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SEMI-CLASSICAL BEHAVIOUR OF SCHRÖDINGER'S DYNAMICS : REVIVALS OF WAVE PACKETS ON HYPERBOLIC TRAJECTORY

OLIVIER LABLÉE

ABSTRACT. The aim of this paper is to study the semi-classical behaviour of Schrödinger's dynamics for an one-dimensional quantum Hamiltonian with a classical hyperbolic trajectory. As in the regular case (elliptic trajectory), we prove, that for an initial wave packets localized in energy, the dynamics follows the classical motion during short time. And, for large time, a new period for the dynamics appears : the wave packets revivals.

1. INTRODUCTION

1.1. Motivation and goals. In this paper, we describe the semi-classical behaviour of Schrödinger's dynamics for an one-dimensional quantum Hamiltonian $P_h : D(P_h) \subset L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ with a classical hyperbolic trajectory; ie. the principal symbol $p \in C^\infty(\mathbb{R}^2, \mathbb{R})$ of P_h has a hyperbolic non-degenerate singularity. Dynamics in the regular case and for elliptic non-degenerate singularity has been the subject of many research in physics [Av-Pe], [LAS], [Robi1], [Robi2], [BKP], [Bl-Ko] and, more recently in mathematics [Co-Ro], [Rob], [Pau1], [Pau2], [Lab2]. The strategy for understand the long times behaviour of dynamics is to use the spectrum of the operator P_h . In the regular case, the spectrum of P_h is given by the famous Bohr-Sommerfeld rules (see for example [He-Ro], [Ch-VuN], [Col8]) : in first approximation, the spectrum in a compact set is a sequence of real numbers with a gap of size h . The classical trajectories are periodics and supported on elliptic curves. In the case of hyperbolic singularity we have a non-periodic trajectory supported on a "height" figure. The spectrum near this singularity is more complicated than in the regular case. Y. Colin de Verdière and B. Parisse gives an implicit singular Bohr-Sommerfeld rules for hyperbolic singularity ([Co-Pa1], [Co-Pa2] and [Co-Pa3]). This quantization formula is too implicit for use it directly in our motivation. In [Lab3] we have an explicit description of the spectrum for an one-dimensional pseudo-differential operator near a hyperbolic non-degenerate singularity. With this description we propose a study of quantum dynamics for large times ($\gg |\ln h|$). We prove that for a localized initial state, at the beginning the dynamics is periodic with a period equal to $T_{hyp} = C |\ln h|$. Next for large time, a new period T_{rev} appears : this is the revivals phenomenon (like in regular case [Co-Ro], [Rob], [Pau1], [Pau2], [Lab2]). We have also the phenomenon of fractional revivals of initial wave packets for time $t = \frac{p}{q} T_{rev}$.

1.2. Plan of the paper. In the part 2 this paper we give some preliminaries about the strategy for analyse the dynamics of a quantum Hamiltonian. The part 3 present the context. We also recall the principal theorem of [Lab3]. The aim of the last parts is to analyse the dynamics in short and large times scales.

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Key words and phrases. Schrödinger's dynamics, revivals of wave packets, semi-classical analysis, hyperbolic trajectory, Schrödinger operator with double wells potential.

2. QUANTUM DYNAMICS AND AUTOCORRELATION FUNCTION

2.1. The dynamics. Generally, for a quantum Hamiltonian $P_h : D(P_h) \subset \mathcal{H} \rightarrow \mathcal{H}$, \mathcal{H} is a Hilbert space, the Schrödinger dynamics is governed by the famous Schrödinger equation :

$$\frac{d}{dt}\psi(t) = -\frac{i}{\hbar}P_h\psi(t).$$

With the functional calculus, we can reformulate this equation with the unitary group $U(t) = \left\{ e^{-i\frac{t}{\hbar}P_h} \right\}_{t \in \mathbb{R}}$. Indeed, for an initial state $\psi_0 \in \mathcal{H}$, the evolution given by :

$$\psi(t) = U(t)\psi_0 \in \mathcal{H}.$$

2.2. Autocorrelation function of dynamics. Now, let us define a quantum analog of the Poincaré return function.

Definition. The quantum return function of the operator P_h and for an initial state ψ_0 is defined by :

$$\mathbf{r}(t) := \langle \psi(t), \psi_0 \rangle_{\mathcal{H}};$$

and the autocorrelation function is defined by :

$$\mathbf{c}(t) := |\mathbf{r}(t)| = |\langle \psi(t), \psi_0 \rangle_{\mathcal{H}}|.$$

The previous function measures the return on the initial state. Since the initial state ψ_0 is normalized, the autocorrelation function takes values in the set $[0, 1]$.

Then, if we have an orthonormal basis of eigenvectors $(e_n)_{n \in \mathbb{N}} : P_h e_n = \lambda_n(h) e_n$ with

$$\lambda_1(h) \leq \lambda_2(h) \leq \dots \leq \lambda_n(h) \rightarrow +\infty;$$

we get, for all integer n

$$\left(e^{-i\frac{t}{\hbar}P_h} \right) e_n = \left(e^{-i\frac{t}{\hbar}\lambda_n(h)} \right) e_n.$$

So for an initial vector $\psi_0 \in D(P_h) \subset \mathcal{H}$, let us denote by $(c_n)_{n \in \mathbb{N}} = (c_n(h))_{n \in \mathbb{N}}$ the sequence of $\ell^2(\mathbb{N})$ given $(c_n)_n = \pi(\psi_0)$, where π is the projector operator (unitary operator) :

$$\pi : \begin{cases} \mathcal{H} \rightarrow \ell^2(\mathbb{N}) \\ \psi \mapsto \langle \psi, e_n \rangle_{\mathcal{H}} \end{cases}$$

Then, for all $t \geq 0$ we have

$$\begin{aligned} \psi(t) &= U(t)\psi_0 = \left(e^{-i\frac{t}{\hbar}P_h} \right) \left(\sum_{n \in \mathbb{N}} c_n e_n \right) \\ &= \sum_{n \in \mathbb{N}} c_n e^{-i\frac{t}{\hbar}\lambda_n(h)} e_n. \end{aligned}$$

So, for all $t \geq 0$ we obtain

$$\mathbf{r}(t) = \sum_{n \in \mathbb{N}} |c_n|^2 e^{-i\frac{t}{\hbar}\lambda_n(h)}; \quad \mathbf{c}(t) = \left| \sum_{n \in \mathbb{N}} |c_n|^2 e^{-i\frac{t}{\hbar}\lambda_n(h)} \right|.$$

2.3. Strategy for study the autocorrelation function. The strategy, performed by the physicists ([Av-Pe], [LAS], [Robi1], [Robi2], [BKP], [Bl-Kol]) is the following :

- (1) We define a initial vector $\psi_0 = \sum_{n \in \mathbb{N}} c_n e_n$ localized in the following sense :
the sequence $(c_n)_{n \in \mathbb{N}}$ is localized close to a quantum number n_0 (depends on h and a energy level $E \in \mathbb{R}$).
- (2) Next, the idea is to expand by a Taylor formula's the eigenvalues $\lambda_n(h)$ around the energy level E :

$$\lambda_n(h) = \lambda_{n_0}(h) + \lambda'_{n_0}(h) (n - n_0) + \frac{\lambda''_{n_0}(h)}{2} (n - n_0)^2 + \frac{\lambda^{(3)}_{n_0}(h)}{6} (n - n_0)^3 + \dots$$

hence we get

$$\mathbf{c}(t) = \left| \sum_{n \in \mathbb{N}} |c_n|^2 e^{-it \left[\frac{\lambda'_{n_0}(h)}{h} (n - n_0) + \frac{\lambda''_{n_0}(h)}{2h} (n - n_0)^2 + \frac{\lambda^{(3)}_{n_0}(h)}{6} (n - n_0)^3 + \dots \right]} \right|.$$

- (3) And, for small values of t , the first approximation of the autocorrelation function $\mathbf{c}(t)$ is the function

$$\mathbf{c}_1(t) = \left| \sum_{n \in \mathbb{N}} |c_n|^2 e^{-it \frac{\lambda'_{n_0}(h)}{h} (n - n_0)} \right|;$$

and for larger values of t , the approximation is

$$\mathbf{c}_2(t) = \left| \sum_{n \in \mathbb{N}} |c_n|^2 e^{-it \left[\frac{\lambda'_{n_0}(h)}{h} (n - n_0) + \frac{\lambda''_{n_0}(h)}{2h} (n - n_0)^2 \right]} \right|.$$

3. THE CONTEXT OF HYPERBOLIC SINGULARITY

3.1. Link between spectrum and geometry : semi-classical analysis. Start by an example : for a real number $E > 0$; the equation

$$-\frac{h^2}{2} \Delta_g \varphi = E \varphi$$

(where Δ_g denotes the Laplace-Beltrami operator on a Riemannian manifold (M, g)) admits the eigenvectors φ_k for solution if

$$-\frac{h^2}{2} \lambda_k = E.$$

Hence if $h \rightarrow 0^+$ then $\lambda_k \rightarrow +\infty$. So there exist a correspondance between the semi-classical limit ($h \rightarrow 0^+$) and large eigenvalues.

The asymptotic's of large eigenvalues for the Laplace-Beltrami operator Δ_g on a Riemannian manifold (M, g) , or more generally for a pseudo-differential operator P_h , is linked to a symplectic geometry : phase space geometry. This is the same phenomenon between quantum mechanics (spectrum, operator algebra) and classical mechanics (length of periodic geodesics, symplectic geometry). More precisely, for a pseudo-differential operator P_h on $L^2(M)$ with a principal symbol $p \in C^\infty(T^*M)$, there exist a link between the geometry of the foliation $(p^{-1}(\lambda))_{\lambda \in \mathbb{R}}$ and the spectrum of the operator P_h . Indeed, if we have :

$$(P_h - \lambda I_d) u_h = O(h^\infty)$$

then

$$MS(u_h) \subset p^{-1}(\lambda);$$

where $MS(u_h) \subset T^*M$ denote the microsupport of the function $u_h \in L^2(M)$.

3.2. Hyperbolic singularity. In dimension one, $(x_0, \xi_0) \in T^*M$ is a non-degenerate hyperbolic singularity if and only if :

- (1) $dp(x_0, \xi_0) = 0$;
- (2) the eigenvalues of the Hessian matrix $\nabla^2 p(x_0, \xi_0)$ are pairwise distinct;
- (3) if, in some local symplectic coordinates (x, ξ) the algebra spanned by $\nabla^2 p(x_0, \xi_0)$ has a basis of the form $q = x\xi$.

Remark 3.1. There exist analogue definition for completely integrable systems, see for example [VuN].

The canonical example in dimension one is the Schrödinger operator with double wells potential :

$$P_h = -\frac{h^2}{2}\Delta + V;$$

we assume $V \in C^\infty(\mathbb{R})$, for all $x \in \mathbb{R}$, $V(x) \geq C$ and $\lim_{|x| \rightarrow \infty} V(x) = +\infty$. With this hypotheses, the operator is self-adjoint, the spectrum is a sequence of real numbers $(\lambda_n(h))_{n \geq 0}$ and the eigenvectors $(e_n)_{n \geq 0}$ be an orthonormal basis of the Hilbert space $L^2(\mathbb{R})$ (for example see the survey [Lab1]). We also suppose that the potential V admits exactly one local non-degenerate maximum. Without loss generality, we may suppose

$$V(0) = 0; V'(0) = 0; V''(0) < 0.$$

Then the foliation associated to the principal symbol $p(x, \xi) = \xi^2/2 + V(x)$ admits a singular fiber $\Lambda_0 = p^{-1}(0)$.

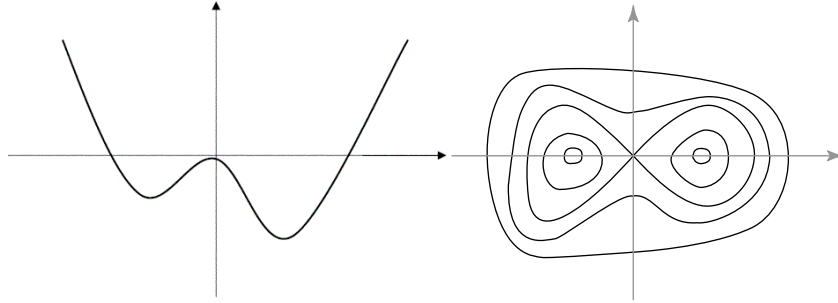


Fig. 1. & 2. On figure 1 (left) the potential function $V(x)$. On figure 2 (right) the associated foliation in the phase plane.

3.3. Spectrum near the singularity. In [Co-Pa1], [Co-Pa2] and [Co-Pa3], Y. Colin de Verdière and B. Parisse gives an implicit singular Bohr-Sommerfeld rules. An explicit description is given in [Lab3]. Record here this description. We take the presentation from the paper [Lab3] and refer to this paper for more details.

3.3.1. Some notations (see [Lab3]). In [Co-Pa1], [Co-Pa2], [Co-Pa3] and [Lab3] we have the smooth function $E \mapsto \varepsilon(E)$ defined for $|E| \leq \delta$, where δ is a real constant and not depends to h ; (this function is linked to a normal form [CLP], [Co-Pa1]) defined by

$$\varepsilon(E) = \sum_{j=0}^{+\infty} \varepsilon_j(E) h^j$$

We have (see [Co-Pa1], [Co-Pa2], [Co-Pa3] and [Lab3]) $\varepsilon_0(0) = 0$ and $\varepsilon'_0(0) = \frac{1}{\sqrt{-V''(0)}}$. Hence, if we use the Taylor formula on the smooth function ε_0 ; for all $E \in [-\delta, \delta]$ we get

$$\varepsilon(E) = \frac{E}{\sqrt{-V''(0)}} + O(E^2) + \sum_{j=1}^{+\infty} \varepsilon_j(E) h^j.$$

For $\lambda \in [-1, 1]$; and h small enough for have $[-h, h] \subset [-\delta, \delta]$ we get

$$\varepsilon(\lambda h^\alpha) = \frac{\lambda h^\alpha}{\sqrt{-V''(0)}} + O(h^{2\alpha}) + \sum_{j=1}^{+\infty} \varepsilon_j(\lambda h^\alpha) h^j.$$

In the papers [Co-Pa1], [Co-Pa2], [Co-Pa3] and [Lab3] we also use two smooth functions S^+ and S^- :

$$S^{+/-}(E) = \sum_{i=0}^{+\infty} S_i^{+/-}(E) h^i;$$

where the functions are $E \rightarrow S_j^{+/-}(E)$ are C^∞ smooth. This functions are usually called singular actions. Let us also denote

$$\theta_{+/-}(E) := \frac{S^{+/-}(E)}{h}.$$

This functions have a holonomy interpretation.

Let us consider the functions $E \mapsto F_h(E)$ and $E \mapsto G_h(E)$ defined on the compact set $[-\delta, \delta]$ by :

$$F_h(E) := -\frac{\theta_+(E) + \theta_-(E)}{2} + \frac{\pi}{2} + \frac{\varepsilon(E)}{h} \ln(h) + \arg \left(\Gamma \left(\frac{1}{2} + i \frac{\varepsilon(E)}{h} \right) \right);$$

Γ is the Gamma function, and by

$$G_h(E) := \frac{\theta_+(E) - \theta_-(E)}{2}.$$

On the compact set $[-1, 1]$, let us consider the functions $\lambda \mapsto f_h(\lambda)$ and $\lambda \mapsto g_h(\lambda)$ defined by :

$$f_h(\lambda) := F_h(\lambda h^\alpha)$$

and

$$g_h(\lambda) := G_h(\lambda h^\alpha) = \frac{\theta_+(\lambda h^\alpha) - \theta_-(\lambda h^\alpha)}{2}.$$

For finish, let us consider the functions $\lambda \mapsto \mathcal{Y}_h(\lambda)$ and $\lambda \mapsto \mathcal{Z}_h(\lambda)$ defined on the compact set $[-1, 1]$ by

$$\mathcal{Y}_h(\lambda) := f_h(\lambda) - \arccos \left(\frac{\cos(g_h(\lambda))}{\sqrt{1 + \exp(2\pi\varepsilon(\lambda h^\alpha)/h)}} \right);$$

$$\mathcal{Z}_h(\lambda) := f_h(\lambda) + \arccos \left(\frac{\cos(g_h(\lambda))}{\sqrt{1 + \exp(2\pi\varepsilon(\lambda h^\alpha)/h)}} \right).$$

Since the function \mathcal{Y}_h et \mathcal{Z}_h are bijectives, we can consider

$$\mathcal{A}_h := \mathcal{Y}_h^{-1} : \mathcal{Y}_h([-1, 1]) \rightarrow [-1, 1];$$

$$\mathcal{B}_h := \mathcal{Z}_h^{-1} : \mathcal{Z}_h([-1, 1]) \rightarrow [-1, 1].$$

Notation 3.2. Let us denotes :

$$I_h := \{k \in \mathbb{Z}, 2\pi k \in \mathcal{Y}_h([-1, 1])\} = \frac{\mathcal{Y}_h([-1, 1])}{2\pi} \cap \mathbb{Z},$$

$$J_h := \{\ell \in \mathbb{Z}, 2\pi\ell \in \mathcal{Z}_h([-1, 1])\} = \frac{\mathcal{Z}_h([-1, 1])}{2\pi} \cap \mathbb{Z}.$$

3.3.2. *The main theorem.* The main theorem of the paper [Lab3] is the the following :

Theorem 3.3. [Lab3]. The semi-classical spectrum of P_h in the compact set $[-\sqrt{h}, \sqrt{h}]$ is the disjoint union

$$(\alpha_k(h))_{k \in I_h} \sqcup (\beta_\ell(h))_{\ell \in J_h}$$

of two families $(\alpha_k(h))_k$ and $(\beta_\ell(h))_\ell$ such that $\alpha_k(h) := h\mathcal{A}_h(2\pi k) \in \mathbb{R}$, $\beta_\ell(h) := h\mathcal{B}_h(2\pi\ell) \in \mathbb{R}$. The functions \mathcal{A}_h and \mathcal{B}_h are C^∞ smooth. The families $(\alpha_k(h))_k$, $(\beta_\ell(h))_\ell$ are strictly non-increasing and :

$$\beta_{k+1}(h) < \alpha_k(h) < \beta_k(h) < \alpha_{k-1}(h).$$

Moreover, the spectral gap is of order $O(h/|\ln(h)|)$; e.g : there exists $C, C' > 0$ such that :

$$\frac{Ch}{|\ln(h)|} \leq |\alpha_{k+1}(h) - \alpha_k(h)| \leq \frac{C'h}{|\ln(h)|}, \quad \frac{Ch}{|\ln(h)|} \leq |\beta_{k+1}(h) - \beta_k(h)| \leq \frac{C'h}{|\ln(h)|}.$$

Corollary 3.4. *The number of eigenvalues in the compact set $[-\sqrt{h}, \sqrt{h}]$ is of order $|\ln(h)| / \sqrt{h}$.*

In this paper, for technical reason, we just use the shape of the spectrum in the compact set $[-h, h] \subset [-\sqrt{h}, \sqrt{h}]$. More precisely we choose an initial wave packet localized in a compact set of size h . Indeed, the estimates in lemma 7.1 and t 9.1 are very difficult to enable in the compact set $[-\sqrt{h}, \sqrt{h}]$.

Notation 3.5. Let us denote by Θ_h the set of index for eigenvalues into the compact set $[-h, h]$; e.g

$$\Theta_h := \{n \in \mathbb{N}, \lambda_n(h) \in [-h, h]\}.$$

4. AUTOCORRÉLATIONS FUNCTIONS

Now, let be a initial vector $\psi_0 = \sum_{n \in \mathbb{N}} c_n e_n$, then we have for all $t \geq 0$

$$\mathbf{r}(t) = \sum_{n \in \Theta_h} |c_n|^2 e^{-i\frac{t}{h}\lambda_n(h)} + \sum_{n \in \mathbb{N} - \Theta_h} |c_n|^2 e^{-i\frac{t}{h}\lambda_n(h)}.$$

After, we let us consider an initial vector localized near the singularity (e.g. in 0) in a set of size h , hence the second serie $\sum_{n \in \mathbb{N} - \Theta_h} |c_n|^2 e^{-i\frac{t}{h}\lambda_n(h)}$ will be equal to $O\left|\frac{1}{\ln(h)^\infty}\right|$. Next, the idea is to use the families $(\alpha_k(h))_k$ and $(\beta_\ell(h))_\ell$: as the eigenvalues of P_h in the compact set $[-h, h]$ are distinct; for all indices $k \in I_h$ and $\ell \in J_h$ there exist a unique pair $(\sigma_1(k), \sigma_2(\ell)) \in \Theta_h^2$ such that :

$$\begin{cases} h\mathcal{A}_h(2\pi k) = \lambda_{\sigma_1(k)}(h) \\ h\mathcal{B}_h(2\pi\ell) = \lambda_{\sigma_2(\ell)}(h). \end{cases}$$

So, we can consider the applications :

$$\sigma_1 : \begin{cases} \mathbf{I}_h \rightarrow \Theta_h \\ k \mapsto \sigma_1(k) \end{cases} \quad \text{and} \quad \sigma_2 : \begin{cases} \mathbf{J}_h \rightarrow \Theta_h \\ \ell \mapsto \sigma_2(\ell). \end{cases}$$

Clearly, these applications are injectives; hence \mathbf{I}_h is isomorphic to $\sigma_1(\mathbf{I}_h)$ (similarly for \mathbf{J}_h and $\sigma_2(\mathbf{J}_h)$). Moreover we have

$$\Theta_h = \sigma_1(\mathbf{I}_h) \sqcup \sigma_2(\mathbf{J}_h).$$

So, we get the equality :

$$\begin{aligned} \mathbf{r}(t) &= \sum_{k \in \sigma_1(\mathbf{I}_h)} |c_k|^2 e^{-i\frac{t}{h}\lambda_k(h)} + \sum_{\ell \in \sigma_2(\mathbf{J}_h)} |c_\ell|^2 e^{-i\frac{t}{h}\lambda_\ell(h)} \\ &\quad + \sum_{n \in \mathbb{N} - \Theta_h} |c_n|^2 e^{-i\frac{t}{h}\lambda_n(h)} \\ &= \sum_{k \in \mathbf{I}_h} |c_{\sigma_1(k)}|^2 e^{-it\mathcal{A}_h(2\pi k)} + \sum_{\ell \in \mathbf{J}_h} |c_{\sigma_2(\ell)}|^2 e^{-it\mathcal{B}_h(2\pi\ell)} \\ &\quad + \sum_{n \in \mathbb{N} - \Theta_h} |c_n|^2 e^{-i\frac{t}{h}\lambda_n(h)}. \end{aligned}$$

Now, our goal here is to study the series :

Definition 4.1. Let us consider partial autocorrelation functions :

$$\mathbf{a}(t) := \sum_{k \in \mathbf{I}_h} |c_{\sigma_1(k)}|^2 e^{-it\mathcal{A}_h(2\pi k)}, \quad \mathbf{b}(t) := \sum_{\ell \in \mathbf{J}_h} |c_{\sigma_2(\ell)}|^2 e^{-it\mathcal{B}_h(2\pi\ell)}.$$

5. CHOICE OF INITIAL STATE

5.1. Preliminaries. Let us define an initial vector $\psi_0 = \sum_{n \in \mathbb{N}} c_n e_n$ localized near the real number E_h (where $E \in [-1, 1]$ and $E_h := hE$).

Definition 5.1. Let us consider the quantum integers $n_0 = n_0(h, E)$ and $m_0 = m_0(h, E)$ defined by

$$n_0 := \arg \min_{n \in \mathbf{I}_h} |h\mathcal{A}_h(2\pi n) - hE| \quad \text{and} \quad m_0 := \arg \min_{m \in \mathbf{J}_h} |h\mathcal{B}_h(2\pi m) - hE|.$$

Remark 5.2. Without loss generality, we may suppose n_0 and m_0 are uniques.

The integer n_0 (resp. m_0) is the indice of the eigenvalues from the family $(\alpha_k(h))_k$ (resp. $(\beta_\ell(h))_\ell$) the more close to the real number hE . As the spectral gap is of order $O(h/|\ln(h)|)$ there exist $C > 0$ such that :

$$|h\mathcal{A}_h(2\pi n_0) - hE| \leq \frac{Ch}{|\ln h|}, \quad |h\mathcal{B}_h(2\pi m_0) - hE| \leq \frac{Ch}{|\ln h|}.$$

So, we get :

Lemma 5.3. As $h \rightarrow 0$, we have

$$n_0 \sim \frac{N}{h} \quad \text{and} \quad m_0 \sim \frac{M}{h}$$

where N and M non-null real numbers.

Proof. We give the proof for the integer n_0 . As

$$|\mathcal{A}_h(2\pi n_0) - E| \leq \frac{C}{|\ln h|};$$

e.g.

$$\mathcal{A}_h(2\pi n_0) = E + O\left(\frac{1}{|\ln h|}\right),$$

so we get

$$\begin{aligned} 2\pi n_0 &= \mathcal{Y}_h\left(E + O\left(\frac{1}{|\ln h|}\right)\right) \\ &= f_h\left(E + O\left(\frac{1}{|\ln h|}\right)\right) - \arccos\left(\frac{\cos\left(g_h\left(E + O\left(\frac{1}{|\ln h|}\right)\right)\right)}{\sqrt{1 + \exp\left(2\pi\varepsilon\left(hE + O\left(\frac{h}{|\ln h|}\right)\right)/h\right)}}\right). \end{aligned}$$

By definition :

$$\begin{aligned} f_h\left(E + O\left(\frac{1}{|\ln h|}\right)\right) &= F_h\left(hE + O\left(\frac{h}{|\ln h|}\right)\right) \\ &= -\frac{\theta_+(hE + O(h/|\ln(h)|)) + \theta_-(hE + O(h/|\ln(h)|))}{2} + \frac{\pi}{2} \\ &\quad + \frac{\varepsilon(hE + O(h/|\ln(h)|))}{h} \ln(h) + \arg\left(\Gamma\left(\frac{1}{2} + i\frac{\varepsilon(hE + O(h/|\ln(h)|))}{h}\right)\right). \end{aligned}$$

Hence, if we multiply $2\pi n_0$ by h we obtain

$$\begin{aligned} 2\pi n_0 h &= -h \frac{\theta_+(hE + O(h/|\ln(h)|)) + \theta_-(hE + O(h/|\ln(h)|))}{2} \\ &\quad + \frac{\pi}{2} h + \varepsilon(hE + O(h/|\ln(h)|)) \ln(h) + h \arg\left(\Gamma\left(\frac{1}{2} + i\frac{\varepsilon(hE + O(h/|\ln(h)|))}{h}\right)\right) \\ &\quad - h \arccos\left(\frac{\cos\left(g_h\left(E + O\left(\frac{1}{|\ln h|}\right)\right)\right)}{\sqrt{1 + \exp\left(2\pi\varepsilon\left(hE + O\left(\frac{h}{|\ln h|}\right)\right)/h\right)}}\right). \end{aligned}$$

Let us evaluate this five terms. As the function

$$E \mapsto -\frac{\theta_+(hE + O(h/|\ln(h)|)) + \theta_-(hE + O(h/|\ln(h)|))}{2}$$

admit a asymptotic expansion in power of h from -1 to $+\infty$, we have

$$-h \frac{\theta_+(hE + O(h/|\ln(h)|)) + \theta_-(hE + O(h/|\ln(h)|))}{2} = O(1);$$

more precisely, for $h \rightarrow 0$

$$-h \frac{\theta_+(hE + O(h/|\ln(h)|)) + \theta_-(hE + O(h/|\ln(h)|))}{2} \rightarrow -\frac{1}{2} (S_{-1}^+(0) + S_{-1}^-(0)) \neq 0$$

Next :

$$\varepsilon(hE + O(h/|\ln(h)|)) \ln(h) = \left[\frac{hE + O(h/|\ln(h)|)}{\sqrt{-V''(0)}} + O(h^2) \right] \ln(h) \rightarrow 0.$$

For finish, as

$$\arg\left(\Gamma\left(\frac{1}{2} + i\frac{\varepsilon(hE + O(h/|\ln(h)|))}{h}\right)\right) = O(1)$$

and

$$\arccos \left(\frac{\cos \left(g_h \left(E + O \left(\frac{1}{|\ln h|} \right) \right) \right)}{\sqrt{1 + \exp \left(2\pi\varepsilon \left(hE + O \left(\frac{h}{|\ln h|} \right) \right) / h \right)}} \right) = O(1);$$

we get, for $h \rightarrow 0$:

$$h \arg \left(\Gamma \left(\frac{1}{2} + i \frac{\varepsilon (hE + O(h/|\ln h|))}{h} \right) \right) \rightarrow 0$$

and

$$h \arccos \left(\frac{\cos \left(g_h \left(E + O \left(\frac{1}{|\ln h|} \right) \right) \right)}{\sqrt{1 + \exp \left(2\pi\varepsilon \left(hE + O \left(\frac{h}{|\ln h|} \right) \right) / h \right)}} \right) \rightarrow 0.$$

Hence,

$$\lim_{h \rightarrow 0} 2\pi n_0 h = -\frac{1}{2} (S_{-1}^+(0) + S_{-1}^-(0));$$

So we prove the lemma. \square

5.2. First definition (non definitive). For technical reason, let us introduce the following sequence :

Definition 5.4. Let us consider :

$$\mu_{n,0} := \begin{cases} h\mathcal{A}_h(2\pi n_0) & \text{if } n \in \sigma_1(\mathbf{I}_h) \\ h\mathcal{B}_h(2\pi m_0) & \text{if } n \in \sigma_2(\mathbf{J}_h) \\ 0 & \text{if } n \in \mathbb{N} - \Theta_h. \end{cases}$$

For $n \in \Theta_h = \sigma_1(\mathbf{I}_h) \sqcup \sigma_2(\mathbf{J}_h)$, the sequence $(\mu_{n,0})_n$ take only two values :

- the closest eigenvalues from the family $(\alpha_k(h))_k$ to the real number hE ;
- or the closest eigenvalues from the family $(\beta_\ell(h))_\ell$ to the real number hE .

Now, we can give the first (non definitive) definition of our initial state.

Definition 5.5. Let us consider the sequence $(c_n(h))_{n \in \mathbb{Z}}$ defined by :

$$c_n := c_n(h) = K_h \chi \left(\frac{\mu_n - \mu_{n,0}}{h^{\alpha'}} |\ln h|^{\gamma'} \right) \chi_0 \left(\frac{\mu_n - \mu_{n,0}}{2h} \right), n \in \mathbb{Z}$$

where $\chi \in \mathcal{S}(\mathbb{R})$, non null, positive and pair, α' and γ' are two reals numbers; $\chi_0 \in \mathcal{D}(\mathbb{R})$ such that $\chi_0 \equiv 1$ on the set $] -1, 1[$ and $\text{supp}(\chi_0) \subset [-1, 1]$. We also denote

$$K_h := \left\| \chi \left(\frac{\mu_n - \mu_{n,0}}{h^{\alpha'}} |\ln h|^{\gamma'} \right) \chi_0 \left(\frac{\mu_n - \mu_{n,0}}{2h} \right) \right\|_{\ell^2(\mathbb{N})}.$$

Let us details this choice :

- (1) The term $\chi \left(\frac{\mu_n - \mu_{n,0}}{h^{\alpha'}} |\ln h|^{\gamma'} \right)$ localize around the energy level E_h (for technical reason we localize around the closest eigenvalues to E_h).
- (2) Constants α' et γ' are coefficients for dilate the function χ .
- (3) The function χ_0 is a cut-off for eigenvalues out of the compact set $[-h, h]$.

5.3. Choice for parameter α' and γ' . The only way to have a localisation for initial state more larger than the spectral gap, and in the same time, for have a localisation in the compact set $[-h, h]$; e.g to have in the same time :

$$\frac{h}{|\ln h|} \ll \frac{h^{\alpha'}}{|\ln h|^{\gamma'}} \leq h;$$

is to take

$$\alpha' = 1; 0 \leq \gamma' < 1.$$

Remark 5.6. The larger choice for $h^{\alpha'} / |\ln h|^{\gamma'} = h / |\ln h|^{\gamma'}$ is to take $\gamma' = 0$.

Under this assumptions, we can forget the function χ_0 and we get :

5.4. Second definition.

Definition 5.7. Let us consider the sequence $(c_n(h))_{n \in \mathbb{Z}}$ defined by :

$$c_n := K_h \chi \left(\frac{\mu_n - \mu_{n,0}}{h} |\ln h|^{\gamma'} \right), n \in \mathbb{Z}$$

where $\chi \in \mathcal{S}(\mathbb{R})$, non null, positive and pair; $0 \leq \gamma' < 1$; and

$$K_h := \left\| \chi \left(\frac{\mu_n - \mu_{n,0}}{h} |\ln h|^{\gamma'} \right) \right\|_{\ell^2(\mathbb{N})}.$$

Now, in the partial autocorrelations functions $\mathbf{a}(t)$ and $\mathbf{b}(t)$ appears the sequence $(c_{\sigma_1(k)})_k$ and $(c_{\sigma_2(\ell)})_\ell$; for simplicity let us consider the following sequences :

Notation 5.8. Let us denote

$$a_n := c_{\sigma_1(n)} = K_h \chi \left((\mathcal{A}_h(2\pi n) - \mathcal{A}_h(2\pi n_0)) |\ln h|^{\gamma'} \right);$$

$$b_m := c_{\sigma_2(m)} = K_h \chi \left((\mathcal{B}_h(2\pi m) - \mathcal{B}_h(2\pi m_0)) |\ln h|^{\gamma'} \right).$$

6. BEHAVIOUR OF A PARTIAL AUTOCORRELATION FUNCTION

The aim of this section is to understand time evolution for one partial autocorrelation function; for example the function $t \mapsto \mathbf{a}(t)$. First, let us define a definitive definition for initial state. By Lagrange's theorem¹ there exists a real number $\zeta = \zeta(n, h, E) \in \mathcal{Y}_h([-1, 1])$ such that

$$\mathcal{A}_h(2\pi n) - \mathcal{A}_h(2\pi n_0) = \mathcal{A}'_h(\zeta) 2\pi(n - n_0);$$

since

$$\mathcal{A}'_h(\zeta) = \frac{2\pi}{\frac{|\ln(h)|}{\sqrt{-V''(0)}} + O(1)};$$

we get

$$\mathcal{A}_h(2\pi n) - \mathcal{A}_h(2\pi n_0) = \frac{2\pi(n - n_0) \sqrt{-V''(0)}}{|\ln h|} \left(1 + O \left| \frac{1}{|\ln h|} \right| \right).$$

¹In french litterature : le théorème des accroissements finis.

Definition 6.1. Let us consider the sequence $(a_n)_{n \in \mathbb{Z}} = (a_n(h))_{n \in \mathbb{Z}}$ defined by :

$$a_n := K_h \chi \left(\frac{n - n_0}{|\ln h|^{1-\gamma'}} \right), \quad n \in \mathbb{Z}$$

where $\chi \in \mathcal{S}(\mathbb{R})$, non null, positive and pair; $0 \leq \gamma' < 1$; and

$$K_h := \left\| \chi \left(\frac{n - n_0}{|\ln h|^{1-\gamma'}} \right) \right\|_{\ell^2(\mathbb{N})}.$$

So, clearly the sequence $(a_n)_n \in \ell^2(\mathbb{Z})$. Now, let us evaluate the constant of normalization K_h . Start by the :

Lemma 6.2. For a function $\varphi \in \mathcal{S}(\mathbb{R})$ and $\varepsilon \in]0, 1]$; we have :

$$\sum_{\ell \in \mathbb{Z}, |\ell| \geq 1} \left| \varphi \left(\frac{\ell}{\varepsilon} \right) \right| = O(\varepsilon^\infty).$$

Proof. The starting point is the following remark : for all function $\psi \in \mathcal{S}(\mathbb{R})$ and for all $\varepsilon \in]0, 1]$ we have

$$\sum_{\ell \in \mathbb{Z}, |\ell| \geq 1} \left| \psi \left(\frac{\ell}{\varepsilon} \right) \right| = O(1).$$

Indeed, for all $\ell \in \mathbb{Z}$, $|\ell| \geq 1$

$$\begin{aligned} \left| \psi \left(\frac{\ell}{\varepsilon} \right) \right| &\leq \frac{M}{1 + \left(\frac{\ell}{\varepsilon} \right)^2} = \frac{\varepsilon^2 M}{\varepsilon^2 + \ell^2} \\ &\leq \frac{M}{\ell^2} \end{aligned}$$

and since

$$\sum_{\ell \in \mathbb{Z}, |\ell| \geq 1} \frac{M}{\ell^2} < +\infty.$$

we conclude.

Next, we apply this to the function $\psi(x) := x^{2N} \varphi(x)$, où $N \in \mathbb{N}$; and we get for all $N \geq 1$:

$$\sum_{\ell \in \mathbb{Z}, |\ell| \geq 1} \left| \varphi \left(\frac{\ell}{\varepsilon} \right) \right| \leq \varepsilon^{2N} \sum_{\ell \in \mathbb{Z}} \left| \psi \left(\frac{\ell}{\varepsilon} \right) \right| = O(\varepsilon^{2N}).$$

So we prove the lemma. □

Theorem 6.3. We have

$$K_h = \frac{1}{\sqrt{\mathfrak{F}(\chi^2)(0)} |\ln h|^{\frac{1-\gamma'}{2}}} + O \left| \frac{1}{\ln(h)^\infty} \right|;$$

so

$$\|a_n\|_{\ell^2(\mathbb{N})} = 1 + O \left| \frac{1}{\ln(h)^\infty} \right|.$$

Proof. By Poisson formula and the lemma 6.2 we get the equality

$$\begin{aligned} \sum_{n \in \mathbb{Z}} \chi^2 \left(\frac{n - n_0}{|\ln h|^{1-\gamma'}} \right) &= |\ln h|^{1-\gamma'} \sum_{\ell \in \mathbb{Z}} \mathfrak{F}(\chi^2) \left(-\ell |\ln h|^{1-\gamma'} \right) \\ &= |\ln h|^{1-\gamma'} \left[\mathfrak{F}(\chi^2)(0) + O \left| \frac{1}{\ln(h)^\infty} \right| \right] \end{aligned}$$

Now, start with the equality

$$\begin{aligned} \sum_{n \in \mathbb{N}} \chi^2 \left(\frac{n - n_0}{|\ln h|^{1-\gamma'}} \right) &= \sum_{n \in \mathbb{Z}} \chi^2 \left(\frac{n - n_0}{|\ln h|^{1-\gamma'}} \right) - \sum_{n=-\infty}^{-1} \chi^2 \left(\frac{n - n_0}{|\ln h|^{1-\gamma'}} \right) \\ &= |\ln h|^{1-\gamma'} \mathfrak{F}(\chi^2)(0) + O \left| \frac{1}{\ln(h)^\infty} \right| - \sum_{n=-\infty}^{-1} \chi^2 \left(\frac{n - n_0}{|\ln h|^{1-\gamma'}} \right). \end{aligned}$$

Since the function $\mathfrak{F}(\chi^2)$ is pair :

$$\sum_{n=-\infty}^{-1} \chi^2 \left(\frac{n - n_0}{|\ln h|^{1-\gamma'}} \right) = \sum_{n=1}^{+\infty} \chi^2 \left(\frac{n + n_0}{|\ln h|^{1-\gamma'}} \right) \leq B_k |\ln h|^{(1-\gamma')k} \sum_{n=1}^{+\infty} \frac{1}{|n + n_0|^k}.$$

And,

$$\begin{aligned} B_k |\ln h|^{(1-\gamma')k} \sum_{n=1}^{+\infty} \frac{1}{|n + n_0|^k} &\leq B_k |\ln h|^{(1-\gamma')k} \int_0^\infty \frac{du}{(u + n_0)^k} \\ &= |\ln h|^{(1-\gamma')(k-1)} \frac{B_k}{(k-1)} \frac{1}{n_0^{k-1}}. \end{aligned}$$

Now, since for $h \rightarrow 0$ we have $n_0 \sim \frac{N}{h}$, we get, for $h \rightarrow 0$

$$|\ln h|^{(1-\gamma')k} \frac{B_k}{(k-1)} \frac{1}{n_0^{k-1}} \sim \frac{B_k}{N^{k-1}(k-1)} h^{k-1} |\ln h|^{(1-\gamma')k}$$

consequently

$$\sum_{n=-\infty}^{-1} \chi^2 \left(\frac{n - n_0}{|\ln h|^{1-\gamma'}} \right) = O \left| \frac{1}{\ln(h)^\infty} \right|.$$

So, we prove that

$$\left\| \chi \left(\frac{n - n_0}{|\ln h|^{1-\gamma'}} \right) \right\|_{\ell^2(\mathbb{N})}^2 = |\ln h|^{1-\gamma'} \mathfrak{F}(\chi^2)(0) + O \left| \frac{1}{\ln(h)^\infty} \right|;$$

hence $K_h = \frac{1}{\sqrt{\mathfrak{F}(\chi^2)(0)|\ln h|^{\frac{1-\gamma'}{2}}}} + O \left| \frac{1}{\ln(h)^\infty} \right|$. For finish, we have

$$\|a_n\|_{\ell^2(\mathbb{N})}^2 = K_h^2 |\ln h|^{1-\gamma'} \left[\mathfrak{F}(\chi^2)(0) + O \left| \frac{1}{\ln(h)^\infty} \right| \right] = 1 + O \left| \frac{1}{\ln(h)^\infty} \right|.$$

□

6.1. The set Δ .

Definition 6.4. Let us define the sets of integers $\Delta = \Delta(h, E)$ and $\Gamma = \Gamma(h, E)$ by :

$$\Delta := \{n \in \mathbb{N}, |n - n_0| \leq |\ln(h)|^\gamma\} \subset \mathbb{N}$$

and

$$\Gamma := \mathbb{N} - \Delta$$

where γ is a real number such that $\gamma < 1$ and $\gamma + \gamma' > 1$.

Remark 6.5. Since $\gamma < 1$, we have $|\ln(h)|^\gamma < |\ln(h)|$, hence for $h \rightarrow 0$ we have

$$\text{Card}(\mathbf{I}_h) \sim |\ln(h)|.$$

So, for h small enough we obtain $\Delta \subset \mathbf{I}_h$. On the other hand, since $\gamma + \gamma' > 1$ we have $|\ln(h)|^{1-\gamma'} \ll |\ln(h)|^\gamma$; this mean that the set Δ is more larger than the localization of initial state.

Lemma 6.6. *We have*

$$\sum_{n \in \Gamma} |a_n|^2 = O \left| \frac{1}{\ln(h)^\infty} \right|.$$

Proof. The starting point is the following inequality :

$$\sum_{n \in \Gamma} |a_n|^2 \leq \sum_{n \in \mathbb{Z}, |n-n_0| > h^{\delta-1}} |a_n|^2.$$

Next, with the same argument as in the lemma 6.2, we show that for all integer $N \geq 1$

$$\sum_{n \in \mathbb{Z}} \left(\frac{n-n_0}{h^{\delta'-1}} \right)^{2N} |a_n|^2 = O(1).$$

Without loss generality, we may suppose that $n_0 = 0$. Next we write :

$$\begin{aligned} \sum_{n \in \mathbb{Z}, |n| > h^{\delta-1}} |a_n|^2 &= h^{2N(\delta'-1)} \sum_{n \in \mathbb{Z}, |n| > h^{\delta-1}} |a_n|^2 \left(\frac{n}{h^{\delta'-1}} \right)^{2N} \frac{1}{n^{2N}} \\ &\leq \frac{h^{2N(\delta'-1)}}{h^{2N(\delta-1)}} \sum_{n \in \mathbb{Z}} |a_n|^2 \left(\frac{n}{h^{\delta'-1}} \right)^{2N} = O \left(h^{2N(\delta'-\delta)} \right). \end{aligned}$$

Since $\delta' > \delta$, we obtain $\sum_{n \in \Gamma} |a_n|^2 = O(h^\infty)$. □

7. ORDER 1 APPROXIMATION : HYPERBOLIC PERIOD

7.1. Goal. The aim of this section is to study the series :

$$\mathbf{a} : t \mapsto \sum_{n \in \mathbb{N}} |a_n|^2 e^{-it \mathcal{A}_h(2\pi n)}.$$

By a Taylor's formula we get :

$$\mathbf{a}(t) = \sum_{n \in \mathbb{N}} |a_n|^2 e^{-it(\mathcal{A}_h(2\pi n_0) + \mathcal{A}'_h(2\pi n_0)2\pi(n-n_0) + \mathcal{A}''_h(\zeta)2\pi^2(n-n_0)^2)}$$

where $\zeta = \zeta(n, h, E) \in \mathcal{Y}_h([-1, 1])$.

Lemma 7.1. *Uniformly, on the compact set $[-1, 1]$ we have :*

$$\lambda \mapsto (\mathcal{A}''_h \circ \mathcal{Y}_h)(\lambda) = O \left| \frac{1}{\ln(h)^3} \right|.$$

Proof. Derivatives formulas gives for all $x \in \mathcal{Y}_h([-1, 1])$, the equality

$$\mathcal{A}''_h(x) = \frac{-(\mathcal{Y}''_h \circ \mathcal{A}_h)(x)}{(\mathcal{Y}'_h \circ \mathcal{A}_h)^3(x)};$$

hence, for $\lambda \in [-1, 1]$

$$(\mathcal{A}''_h \circ \mathcal{Y}_h)(\lambda) = \frac{-\mathcal{Y}''_h(\lambda)}{(\mathcal{Y}'_h)^3(\lambda)}.$$

First, for all $\lambda \in [-1, 1]$

$$\begin{aligned} \mathcal{Y}''_h(\lambda) &= -h^2 \frac{\theta''_+(\lambda h) + \theta''_-(\lambda h)}{2} + h \varepsilon''(\lambda h) \ln(h) \\ &+ \frac{\partial^2}{\partial \lambda^2} \left[\arg \left(\Gamma \left(\frac{1}{2} + i \frac{\varepsilon(\lambda h)}{h} \right) \right) \right] - \frac{\partial^2}{\partial \lambda^2} \left[\arccos \left(\frac{\cos(g_h(\lambda))}{\sqrt{1 + \exp(2\pi \varepsilon(\lambda h)/h)}} \right) \right]. \end{aligned}$$

And, for all $\lambda \in [-1, 1]$ we have

$$-h^2 \frac{\theta''_+(\lambda h) + \theta''_-(\lambda h)}{2} = O(h)$$

and

$$h\varepsilon''(\lambda h) \ln(h) = O(h \ln(h)).$$

Next, we study the term $\frac{\partial^2}{\partial \lambda^2} \left[\arg \left(\Gamma \left(\frac{1}{2} + i \frac{\varepsilon(\lambda h)}{h} \right) \right) \right]$: for all $\lambda \in [-1, 1]$ we have :

$$\begin{aligned} & \frac{\partial}{\partial \lambda} \left[\arg \left(\Gamma \left(\frac{1}{2} + i \frac{\varepsilon(\lambda h)}{h} \right) \right) \right] = \frac{\partial}{\partial \lambda} \left[\operatorname{Im} \left(\ln \left(\Gamma \left(\frac{1}{2} + i \frac{\varepsilon(\lambda h)}{h} \right) \right) \right) \right] \\ &= \operatorname{Im} \left[\frac{\partial}{\partial \lambda} \left(\ln \left(\Gamma \left(\frac{1}{2} + i \frac{\varepsilon(\lambda h)}{h} \right) \right) \right) \right] = \operatorname{Im} \left[\frac{\Gamma' \left(\frac{1}{2} + i \frac{\varepsilon(\lambda h)}{h} \right) i \varepsilon'(\lambda h)}{\Gamma \left(\frac{1}{2} + i \frac{\varepsilon(\lambda h)}{h} \right)} \right] \\ &= \varepsilon'(\lambda h) \operatorname{Re} \left[\frac{\Gamma' \left(\frac{1}{2} + i \frac{\varepsilon(\lambda h)}{h} \right)}{\Gamma \left(\frac{1}{2} + i \frac{\varepsilon(\lambda h)}{h} \right)} \right] = \varepsilon'(\lambda h) \operatorname{Re} \left(\Psi \left(\frac{1}{2} + i \frac{\varepsilon(\lambda h)}{h} \right) \right); \end{aligned}$$

hence, for all $\lambda \in [-1, 1]$

$$\begin{aligned} & \frac{\partial^2}{\partial \lambda^2} \left[\arg \left(\Gamma \left(\frac{1}{2} + i \frac{\varepsilon(\lambda h)}{h} \right) \right) \right] \\ &= \frac{\partial}{\partial \lambda} \left[\varepsilon'(\lambda h) \operatorname{Re} \left(\Psi \left(\frac{1}{2} + i \frac{\varepsilon(\lambda h)}{h} \right) \right) \right] \\ &= h\varepsilon''(\lambda h) \operatorname{Re} \left(\Psi \left(\frac{1}{2} + i \frac{\varepsilon(\lambda h)}{h} \right) \right) - (\varepsilon'(\lambda h))^2 \operatorname{Im} \left(\Psi^{(1)} \left(\frac{1}{2} + i \frac{\varepsilon(\lambda h)}{h} \right) \right), \end{aligned}$$

wher $\Psi^{(1)}$ is the first-derivative of di-Gamma function ($\Psi(z) := \frac{\Gamma'(z)}{\Gamma(z)}$, see **[Ab-St]**).

Hence the function

$$\lambda \mapsto \frac{\partial}{\partial \lambda} \left[\varepsilon'(\lambda h) \operatorname{Re} \left(\Psi \left(\frac{1}{2} + i \frac{\varepsilon(\lambda h)}{h} \right) \right) \right]$$

is equal to $O(1)$ on the compact set $[-1, 1]$. For finish, we estimate the term :

$$\lambda \mapsto \frac{\partial^2}{\partial \lambda^2} \left[\arccos \left(\frac{\cos(g_h(\lambda))}{\sqrt{1 + \exp(2\pi\varepsilon(\lambda h)/h)}} \right) \right];$$

for all $\lambda \in [-1, 1]$ we obtain

$$\frac{\partial^2}{\partial \lambda^2} \left[\arccos \left(\frac{\cos(g_h(\lambda))}{\sqrt{1 + \exp(2\pi\varepsilon(\lambda h)/h)}} \right) \right] = \frac{N_h(\lambda)}{D_h(\lambda)}$$

where $N_h(\lambda)$ is a polynomial (not depend on h) in the variables :

$$\begin{aligned} & \left(\cos(g_h(\lambda)), \sin(g_h(\lambda)), \frac{\partial}{\partial \lambda}(g_h(\lambda)), \frac{\partial^2}{\partial \lambda^2}(g_h(\lambda)) \right) \\ & \left(\frac{\partial}{\partial \lambda} \left(\frac{\varepsilon(\lambda h)}{h} \right), \frac{\partial^2}{\partial \lambda^2} \left(\frac{\varepsilon(\lambda h)}{h} \right), \exp(2\pi\varepsilon(\lambda h)/h) \right). \end{aligned}$$

Hence $\lambda \mapsto N_h(\lambda) = O(1)$ on the compact set $[-1, 1]$. Next we have

$$D_h(\lambda) = (1 + \exp(2\pi\varepsilon(\lambda h)/h))^2 \left(1 + \exp(2\pi\varepsilon(\lambda h)/h) - \cos^2(g_h(\lambda)) \right)^{\frac{3}{2}};$$

since the function $\lambda \mapsto \exp(2\pi\varepsilon(\lambda h)/h) > 0$ is equal to $O(1)$ on the compact set $[-1, 1]$, we see easily that on the compact set $[-1, 1]$ we have

$$\lambda \mapsto \frac{1}{D_h(\lambda)} = O(1).$$

Next, on the set $[-1, 1]$

$$\lambda \mapsto \frac{\partial^2}{\partial \lambda^2} \left[\arccos \left(\frac{\cos(g_h(\lambda))}{\sqrt{1 + \exp(2\pi\varepsilon(\lambda h)/h)}} \right) \right] = O(1).$$

Then on the compact set $[-1, 1]$ we obtain that $\lambda \mapsto \mathcal{Y}_h''(\lambda) = O(1)$. For finish, since, uniformly on the compact set $[-1, 1]$ we have the equality

$$\mathcal{Y}_h'(\lambda) = \frac{\ln(h)}{\sqrt{-V''(0)}} + O(1);$$

then, we deduce that $\lambda \mapsto (\mathcal{A}_h'' \circ \mathcal{Y}_h)(\lambda) = O\left|\frac{1}{\ln(h)^3}\right|$ on $[-1, 1]$. \square

7.2. Definition of a first order approximation time scale.

Proposition 7.2. *Let α a real number such that $\alpha < 3 - 2\gamma$. Then, uniformly for all $t \in [0, |\ln(h)|^\alpha]$ we have :*

$$\sum_{n \in \mathbb{N}} |a_n|^2 e^{-it\mathcal{A}_h(2\pi n)} = \sum_{n \in \mathbb{N}} |a_n|^2 e^{-it(\mathcal{A}_h(2\pi n_0) + \mathcal{A}_h'(2\pi n_0)2\pi(n-n_0))} + O\left(|\ln(h)|^{\alpha+2\gamma-3}\right).$$

Proof. Let us introduce the difference $\varepsilon(t) := \varepsilon(t, h)$ defined by :

$$\varepsilon(t) := \left| \sum_{n \in \mathbb{N}} |a_n|^2 e^{-it\mathcal{A}_h(2\pi n)} - \sum_{n \in \mathbb{N}} |a_n|^2 e^{-it(\mathcal{A}_h(2\pi n_0) + \mathcal{A}_h'(2\pi n_0)2\pi(n-n_0))} \right|.$$

Taylor-Lagrange's formula give the existence of a real number $\zeta = \zeta(n, h, E) \in \mathcal{Y}_h([-1, 1])$ such that

$$\mathcal{A}_h(2\pi n) = \mathcal{A}_h(2\pi n_0) + \mathcal{A}_h'(2\pi n_0)2\pi(n-n_0) + \mathcal{A}_h''(\zeta)2\pi^2(n-n_0)^2;$$

hence for all $t \geq 0$ we get

$$\begin{aligned} \varepsilon(t) &= \left| \sum_{n \in \mathbb{N}} |a_n|^2 e^{-it(\mathcal{A}_h(2\pi n_0) + \mathcal{A}_h'(2\pi n_0)2\pi(n-n_0) + \mathcal{A}_h''(\zeta)2\pi^2(n-n_0)^2)} \right. \\ &\quad \left. - \sum_{n \in \mathbb{N}} |a_n|^2 e^{-it(\mathcal{A}_h(2\pi n_0) + \mathcal{A}_h'(2\pi n_0)2\pi(n-n_0))} \right| \\ &= \left| \sum_{n \in \mathbb{N}} |a_n|^2 e^{-it(\mathcal{A}_h(2\pi n_0) + \mathcal{A}_h'(2\pi n_0)2\pi(n-n_0))} \left[e^{-it\mathcal{A}_h''(\zeta)2\pi^2(n-n_0)^2} - 1 \right] \right|. \end{aligned}$$

With the sets Γ, Δ and by triangular inequality, we obtain for all $t \geq 0$

$$\begin{aligned} \varepsilon(t) &\leq \left| \sum_{n \in \Delta} |a_n|^2 e^{-it(\mathcal{A}_h(2\pi n_0) + \mathcal{A}_h'(2\pi n_0)2\pi(n-n_0))} \left[e^{-it\mathcal{A}_h''(\zeta)2\pi^2(n-n_0)^2} - 1 \right] \right| \\ (7.1) \quad &+ \left| \sum_{n \in \Gamma} |a_n|^2 e^{-it(\mathcal{A}_h(2\pi n_0) + \mathcal{A}_h'(2\pi n_0)2\pi(n-n_0))} \left[e^{-it\mathcal{A}_h''(\zeta)2\pi^2(n-n_0)^2} - 1 \right] \right|. \end{aligned}$$

First, look at the term the right part of the inequality 7.1, for all $t \geq 0$ we have

$$\begin{aligned} &\left| \sum_{n \in \Gamma} |a_n|^2 e^{-it(\mathcal{A}_h(2\pi n_0) + \mathcal{A}_h'(2\pi n_0)2\pi(n-n_0))} \left[e^{-it\mathcal{A}_h''(\zeta)2\pi^2(n-n_0)^2} - 1 \right] \right| \\ &\leq 2 \sum_{n \in \Gamma} |a_n|^2 = O\left|\frac{1}{\ln(h)^\infty}\right| \end{aligned}$$

by the lemma 6.6.

Now look at the left term of the inequality 7.1. With the lemma 7.1: for all integer $n \in \Delta$ and for all real number $t \in [0, |\ln(h)|^\alpha]$ we have

$$t\mathcal{A}_h''(\zeta)2\pi^2(n-n_0)^2 \leq M |\ln(h)|^{\alpha+2\gamma-3}$$

where $M > 0$. Consequently, for all integer $n \in \Delta$ and for all $t \in [0, |\ln(h)|^\alpha]$ we have

$$e^{-it\mathcal{A}_h''(\zeta)2\pi^2(n-n_0)^2} - 1 = O\left(|\ln(h)|^{\alpha+2\gamma-3}\right);$$

hence for all $t \in [0, |\ln(h)|^\alpha]$

$$\begin{aligned} & \left| \sum_{n \in \Delta} |a_n|^2 e^{-it(\mathcal{A}_h(2\pi n_0) + \mathcal{A}_h'(2\pi n_0)2\pi(n-n_0))} \left[e^{-it\mathcal{A}_h''(\zeta)2\pi^2(n-n_0)^2} - 1 \right] \right| \\ & \leq O\left(|\ln(h)|^{\alpha+2\gamma-3}\right) \sum_{n \in \Delta} |a_n|^2 \\ & \leq O\left(|\ln(h)|^{\alpha+2\gamma-3}\right) \sum_{n \in \mathbb{N}} |a_n|^2 = O\left(|\ln(h)|^{\alpha+2\gamma-3}\right). \end{aligned}$$

So, for all $t \in [0, |\ln(h)|^\alpha]$ we get finally

$$\varepsilon(t) = O\left(|\ln(h)|^{\alpha+2\gamma-3}\right).$$

□

We conclude that, the principal term of the partial autocorrelation $\mathbf{a}(t)$ in the time-scale $[0, |\ln(h)|^\alpha]$ is:

Definition 7.3. The principal term of the partial autocorrelation $\mathbf{a}(t)$ is defined by :

$$\mathbf{a}_1 : t \mapsto \sum_{n \in \mathbb{N}} |a_n|^2 e^{-it(\mathcal{A}_h(2\pi n_0) + \mathcal{A}_h'(2\pi n_0)2\pi(n-n_0))}.$$

We also define the function $\tilde{\mathbf{a}}_1$ by

$$\tilde{\mathbf{a}}_1 : t \mapsto \sum_{n \in \mathbb{N}} |a_n|^2 e^{-it\mathcal{A}_h'(2\pi n_0)2\pi(n-n_0)}.$$

Hence

$$\mathbf{a}_1(t) = e^{-it\mathcal{A}_h(2\pi n_0)} \tilde{\mathbf{a}}_1(t).$$

Now, we study this serie in details.

8. SEMI-CLASSICAL' S BEHAVIOUR OF THE PRINCIPAL TERM

8.1. Periodicity of the principal term. Clearly, the function $t \mapsto |\mathbf{a}_1(t)|$ is $\frac{1}{\mathcal{A}_h'(2\pi n_0)}$ -periodic.

Definition 8.1. Let us define the hyperbolic period $T_{hyp} = T_{hyp}(h, E)$ by

$$T_{hyp} := \frac{1}{\mathcal{A}_h'(2\pi n_0)}.$$

Since

$$\mathcal{A}_h'(\zeta) = \frac{2\pi}{\frac{|\ln(h)|}{\sqrt{-V''(0)}} + O(1)};$$

for $h \rightarrow 0$ we have $T_{hyp} \sim K |\ln(h)|$; where $K > 0$ (not depend on h).

8.2. Geometric interpretation of the period. The period T_{hyp} correspond to the period of the classical flow. The term $|\ln(h)|$ is the signature of the hyperbolic singularity : indeed, the term $|\ln(h)|$ correspond to the period of the classical flow with a initial point on the disc $D(O, \sqrt{h}) := \{(x, \zeta) \in \mathbb{R}^2, \sqrt{x^2 + \zeta^2} = \sqrt{h}\}$.

Theorem 8.2. [Lab3]. Let us consider the point $m_h = (\sqrt{h}, 0) \in T^*\mathbb{R}$. Then the Hamiltonian's flow associated to the function p and with initial point m_h is periodic and the period τ_h verify, for $h \rightarrow 0$, the following equivalent :

$$\tau_h \sim \frac{|\ln(h)|}{K}$$

where the constant $K \neq 0$ and not depend on h .

Remark 8.3. See also the paper [DB-Ro] for a similar result.

8.3. Hyperbolic period and the time scale $[0, |\ln(h)|^\alpha]$. Since $\gamma < 1$, we have $1 < 3 - 2\gamma$; hence there exist a real number α such that $\alpha \in]1, 3 - 2\gamma[$. Consequently, we get

$$|\ln(h)| < |\ln(h)|^\alpha < |\ln(h)|^{3-2\gamma}.$$

So, we can make a "good choice" for α : for h small enough we have :

$$[0, T_{hyp}] \subset [0, |\ln(h)|^\alpha].$$

8.4. Behaviour of autocorrelation function on a hyperbolic period. Let us study the function $\mathbf{a}_1(t)$ on the period $[0, T_{hyp}]$. Start by a technical proposition :

Proposition 8.4. For all $t \geq 0$, we have the equality :

$$\sum_{n \in \mathbb{Z}} |a_n|^2 e^{-it2\pi(n-n_0)\frac{1}{T_{hyp}}} = \frac{1}{\mathfrak{F}(\chi^2)(0)} \sum_{\ell \in \mathbb{Z}} \mathfrak{F}(\chi^2) \left(-|\ln h|^{1-\gamma'} \left(\ell + \frac{t}{T_{hyp}} \right) \right).$$

Proof. Let us consider the function Ω_t defined by

$$\Omega_t : \begin{cases} \mathbb{R} \rightarrow \mathbb{C} \\ x \mapsto |a_x|^2 e^{-it2\pi(x-n_0)\frac{1}{T_{hyp}}} \end{cases}$$

where $t \in \mathbb{R}$ is a parameter, and let us recall a_x is defined by

$$a_x = \frac{1}{\sqrt{\mathfrak{F}(\chi^2)(0)} |\ln h|^{\frac{1-\gamma'}{2}}} \chi \left(\frac{x - n_0}{|\ln h|^{1-\gamma'}} \right);$$

thus

$$\sum_{n \in \mathbb{Z}} |a_n|^2 e^{-it2\pi(n-n_0)\frac{1}{T_{hyp}}} = \sum_{n \in \mathbb{Z}} \Omega_t(n).$$

So clearly, the function $\Omega_t \in \mathcal{S}(\mathbb{R})$, then the Fourier transform $\mathfrak{F}(\Omega_t)$ is equal, for all $\zeta \in \mathbb{R}$, to

$$\begin{aligned} \mathfrak{F}(\Omega_t)(\zeta) &= \frac{1}{\mathfrak{F}(\chi^2)(0) |\ln h|^{1-\gamma'}} e^{it2\pi n_0 \frac{1}{T_{hyp}}} \mathfrak{F} \left(\chi^2 \left(\frac{x - n_0}{|\ln h|^{1-\gamma'}} \right) e^{-it2\pi x \frac{1}{T_{hyp}}} \right) (\zeta). \\ &= \frac{1}{\mathfrak{F}(\chi^2)(0)} e^{-2i\pi n_0 \zeta} \mathfrak{F}(\chi^2) \left(-|\ln h|^{1-\gamma'} \left(\zeta + \frac{t}{T_{hyp}} \right) \right). \end{aligned}$$

With the Poisson formula we get

$$\sum_{n \in \mathbb{Z}} \Omega_t(n) = \sum_{\ell \in \mathbb{Z}} \mathfrak{F}(\Omega_t)(\ell)$$

$$= \frac{1}{\mathfrak{F}(\chi^2)(0)} \sum_{\ell \in \mathbb{Z}} \mathfrak{F}(\chi^2) \left(-|\ln h|^{1-\gamma'} \left(\ell + \frac{t}{T_{hyp}} \right) \right).$$

□

So, now, our goal is to study the behaviour of the serie :

$$t \mapsto \frac{1}{\mathfrak{F}(\chi^2)(0)} \sum_{\ell \in \mathbb{Z}} \mathfrak{F}(\chi^2) \left(-|\ln h|^{1-\gamma'} \left(\ell + \frac{t}{T_{hyp}} \right) \right).$$

Since the function $\mathfrak{F}(\chi^2) \in \mathcal{S}(\mathbb{R})$, it's clear that only indices $\ell \in \mathbb{Z}$ such that $\left(\ell + \frac{t}{T_{hyp}} \right)$ is close to zero are important. More precisely :

Definition 8.5. For all $t \in \mathbb{R}$, let us define $\ell(t) = \ell(t, h, E)$ the closest integer to the real number $\frac{-t}{T_{hyp}}$; e.g.

$$\ell(t) + \frac{t}{T_{hyp}} = d\left(t, T_{hyp}\mathbb{Z}\right);$$

where $d(., .)$ denote the Euclidian distance on \mathbb{R} .

Remark 8.6. Without loss generality, we may suppose $\ell(t)$ is unique. On the other hand, for all integer $\ell \in \mathbb{Z}$ such that $\ell \neq \ell(t)$ we get :

$$\left| \ell + \frac{t}{T_{hyp}} \right| \geq \frac{1}{2}.$$

Lemma 8.7. For a function $\varphi \in \mathcal{S}(\mathbb{R})$ and $\varepsilon \in]0, 1]$ then, uniformly for $u \in \mathbb{R}$ we have :

$$\sum_{\ell \in \mathbb{Z}, |\ell+u| \geq \frac{1}{2}} \left| \varphi\left(\frac{\ell+u}{\varepsilon}\right) \right| = O(\varepsilon^\infty).$$

Proof. We see easily that, uniformly for $u \in \mathbb{R}$ we have

$$\sum_{\ell \in \mathbb{Z}, |\ell+u| \geq \frac{1}{2}} \left| \varphi\left(\frac{\ell+u}{\varepsilon}\right) \right| = O(1).$$

Indeed, since $\varepsilon \in]0, 1]$, without loss generality, we may suppose $u \in [-1, 1]$. And since for all $u \in [-1, 1]$ we have

$$\begin{aligned} \left| \varphi\left(\frac{\ell+u}{\varepsilon}\right) \right| &\leq \frac{M}{1 + \frac{(\ell+u)^2}{\varepsilon^2}} = \frac{\varepsilon^2 M}{\varepsilon^2 + (\ell+u)^2} \\ &\leq \frac{M}{(\ell+u)^2} = \frac{M}{|\ell^2 - 2|\ell||}; \end{aligned}$$

and we conclude.

Next,

$$\begin{aligned} \sum_{\ell \in \mathbb{Z}, |\ell+u| \geq \frac{1}{2}} \left| \varphi\left(\frac{\ell+u}{\varepsilon}\right) \right| &= \sum_{\ell \in \mathbb{Z}, |\ell+u| \geq \frac{1}{2}} \left(\frac{\ell+u}{\varepsilon}\right)^{2N} \left| \varphi\left(\frac{\ell+u}{\varepsilon}\right) \right| \frac{\varepsilon^{2N}}{(\ell+u)^{2N}} \\ &\leq \varepsilon^{2N} 4^N \sum_{\ell \in \mathbb{Z}} \left(\frac{\ell+u}{\varepsilon}\right)^{2N} \left| \varphi\left(\frac{\ell+u}{\varepsilon}\right) \right|. \end{aligned}$$

For conclude the proof, we apply that to the function $\psi(x) := x^{2N} \varphi(x)$. □

Theorem 8.8. Uniformly for $t \geq 0$ we have :

$$\sum_{n \in \mathbb{N}} |a_n|^2 e^{-\frac{it2\pi(n-n_0)}{T_{hyp}}} = \frac{1}{\mathfrak{F}(\chi^2)(0)} \mathfrak{F}(\chi^2) \left(-|\ln h|^{1-\gamma'} d \left(t, T_{hyp} \mathbb{Z} \right) \right) + O \left| \frac{1}{\ln(h)^\infty} \right|.$$

Proof. Since the proposition 8.4 and with the lemma 8.7 we get for all $t \geq 0$

$$\begin{aligned} \sum_{n \in \mathbb{Z}} |a_n|^2 e^{-\frac{it2\pi(n-n_0)}{T_{hyp}}} &= \frac{1}{\mathfrak{F}(\chi^2)(0)} \sum_{\ell \in \mathbb{Z}} \mathfrak{F}(\chi^2) \left(-|\ln h|^{1-\gamma'} \left(\ell + \frac{t}{T_{hyp}} \right) \right) \\ &= \frac{1}{\mathfrak{F}(\chi^2)(0)} \mathfrak{F}(\chi^2) \left(-|\ln h|^{1-\gamma'} d \left(t, T_{hyp} \mathbb{Z} \right) \right) + O \left| \frac{1}{\ln(h)^\infty} \right|. \end{aligned}$$

Next, for all $t \geq 0$

$$\begin{aligned} &\sum_{n \in \mathbb{N}} |a_n|^2 e^{-\frac{it2\pi(n-n_0)}{T_{hyp}}} \\ &= \sum_{n \in \mathbb{Z}} |a_n|^2 e^{-\frac{it2\pi(n-n_0)}{T_{hyp}}} - \sum_{n=-\infty}^{-1} |a_n|^2 e^{-\frac{it2\pi(n-n_0)}{T_{hyp}}}; \\ &= \frac{1}{\mathfrak{F}(\chi^2)(0)} \sum_{\ell \in \mathbb{Z}} \mathfrak{F}(\chi^2) \left(-|\ln h|^{1-\gamma'} \left(\ell + \frac{t}{T_{hyp}} \right) \right) \\ &\quad - \sum_{n=-\infty}^{-1} |a_n|^2 e^{-\frac{it2\pi(n-n_0)}{T_{hyp}}}. \end{aligned}$$

Hence, by triangular inequality, we have for all $t \geq 0$

$$\begin{aligned} &\left| \sum_{n \in \mathbb{N}} |a_n|^2 e^{-\frac{it2\pi(n-n_0)}{T_{hyp}}} - \frac{1}{\mathfrak{F}(\chi^2)(0)} \mathfrak{F}(\chi^2) \left(-|\ln h|^{1-\gamma'} d \left(t, T_{hyp} \mathbb{Z} \right) \right) \right| \\ &\leq \sum_{\ell \neq \ell(t)} \left| \frac{1}{\mathfrak{F}(\chi^2)(0)} \mathfrak{F}(\chi^2) \left(-|\ln h|^{1-\gamma'} \left(\ell + \frac{t}{T_{hyp}} \right) \right) \right| + \sum_{n=1}^{+\infty} |a_{-n}|^2. \end{aligned}$$

Since, for all $t \geq 0$, we have

$$\frac{1}{\mathfrak{F}(\chi^2)(0)} \sum_{\ell \neq \ell(t)} \mathfrak{F}(\chi^2) \left(-|\ln h|^{1-\gamma'} \left(\ell + \frac{t}{T_{hyp}} \right) \right) = O \left| \frac{1}{\ln(h)^\infty} \right|;$$

and

$$\sum_{n=1}^{+\infty} |a_{-n}|^2 = O \left| \frac{1}{\ln(h)^\infty} \right|.$$

So we prove the theorem. \square

Corollary 8.9. We have :

(i) for t such that $t \in T_{hyp} \mathbb{N}$ we get

$$\sum_{n \in \mathbb{N}} |a_n|^2 e^{-\frac{it2\pi(n-n_0)}{T_{hyp}}} = 1.$$

(ii) For all $\varepsilon > 0$ such that $\varepsilon < 1 - \gamma'$, and for t such that $\left| d \left(t, T_{hyp} \mathbb{Z} \right) \right| > |\ln h|^{\gamma'-1+\varepsilon}$; we get

$$\sum_{n \in \mathbb{N}} |a_n|^2 e^{-\frac{it2\pi(n-n_0)}{T_{hyp}}} = O \left| \frac{1}{\ln(h)^\infty} \right|.$$

Proof. The first point is clear. For the second : if

$$\left| -|\ln h|^{1-\gamma'} (\ell(t) + t\mathcal{A}'_h(2\pi n_0)) \right| > |\ln h|^\varepsilon$$

thus, we have :

$$\begin{aligned} \left| \mathfrak{F}(\chi^2) \left(-|\ln h|^{1-\gamma'} (\ell(t) + t\mathcal{A}'_h(2\pi n_0)) \right) \right| &\leq \frac{B_k}{(1 + |\ln h|^\varepsilon)^k} \\ &\leq B_k |\ln h|^{-\varepsilon k}. \end{aligned}$$

for all integer $k \geq 1$; so we prove the second point. \square

9. SECOND ORDER SEMI-CLASSICAL' S APPROXIMATION

By a Taylor's formula we have for all $t \geq 0$

$$\mathbf{a}(t) = \sum_{n \in \mathbb{N}} |a_n|^2 e^{-it \left(\mathcal{A}_h(2\pi n_0) + \mathcal{A}'_h(2\pi n_0) 2\pi(n-n_0) + \mathcal{A}''(2\pi n_0) 2\pi^2(n-n_0)^2 + \mathcal{A}_h^{(3)}(\zeta) \frac{(2\pi)^3}{8} (n-n_0)^3 \right)}$$

where $\zeta = \zeta(n, h, E)$.

Lemma 9.1. *Uniformly, on the compact set $[-1, 1]$ we have*

$$\lambda \mapsto \left(\mathcal{A}_h^{(3)} \circ \mathcal{Y}_h \right) (\lambda) = O \left| \frac{1}{\ln(h)^4} \right|.$$

Proof. With the derivatives formulas, we have for all $x \in \mathcal{Y}_h([-1, 1])$

$$\mathcal{A}_h^{(3)}(x) = \frac{-\left(\mathcal{Y}_h^{(3)} \circ \mathcal{A}_h \right) (x)}{\left(\mathcal{Y}'_h \circ \mathcal{A}_h \right)^4 (x)} + \frac{3 \left(\mathcal{Y}_h'' \circ \mathcal{A}_h \right)^2 (x)}{\left(\mathcal{Y}'_h \circ \mathcal{A}_h \right)^5 (x)};$$

hence, for all $\lambda \in [-1, 1]$ we get

$$\left(\mathcal{A}_h^{(3)} \circ \mathcal{Y}_h \right) (\lambda) = \frac{-\mathcal{Y}_h^{(3)}(\lambda)}{\left(\mathcal{Y}'_h \right)^4 (\lambda)} + \frac{3 \left(\mathcal{Y}_h'' \right)^2 (\lambda)}{\left(\mathcal{Y}'_h \right)^5 (\lambda)}.$$

First, let us estimate the function $\lambda \mapsto \mathcal{Y}_h^{(3)}(\lambda)$. For all $\lambda \in [-1, 1]$ we have

$$\begin{aligned} \mathcal{Y}_h^{(3)}(\lambda) &= -h^3 \frac{\theta_+^{(3)}(\lambda h) + \theta_-^{(3)}(\lambda h)}{2} + h^2 \varepsilon^{(3)}(\lambda h) \ln(h) \\ &+ \frac{\partial^3}{\partial \lambda^3} \left[\arg \left(\Gamma \left(\frac{1}{2} + i \frac{\varepsilon(\lambda h)}{h} \right) \right) \right] - \frac{\partial^3}{\partial \lambda^3} \left[\arccos \left(\frac{\cos(g_h(\lambda))}{\sqrt{1 + \exp(2\pi \varepsilon(\lambda h)/h)}} \right) \right]. \end{aligned}$$

Uniformly, on the compact set $[-1, 1]$ we have

$$\lambda \mapsto -h^3 \frac{\theta_+^{(3)}(\lambda h) + \theta_-^{(3)}(\lambda h)}{2} = O(h^2)$$

and

$$\lambda \mapsto h^2 \varepsilon^{(3)}(\lambda h) \ln(h) = O(h^2 \ln(h)).$$

Next, let us estimate the function $\lambda \mapsto \frac{\partial^3}{\partial \lambda^3} \left[\arg \left(\Gamma \left(\frac{1}{2} + i \frac{\varepsilon(\lambda h)}{h} \right) \right) \right]$: for all $\lambda \in [-1, 1]$

$$\begin{aligned} &\frac{\partial^3}{\partial \lambda^3} \left[\arg \left(\Gamma \left(\frac{1}{2} + i \frac{\varepsilon(\lambda h)}{h} \right) \right) \right] \\ &= \frac{\partial}{\partial \lambda} \left[h \varepsilon''(\lambda h) \operatorname{Re} \left(\Psi \left(\frac{1}{2} + i \frac{\varepsilon(\lambda h)}{h} \right) \right) - (\varepsilon'(\lambda h))^2 \operatorname{Im} \left(\Psi^{(1)} \left(\frac{1}{2} + i \frac{\varepsilon(\lambda h)}{h} \right) \right) \right] \\ &= h^2 \varepsilon^{(3)}(\lambda h) \operatorname{Re} \left(\Psi \left(\frac{1}{2} + i \frac{\varepsilon(\lambda h)}{h} \right) \right) + h \varepsilon''(\lambda h) \frac{\partial}{\partial \lambda} \left[\operatorname{Re} \left(\Psi \left(\frac{1}{2} + i \frac{\varepsilon(\lambda h)}{h} \right) \right) \right] \end{aligned}$$

$$\begin{aligned}
& -2h\varepsilon'(\lambda h)\varepsilon''(\lambda h)\operatorname{Im}\left(\Psi^{(1)}\left(\frac{1}{2}+i\frac{\varepsilon(\lambda h)}{h}\right)\right) - (\varepsilon'(\lambda h))^2\frac{\partial}{\partial\lambda}\left[\operatorname{Im}\left(\Psi^{(1)}\left(\frac{1}{2}+i\frac{\varepsilon(\lambda h)}{h}\right)\right)\right] \\
& = h^2\varepsilon^{(3)}(\lambda h)\operatorname{Re}\left(\Psi\left(\frac{1}{2}+i\frac{\varepsilon(\lambda h)}{h}\right)\right) - 3h\varepsilon''(\lambda h)\varepsilon'(\lambda h)\operatorname{Im}\left(\Psi^{(1)}\left(\frac{1}{2}+i\frac{\varepsilon(\lambda h)}{h}\right)\right) \\
& \quad + (\varepsilon'(\lambda h))^3\operatorname{Re}\left(\Psi^{(2)}\left(\frac{1}{2}+i\frac{\varepsilon(\lambda h)}{h}\right)\right);
\end{aligned}$$

thus, the function $\lambda \mapsto \frac{\partial^3}{\partial\lambda^3}\left[\arg\left(\Gamma\left(\frac{1}{2}+i\frac{\varepsilon(\lambda h)}{h}\right)\right)\right]$ is equal to a $O(1)$ on the compact set $[-1,1]$. Now, for finish, let us estimate the function

$$\lambda \mapsto \frac{\partial^3}{\partial\lambda^3}\left[\arccos\left(\frac{\cos(g_h(\lambda))}{\sqrt{1+\exp(2\pi\varepsilon(\lambda h)/h)}}\right)\right],$$

with the notations introduced in the proof of lemma 7.1, for all $\lambda \in [-1,1]$ we have

$$\begin{aligned}
\frac{\partial^3}{\partial\lambda^3}\left[\arccos\left(\frac{\cos(g_h(\lambda))}{\sqrt{1+\exp(2\pi\varepsilon(\lambda h)/h)}}\right)\right] & = \frac{\partial}{\partial\lambda}\left(\frac{N_h}{D_h}\right) \\
& = \frac{N'_h(\lambda)D_h(\lambda) - N_h(\lambda)D'_h(\lambda)}{D_h^2(\lambda)};
\end{aligned}$$

we have seen into the proof of lemma 7.1, that, on the compact set $[-1,1]$

$$\lambda \mapsto \frac{1}{D_h^2(\lambda)} = O(1).$$

By the same argument as into the prof of lemma 7.1, we show that $\lambda \mapsto N'_h(\lambda)D_h(\lambda) - N_h(\lambda)D'_h(\lambda)$ is equal to a $O(1)$ on the compact set $[-1,1]$. Consequently for all $\lambda \in [-1,1]$ the function

$$\lambda \mapsto \frac{\partial^3}{\partial\lambda^3}\left[\arccos\left(\frac{\cos(g_h(\lambda))}{\sqrt{1+\exp(2\pi\varepsilon(\lambda h)/h)}}\right)\right]$$

is to equal to a $O(1)$ on the compact set $[-1,1]$. Thus, on $[-1,1]$ we have $\lambda \mapsto \mathcal{Y}_h^{(3)}(\lambda) = O(1)$.

Since, we have, uniformly, on the compact set $[-1,1]$ the equality

$$\mathcal{Y}'_h(\lambda) = \frac{\ln(h)}{\sqrt{-V''(0)}} + O(1),$$

we deduce, that the function

$$\lambda \mapsto \frac{-\mathcal{Y}_h^{(3)}(\lambda)}{(\mathcal{Y}'_h)^4(\lambda)} = O\left|\frac{1}{\ln(h)^4}\right|.$$

on the compact set $[-1,1]$.

Into the proof of lemme 7.1 we have seen, that, on the compact set $[-1,1]$:

$$\lambda \mapsto \mathcal{Y}''_h(\lambda) = O\left|\frac{1}{\ln(h)^3}\right|,$$

hence, on the compact set $[-1,1]$, we get

$$\lambda \mapsto \frac{(\mathcal{Y}''_h)^2(\lambda)}{(\mathcal{Y}'_h)^5(\lambda)} = O\left|\frac{1}{\ln(h)^5}\right|;$$

thus

$$\lambda \mapsto \left(\mathcal{A}_h^{(3)} \circ \mathcal{Y}_h \right) (\lambda) = \frac{-\mathcal{Y}_h^{(3)}(\lambda)}{(\mathcal{Y}_h'(\lambda))^4} + \frac{3(\mathcal{Y}_h''(\lambda))^2}{(\mathcal{Y}_h'(\lambda))^5}$$

is equal to a $O\left|\frac{1}{\ln(h)^4}\right|$ on $[-1, 1]$. \square

Notation 9.2. Let us denote by $R_2(X) = R_2(h, n_0, X)$ the polynomial of $\mathbb{R}_2[X]$ defined by :

$$R_2(X) := \mathcal{A}_h(2\pi n_0) + \mathcal{A}_h'(2\pi n_0)2\pi(X - n_0) + \mathcal{A}_h''(2\pi n_0)2\pi^2(X - n_0)^2.$$

9.1. Definition of a new time scale.

Proposition 9.3. *Let β a real number such that $\beta < 4 - 3\gamma$. Then, uniformly for all $t \in [0, |\ln(h)|^\beta]$ we have :*

$$\mathbf{a}(t) = \sum_{n \in \mathbb{N}} |a_n|^2 e^{-itQ_2(n)} + O\left(|\ln(h)|^{\beta+3\gamma-4}\right).$$

Proof. Let us introduce the difference $\varepsilon(t) := \varepsilon(t, h)$ defined by :

$$\varepsilon(t) := \left| \mathbf{a}(t) - \sum_{n \in \mathbb{N}} |a_n|^2 e^{-itQ_2(n)} \right|.$$

Taylor-Lagrange's formula give the existence of a real number $\zeta = \zeta(n, h, E) \in \mathcal{Y}_h([-1, 1])$ such that :

$$\begin{aligned} \mathcal{A}_h(2\pi n) &= \mathcal{A}_h(2\pi n_0) + \mathcal{A}_h'(2\pi n_0)2\pi(n - n_0) \\ &+ \mathcal{A}_h''(2\pi n_0)2\pi^2(n - n_0)^2 + \mathcal{A}_h^{(3)}(\zeta) \frac{(2\pi)^3}{8}(n - n_0)^3; \end{aligned}$$

hence, for all $t \geq 0$ we get

$$\begin{aligned} \varepsilon(t) &= \left| \sum_{n \in \mathbb{N}} |a_n|^2 e^{-it\left(Q_2(n) + \mathcal{A}_h^{(3)}(\zeta) \frac{(2\pi)^3}{8}(n - n_0)^3\right)} - \sum_{n \in \mathbb{N}} |a_n|^2 e^{-itQ_2(n)} \right| \\ &= \left| \sum_{n \in \mathbb{N}} |a_n|^2 e^{-itQ_2(n)} \left[e^{-it\mathcal{A}_h^{(3)}(\zeta) \frac{(2\pi)^3}{8}(n - n_0)^3} - 1 \right] \right|. \end{aligned}$$

With the sets Γ, Δ and by triangular inequality, we obtain for all $t \geq 0$

$$\begin{aligned} \varepsilon(t) &\leq \left| \sum_{n \in \Delta} |a_n|^2 e^{-itQ_2(n)} \left[e^{-it\mathcal{A}_h^{(3)}(\zeta) \frac{(2\pi)^3}{8}(n - n_0)^3} - 1 \right] \right| \\ &+ \left| \sum_{n \in \Gamma} |a_n|^2 e^{-itQ_2(n)} \left[e^{-it\mathcal{A}_h^{(3)}(\zeta) \frac{(2\pi)^3}{8}(n - n_0)^3} - 1 \right] \right|. \end{aligned}$$

Since :

$$\left| \sum_{n \in \Gamma} |a_n|^2 e^{-itQ_2(n)} \left[e^{-it\mathcal{A}_h^{(3)}(\zeta) \frac{(2\pi)^3}{8}(n - n_0)^3} - 1 \right] \right| \leq 2 \sum_{n \in \Gamma} |a_n|^2 = O\left|\frac{1}{\ln(h)^\infty}\right|$$

by the lemma 6.6.

On the other hand, with the lemma 9.1: for all integer $n \in \Delta$ and for all real number $t \in [0, |\ln(h)|^\beta]$

$$\left| t\mathcal{A}_h^{(3)}(\zeta) \frac{(2\pi)^3}{8}(n - n_0)^3 \right| \leq M |\ln(h)|^{\beta+3\gamma-4}$$

where $M > 0$. Thus, for all integer $n \in \Delta$ and for all real number $t \in [0, |\ln(h)|^\beta]$ we have

$$e^{-it\mathcal{A}_h^{(3)}(\zeta)\frac{(2\pi)^3}{8}(n-n_0)^3} - 1 = O\left(|\ln(h)|^{\beta+3\gamma-4}\right)$$

hence for all $t \in [0, |\ln(h)|^\beta]$ we get

$$\begin{aligned} & \left| \sum_{n \in \Delta} |a_n|^2 e^{-itQ_2(n)} \left[e^{-it\mathcal{A}_h^{(3)}(\zeta)\frac{(2\pi)^3}{8}(n-n_0)^3} - 1 \right] \right| \\ & \leq O\left(|\ln(h)|^{\beta+3\gamma-4}\right) \sum_{n \in \Delta} |a_n|^2 \\ & \leq O\left(|\ln(h)|^{\beta+3\gamma-4}\right) \sum_{n \in \mathbb{N}} |a_n|^2 = O\left(|\ln(h)|^{\beta+3\gamma-4}\right). \end{aligned}$$

So, for all $t \in [0, |\ln(h)|^\beta]$ we get finally

$$\varepsilon(t) = O\left(|\ln(h)|^{\beta+3\gamma-4}\right).$$

□

Definition 9.4. Let us define the second order approximation of the partial auto-correlation $\mathbf{a}(t)$ by :

$$\mathbf{a}_2 : t \mapsto \sum_{n \in \mathbb{N}} |a_n|^2 e^{-itQ_2(n)}.$$

And we also define the function

$$\tilde{\mathbf{a}}_2 : t \mapsto \sum_{n \in \mathbb{N}} |a_n|^2 e^{-it(\mathcal{A}'_h(2\pi n_0)2\pi(n-n_0) + \mathcal{A}''_h(2\pi n_0)2\pi^2(n-n_0)^2)}.$$

Thus, we get

$$\mathbf{a}_2(t) = e^{-it\mathcal{A}_h(2\pi n_0)} \tilde{\mathbf{a}}_2(t)$$

and, we have also

$$|\mathbf{a}_2(t)| = |\tilde{\mathbf{a}}_2(t)| = \left| \sum_{n \in \mathbb{N}} |a_n|^2 e^{-it(\mathcal{A}'_h(2\pi n_0)2\pi(n-n_0) + \mathcal{A}''_h(2\pi n_0)2\pi^2(n-n_0)^2)} \right|.$$

10. FULL REVIVALS THEOREM

Let us start by some notations.

Definition 10.1. Let us define the revival time $T_{rev} = T_{rev}(h, E)$ by :

$$T_{rev} := \frac{1}{\pi \mathcal{A}''_h(2\pi n_0)}$$

and we also denote the integer $N_h = N(h)$ defined by :

$$N_h := E \left\lfloor \frac{T_{rev}}{T_{hyp}} \right\rfloor \in \mathbb{N}.$$

Hence, there exist an unique real number $\Theta_h \in [0, 1[$ such that :

$$\frac{T_{rev}}{T_{hyp}} = N_h + \Theta_h.$$

Proposition 10.2. *If we suppose that*

$$\lim_{h \rightarrow 0} (\mathcal{Y}'_h \circ \mathcal{A}_h) (2\pi n_0) = K$$

where $K \neq 0$ is a constant, then

$$T_{rev} = \frac{|\ln(h)|^3}{K (-V''(0))^{\frac{3}{2}}} + O(1).$$

Proof. Since

$$T_{rev} = -\frac{(\mathcal{Y}'_h \circ \mathcal{A}_h)^3 (2\pi n_0)}{\pi (\mathcal{Y}''_h \circ \mathcal{A}_h) (2\pi n_0)};$$

if we suppose that

$$\lim_{h \rightarrow 0} (\mathcal{Y}''_h \circ \mathcal{A}_h) (2\pi n_0) = K;$$

then we deduce

$$T_{rev} = \frac{-\ln(h)^3}{K (-V''(0))^{\frac{3}{2}}} + O(1).$$

□

Theorem 10.3. With the previous notations we have :

(i) if $\Theta_h = 0$; then for all $t \geq 0$:

$$\tilde{\mathbf{a}}_2(t + N_h T_{hyp}) = \tilde{\mathbf{a}}_2(t + T_{ren}) = \tilde{\mathbf{a}}_2(t);$$

(ii) if $\Theta_h \in]0, 1[$, then for all $t \geq 0$

$$\tilde{\mathbf{a}}_2(t + N_h T_{hyp}) = \tilde{\mathbf{a}}_2(t) + O\left|\frac{1}{\ln(h)^{2-2\gamma}}\right|;$$

in particular, we have

$$\left|\mathbf{a}_2(t + N_h T_{hyp})\right| = \left|\mathbf{a}_2(t)\right| + O\left|\frac{1}{\ln(h)^{2-2\gamma}}\right|.$$

Proof. For all $t \geq 0$

$$\begin{aligned} \tilde{\mathbf{a}}_2(t + N_h T_{hyp}) &= \sum_{n \in \mathbb{N}} |a_n|^2 e^{-2i\pi \frac{t + N_h T_{hyp}}{T_{hyp}} (n - n_0)} e^{-2i\pi \frac{(t + T_{ren} - T_{hyp} \Theta_h)}{T_{ren}} (n - n_0)^2} \\ &= \sum_{n \in \mathbb{N}} |a_n|^2 e^{-2i\pi \frac{t}{T_{hyp}} (n - n_0)} e^{-2i\pi \frac{(t - T_{hyp} \Theta_h)}{T_{ren}} (n - n_0)^2}. \end{aligned}$$

If we suppose that $\Theta_h = 0$; then for all $t \geq 0$

$$\tilde{\mathbf{a}}_2(t + N_h T_{hyp}) = \tilde{\mathbf{a}}_2(t).$$

Suppose that $\Theta_h \in]0, 1[$, then for all $t \in \mathbb{R}_+$

$$\begin{aligned} &\left|\tilde{\mathbf{a}}_2(t + N_h T_{hyp}) - \tilde{\mathbf{a}}_2(t)\right| \\ &= \left|\sum_{n \in \mathbb{N}} |a_n|^2 e^{-2i\pi \frac{t}{T_{hyp}} (n - n_0)} e^{-2i\pi \frac{(t - T_{hyp} \Theta_h)}{T_{ren}} (n - n_0)^2} \right. \\ &\quad \left. - \sum_{n \in \mathbb{N}} |a_n|^2 e^{-2i\pi \frac{t}{T_{hyp}} (n - n_0)} e^{-2i\pi \frac{t}{T_{ren}} (n - n_0)^2}\right| \\ &= \left|\sum_{n \in \mathbb{N}} |a_n|^2 e^{-2i\pi \frac{t}{T_{hyp}} (n - n_0)} e^{-2i\pi \frac{t}{T_{ren}} (n - n_0)^2} \left(e^{2i\pi \Theta_h \frac{T_{hyp}}{T_{ren}} (n - n_0)^2} - 1\right)\right| \end{aligned}$$

$$\begin{aligned} &\leq \left| \sum_{n \in \Delta} |a_n|^2 e^{-2i\pi \frac{t}{T_{hyp}}(n-n_0)} e^{-2i\pi \frac{t}{T_{rev}}(n-n_0)^2} \left(e^{2i\pi \Theta_h \frac{T_{hyp}}{T_{rev}}(n-n_0)^2} - 1 \right) \right| \\ &+ \left| \sum_{n \in \Gamma} |a_n|^2 e^{-2i\pi \frac{t}{T_{hyp}}(n-n_0)} e^{-2i\pi \frac{t}{T_{rev}}(n-n_0)^2} \left(e^{2i\pi \Theta_h \frac{T_{hyp}}{T_{rev}}(n-n_0)^2} - 1 \right) \right|. \end{aligned}$$

For all integer $n \in \Delta$ we have

$$(n - n_0)^2 \leq |\ln(h)|^{2\gamma}$$

hence, for $h \rightarrow 0$, we get, for all integer $n \in \Delta$

$$e^{2i\pi \Theta_h \frac{T_{hyp}}{T_{rev}}(n-n_0)^2} = 1 + O\left(|\ln(h)|^{2\gamma-2}\right).$$

Thus, for all $t \geq 0$

$$\begin{aligned} &\left| \sum_{n \in \Delta} |a_n|^2 e^{-2i\pi \frac{t}{T_{hyp}}(n-n_0)} e^{-2i\pi \frac{t}{T_{rev}}(n-n_0)^2} \left(e^{2i\pi \Theta_h \frac{T_{hyp}}{T_{rev}}(n-n_0)^2} - 1 \right) \right| \\ &\leq O\left(|\ln(h)|^{2\gamma-2}\right) \sum_{n \in \Delta} |a_n|^2 \\ &\leq O\left(|\ln(h)|^{2\gamma-2}\right) \sum_{n \in \mathbb{N}} |a_n|^2 = O\left(|\ln(h)|^{2\gamma-2}\right). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &\left| \sum_{n \in \Gamma} |a_n|^2 e^{-2i\pi \frac{t}{T_{hyp}}(n-n_0)} e^{-2i\pi \frac{t}{T_{rev}}(n-n_0)^2} \left(e^{2i\pi \Theta_h \frac{T_{hyp}}{T_{rev}}(n-n_0)^2} - 1 \right) \right| \\ &\leq \sum_{n \in \Gamma} 2 |a_n|^2 = O\left|\frac{1}{\ln(h)^\infty}\right|. \end{aligned}$$

□

Remark 10.4. For $h \rightarrow 0$, we have

$$\frac{N_h T_{hyp}}{T_{rev}} = \frac{T_{rev} - \Theta_h T_{hyp}}{T_{rev}} = 1 - \frac{\Theta_h T_{hyp}}{T_{rev}} \rightarrow 1$$

thus, for $h \rightarrow 0$ we have also

$$N_h T_{hyp} \sim T_{rev}.$$

The previous proposition show that the function $t \mapsto \tilde{\mathbf{a}}_2(t)$, is, modulo $O\left(|\ln(h)|^{2\gamma-2}\right)$, periodic, with a period equivalent (for $h \rightarrow 0$) to T_{rev} . This is the full revival phenomenon.

11. FRACTIONAL REVIVALS THEOREM

The aim of this section is to study the dynamics for time close to $\frac{p}{q} T_{rev}$, where $\frac{p}{q} \in \mathbb{Q}$.

11.1. Preliminaries.

Notation 11.1. For all $(p, q) \in \mathbb{Z} \times \mathbb{N}^*$ let us consider the sequence $(\sigma_h(p, q))$ defined by :

$$(\sigma_h(p, q))_n := e^{-2i\pi \frac{p}{q}(n-n_0)^2}, \quad n \in \mathbb{Z}.$$

The periodicity of this sequence is characterised by the following easy proposition.

Proposition 11.2. *For all $(p, q) \in \mathbb{Z} \times \mathbb{N}^*$, the sequence $(\sigma_h(p, q))_n$ is ℓ -periodic if and only if the integer ℓ satisfy the equation :*

$$\forall m \in \mathbb{Z}, \frac{2p\ell}{q}m + \frac{p\ell^2}{q} \equiv 0 \pmod{1}.$$

Now, we solve this equation.

Proposition 11.3. *Suppose $p \wedge q = 1$, then the set \mathcal{E} of solutions ℓ such that :*

$$\forall m \in \mathbb{Z}, \frac{2p\ell}{q}m + \frac{p\ell^2}{q} \equiv 0 \pmod{1}$$

is characterised by :

- (i) if q is odd then $\mathcal{E} = \{q\mathbb{Z}\}$;
- (ii) si q even and $\frac{q}{2}$ odd then $\mathcal{E} = \{q\mathbb{Z}\}$;
- (iii) si q even et $\frac{q}{2}$ even then $\mathcal{E} = \{\frac{q}{2}\mathbb{Z}\}$.

Proof. Let us start by a remark. To be a multiple of q is always a necessary condition to be a solution. Now, let study sufficient conditions. We have

$$\begin{aligned} \ell \in \mathcal{E} &\Leftrightarrow \forall m \in \mathbb{Z}, \frac{2p\ell}{q}m + \frac{p\ell^2}{q} \equiv 0 \pmod{1} \\ &\Leftrightarrow \forall m \in \mathbb{Z}, q|(2p\ell m + p\ell^2); \end{aligned}$$

and, since $p \wedge q = 1$, by Gauss lemma we have

$$\ell \in \mathcal{E} \Leftrightarrow \forall m \in \mathbb{Z}, q|(2\ell m + \ell^2).$$

Let ℓ an integer solution of \mathcal{E} ; then if we take $m = 0$ we see that $q|\ell^2$, hence there exist $\alpha \in \mathbb{Z}$ such that $\ell^2 = \alpha q$. On the other hand, since for all $m \in \mathbb{Z}$ we have $q|(2\ell m + \ell^2)$, we deduce that for all $m \in \mathbb{Z}$, there exist $\beta_m \in \mathbb{Z}$ such that $2\ell m + \ell^2 = \beta_m q$. Thus, since $\ell^2 = \alpha q$, we get for all $m \in \mathbb{Z}$ the equality $2\ell m + \alpha q = \beta_m q$. In particular, with $m = 1$ we deduce that :

$$2\ell = (\beta_1 - \alpha)q.$$

- If q is odd : then a $\beta_1 - \alpha$ is even; thus $\ell = \underbrace{(\beta_1 - \alpha)}_{\in \mathbb{Z}} q$, and $q|\ell$, thus $\mathcal{E} = \{q\mathbb{Z}\}$.

- If q is even : then $\ell = (\beta_1 - \alpha)\frac{q}{2}$; thus $\frac{q}{2}|\ell$. If we write $\ell = k\frac{q}{2}$ with $k \in \mathbb{Z}$, then :

$$\forall m \in \mathbb{Z}, \frac{2p\ell}{q}m + \frac{p\ell^2}{q} = kpm + \frac{pk^2q}{4}.$$

Thus

$$\forall m \in \mathbb{Z}, kpm + \frac{pk^2q}{4} \in \mathbb{Z} \Leftrightarrow pk^2q \in 4\mathbb{Z}.$$

And, since p is odd, finally we get, with $\ell = k\frac{q}{2}$:

$$\forall m \in \mathbb{Z}, \frac{2p\ell}{q}m + \frac{p\ell^2}{q} \in \mathbb{Z} \Leftrightarrow k^2q \in 4\mathbb{Z}.$$

Then, we have two cases :

- if $\frac{q}{2}$ is odd : then $k^2q \in 4\mathbb{Z} \Leftrightarrow k^2$ is even, equivalent to k even, hence $\ell = k^2q$ is a multiple of q , hence $\mathcal{E} = \{q\mathbb{Z}\}$.
- If $\frac{q}{2}$ is even : then $q = 4q'$ where $q' \in \mathbb{Z}^*$, hence for all $k \in \mathbb{Z}$; $k^2q = k^24q' \in 4\mathbb{Z}$, thus $\mathcal{E} = \{\frac{q}{2}\mathbb{Z}\}$. \square

For a period ℓ , let us defined the set of sequences ℓ -periodic with a scalar product.

Definition 11.4. For a integer $\ell \in \mathbb{Z}^*$; let us denote by $\mathfrak{S}_\ell(\mathbb{Z})$ set of sequences ℓ -periodic :

$$\mathfrak{S}_\ell(\mathbb{Z}) := \left\{ u_n \in \mathbb{C}^{\mathbb{Z}}; \forall n \in \mathbb{Z}, u_{n+\ell} = u_n \right\}.$$

So we have the :

Proposition 11.5. *The application*

$$\langle \cdot, \cdot \rangle_{\mathfrak{S}_\ell} : \begin{cases} \mathfrak{S}_\ell(\mathbb{Z})^2 \rightarrow \mathbb{C} \\ (u, v) \mapsto \langle u, v \rangle_{\mathfrak{S}_\ell} := \frac{1}{|\ell|} \sum_{k=0}^{|\ell|-1} u_k \overline{v_k} \end{cases}$$

is a Hermitean product on the space $\mathfrak{S}_\ell(\mathbb{Z})$.

Proof. The proof is elementary. \square

Proposition 11.6. *Let us consider $\phi_n^k := e^{-\frac{2i\pi kn}{\ell}}$ where $(k, n) \in \mathbb{Z}^2$; then the family $\left\{ \left(\phi_n^k \right)_{n \in \mathbb{Z}} \right\}_{k=0 \dots \ell-1}$ is an orthonormal basis of the space vector $\mathfrak{S}_\ell(\mathbb{Z})$.*

Proof. For $\ell > 0$, the family $\left\{ \left(\phi_n^k \right)_{n \in \mathbb{Z}} \right\}_{k=0 \dots \ell-1}$ is clearly a family of ℓ vectors of $\mathfrak{S}_\ell(\mathbb{Z})$. Next, for all pair $(p, q) \in \{0 \dots \ell-1\}^2$ we have

$$\langle \phi^p, \phi^q \rangle_{\mathfrak{S}_\ell} = \frac{1}{\ell} \sum_{k=0}^{\ell-1} \phi_k^p \overline{\phi_k^q} = \frac{1}{\ell} \sum_{k=0}^{\ell-1} \left(e^{\frac{2i\pi k(q-p)}{\ell}} \right)^k = \delta_{p,q}.$$

So $\left\{ \left(\phi_n^k \right)_{n \in \mathbb{Z}} \right\}_{k=0 \dots \ell-1}$ is an orthonormal basis of the space vector $\mathfrak{S}_\ell(\mathbb{Z})$. \square

11.2. The main theorem.

Theorem 11.7. For all $(p, q) \in \mathbb{Z} \times \mathbb{N}^*$ such that $p \wedge q = 1$; there exist a family of ℓ complex numbers $\left(\tilde{b}_k(\ell) \right)_{k \in \{0 \dots p-1\}}$ where the integer $\ell \in \mathbb{Z}$ is solution of :

$$\forall m \in \mathbb{Z}, \frac{2p\ell}{q}m + \frac{p\ell^2}{q} \equiv 0 \pmod{1};$$

such that for all $t \in [0, |\ln(h)|^\alpha]$ we get :

$$\tilde{\mathbf{a}}_2 \left(t + \frac{p}{q} N_h T_{hyp} \right) = \sum_{k=0}^{\ell-1} \tilde{b}_k(\ell) \tilde{\mathbf{a}}_1 \left(t + T_{hyp} \left(\frac{k}{\ell} + \frac{p}{q} N_h \right) \right) + O \left(|\ln(h)|^{\alpha+2\gamma-3} \right)$$

The numbers $\tilde{b}_k(\ell)$ are called fractionnals coefficients; and for all $k \in \{0 \dots \ell-1\}$

$$\tilde{b}_k(\ell) = e^{-\frac{2i\pi k n_0}{\ell}} b_k(\ell)$$

where

$$b_k(\ell) = b_k(h, \ell) = \left\langle \sigma_h(p, q), \phi^k \right\rangle_{\mathfrak{S}_\ell} = \frac{1}{\ell} \sum_{n=0}^{\ell-1} e^{-2i\pi \frac{p}{q} (n-n_0)^2} e^{-\frac{2i\pi kn}{\ell}}.$$

Moreover, if we suppose $\frac{T_{rev}}{T_{hyp}} \in \mathbb{Q}$, then we have

$$\tilde{\mathbf{a}}_2 \left(t + \frac{p}{q} T_{rev} \right) = \sum_{k=0}^{\ell-1} \tilde{b}_k(\ell) \tilde{\mathbf{a}}_1 \left(t + T_{hyp} \left(\frac{k}{\ell} + \frac{p}{q} N_h \right) \right).$$

Proof. Let us denote the integer $\tilde{n} := n - n_0$; and let us consider $\ell \in \mathbb{Z}$ a solution of the equation \mathcal{E} . So, we have for all $t \geq 0$

$$\tilde{\mathbf{a}}_2 \left(t + \frac{p}{q} N_h T_{cl} \right) = \sum_{n \in \mathbb{N}} |a_n|^2 e^{-2i\pi \frac{t}{T_{hyp}} \tilde{n}} e^{-2i\pi \frac{t}{T_{rev}} \tilde{n}^2} e^{-2i\pi \frac{p}{q} \tilde{n} N_h} e^{-2i\pi \frac{p}{q} \frac{N_h T_{hyp}}{T_{rev}} \tilde{n}^2}.$$

And, since $T_{rev} = N_h T_{hyp} + \Theta_h T_{hyp}$ we have

$$e^{-2i\pi \frac{p}{q} \frac{N_h T_{hyp}}{T_{rev}} \tilde{n}^2} = e^{-2i\pi \frac{p}{q} \tilde{n}^2} e^{2i\pi \frac{p}{q} \frac{\Theta_h T_{hyp}}{T_{rev}} \tilde{n}^2}$$

and

$$e^{-2i\pi \frac{p}{q} \tilde{n}^2} = e^{-2i\pi \frac{p}{q} (n-n_0)^2} \in \mathfrak{S}_\ell(\mathbb{Z});$$

hence, there exist a unique decomposition of the sequence $\left(e^{-2i\pi \frac{p}{q} (n-n_0)^2} \right)_n$ on the basis $\left(\phi^k \right)_{k \in \{0 \dots \ell-1\}}$:

$$\begin{aligned} e^{-2i\pi \frac{p}{q} \tilde{n}^2} &= e^{-2i\pi \frac{p}{q} (n-n_0)^2} = \sum_{k=0}^{\ell-1} \left\langle \sigma_h(p, q), \phi^k \right\rangle_{\mathfrak{S}_\ell} \phi_n^k \\ &= \sum_{k=0}^{\ell-1} b_k(\ell) e^{-2i\pi \frac{k}{\ell} n}. \end{aligned}$$

Thus, for all $t \geq 0$

$$\begin{aligned} &\tilde{\mathbf{a}}_2 \left(t + \frac{p}{q} N_h T_{hyp} \right) \\ &= \sum_{n \in \mathbb{N}} |a_n|^2 e^{-2i\pi \frac{t}{T_{hyp}} \tilde{n}} e^{-2i\pi \frac{t}{T_{rev}} \tilde{n}^2} e^{-2i\pi \frac{p}{q} \tilde{n} N_h} \left(\sum_{k=0}^{\ell-1} b_k(\ell) e^{-2i\pi \frac{k}{\ell} n} \right) e^{2i\pi \frac{p}{q} \frac{\Theta_h T_{hyp}}{T_{rev}} \tilde{n}^2} \\ &= \sum_{n \in \mathbb{N}} \sum_{k=0}^{\ell-1} b_k(\ell) |a_n|^2 e^{-2i\pi \frac{t}{T_{hyp}} \tilde{n}} e^{-2i\pi \frac{t}{T_{rev}} \tilde{n}^2} e^{-2i\pi \frac{p}{q} \tilde{n} N_h} e^{-2i\pi \frac{k}{\ell} n} e^{2i\pi \frac{p}{q} \frac{\Theta_h T_{hyp}}{T_{rev}} \tilde{n}^2}. \end{aligned}$$

For all $t \geq 0$ we have

$$\tilde{\mathbf{a}}_1 \left(t + T_{hyp} \left(\frac{k}{\ell} + \frac{p}{q} N_h \right) \right) = \sum_{n \in \mathbb{N}} |a_n|^2 e^{-2i\pi \frac{t}{T_{hyp}} \tilde{n}} e^{-2i\pi \frac{k}{\ell} \tilde{n}} e^{-2i\pi \frac{p}{q} N_h \tilde{n}}$$

thus,

$$\begin{aligned} &\sum_{k=0}^{\ell-1} \tilde{b}_k(\ell) \tilde{\mathbf{a}}_1 \left(t + T_{hyp} \left(\frac{k}{\ell} + \frac{p}{q} N_h \right) \right) \\ &= \sum_{k=0}^{\ell-1} e^{-\frac{2i\pi k n_0}{\ell}} b_k(\ell) \sum_{n \in \mathbb{N}} |a_n|^2 e^{-2i\pi \frac{t}{T_{hyp}} \tilde{n}} e^{-2i\pi \frac{k}{\ell} \tilde{n}} e^{-2i\pi \frac{p}{q} N_h \tilde{n}} \\ &= \sum_{n \in \mathbb{N}} \sum_{k=0}^{\ell-1} |a_n|^2 b_k(\ell) e^{-2i\pi \frac{t}{T_{hyp}} \tilde{n}} e^{-2i\pi \frac{k n}{\ell}} e^{-2i\pi \frac{p}{q} N_h \tilde{n}}. \end{aligned}$$

Hence

$$\begin{aligned} &\left| \tilde{\mathbf{a}}_2 \left(t + \frac{p}{q} N_h T_{hyp} \right) - \sum_{k=0}^{\ell-1} \tilde{b}_k(\ell) \tilde{\mathbf{a}}_1 \left(t + T_{hyp} \left(\frac{k}{\ell} + \frac{p}{q} N_h \right) \right) \right| \\ &= \left| \sum_{n \in \mathbb{N}} \sum_{k=0}^{\ell-1} |a_n|^2 b_k(\ell) e^{-2i\pi \frac{t}{T_{hyp}} \tilde{n}} e^{-2i\pi \frac{p}{q} \tilde{n} N_h} e^{-\frac{2i\pi k n}{\ell}} \left(e^{-2i\pi \frac{t}{T_{rev}} \tilde{n}^2} e^{2i\pi \frac{p}{q} \frac{\Theta_h T_{hyp}}{T_{rev}} \tilde{n}^2} - 1 \right) \right|; \end{aligned}$$

with the sets Δ , Γ and by triangular inequality we get :

$$\begin{aligned} &\leq \left| \sum_{n \in \Gamma} \sum_{k=0}^{\ell-1} |a_n|^2 b_k(\ell) e^{-2i\pi \frac{t}{T_{hyp}} \tilde{n}} e^{-2i\pi \frac{p}{q} \tilde{n} N_h} e^{-\frac{2i\pi k n}{\ell}} \left(e^{-2i\pi \frac{t}{T_{rev}} \tilde{n}^2} e^{+2i\pi \frac{p}{q} \frac{\Theta_h T_{hyp} p}{T_{rev}} \tilde{n}^2} - 1 \right) \right| \\ &+ \left| \sum_{n \in \Delta} \sum_{k=0}^{\ell-1} |a_n|^2 b_k(\ell) e^{-2i\pi \frac{t}{T_{hyp}} \tilde{n}} e^{-2i\pi \frac{p}{q} \tilde{n} N_h} e^{-\frac{2i\pi k n}{\ell}} \left(e^{-2i\pi \frac{t}{T_{rev}} \tilde{n}^2} e^{+2i\pi \frac{p}{q} \frac{\Theta_h T_{hyp} p}{T_{rev}} \tilde{n}^2} - 1 \right) \right|. \end{aligned}$$

Increase the first serie :

$$\begin{aligned} &\left| \sum_{n \in \Gamma} \sum_{k=0}^{\ell-1} |a_n|^2 b_k(\ell) e^{-2i\pi \frac{t}{T_{hyp}} \tilde{n}} e^{-2i\pi \frac{p}{q} \tilde{n} N_h} e^{-\frac{2i\pi k n}{\ell}} \left(e^{-2i\pi \frac{t}{T_{rev}} \tilde{n}^2} e^{+2i\pi \frac{p}{q} \frac{\Theta_h T_{hyp} p}{T_{rev}} \tilde{n}^2} - 1 \right) \right| \\ &\leq 2 \left(\sum_{n \in \Gamma} |a_n|^2 \right) \left(\sum_{k=0}^{\ell-1} |b_k(\ell)| \right) = O \left| \frac{1}{\ln(h)^\infty} \right|. \end{aligned}$$

Next, for the second serie, observe the following term :

$$e^{-2i\pi \frac{t}{T_{rev}} \tilde{n}^2} e^{+2i\pi \frac{p}{q} \frac{\Theta_h T_{hyp} p}{T_{rev}} \tilde{n}^2} = e^{-2i\pi \left(t + \Theta_h T_{hyp} \frac{p}{q} \right) \frac{1}{T_{rev}} \tilde{n}^2}.$$

For all $n \in \Delta$ and for all $t \in [0, |\ln(h)|^\alpha]$ there exist a constant $C > 0$ such that

$$\left| \left(t + \frac{\Theta_h T_{hyp} p}{q} \right) \frac{(n - n_0)^2}{T_{rev}} \right| \leq C \left(|\ln(h)|^{2\gamma-3+\alpha} + |\ln(h)|^{2\gamma-2} \right) \leq 2C |\ln(h)|^{2\gamma-2}.$$

Hence, there exist $A > 0$ such that for all $t \in [0, |\ln(h)|^\alpha]$

$$\begin{aligned} &\left| \sum_{n \in \Delta} \sum_{k=0}^{\ell-1} |a_n|^2 b_k(\ell) e^{-2i\pi \frac{t}{T_{hyp}} \tilde{n}} e^{-2i\pi \frac{p}{q} \tilde{n} N_h} e^{-\frac{2i\pi k n}{\ell}} \left(e^{-2i\pi \frac{t}{T_{rev}} \tilde{n}^2} e^{+2i\pi \frac{p}{q} \frac{\Theta_h T_{hyp} p}{T_{rev}} \tilde{n}^2} - 1 \right) \right| \\ &\leq A |\ln(h)|^{2\gamma-2} \sum_{n \in \Delta} |a_n|^2 \leq A |\ln(h)|^{2\gamma-2} \sum_{n=0}^{+\infty} |a_n|^2. \end{aligned}$$

For finish, the case where $t = 0$ and $\frac{T_{rev}}{T_{cl}} \in \mathbb{Q}$ is clear. \square

Corollary 11.8. *We have the equality :*

$$\sum_{k=0}^{\ell-1} |b_k(\ell)|^2 = 1.$$

Proof. Since $b_k(\ell) = \langle \sigma_h(p, q), \phi^k \rangle_{\mathfrak{S}_\ell}$ and $(\phi^k)_{k \in \{0, \dots, \ell-1\}}$ is a orthonormal basis of the space vector $\mathfrak{S}_\ell(\mathbb{Z})$, by Pythagorean equality we get

$$\sum_{k=0}^{\ell-1} |b_k(\ell)|^2 = \|\sigma_h(p, q)\|_{\mathfrak{S}_\ell}^2 = 1. \quad \square$$

Now, let us examine the case $\frac{p}{q} = 1$ and the case $\frac{p}{q} = \frac{1}{2}$.

Corollary 11.9. *With the same notation as in the theorem 11.7, we have*

(i) for all $t \in [0, |\ln(h)|^\alpha]$

$$\tilde{\mathbf{a}}_2(t + N_h T_{hyp}) = \tilde{\mathbf{a}}_1(t) + O \left(|\ln(h)|^{\alpha+2\gamma-3} \right);$$

(ii) for all $t \in [0, |\ln(h)|^\alpha]$

$$\tilde{\mathbf{a}}_2 \left(t + \frac{N_h T_{hyp}}{2} \right) = \tilde{\mathbf{a}}_1 \left(t + \left(\frac{N_h + 1}{2} \right) T_{hyp} \right) + O \left(|\ln(h)|^{\alpha+2\gamma-3} \right);$$

in particular, if N_h is odd, then :

$$\tilde{\mathbf{a}}_2 \left(t + \frac{N_h T_{hyp}}{2} \right) = \tilde{\mathbf{a}}_1(t) + O \left(|\ln(h)|^{\alpha+2\gamma-3} \right)$$

and if N_h is even, then :

$$\tilde{\mathbf{a}}_2 \left(t + \frac{N_h T_{hyp}}{2} \right) = \tilde{\mathbf{a}}_1 \left(t + \frac{T_{hyp}}{2} \right) + O \left(|\ln(h)|^{\alpha+2\gamma-3} \right).$$

Proof. In the case (i) : $p = 1, q = \ell = 1$; thus by the previous theorem we get, for all $t \in [0, |\ln(h)|^\alpha]$:

$$\begin{aligned} \tilde{\mathbf{a}}_2 \left(t + \frac{p}{q} N_h T_{hyp} \right) &= \tilde{b}_0(1) \tilde{\mathbf{a}}_1 \left(t + T_{hyp} N_h \right) + O \left(|\ln(h)|^{\alpha+2\gamma-3} \right) \\ &= \tilde{b}_0(1) \tilde{\mathbf{a}}_1(t) + O \left(|\ln(h)|^{\alpha+2\gamma-3} \right) \end{aligned}$$

and $b_0(1) = \frac{1}{1} e^{-2i\pi n_0^2} = 1$, thus

$$\tilde{b}_0(1) = 1.$$

Next, for the case (ii) : $p = 1, q = \ell = 2$, then

$$\begin{aligned} \tilde{\mathbf{a}}_2 \left(t + \frac{p}{q} N_h T_{hyp} \right) &= \\ \tilde{b}_0(2) \tilde{\mathbf{a}}_1 \left(t + \frac{T_{hyp} N_h}{2} \right) &+ \tilde{b}_1(2) \tilde{\mathbf{a}}_1 \left(t + T_{hyp} \left(\frac{1}{2} + \frac{N_h}{2} \right) \right) + O \left(|\ln(h)|^{\alpha+2\gamma-3} \right). \end{aligned}$$

We have

$$b_0(2) = \frac{1}{2} e^{-2i\pi \frac{1}{2} n_0^2} + \frac{1}{2} e^{-2i\pi \frac{1}{2} (1-n_0)^2} = 0$$

thus

$$\tilde{b}_0(2) = 0.$$

And

$$\begin{aligned} b_1(2) &= \frac{1}{2} e^{-2i\pi \frac{1}{2} n_0^2} e^{\frac{2i\pi 0}{2}} + \frac{1}{2} e^{-2i\pi \frac{1}{2} (1-n_0)^2} e^{\frac{2i\pi}{2}} \\ &= \frac{1}{2} \left(e^{-i\pi n_0^2} + (-1) e^{-i\pi (n_0-1)^2} \right) = (-1)^{n_0} \end{aligned}$$

then

$$\tilde{b}_1(2) = e^{-i\pi n_0} (-1)^{n_0} = 1.$$

□

11.3. Explicit values of modulus for revivals coefficients. About the coefficients $\tilde{b}_k(\ell)$ we just know that :

$$\sum_{k=0}^{\ell-1} \left| \tilde{b}_k(\ell) \right|^2 = 1.$$

But we can say more. We can calculate $\left| \tilde{b}_k(\ell) \right| = |b_k(\ell)|$ for all k . From the proposition 11.3 we consider two cases : the case $\ell = q$ and the case $\ell = \frac{q}{2}$.

11.3.1. *Case $\ell = q$.* If the integer q is odd we have :

Theorem 11.10. [Lab2]. For all integers p and q , such that $p \wedge q = 1$ and q odd, then for all $k \in \{0 \dots q - 1\}$ we get :

$$|b_k(q)|^2 = \frac{1}{q}.$$

And in the case q even :

Theorem 11.11. [Lab2]. For all integers p and q , such that $p \wedge q = 1$ and q even, then for all $k \in \{0 \dots q - 1\}$ we get :

$$\begin{aligned} \text{If } \frac{q}{2} \text{ is even, then : } |b_k(q)|^2 &= \begin{cases} \frac{2}{q} & \text{if } k \text{ is even} \\ 0 & \text{else.} \end{cases} \\ \text{If } \frac{q}{2} \text{ is odd, then : } |b_k(q)|^2 &= \begin{cases} 0 & \text{if } k \text{ is even} \\ \frac{2}{q} & \text{else.} \end{cases} \end{aligned}$$

11.3.2. *Case $\ell = \frac{q}{2}$.* It is the case $q \in 4\mathbb{Z}$.

Theorem 11.12. [Lab2]. For all integers p and q , such that $p \wedge q = 1$ and $q \in 4\mathbb{Z}^*$; for all $k \in \{0 \dots \frac{q}{2} - 1\}$ we have

$$\left| b_k\left(\frac{q}{2}\right) \right|^2 = \frac{2}{q}.$$

11.4. Comparison between time scale approximation, hyperbolic period and revivals periods. Since $\gamma < 1$ we have $1 < 3 - 2\gamma$, thus there exist α such that $\alpha \in]1, 3 - 2\gamma[$; hence :

$$|\ln(h)| < |\ln(h)|^\alpha < |\ln(h)|^{3-2\gamma}.$$

So we can choose α such that : for h small enough :

$$\left[0, T_{hyp}\right] \subset \left[0, |\ln(h)|^\alpha\right].$$

Next, with $\gamma < \frac{1}{3}$ we get $3 < 4 - 3\gamma$ and there exist β such that $\beta \in]1, 4 - 3\gamma[$ hence :

$$|\ln(h)|^3 < |\ln(h)|^\beta < |\ln(h)|^{4-3\gamma};$$

so, for h small enough we have :

$$\left[0, T_{rev}\right] \subset \left[0, |\ln(h)|^\beta\right].$$

REFERENCES

- [Ab-St] M. ABRAMOWITZ & I.A STEGUN, *Handbook of Mathematical functions with formulas, graphs and mathematical tables*. New York: Dover, pp. 255-258 and 260-263, 1972.
- [Av-Pe] I. Sh. AVERBUKH & N. F PERELMAN, *Fractional revivals : universality in the long-term evolution of quantum wave packets beyond the correspondance principle dynamics*. Physics Letters A, 139, p. 449-453, 1989.
- [Berr] M. BERGER, P. GAUDUCHON & E. MAZET, *Le spectre d'une variété Riemannienne*, Lectures Notes in Mathematics 194, Springer-Verlag 1971M. BERRY, *Some quantum-to-classical asymptotics*. In Gionnoni & al [GVZJ], pages 251-303.
- [BKP] R. BLUHM, V. A. KOSTELECKY & J. A. PORTER, *The evolution and revival structure of localized quantum wave packets*. American Journal of Physics 64, 944, 1996.
- [Bl-Ko] R. BLUHM & V. A. KOSTELECKY, *Long term evolution and revival structure of Rydberg wave packets for hydrogen and alkali-metal atoms*. Phys. Rev A51, 4767-4786, 1995.
- [Ch-VuN] L. CHARLES & S. VU NGOC, *Spectral asymptotics via the semi-classical birkhoff normal form*, math.SP.0605096.

- [Co-Pa1] Y. COLIN DE VERDIERE & B. PARISSÉ, *Equilibre instable en régime semi-classique I: Concentration microlocale*, Comm PDE, 19 : 1535-1564, 1994.
- [Co-Pa2] Y. COLIN DE VERDIERE & B. PARISSÉ, *Equilibre instable en régime semi-classique II: Conditions de Bohr-Sommerferld*, Ann IHP, 61 : 347-367, 1994.
- [CLP] Y. COLIN DE VERDIERE, M. LOMBARDI & J. POLLET, *The microlocal Landau-Zener formula*, Ann. IHP, 71 : 95-127, 1999.
- [Co-Pa3] Y. COLIN DE VERDIERE & B. PARISSÉ, *Singular Bohr-Sommerfeld rules*, Commun. Math. Phys, 205 : 459-500, 2000.
- [Col] Y. COLIN DE VERDIERE, *Bohr-Sommerfeld rules to all orders*, Henri Poincaré Acta, 6, 925-936, 2005.
- [Co-Ro] M. COMBESURE & D. ROBERT, *A phase study of the quantum Loschmidt Echo in the semi-classical limit*, [arXiv : quant-ph0510151], 2005.
- [DB-Ro] S. DE BIEVRE & D. ROBERT, *Semi-classical propagation on $\log(\hbar)$ time scales*. Int. Math. Res. Not, No 12, 667-696, 2003.
- [He-Ro] B. HELFFER & D. ROBERT, *Puit de potentiel généralisé et asymptotique semi-classique*, Annales de l'IHP Physique Théorique 41(3):291-331, 1984.
- [Lab1] O. LABLÉE, *Spectre du Laplacien et de l'opérateur de Schrödinger sur une variété : de la géométrie spectrale à l'analyse semi-classique*, Gazette des Mathématiciens, Société Mathématique de France :116, 2008.
- [Lab2] O. LABLÉE, *Autour de la dynamique semi-classique de certains systèmes intégrables*. Thèse de doctorat en mathématiques, Université de Grenoble 1, 2009.
- [Lab3] O. LABLÉE, *Sur le spectre semi-classique d'un système intégrable de dimension 1 autour d'une singularité hyperbolique*. Annales de la faculté des sciences de Toulouse, Mathématiques, vol. 19, 1., pp. 85-123, 2010.
- [Lab4] O. LABLÉE, *"Dynamique semi-classique d'un système intégrable de dimension deux : renaissances et clonages de l'état initial"*, (pre-print), 2009.
- [LAS] C. LEICHTLE, I. Sh. AVERBUKH & W. SCHLEICH, *Multilevel quantum beats : An analytical approach..* Phys. Rev. A, 54, pp. 5299-5312, 1996.
- [Pau1] T. PAUL, *Echelles de temps pour l'évolution quantique à petite constante de planck*, Séminaire X-EDP, 2007.
- [Pau2] T. PAUL, *Reconstruction and non-reconstruction of wave packets*, Preprint, 2008.
- [Pau3] T. PAUL, *Reconstruction of wave packets on a hyperbolic trajectory*, Preprint, 2008.
- [Ram] T. RAMOND, *Semiclassical study of quantum scattering on the line*, Comm. Math. Phys., 177, p.221-254, 1996.
- [Rob] D. ROBERT, *Revivals of wave packets and Bohr-Sommerfeld quantization rules*, Adventures in mathematical physics, 219-235, Contemp. Math., 447, Amer. Math. Soc., Providence, RI, 2007.
- [Rob1] R. W. ROBINNET, *Quantum wave packet revivals*, Physics Reports. 392, 1-119, 2005.
- [Rob2] R. W. ROBINNET, *Wave packets revivals and quasirevivals in one-dimensionnal power law potentials*, Journal of Mathematical Physics, Volume 41(4), 2000.
- [VuN] S. VU NGOC, *Systèmes intégrables semi-classiques: du local au global*, Panoramas et synthèses 22, SMF, 2006.

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