\ln(h)|^{\beta+3\gamma-4}$$

where $M > 0$. Thus, for all integer $n \in \Delta$ and for all real number $t \in \left[0, |\ln(h)|^{\beta}\right]$ we have

$$e^{-it\mathcal{A}_h^{(3)}(\zeta)\frac{(2\pi)^3}{6}(n-n_0)^3} - 1 = O\left(|\ln(h)|^{\beta+3\gamma-4}\right)$$

hence for all $t \in \left[0, |\ln(h)|^{\beta}\right]$ we get

$$\left|\sum_{n\in\Delta} |a_n|^2 e^{-itQ_2(n)} \left[e^{-it\mathcal{A}_h^{(3)}(\zeta)\frac{(2\pi)^3}{6}(n-n_0)^3} - 1\right]\right|$$

$$\leq O\left(|\ln(h)|^{\beta+3\gamma-4}\right) \sum_{n\in\Delta} |a_n|^2$$

$$\leq O\left(|\ln(h)|^{\beta+3\gamma-4}\right) \sum_{n\in\mathbb{N}} |a_n|^2 = O\left(|\ln(h)|^{\beta+3\gamma-4}\right).$$

So, for all $t \in \left[0, |\ln(h)|^{\beta}\right]$ we get finally

$$\varepsilon(t) = O\left(|\ln(h)|^{\beta+3\gamma-4}\right).$$

□

Definition 6.4. Let us define the second order approximation of the partial autocorrelation $\mathbf{a}(t)$ by :

$$\mathbf{a_2} : t \mapsto \sum_{n\in\mathbb{N}} |a_n|^2 e^{-itQ_2(n)}.$$

And we also define the function

$$\widetilde{\mathbf{a_2}} : t \mapsto \sum_{n\in\mathbb{N}} |a_n|^2 e^{-it\left(\mathcal{A}'_h(2\pi n_0)2\pi(n-n_0)+\mathcal{A}''_h(2\pi n_0)2\pi^2(n-n_0)^2\right)}.$$

Thus, we get

$$\mathbf{a_2}(t) = e^{-it\mathcal{A}_h(2\pi n_0)}\widetilde{\mathbf{a_2}}(t)$$

and, we have also

$$|\mathbf{a_2}(t)| = |\widetilde{\mathbf{a_2}}(t)| = \left|\sum_{n\in\mathbb{N}} |a_n|^2 e^{-it\left(\mathcal{A}'_h(2\pi n_0)2\pi(n-n_0)+\mathcal{A}''_h(2\pi n_0)2\pi^2(n-n_0)^2\right)}\right|.$$

6.3 Full revival theorem

Let us start by some notations.

Definition 6.5. Let us define the revival time $T_{rev} = T_{rev}(h, E)$ by :

$$T_{rev} := \frac{1}{\pi\mathcal{A}''_h(2\pi n_0)}$$

and we also denote the integer $N_h = N(h)$ defined by :

$$N_h := E\left[\frac{T_{rev}}{T_{hyp}}\right] \in \mathbb{N}.$$

Hence, there exists an unique real number $\Theta_h \in [0,1[$ such that :

$$\frac{T_{rev}}{T_{hyp}} = N_h + \Theta_h.$$

Proposition 6.6. *If we suppose that*

$$\lim_{h\to 0} \left(\mathcal{Y}_h'' \circ \mathcal{A}_h\right)(2\pi n_0) = K$$

where $K \neq 0$ is a constant, then

$$T_{rev} = \frac{|\ln(h)|^3}{K\left(-V''(0)\right)^{\frac{3}{2}}} + O(1).$$

Proof. Since

$$T_{rev} = -\frac{\left(\mathcal{Y}_h' \circ \mathcal{A}_h\right)^3(2\pi n_0)}{\pi\left(\mathcal{Y}_h'' \circ \mathcal{A}_h\right)(2\pi n_0)};$$

if we suppose that

$$\lim_{h\to 0} \left(\mathcal{Y}_h'' \circ \mathcal{A}_h\right)(2\pi n_0) = K;$$

then we deduce

$$T_{rev} = \frac{-\ln(h)^3}{K\left(-V''(0)\right)^{\frac{3}{2}}} + O(1).$$

□

The full revival theorem is the following :

Theorem 6.7. With the previous notations we have :
(i) if $\Theta_h = 0$; then for all $t \geq 0$:

$$\widetilde{\mathbf{a_2}}(t + N_h T_{hyp}) = \widetilde{\mathbf{a_2}}(t + T_{rev}) = \widetilde{\mathbf{a_2}}(t);$$

(ii) if $\Theta_h \in]0,1[$, then for all $t \geq 0$

$$\widetilde{\mathbf{a_2}}(t + N_h T_{hyp}) = \widetilde{\mathbf{a_2}}(t) + O\left|\frac{1}{\ln(h)^{2-2\gamma}}\right|;$$

in particular, we have

$$\left|\mathbf{a_2}(t + N_h T_{hyp})\right| = |\mathbf{a_2}(t)| + O\left|\frac{1}{\ln(h)^{2-2\gamma}}\right|.$$

Proof. For all $t \geq 0$

$$\widetilde{\mathbf{a_2}}(t + N_h T_{hyp}) = \sum_{n\in\mathbb{N}} |a_n|^2 e^{-2i\pi\frac{t+N_h T_{hyp}}{T_{hyp}}(n-n_0)} e^{-2i\pi\frac{(t+T_{rev}-T_{hyp}\Theta_h)}{T_{rev}}(n-n_0)^2}$$

$$= \sum_{n\in\mathbb{N}} |a_n|^2 e^{-2i\pi\frac{t}{T_{hyp}}(n-n_0)} e^{-2i\pi\frac{(t-T_{hyp}\Theta_h)}{T_{rev}}(n-n_0)^2}.$$

If we suppose that $\Theta_h = 0$; then for all $t \geq 0$

$$\widetilde{\mathbf{a_2}}(t + N_h T_{hyp}) = \widetilde{\mathbf{a_2}}(t).$$

Suppose that $\Theta_h \in]0,1[$, then for all $t \in \mathbb{R}_+$

$$\left|\widetilde{\mathbf{a_2}}(t + N_h T_{hyp}) - \widetilde{\mathbf{a_2}}(t)\right|$$

$$= \left| \sum_{n\in\mathbb{N}} |a_n|^2 e^{-2i\pi\frac{t}{T_{hyp}}(n-n_0)} e^{-2i\pi\frac{(t-T_{hyp}\Theta_h)}{T_{rev}}(n-n_0)^2} \right.$$

$$\left. - \sum_{n\in\mathbb{N}} |a_n|^2 e^{-2i\pi\frac{t}{T_{hyp}}(n-n_0)} e^{-2i\pi\frac{t}{T_{rev}}(n-n_0)^2} \right|$$

$$= \left| \sum_{n\in\mathbb{N}} |a_n|^2 e^{-2i\pi\frac{t}{T_{hyp}}(n-n_0)} e^{-2i\pi\frac{t}{T_{rev}}(n-n_0)^2} \left(e^{2i\pi\Theta_h\frac{T_{hyp}}{T_{rev}}(n-n_0)^2} - 1 \right) \right|$$

$$\leq \left| \sum_{n\in\Delta} |a_n|^2 e^{-2i\pi\frac{t}{T_{hyp}}(n-n_0)} e^{-2i\pi\frac{t}{T_{rev}}(n-n_0)^2} \left(e^{2i\pi\Theta_h\frac{T_{hyp}}{T_{rev}}(n-n_0)^2} - 1 \right) \right|$$

$$+ \left| \sum_{n\in\Gamma} |a_n|^2 e^{-2i\pi\frac{t}{T_{hyp}}(n-n_0)} e^{-2i\pi\frac{t}{T_{rev}}(n-n_0)^2} \left(e^{2i\pi\Theta_h\frac{T_{hyp}}{T_{rev}}(n-n_0)^2} - 1 \right) \right|.$$

For all integer $n \in \Delta$ we have $(n-n_0)^2 \leq |\ln(h)|^{2\gamma}$, hence, for $h \to 0$, we get, for all integer $n \in \Delta$

$$e^{2i\pi\Theta_h\frac{T_{hyp}}{T_{rev}}(n-n_0)^2} = 1 + O\left(|\ln(h)|^{2\gamma-2}\right).$$

Thus, for all $t \geq 0$

$$\left| \sum_{n\in\Delta} |a_n|^2 e^{-2i\pi\frac{t}{T_{hyp}}(n-n_0)} e^{-2i\pi\frac{t}{T_{rev}}(n-n_0)^2} \left(e^{2i\pi\Theta_h\frac{T_{hyp}}{T_{rev}}(n-n_0)^2} - 1 \right) \right|$$

$$\leq O\left(|\ln(h)|^{2\gamma-2}\right) \sum_{n\in\Delta} |a_n|^2$$

$$\leq O\left(|\ln(h)|^{2\gamma-2}\right) \sum_{n\in\mathbb{N}} |a_n|^2 = O\left(|\ln(h)|^{2\gamma-2}\right).$$

On the other hand, we have

$$\left| \sum_{n\in\Gamma} |a_n|^2 e^{-2i\pi\frac{t}{T_{hyp}}(n-n_0)} e^{-2i\pi\frac{t}{T_{rev}}(n-n_0)^2} \left(e^{2i\pi\Theta_h\frac{T_{hyp}}{T_{rev}}(n-n_0)^2} - 1 \right) \right|$$

$$\leq \sum_{n\in\Gamma} 2\,|a_n|^2 = O\left|\frac{1}{\ln(h)^\infty}\right|.$$

□

Remark 6.8. For $h \to 0$, we have

$$\frac{N_h T_{hyp}}{T_{rev}} = \frac{T_{rev} - \Theta_h T_{hyp}}{T_{rev}} = 1 - \frac{\Theta_h T_{hyp}}{T_{rev}} \to 1$$

thus, for $h \to 0$ we have also

$$N_h T_{hyp} \sim T_{rev}.$$

The previous theorem show that the function $t \mapsto \widetilde{\mathbf{a_2}}(t)$, is, modulo $O\left(|\ln(h)|^{2\gamma-2}\right)$, periodic, with a period equivalent (for $h \to 0$) to T_{rev}. This is the full revival phenomenon.

6.4 Fractional revivals theorem

The aim of this section is to study the dynamics for time close to $\frac{p}{q}T_{rev}$, where $\frac{p}{q} \in \mathbb{Q}$.

6.4.1 Preliminaries

Notation 6.9. For all $(p,q) \in \mathbb{Z} \times \mathbb{N}^*$ let us consider the sequence $(\sigma_h(p,q))$ defined by :

$$(\sigma_h(p,q))_n := e^{-2i\pi\frac{p}{q}(n-n_0)^2} \ , \ n \in \mathbb{Z}.$$

The periodicity of this sequence is caracterised by the following easy proposition.

Proposition 6.10. *For all $(p,q) \in \mathbb{Z} \times \mathbb{N}^*$, the sequence $(\sigma_h(p,q))_n$ is ℓ-periodic if and only if the integer ℓ satisfy the equation :*

$$\forall m \in \mathbb{Z},\ \frac{2p\ell}{q}m + \frac{p\ell^2}{q} \equiv 0\ (mod\, 1).$$

Now, we solve this equation.

Proposition 6.11. *Suppose $p \wedge q = 1$, then the set $\mathcal{E}$ of solutions ℓ such that :*

$$\forall m \in \mathbb{Z},\ \frac{2p\ell}{q}m + \frac{p\ell^2}{q} \equiv 0\ (mod\, 1)$$

is caracterised by :
(i) *if q is odd then $\mathcal{E} = \{q\mathbb{Z}\}$;*
(ii) *si q even and $\frac{q}{2}$ odd then $\mathcal{E} = \{q\mathbb{Z}\}$;*
(iii) *si q even et $\frac{q}{2}$ even then $\mathcal{E} = \left\{\frac{q}{2}\mathbb{Z}\right\}$.*

Proof. Let us start by a remark : to be a multiple of q is always a necessary condition to be a solution. Now, let us study sufficient conditions. We have

$$\ell \in \mathcal{E} \Leftrightarrow \forall m \in \mathbb{Z},\ \frac{2p\ell}{q}m + \frac{p\ell^2}{q} \equiv 0\ (mod\, 1)$$

$$\Leftrightarrow \forall m \in \mathbb{Z},\ q|(2p\ell m + p\ell^2);$$

and, since $p \wedge q = 1$, by Gauss lemma we have

$$\ell \in \mathcal{E} \Leftrightarrow \forall m \in \mathbb{Z},\ q|(2\ell m + \ell^2).$$

Let ℓ an integer solution of $\mathcal{E}$; then if we take $m = 0$ we see that $q|\ell^2$, hence there exist $\alpha \in \mathbb{Z}$ such that $\ell^2 = \alpha q$. On the other hand, since for all $m \in \mathbb{Z}$ we have $q|(2\ell m + \ell^2)$ we deduce that for all $m \in \mathbb{Z}$, there exists $\beta_m \in \mathbb{Z}$ such that $2\ell m + \ell^2 = \beta_m q$. Thus, since $\ell^2 = \alpha q$, we get for all $m \in \mathbb{Z}$ the equality $2\ell m + \alpha q = \beta_m q$. In particular, with $m = 1$ we deduce that :

$$2\ell = (\beta_1 - \alpha)q.$$

• If q is odd : then a $\beta_1 - \alpha$ is even; thus $\ell = \underbrace{\frac{(\beta_1 - \alpha)}{2}}_{\in \mathbb{Z}} q$, and $q|\ell$, thus $\mathcal{E} = \{q\mathbb{Z}\}$.

•If q is even : then $\ell = (\beta_1 - \alpha)\frac{q}{2}$; thus $\frac{q}{2}|\ell$. If we write $\ell = k\frac{q}{2}$ with $k \in \mathbb{Z}$, then

$$\forall m \in \mathbb{Z},\ \frac{2p\ell}{q}m + \frac{p\ell^2}{q} = kpm + \frac{pk^2q}{4}.$$

Thus

$$\forall m \in \mathbb{Z},\ kpm + \frac{pk^2q}{4} \in \mathbb{Z} \Leftrightarrow pk^2q \in 4\mathbb{Z}.$$

And, since p is odd, finaly we get, with $\ell = k\frac{q}{2}$:

$$\forall m \in \mathbb{Z},\ \frac{2p\ell}{q}m + \frac{p\ell^2}{q} \in \mathbb{Z} \Leftrightarrow k^2q \in 4\mathbb{Z}.$$

Then, we have two cases :

• if $\frac{q}{2}$ is odd : then $k^2q \in 4\mathbb{Z} \Leftrightarrow k^2$ is even, equivalent to k even, hence $\ell = k\frac{q}{2}$ is a multiple of q, hence $\mathcal{E} = \{q\mathbb{Z}\}$.

• If $\frac{q}{2}$ is even : then $q = 4q'$ where $q' \in \mathbb{Z}^*$, hence for all $k \in \mathbb{Z}$; $k^2q = k^2 4q' \in 4\mathbb{Z}$, thus $\mathcal{E} = \left\{\frac{q}{2}\mathbb{Z}\right\}$. □

For a period $\ell \in \mathbb{Z}$, let us defined the set of sequences $\ell-$periodic with a scalar product.

Definition 6.12. For a integer $\ell \in \mathbb{Z}^*$; let us denote by $\mathfrak{S}_\ell(\mathbb{Z})$ set of sequences $\ell-$periodic :

$$\mathfrak{S}_\ell(\mathbb{Z}) := \left\{u_n \in \mathbb{C}^{\mathbb{Z}};\ \forall n \in \mathbb{Z},\ u_{n+\ell} = u_n\right\}.$$

So we have the :

Proposition 6.13. *The application*

$$\langle\ ,\ \rangle_{\mathfrak{S}_\ell} : \begin{cases} \mathfrak{S}_\ell(\mathbb{Z})^2 \to \mathbb{C} \\ (u,v) \mapsto \langle u,v\rangle_{\mathfrak{S}_\ell} := \frac{1}{|\ell|}\sum_{k=0}^{|\ell|-1} u_k\overline{v_k} \end{cases}$$

is a Hermitean product on the space $\mathfrak{S}_\ell(\mathbb{Z})$.

Proof. The proof is elementary. □

Proposition 6.14. *Let us consider* $\phi_n^k := e^{-\frac{2i\pi kn}{\ell}}$ *where* $(k,n) \in \mathbb{Z}^2$*; then the family* $\left\{\left(\phi_n^k\right)_{n\in\mathbb{Z}}\right\}_{k=0\ldots\ell-1}$ *is an orthonormal basis of the space vector* $\mathfrak{S}_\ell(\mathbb{Z})$.

Proof. For $\ell > 0$, the family $\left\{\left(\phi_n^k\right)_{n\in\mathbb{Z}}\right\}_{k=0\ldots\ell-1}$ is clearly a familly of ℓ vectors of $\mathfrak{S}_\ell(\mathbb{Z})$. Next, for all pair $(p,q) \in \{0\ldots\ell-1\}^2$ we have

$$\langle \phi^p, \phi^q \rangle_{\mathfrak{S}_\ell} = \frac{1}{\ell}\sum_{k=0}^{\ell-1} \phi_k^p \overline{\phi_k^q} = \frac{1}{\ell}\sum_{k=0}^{\ell-1} \left(e^{\frac{2i\pi(q-p)}{\ell}}\right)^k = \delta_{p,q}.$$

So $\left\{\left(\phi_n^k\right)_{n\in\mathbb{Z}}\right\}_{k=0\ldots\ell-1}$ is a orthonormal basis of the space vector $\mathfrak{S}_\ell(\mathbb{Z})$. □

6.4.2 The main theorem

Theorem 6.15. For all $(p,q) \in \mathbb{Z}\times\mathbb{N}^*$ such that $p\wedge q = 1$; there exists a family of ℓ complex numbers $\left(\widetilde{b_k}(\ell)\right)_{k\in\{0\ldots\ell-1\}}$ where the integer $\ell \in \mathbb{Z}$ is solution of :

$$\forall m \in \mathbb{Z},\ \frac{2p\ell}{q}m + \frac{pl^2}{q} \equiv 0\ [1];$$

such that for all $t \in \left[0, |\ln(h)|^\alpha\right]$ we get :

$$\widetilde{\mathbf{a_2}}\left(t + \frac{p}{q}N_hT_{hyp}\right) = \sum_{k=0}^{\ell-1} \widetilde{b_k}(\ell)\widetilde{\mathbf{a_1}}\left(t + T_{hyp}\left(\frac{k}{\ell} + \frac{p}{q}N_h\right)\right) + O\left(|\ln(h)|^{\alpha+2\gamma-3}\right)$$

The numbers $\widetilde{b_k}(\ell)$ are called fractionnals coefficients; and for all $k \in \{0\ldots\ell-1\}$

$$\widetilde{b_k}(\ell) = e^{-\frac{2i\pi kn_0}{\ell}} b_k(\ell)$$

where

$$b_k(\ell) = b_k(h,\ell) = \left\langle \sigma_h(p,q), \phi^k \right\rangle_{\mathfrak{S}_\ell} = \frac{1}{\ell}\sum_{n=0}^{\ell-1} e^{-2i\pi\frac{p}{q}(n-n_0)^2} e^{-\frac{2i\pi kn}{\ell}}.$$

Moreover, if we suppose $\frac{T_{rev}}{T_{hyp}} \in \mathbb{Q}$, then we have

$$\widetilde{\mathbf{a_2}}\left(t + \frac{p}{q}T_{rev}\right) = \sum_{k=0}^{\ell-1} \widetilde{b_k}(l)\widetilde{\mathbf{a_1}}\left(t + T_{hyp}\left(\frac{k}{\ell} + \frac{p}{q}N_h\right)\right).$$

Proof. Let us denote the integer $\widetilde{n} := n - n_0$; and let us consider $\ell \in \mathbb{Z}$ a solution of the equation $\mathcal{E}$. So, we have for all $t \geq 0$

$$\widetilde{\mathbf{a_2}}\left(t + \frac{p}{q}N_hT_{cl}\right) = \sum_{n\in\mathbb{N}} |a_n|^2 e^{-2i\pi\frac{t}{T_{hyp}}\widetilde{n}} e^{-2i\pi\frac{t}{T_{rev}}\widetilde{n}^2} e^{-2i\pi\frac{p}{q}\widetilde{n}N_h} e^{-2i\pi\frac{p}{q}\frac{N_hT_{hyp}}{T_{rev}}\widetilde{n}^2}.$$

And, since $T_{rev} = N_hT_{hyp} + \Theta_hT_{hyp}$ we have

$$e^{-2i\pi\frac{p}{q}\frac{N_hT_{hyp}}{T_{rev}}\widetilde{n}^2} = e^{-2i\pi\frac{p}{q}\widetilde{n}^2} e^{2i\pi\frac{p}{q}\frac{\Theta_hT_{hyp}}{T_{rev}}\widetilde{n}^2}$$

and

$$e^{-2i\pi\frac{p}{q}\widetilde{n}^2} = e^{-2i\pi\frac{p}{q}(n-n_0)^2} \in \mathfrak{S}_\ell(\mathbb{Z});$$

hence, there exist a unique decomposition of the sequence $\left(e^{-2i\pi\frac{p}{q}(n-n_0)^2}\right)_n$ on the basis $\left(\phi^k\right)_{k\in\{0\dots\ell-1\}}$:

$$e^{-2i\pi\frac{p}{q}\widetilde{n}^2} = e^{-2i\pi\frac{p}{q}(n-n_0)^2} = \sum_{k=0}^{\ell-1} \left\langle \sigma_h(p,q), \phi^k \right\rangle_{\mathfrak{S}_\ell} \phi_n^k$$

$$= \sum_{k=0}^{\ell-1} b_k(\ell) e^{-2i\pi\frac{k}{\ell}n}.$$

Thus, for all $t \geq 0$

$$\widetilde{\mathbf{a_2}}\left(t + \frac{p}{q}N_h T_{hyp}\right)$$

$$= \sum_{n\in\mathbb{N}} |a_n|^2 e^{-2i\pi\frac{t}{T_{hyp}}\widetilde{n}} e^{-2i\pi\frac{t}{T_{rev}}\widetilde{n}^2} e^{-2i\pi\frac{p}{q}\widetilde{n}N_h} \left(\sum_{k=0}^{\ell-1} b_k(\ell) e^{-2i\pi\frac{k}{\ell}n}\right) e^{2i\pi\frac{p}{q}\frac{\Theta_h T_{hyp}}{T_{rev}}\widetilde{n}^2}$$

$$= \sum_{n\in\mathbb{N}} \sum_{k=0}^{\ell-1} b_k(\ell) |a_n|^2 e^{-2i\pi\frac{t}{T_{hyp}}\widetilde{n}} e^{-2i\pi\frac{t}{T_{rev}}\widetilde{n}^2} e^{-2i\pi\frac{p}{q}\widetilde{n}N_h} e^{-2i\pi\frac{k}{\ell}n} e^{2i\pi\frac{p}{q}\frac{\Theta_h T_{hyp}}{T_{rev}}\widetilde{n}^2}.$$

For all $t \geq 0$ we have

$$\widetilde{\mathbf{a_1}}\left(t + T_{hyp}\left(\frac{k}{\ell} + \frac{p}{q}N_h\right)\right) = \sum_{n\in\mathbb{N}} |a_n|^2 e^{-2i\pi\frac{t}{T_{hyp}}\widetilde{n}} e^{-2i\pi\frac{k}{\ell}\widetilde{n}} e^{-2i\pi\frac{p}{q}N_h\widetilde{n}}$$

thus,

$$\sum_{k=0}^{\ell-1} \widetilde{b_k}(\ell)\widetilde{\mathbf{a_1}}\left(t + T_{hyp}\left(\frac{k}{\ell} + \frac{p}{q}N_h\right)\right)$$

$$= \sum_{k=0}^{\ell-1} e^{-\frac{2i\pi k n_0}{\ell}} b_k(\ell) \sum_{n\in\mathbb{N}} |a_n|^2 e^{-2i\pi\frac{t}{T_{hyp}}\widetilde{n}} e^{-2i\pi\frac{k}{\ell}\widetilde{n}} e^{-2i\pi\frac{p}{q}N_h\widetilde{n}}$$

$$= \sum_{n\in\mathbb{N}} \sum_{k=0}^{\ell-1} |a_n|^2 b_k(\ell) e^{-2i\pi\frac{t}{T_{hyp}}\widetilde{n}} e^{-2i\pi\frac{kn}{\ell}} e^{-2i\pi\frac{p}{q}N_h\widetilde{n}}.$$

Hence

$$\left|\widetilde{\mathbf{a_2}}\left(t + \frac{p}{q}N_h T_{hyp}\right) - \sum_{k=0}^{\ell-1} \widetilde{b_k}(\ell)\widetilde{\mathbf{a_1}}\left(t + T_{hyp}\left(\frac{k}{\ell} + \frac{p}{q}N_h\right)\right)\right|$$

$$= \left|\sum_{n\in\mathbb{N}} \sum_{k=0}^{\ell-1} |a_n|^2 b_k(\ell) e^{-2i\pi\frac{t}{T_{hyp}}\widetilde{n}} e^{-2i\pi\frac{p}{q}\widetilde{n}N_h} e^{-\frac{2i\pi k n}{\ell}} \left(e^{-2i\pi\frac{t}{T_{rev}}\widetilde{n}^2} e^{+2i\pi\frac{p}{q}\frac{\Theta_h T_{hyp}}{T_{rev}}\widetilde{n}^2} - 1\right)\right|;$$

with the sets Δ, Γ and by triangular inequality we get :

$$\leq \left|\sum_{n\in\Gamma} \sum_{k=0}^{\ell-1} |a_n|^2 b_k(\ell) e^{-2i\pi\frac{t}{T_{hyp}}\widetilde{n}} e^{-2i\pi\frac{p}{q}\widetilde{n}N_h} e^{-\frac{2i\pi k n}{\ell}} \left(e^{-2i\pi\frac{t}{T_{rev}}\widetilde{n}^2} e^{+2i\pi\frac{p}{q}\frac{\Theta_h T_{hyp}}{T_{rev}}\widetilde{n}^2} - 1\right)\right|$$

$$+\left|\sum_{n\in\Delta}\sum_{k=0}^{\ell-1}|a_n|^2\,b_k(\ell)e^{-2i\pi\frac{t}{T_{hyp}}\widetilde{n}}e^{-2i\pi\frac{p}{q}\widetilde{n}N_h}e^{-\frac{2i\pi kn}{\ell}}\left(e^{-2i\pi\frac{t}{T_{rev}}\widetilde{n}^2}e^{+2i\pi\frac{p}{q}\frac{\Theta_h T_{hyp}}{T_{rev}}\widetilde{n}^2}-1\right)\right|.$$

Increase the first serie :

$$\left|\sum_{n\in\Gamma}\sum_{k=0}^{\ell-1}|a_n|^2\,b_k(\ell)e^{-2i\pi\frac{t}{T_{hyp}}\widetilde{n}}e^{-2i\pi\frac{p}{q}\widetilde{n}N_h}e^{-\frac{2i\pi kn}{\ell}}\left(e^{-2i\pi\frac{t}{T_{rev}}\widetilde{n}^2}e^{+2i\pi\frac{p}{q}\frac{\Theta_h T_{hyp}}{T_{rev}}\widetilde{n}^2}-1\right)\right|$$

$$\leq 2\left(\sum_{n\in\Gamma}|a_n|^2\right)\left(\sum_{k=0}^{\ell-1}|b_k(\ell)|\right)=O\left|\frac{1}{\ln(h)^{\infty}}\right|.$$

Next, for the second serie, observe the following term :

$$e^{-2i\pi\frac{t}{T_{rev}}\widetilde{n}^2}e^{+2i\pi\frac{p}{q}\frac{\Theta_h T_{hyp}}{T_{rev}}\widetilde{n}^2}=e^{-2i\pi\left(t+\Theta_h T_{hyp}\frac{p}{q}\right)\frac{1}{T_{rev}}\widetilde{n}^2}.$$

For all $n\in\Delta$ and for all $t\in\left[0,|\ln(h)|^{\alpha}\right]$ there exist a constant $C>0$ such that

$$\left|\left(t+\frac{\Theta_h T_{hyp}\,p}{q}\right)\frac{(n-n_0)^2}{T_{rev}}\right|\leq C\left(|\ln(h)|^{2\gamma-3+\alpha}+|\ln(h)|^{2\gamma-2}\right)\leq 2C\,|\ln(h)|^{2\gamma-2}.$$

Hence, there exist $A>0$ such that for all $t\in\left[0,|\ln(h)|^{\alpha}\right]$

$$\left|\sum_{n\in\Delta}\sum_{k=0}^{\ell-1}|a_n|^2\,b_k(\ell)e^{-2i\pi\frac{t}{T_{hyp}}\widetilde{n}}e^{-2i\pi\frac{p}{q}\widetilde{n}N_h}e^{-\frac{2i\pi kn}{\ell}}\left(e^{-2i\pi\frac{t\widetilde{n}^2}{Trev}}e^{+2i\pi\frac{p}{q}\frac{\Theta_h T_{hyp}\widetilde{n}^2}{T_{rev}}}-1\right)\right|$$

$$\leq A\,|\ln(h)|^{2\gamma-2}\sum_{n\in\Delta}|a_n|^2\leq A\,|\ln(h)|^{2\gamma-2}\sum_{n=0}^{+\infty}|a_n|^2.$$

For finish, the case where $t=0$ and $\frac{T_{rev}}{T_{hyp}}\in\mathbb{Q}$ is clear. □

Corollary 6.16. *We have the equality :*

$$\sum_{k=0}^{\ell-1}|b_k(\ell)|^2=1.$$

Proof. Since $b_k(\ell)=\left\langle\sigma_h(p,q),\phi^k\right\rangle_{\mathfrak{S}_\ell}$ and $\left(\phi^k\right)_{k\in\{0\ldots\ell-1\}}$ is a orthomoral basis of the space vector $\mathfrak{S}_\ell(\mathbb{Z})$, by Pythagorean equality we get

$$\sum_{k=0}^{\ell-1}|b_k(\ell)|^2=\|\sigma_h(p,q)\|^2_{\mathfrak{S}_\ell}=1.$$

□

Now, let us examine the case $\frac{p}{q}=1$ and the case $\frac{p}{q}=\frac{1}{2}$.

Corollary 6.17. *With the same notation as in the theorem 6.15, we have*
(i) *for all $t\in\left[0,|\ln(h)|^{\alpha}\right]$*

$$\widetilde{\mathbf{a_2}}(t+N_hT_{hyp})=\widetilde{\mathbf{a_1}}(t)+O\left(|\ln(h)|^{\alpha+2\gamma-3}\right);$$

(ii) *for all* $t \in \left[0, |\ln(h)|^{\alpha}\right]$

$$\widetilde{\mathbf{a_2}}\left(t + \frac{N_h T_{hyp}}{2}\right) = \widetilde{\mathbf{a_1}}\left(t + \left(\frac{N_h + 1}{2}\right) T_{hyp}\right) + O\left(|\ln(h)|^{\alpha+2\gamma-3}\right);$$

in particular, if N_h *is odd, then :*

$$\widetilde{\mathbf{a_2}}\left(t + \frac{N_h T_{hyp}}{2}\right) = \widetilde{\mathbf{a_1}}(t) + O\left(|\ln(h)|^{\alpha+2\gamma-3}\right)$$

and if N_h *is even, then :*

$$\widetilde{\mathbf{a_2}}\left(t + \frac{N_h T_{hyp}}{2}\right) = \widetilde{\mathbf{a_1}}\left(t + \frac{T_{hyp}}{2}\right) + O\left(|\ln(h)|^{\alpha+2\gamma-3}\right).$$

Proof. In the case (i) : $p = 1$, $q = \ell = 1$; thus by the previous theorem we get, for all $t \in [0, |\ln(h)|^{\alpha}]$:

$$\widetilde{\mathbf{a_2}}\left(t + \frac{p}{q} N_h T_{hyp}\right) = \widetilde{b_0}(1)\widetilde{\mathbf{a_1}}\left(t + T_{hyp} N_h\right) + O\left(|\ln(h)|^{\alpha+2\gamma-3}\right)$$

$$= \widetilde{b_0}(1)\widetilde{\mathbf{a_1}}(t) + O\left(|\ln(h)|^{\alpha+2\gamma-3}\right)$$

and $b_0(1) = \frac{1}{1} e^{-2i\pi n_0^2} = 1$, thus

$$\widetilde{b_0}(1) = 1.$$

Next, for the case (ii) : $p = 1$, $q = \ell = 2$, then

$$\widetilde{\mathbf{a_2}}\left(t + \frac{p}{q} N_h T_{hyp}\right) =$$

$$\widetilde{b_0}(2)\widetilde{\mathbf{a_1}}\left(t + \frac{T_{hyp} N_h}{2}\right) + \widetilde{b_1}(2)\widetilde{\mathbf{a_1}}\left(t + T_{hyp}\left(\frac{1}{2} + \frac{N_h}{2}\right)\right) + O\left(|\ln(h)|^{\alpha+2\gamma-3}\right).$$

We have

$$b_0(2) = \frac{1}{2} e^{-2i\pi \frac{1}{2} n_0^2} + \frac{1}{2} e^{-2i\pi \frac{1}{2}(1-n_0)^2} = 0$$

thus

$$\widetilde{b_0}(2) = 0.$$

And

$$b_1(2) = \frac{1}{2} e^{-2i\pi \frac{1}{2} n_0^2} e^{\frac{2i\pi 0}{2}} + \frac{1}{2} e^{-2i\pi \frac{1}{2}(1-n_0)^2} e^{\frac{2i\pi}{2}}$$

$$= \frac{1}{2}\left(e^{-i\pi {n_0}^2} + (-1) e^{-i\pi(n_0-1)^2}\right) = (-1)^{n_0}$$

then

$$\widetilde{b_1}(2) = e^{-i\pi n_0}(-1)^{n_0} = 1.$$

□

6.5 Explicit values of modulus for revivals coefficients

About the coefficients $\widetilde{b_k}(\ell)$ we just know that :

$$\sum_{k=0}^{\ell-1} \left|\widetilde{b_k}(\ell)\right|^2 = 1.$$

But we can say more. We can calculate $\left|\widetilde{b_k}(\ell)\right| = |b_k(\ell)|$ for all k. From the proposition 6.11 we consider two cases : the case $\ell = q$ and the case $\ell = \frac{q}{2}$.

6.5.1 Case $\ell = q$

If the integer q is odd we have :

Theorem 6.18. [Lab2]. For all integers p and q, such that $p \wedge q = 1$ and q odd, then for all $k \in \{0...q-1\}$ we get :

$$|b_k(q)|^2 = \frac{1}{q}.$$

And in the case q even :

Theorem 6.19. [Lab2]. For all integers p and q, such that $p \wedge q = 1$ and q even, then for all $k \in \{0...q-1\}$ we get :

$$If\ \frac{q}{2}\ is\ even,\ then :\ |b_k(q)|^2 = \begin{cases} \frac{2}{q} & if\ k\ is\ even \\ \\ 0 & else. \end{cases}$$

$$If\ \frac{q}{2}\ is\ odd,\ then :\ |b_k(q)|^2 = \begin{cases} 0 & if\ k\ is\ even \\ \\ \frac{2}{q} & else. \end{cases}$$

6.5.2 Case $\ell = \frac{q}{2}$

It is the case $q \in 4\mathbb{Z}$.

Theorem 6.20. [Lab2]. For all integers p and q, such that $p \wedge q = 1$ and $q \in 4\mathbb{Z}^*$; for all $k \in \left\{0...\frac{q}{2}-1\right\}$ we have

$$\left|b_k\left(\frac{q}{2}\right)\right|^2 = \frac{2}{q}.$$

6.6 Comparison between time scale approximation, hyperbolic period and revivals periods

Since $\gamma < 1$ we have $1 < 3 - 2\gamma$, thus there exist α such that $\alpha \in]1, 3-2\gamma[$; hence :

$$|\ln(h)| < |\ln(h)|^{\alpha} < |\ln(h)|^{3-2\gamma}.$$

So we can choose α such that : for h small enough :

$$\left[0, T_{hyp}\right] \subset \left[0, |\ln(h)|^{\alpha}\right].$$

Next, with $\gamma < \frac{1}{3}$ we get $3 < 4 - 3\gamma$ and there exist β such that $\beta \in]1, 4 - 3\gamma[$ hence :

$$|\ln(h)|^3 < |\ln(h)|^{\beta} < |\ln(h)|^{4-3\gamma};$$

so, for h small enough we have :

$$[0, T_{rev}] \subset \left[0, |\ln(h)|^{\beta}\right].$$

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Olivier Lablée

Université Grenoble 1-CNRS
Institut Fourier
UFR de Mathématiques
UMR 5582
BP 74 38402 Saint Martin d'Hères
mail: lablee@ujf-grenoble.fr
http://www-fourier.ujf-grenoble.fr/~lablee/