Numerical simulation of BSDEs with drivers of quadratic growth

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Abstract

We consider Markovian backward stochastic differential equations (BSDEs) with drivers of quadratic growth and bounded terminal conditions. We first show some bound estimations on the process Z. Then we give a new time discretization scheme for such BSDEs and we obtain an explicit convergence rate for this scheme.

1 Introduction

Since the early nineties, there has been an increasing interest for backward stochastic differential equations (BSDEs for short). These equations have a wide range of applications in stochastic control, in finance or in partial differential equation theory. A particular class of BSDE is studied since few years: BSDEs with drivers of quadratic growth with respect to the variable z. This class arise, for example, in the context of utility optimization problems with exponential utility functions, or alternatively in questions related to risk minimization for the entropic risk measure (see e.g. [10]). Many papers deal with existence and uniqueness of solution for such BSDEs: we refer to [13], [14], [3], [4] or [7]. Our concern is rather related to the simulation of BSDEs and more precisely time discretization of BSDEs coupled with a forward stochastic differential equation (SDE for short). Actually, the design of efficient algorithms that are able to solve BSDEs in any reasonable dimension has been intensively studied since the first work of Chevance [5], see for instance [15], [1] or [8]. But in all these works, the driver of the BSDE is a Lipschitz function with respect to z and this assumption play a key role in theirs proofs. To the best of our knowledge, the only work where the time approximation of a BSDE with a quadratic growth with respect to z is studied is the one of Imkeller and dos Reis [11].

Our approach for the case of drivers with a quadratic growth contains an essential modification of previous works on time discretization of BSDEs. In a first step, we extend a result of [6] that shows the estimation

$$|Z_t| \leqslant \frac{C}{(T-t)^{1/2}}, \quad 0 \leqslant t < T,$$

without assuming that the terminal condition is a Lipschitz function. Let us notice that this type of estimation is well known in the case of drivers with linear growth as a consequence of the Bismut-Elworthy formula. But in our case, we do not need to suppose that the diffusion part of the SDE is invertible. Then, thanks to this estimation, we modify the classical uniform time net to obtain a convergence speed for a modified time discretization scheme for BSDEs: The idea is to put more discretization points near the final time T than near 0. The same idea is used by Gobet and Makhlouf in [9] for BSDEs with linear growth.

The paper is organized as follows. In the introductory Section 2 we recall some of the well known results concerning SDEs and BSDEs. In Section 3 we establish some estimates concerning the process Z. Finally, in Section 4 we study the convergence speed of a time discretization scheme for BSDEs.

2 Preliminaries

2.1 Notations

Throughout this paper, $(W_t)_{t\geqslant 0}$ will denote a d-dimensional Brownian motion, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For $t \geqslant 0$, let $\mathcal{F}_t$ denote the σ-algebra $\sigma(W_s; 0 \leqslant s \leqslant t)$, augmented with the $\mathbb{P}$-null sets of $\mathcal{F}$. The Euclidian norm on $\mathbb{R}^d$ will be denoted by $|.|$. The operator norm induced by $|.|$ on the space of linear operator is also denoted by $|.|$. For $p \geqslant 2$, $m \in \mathbb{N}$, we denote further

- $\mathcal{S}^p(\mathbb{R}^m)$, or $\mathcal{S}^p$ when no confusion is possible, the space of all adapted processes $(Y_t)_{t\in[0,T]}$ with values in $\mathbb{R}^m$ normed by $\|Y\|_{\mathcal{S}^p} = \mathbb{E}[(\sup_{t\in[0,T]} |Y_t|)^p]^{1/p}$; $\mathcal{S}^\infty(\mathbb{R}^m)$, or $\mathcal{S}^\infty$, the space of bounded measurable processes;
- $\mathcal{M}^p(\mathbb{R}^m)$, or $\mathcal{M}^p$, the space of all progressively measurable processes $(Z_t)_{t\in[0,T]}$ with values in $\mathbb{R}^m$ normed by $\|Z\|_{\mathcal{M}^p} = \mathbb{E}[(\int_0^T |Z_s|^2\, ds)^{p/2}]^{1/p}$.

In the following, we keep the same notation C for all finite, nonnegative constants that appear in our computations: they may depend on known parameters deriving from assumptions and on T, but not on any of the approximation and discretization parameters. In the same spirit, we keep the same notation η for all finite, positive constants that we can take as little as we want independently from the approximation and discretization parameters.

2.2 Some results on BMO martingales

In our work, the space of Bounded Mean Oscillation martingales (BMO martingales for short) play a key role for a priori estimates needed in our analysis of solutions of BSDE. We refer the reader to [12] for the theory of BMO martingales and we just recall the properties that we will use in the sequel. Let $\Phi_t = \int_0^t \phi_s dW_s$, $t \in [0,T]$ be a real square integrable martingale with respect to the Brownian filtration. Then Φ is a BMO martingale if

$$\|\Phi\|_{BMO} = \sup_{\tau\in[0,T]} \mathbb{E}\left[\langle\Phi\rangle_T - \langle\Phi\rangle_\tau|\mathcal{F}_\tau\right]^{1/2} = \sup_{\tau\in[0,T]} \mathbb{E}\left[\int_\tau^T \phi_s^2 ds|\mathcal{F}_\tau\right]^{1/2} < +\infty,$$

where the supremum is taken over all stopping times in $[0,T]$; $\langle\Phi\rangle$ denotes the quadratic variation of Φ. In our case, the very important feature of BMO martingales is the following lemma:

Lemma 2.1. *Let Φ be a BMO martingale. Then we have:*

1. *The stochastic exponential*

$$\mathcal{E}(\Phi)_t = \mathcal{E}_t = \exp\left(\int_0^t \phi_s dW_s - \frac{1}{2}\int_0^t |\phi_s|^2\, ds\right), \quad 0 \leqslant t \leqslant T,$$

   is a uniformly integrable martingale.

2. *Thanks to the reverse Hölder inequality, there exists $p > 1$ such that $\mathcal{E}_T \in L^p$. The maximal p with this property can be expressed in terms of the BMO norm of Φ.*

2.3 The backward-forward system

For functions b, σ, g and f and for $x \in \mathbb{R}^d$ we will deal with the solution processes of the following system of backward-forward stochastic differential equations. For $t \in [0,T]$ they are given by

$$X_t = x + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s)dW_s, \tag{2.1}$$

$$Y_t = g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s. \tag{2.2}$$

For the functions that appear in the above system of equations we give some general assumptions.

(HX0). $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : [0,T] \to \mathbb{R}^{d\times d}$ are measurable functions. There exist four positive constants M_b, K_b, M_σ and K_σ such that $\forall t, t' \in [0,T]$, $\forall x, x' \in \mathbb{R}^d$,

$$\begin{aligned} |b(t,x)| &\leqslant M_b(1+|x|) \\ \left|b(t,x)-b(t',x')\right| &\leqslant K_b(\left|x-x'\right| + \left|t-t'\right|^{1/2}) \\ |\sigma(t)| &\leqslant M_\sigma \\ \left|\sigma(t)-\sigma(t')\right| &\leqslant K_\sigma \left|t-t'\right|. \end{aligned}$$

(HY0). $f : [0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^{1\times d} \to \mathbb{R}$, $g : \mathbb{R}^d \to \mathbb{R}$ are measurable functions. There exist five positive constants M_f, $K_{f,x}$, $K_{f,y}$, $K_{f,z}$ and M_g such that $\forall t \in [0,T]$, $\forall x, x' \in \mathbb{R}^d$, $\forall y, y' \in \mathbb{R}$, $\forall z, z' \in \mathbb{R}^{1\times d}$,

$$\begin{aligned} |f(t,x,y,z)| &\leqslant M_f(1+|y|+|z|^2) \\ \left|f(t,x,y,z)-f(t,x',y',z')\right| &\leqslant K_{f,x}\left|x-x'\right| + K_{f,y}\left|y-y'\right| + (K_{f,z}+L_{f,z}(|z|+\left|z'\right|))\left|z-z'\right| \\ |g(x)| &\leqslant M_g. \end{aligned}$$

We next recall some results on BSDEs with quadratic growth. For their original version and their proof we refer to citer Kobylansky, Briant Confortola et Imkeller dos Reis.

Theorem 2.2. *Under (HX0), (HY0), the system (2.1)-(2.2) has a unique solution $(X,Y,Z) \in \mathcal{S}^2 \times \mathcal{S}^\infty \times \mathcal{M}^2$. The martingale $Z * W$ belongs to the space of BMO martingales and $\|Z*W\|_{BMO}$ only depends on T, M_g and M_f. Moreover, there exists a pair (r,q) such that $1/r + 1/q = 1$ and $\mathcal{E}(Z*W) \in L^r$.*

3 some useful estimates of Z

3.1 A first bound for Z

Theorem 3.1. *Suppose that (HX0), (HY0) hold and g is Lipschitz with Lipschitz constant K_g. Then, $\forall t \in [0,T]$*

$$|Z_t| \leqslant e^{(2K_b+K_{f,y})T} M_\sigma (K_g + TK_{f,x}).$$

Proof. Firstly, we suppose that b, g and f are differentiable with respect to x, y and z. Then (X, Y, Z) is differentiable with respect to x and $(\nabla X, \nabla Y, \nabla Z)$ is solution of

$$\begin{aligned}
\nabla X_t &= I_d + \int_0^t \nabla b(s, X_s)\nabla X_s ds, && (3.1)\\
\nabla Y_t &= \nabla g(X_T)\nabla X_T - \int_t^T \nabla Z_s dW_s && (3.2)\\
&\quad + \int_t^T \nabla_x f(s, X_s, Y_s, Z_s)\nabla X_s + \nabla_y f(s, X_s, Y_s, Z_s)\nabla Y_s + \nabla_z f(s, X_s, Y_s, Z_s)\nabla Z_s ds.
\end{aligned}$$

Where $\nabla X_t = (\partial X_t^i/\partial x^j)_{1\leqslant i,j\leqslant d}$, $\nabla Y_t = {}^t(\partial Y_t/\partial x^j)_{1\leqslant j\leqslant d} \in \mathbb{R}^{1\times d}$, $\nabla Z_t = (\partial Z_t^i/\partial x^j)_{1\leqslant i,j\leqslant d}$ and $\int_t^T \nabla Z_s dW_s$ means

$$\sum_{1\leqslant i\leqslant d} \int_t^T (\nabla Z_s)^i dW_s^i$$

with $(\nabla Z)^i$ denoting the i-th line of the $d \times d$ matrix process ∇Z. Thanks to usual transformations on the BSDE we obtain

$$\begin{aligned}
e^{\int_0^t \nabla_y f(s,X_s,Y_s,Z_s)ds}\nabla Y_t &= e^{\int_0^T \nabla_y f(s,X_s,Y_s,Z_s)ds}\nabla g(X_T)\nabla X_T - \int_t^T e^{\int_0^s \nabla_y f(u,X_u,Y_u,Z_u)du}\nabla Z_s d\tilde{W}_s\\
&\quad + \int_t^T e^{\int_0^s \nabla_y f(u,X_u,Y_u,Z_u)du}\nabla_x f(s, X_s, Y_s, Z_s)\nabla X_s ds,
\end{aligned}$$

with $d\tilde{W}_s = dW_s - \nabla_z f(s, X_s, Y_s, Z_s)ds$. $Z * W$ belongs to the space of BMO martingales so we are able to apply Girsanov theorem: there exists a probability $\mathbb{Q}$ under which $(\tilde{W})_{t\in[0,T]}$ is a Brownian motion. Then,

$$\begin{aligned}
e^{\int_0^t \nabla_y f(s,X_s,Y_s,Z_s)ds}\nabla Y_t &= \mathbb{E}^{\mathbb{Q}}\Big[e^{\int_0^T \nabla_y f(s,X_s,Y_s,Z_s)ds}\nabla g(X_T)\nabla X_T\\
&\quad + \int_t^T e^{\int_0^s \nabla_y f(u,X_u,Y_u,Z_u)du}\nabla_x f(s, X_s, Y_s, Z_s)\nabla X_s ds\Big|\mathcal{F}_t\Big],
\end{aligned}$$

and

$$|\nabla Y_t| \leqslant e^{(K_b+K_{f,y})T}(K_g + TK_{f,x}), \tag{3.3}$$

because $|\nabla X_t| \leqslant e^{K_b T}$. Moreover, thanks to the Malliavin calculus, it is possible to show that $Z_t = \nabla Y_t(\nabla X_t)^{-1}\sigma(t)$. So we obtain

$$|Z_t| \leqslant e^{K_b T} M_\sigma |\nabla Y_t| \leqslant e^{(2K_b+K_{f,y})T} M_\sigma (K_g + TK_{f,x}),$$

because $\left|\nabla X_t^{-1}\right| \leqslant e^{K_b T}$.

When b, g and f are not differentiable, we can also prove the result by a standard approximation and stability results for BSDEs with linear growth. □

3.2 A time dependent estimate of Z

We will introduce two alternative assumptions.

(HX1). b is differentiable with respect to x and σ is differentiable with respect to t. There exists $\lambda \in \mathbb{R}^+$ such that $\forall \eta \in \mathbb{R}^d$

$$\left|{}^t\eta\sigma(s)[{}^t\sigma(s)\,{}^t\nabla b(s,x) - {}^t\sigma'(s)]\eta\right| \leqslant \lambda\left|{}^t\eta\sigma(s)\right|^2. \tag{3.4}$$

(HX1'). σ is invertible and $\forall t \in [0,T]$, $\left|\sigma(t)^{-1}\right| \leqslant M_{\sigma^{-1}}$.

Example. Assumption (HX1) is verified when, $\forall s \in [0,T]$, $\nabla b(s,.)$ commutes with $\sigma(s)$ and $\exists A : [0,T] \to \mathbb{R}^{d\times d}$ bounded such that $\sigma'(t) = \sigma(t)A(t)$.

Theorem 3.2. *Suppose that (HX0), (HY0) hold and (HX1) or (HX1') holds. Moreover, suppose that g is lower (or upper) semi-continuous. Then there exists two constants $C, C' \in \mathbb{R}^+$ that depends only in T, M_g, M_f, $K_{f,x}$, $K_{f,y}$, $K_{f,z}$ and $L_{f,z}$ such that, $\forall t \in [0,T[$*

$$|Z_t| \leqslant C + C'(T-t)^{-1/2}.$$

Proof. In a first time, we will suppose that (HX1) holds and f, g are differentiable with respect to x, y and z. Then (Y,Z) is differentiable with respect to x and $(\nabla Y, \nabla Z)$ is the solution of the BSDE

$$\begin{aligned}\nabla Y_t &= \nabla g(X_T)\nabla X_T - \int_t^T \nabla Z_s dW_s \\ &\quad + \int_t^T \nabla_x f(s,X_s,Y_s,Z_s)\nabla X_s + \nabla_y f(s,X_s,Y_s,Z_s)\nabla Y_s + \nabla_z f(s,X_s,Y_s,Z_s)\nabla Z_s ds.\end{aligned}$$

Thanks to usual transformations we obtain

$$\begin{aligned}&e^{\int_0^t \nabla_y f(s,X_s,Y_s,Z_s)ds}\nabla Y_t + \int_0^t e^{\int_0^s \nabla_y f(u,X_u,Y_u,Z_u)du}\nabla_x f(s,X_s,Y_s,Z_s)\nabla X_s ds = \\ &e^{\int_0^T \nabla_y f(s,X_s,Y_s,Z_s)ds}\nabla g(X_T)\nabla X_T + \int_0^T e^{\int_0^s \nabla_y f(u,X_u,Y_u,Z_u)du}\nabla_x f(s,X_s,Y_s,Z_s)\nabla X_s ds \\ &- \int_t^T e^{\int_0^s \nabla_y f(u,X_u,Y_u,Z_u)du}\nabla Z_s d\tilde{W}_s,\end{aligned}$$

with $d\tilde{W}_s = dW_s - \nabla_z f(s,X_s,Y_s,Z_s)ds$. We can rewrite it

$$F_t = F_T - \int_t^T e^{\int_0^s \nabla_y f(u,X_u,Y_u,Z_u)du}\nabla Z_s d\tilde{W}_s \tag{3.5}$$

with

$$F_t := e^{\int_0^t \nabla_y f(s,X_s,Y_s,Z_s)ds}\nabla Y_t + \int_0^t e^{\int_0^s \nabla_y f(u,X_u,Y_u,Z_u)du}\nabla_x f(s,X_s,Y_s,Z_s)\nabla X_s ds.$$

$Z * W$ belongs to the space of BMO martingales so we are able to apply Girsanov theorem: there exists a probability $\mathbb{Q}$ under which $(\tilde{W})_{t\in[0,T]}$ is a Brownian motion. Thanks to the Malliavin calculus, it is possible to show that $Z_t = \nabla Y_t(\nabla X_t)^{-1}\sigma(t)$. So, we define:

$$\begin{aligned}\alpha_t &:= \int_0^t e^{\int_0^s \nabla_y f(u,X_u,Y_u,Z_u)du}\nabla_x f(s,X_s,Y_s,Z_s)\nabla X_s ds(\nabla X_t)^{-1}\sigma(t) \\ \tilde{Z}_t &:= F_t(\nabla X_t)^{-1}\sigma(t) = e^{\int_0^t \nabla_y f(s,X_s,Y_s,Z_s)ds}Z_t + \alpha_t \\ \tilde{F}_t &:= e^{\lambda t}F_t(\nabla X_t)^{-1}.\end{aligned}$$

Since $d\nabla X_t = \nabla b(t,X_t)\nabla X_t dt$, then $d(\nabla X_t)^{-1} = -(\nabla X_t)^{-1}\nabla b(t,X_t)dt$ and so , thanks to the Itô's formula,

$$d\tilde{Z}_t = dF_t(\nabla X_t)^{-1}\sigma(t) - F_t(\nabla X_t)^{-1}\nabla b(t,X_t)\sigma(t)dt + F_t(\nabla X_t)^{-1}\sigma'(t)dt,$$

and

$$d(e^{\lambda t}\tilde{Z}_t) = \tilde{F}_t(\lambda Id - \nabla b(t, X_t))\sigma(t)dt + \tilde{F}_t\sigma'(t)dt + e^{\lambda t}dF_t(\nabla X_t)^{-1}\sigma(t).$$

Finally,

$$d\left|e^{\lambda t}\tilde{Z}_t\right|^2 = d\langle M\rangle_t + 2\left[\lambda\left|\tilde{F}_t\sigma(t)\right|^2 - \tilde{F}_t\sigma(t)[{}^t\sigma(t)^t\nabla b(t, X_t) - {}^t\sigma'(t)]^t\tilde{F}_t\right]dt + dM_t,$$

with M_t a $\mathbb{Q}$-martingale. Thanks to the assumption (HX1) we are able to conclude that $\left|e^{\lambda t}\tilde{Z}_t\right|^2$ is a $\mathbb{Q}$-submartingale. So

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{2\lambda s}\left|\tilde{Z}_s\right|^2 ds\middle|\mathcal{F}_t\right] &\geqslant e^{2\lambda t}\left|\tilde{Z}_t\right|^2(T-t)\\ &\geqslant e^{2\lambda t}\left|e^{\int_0^t \nabla_y f(s,X_s,Y_s,Z_s)ds}Z_t + \alpha_t\right|^2(T-t)\\ &\geqslant C(|Z_t|^2 - 1)(T-t),\end{aligned}$$

with C a constant that only depends on T, K_b, M_σ, $K_{f,x}$, $K_{f,y}$ and λ. Moreover,

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{2\lambda s}\left|\tilde{Z}_s\right|^2 ds\middle|\mathcal{F}_t\right] &\leqslant C\mathbb{E}^{\mathbb{Q}}\left[\int_t^T |Z_s|^2 + |\alpha_s|^2\, ds\middle|\mathcal{F}_t\right]\\ &\leqslant C(\|Z\|_{BMO(\mathbb{Q})} + 1).\end{aligned}$$

But $\|Z\|_{BMO(\mathbb{Q})}$ does not depend on K_g because (Y, Z) is a solution of such a quadratic BSDE:

$$Y_t = g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s) - Z_s\nabla_z f(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s d\tilde{W}_s. \tag{3.6}$$

Finally $|Z_t| \leqslant C\left(1 + (T-t)^{-1/2}\right)$.

When σ is invertible, the inequality (3.4) is verified with $\lambda := M_{\sigma^{-1}}(M_\sigma K_b + K_\sigma)$. Since this λ does not depend on ∇b and σ', we can prove the result when $b(t,.)$ and σ are not differentiable by a standard approximation and stability results for linear BSDEs. So, we are allowed to replace assumption (HX1) by (HX1').

When f is not differentiable and g is only Lipschitz we can prove the result by a standard approximation and stability results for linear BSDEs. But we notice that our estimation on Z does not depend on K_g. This allows us to weaken the hypothesis on g further: When g is only lower or upper semi-continuous the result stay true. The proof is the same as the proof of Proposition 4.3 in [6]. □

Remark 3.3. *The previous proof gives us a more precise estimation for Z when f is differentiable with respect to z:*

$$|Z_t| \leqslant C + C'\mathbb{E}^{\mathbb{Q}}\left[\int_t^T |Z_s|^2\, ds\middle|\mathcal{F}_t\right]^{1/2}(T-t)^{-1/2}.$$

3.3 Zhang's path regularity Theorem

Let $0 = t_0 < t_1 < \ldots < t_n = T$ be any given partition of $[0, T]$, and denote δ_n the mesh size of this partition. We define a set of random variables

$$\bar{Z}_{t_i} = \frac{1}{t_{i+1} - t_i}\mathbb{E}\left[\int_{t_i}^{t_{i+1}} Z_s ds\middle|\mathcal{F}\right], \quad \forall i \in \{0, \ldots, n-1\}.$$

Then we are able to precise the famous Theorem 3.4.3 in [16]:

Theorem 3.4. *Suppose that (HX0), (HY0) hold and g is a Lipschitz function, with Lipschitz constant K_g. Then we have*

$$\sum_{i=0}^{n-1}\mathbb{E}\left[\int_{t_i}^{t_{i+1}}\left|Z_t-\bar{Z}_{t_i}\right|^2dt\right]\leqslant C(1+K_g^2)\delta_n,$$

where C is a positive constant independent of δ_n and K_g.

Proof. We will follow the proof of Theorem 5.6. in [11]: we just need to specify how the estimate depends on K_g. Firstly, it is not difficult to show that $\bar{Z}_{t_i}$ is the best $\mathcal{F}_{t_i}$-adapted approximation of Z in $\mathcal{M}^2([t_i,t_{i+1}])$, i.e.

$$\mathbb{E}\left[\int_{t_i}^{t_{i+1}}\left|Z_t-\bar{Z}_{t_i}\right|^2dt\right]=\inf_{Z_i\in L^2(\Omega,\mathcal{F}_{t_i})}\mathbb{E}\left[\int_{t_i}^{t_{i+1}}|Z_t-Z_i|^2dt\right].$$

In particular,

$$\mathbb{E}\left[\int_{t_i}^{t_{i+1}}\left|Z_t-\bar{Z}_{t_i}\right|^2dt\right]\leqslant\mathbb{E}\left[\int_{t_i}^{t_{i+1}}|Z_t-Z_{t_i}|^2dt\right].$$

In the same spirit than previous proofs, we suppose in a first time that b, g and f are differentiable with respect to x, y and z. So,

$$Z_t-Z_{t_i}=\nabla Y_t(\nabla X_t)^{-1}\sigma(t)-\nabla Y_{t_i}(\nabla X_{t_i})^{-1}\sigma(t_i)=I_1+I_2+I_3,$$

with $I_1=\nabla Y_t(\nabla X_t)^{-1}(\sigma(t)-\sigma(t_i))$, $I_2=\nabla Y_t((\nabla X_t)^{-1}-(\nabla X_{t_i})^{-1})\sigma(t_i)$ and $I_3=\nabla(Y_t-Y_{t_i})(\nabla X_{t_i})^{-1}\sigma(t_i)$. Firstly, thanks to the estimation (3.3) we have

$$|I_1|^2\leqslant|\nabla Y_t|^2e^{2K_bT}K_\sigma^2|t_{i+1}-t_i|^2\leqslant C(1+K_g^2)\delta_n^2.$$

We obtain the same estimation for $|I_2|$ because

$$\left|(\nabla X_t)^{-1}-(\nabla X_{t_i})^{-1}\right|\leqslant\left|\int_{t_i}^t\nabla b(s,X_s)\nabla X_sds\right|\leqslant K_be^{K_bT}|t-t_i|.$$

Lastly, $I_3\leqslant M_\sigma e^{K_bT}|\nabla Y_t-\nabla Y_{t_i}|$. So,

$$\sum_{i=0}^{n-1}\mathbb{E}\left[\int_{t_i}^{t_{i+1}}|I_3|^2dt\right]\leqslant C\delta_n\sum_{i=0}^{n-1}\mathbb{E}\left[\operatorname*{ess\,sup}_{t\in[t_i,t_{i+1}]}|\nabla Y_t-\nabla Y_{t_i}|^2\right].$$

By using the BSDE (3.2), (HY0), the estimation on ∇X_s and the estimation (3.3), we have

$$\begin{aligned}&|\nabla Y_t-\nabla Y_{t_i}|^2\\&\leqslant C\left(\int_{t_i}^tC(1+K_g)+|\nabla_zf(s,X_s,Y_s,Z_s)|\,|\nabla Z_s|\,ds\right)^2+C\left(\int_{t_i}^t\nabla Z_sdW_s\right)^2.\end{aligned}$$

The inequality of Burkholder-Davis-Gundy gives us

$$\begin{aligned}&\sum_{i=0}^{n-1}\mathbb{E}\left[\operatorname*{ess\,sup}_{t\in[t_i,t_{i+1}]}|\nabla Y_t-\nabla Y_{t_i}|^2\right]\\&\leqslant C(1+K_g^2)+C\sum_{i=0}^{n-1}\mathbb{E}\left(\int_{t_i}^{t_{i+1}}|\nabla_zf(s,X_s,Y_s,Z_s)|\,|\nabla Z_s|\,ds\right)^2+C\mathbb{E}\left(\int_{t_i}^{t_{i+1}}|\nabla Z_s|^2ds\right)\\&\leqslant C(1+K_g^2)+C\mathbb{E}\left[\left(\int_0^T|\nabla_zf(s,X_s,Y_s,Z_s)|\,|\nabla Z_s|\,ds\right)^2+\int_0^T|\nabla Z_s|^2ds\right].\end{aligned}$$

Then, from Corollary 9 in [2] we have

$$\mathbb{E}\left[\int_0^T |\nabla Z_s|^2 \, ds\right] \leqslant C(1+K_g^2).$$

Moreover, by defining $\mathbb{1} \in \mathbb{R}^d$ the vector which all components are equal to 1, we have

$$\begin{aligned}
\mathbb{E}\left[\left(\int_0^T |\nabla_z f(s,X_s,Y_s,Z_s)|\,|\nabla Z_s|\,ds\right)^2\right] &\leqslant C\mathbb{E}\left[\left|\int_0^T |\nabla Z_s|\,|\nabla_z f(s,X_s,Y_s,Z_s)|\,\mathbb{1}ds\right|^2\right] \\
&\leqslant C\mathbb{E}\left[\left|\int_0^T |\nabla Z_s|\,(dW_s - |\nabla_z f(s,X_s,Y_s,Z_s)|\,\mathbb{1}ds)\right|^2\right] \\
&\quad + C\mathbb{E}\left[\left|\int_0^T |\nabla Z_s|\,dW_s\right|^2\right].
\end{aligned}$$

Since $|Z| * W$ belongs to the space of BMO martingales, there exists a probability $\mathbb{Q}$ under which $(\tilde{W}_t := \int_0^t dW_s - |\nabla_z f(s,X_s,Y_s,Z_s)|\,\mathbb{1}ds)_{t\in[0,T]}$ is a Brownian motion. Moreover, there exists a pair $(r,p) \in]1,+\infty[^2$ such that $1/r + 1/p = 1$, $\mathcal{E} \in L^r$ and $\mathcal{E}^{-1} \in L^r$ with $\mathcal{E} := \frac{d\mathbb{Q}}{d\mathbb{P}}$. Then, the inequalities of Hölder and Burkholder-Davis-Gundy give us

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^T |\nabla_z f(s,X_s,Y_s,Z_s)|\,|\nabla Z_s|\,ds\right)^2\right] \\
&\leqslant C\mathbb{E}^{\mathbb{Q}}\left[\left|\int_0^T |\nabla Z_s|\,d\tilde{W}_s\right|^{2p}\right]^{1/p} + C\mathbb{E}\left[\left|\int_0^T |\nabla Z_s|\,dW_s\right|^2\right] \\
&\leqslant C\mathbb{E}^{\mathbb{Q}}\left[\left|\int_0^T |\nabla Z_s|^2\,ds\right|^{p}\right]^{1/p} + C\mathbb{E}\left[\int_0^T |\nabla Z_s|^2\,ds\right].
\end{aligned}$$

Since $(\nabla Y, \nabla Z)$ is also the solution of the BSDE

$$\begin{aligned}
\nabla Y_t &= \nabla g(X_T)\nabla X_T - \int_t^T \nabla Z_s d\tilde{W}_s + \int_t^T \nabla Z_s\,|\nabla_z f(s,X_s,Y_s,Z_s)|\,\mathbb{1}ds \\
&\quad + \int_t^T \nabla_x f(s,X_s,Y_s,Z_s)\nabla X_s + \nabla_y f(s,X_s,Y_s,Z_s)\nabla Y_s + \nabla_z f(s,X_s,Y_s,Z_s)\nabla Z_s ds,
\end{aligned}$$

we are allowed to use again Corollary 9 in [2] to obtain

$$\mathbb{E}\left[\left(\int_0^T |\nabla_z f(s,X_s,Y_s,Z_s)|\,|\nabla Z_s|\,ds\right)^2\right] \leqslant C(1+K_g^2).$$

Finally,

$$\sum_{i=0}^{n-1} \mathbb{E}\left[\int_{t_i}^{t_{i+1}} |I_3|^2\,dt\right] \leqslant C(1+K_g^2)\delta_n$$

and

$$\begin{aligned}
\sum_{i=0}^{n-1} \mathbb{E}\left[\int_{t_i}^{t_{i+1}} \left|Z_t - \bar{Z}_{t_i}\right|^2 dt\right] &\leqslant \sum_{i=0}^{n-1} \mathbb{E}\left[\int_{t_i}^{t_{i+1}} |I_1|^2 + |I_2|^2 + |I_3|^2\,dt\right] \\
&\leqslant C(1+K_g^2)\delta_n.
\end{aligned}$$

□

4 Convergence of a modified time discretization scheme for the BSDE

In all this section we suppose that (HX0), (HY0) and (HX1) or (HX1') hold.

4.1 An approximation of the quadratic BSDE

In a first time we will approximate our quadratic BSDE (2.2) by an another one. We set $\varepsilon \in]0,T[$ and $N \in \mathbb{N}$. Let $(Y_t^{N,\varepsilon}, Z_t^{N,\varepsilon})$ the solution of the BSDE

$$Y_t^{N,\varepsilon} = g_N(X_T) + \int_t^T f^\varepsilon(s, X_s, Y_s^{N,\varepsilon}, Z_s^{N,\varepsilon})ds - \int_t^T Z_s^{N,\varepsilon} dW_s, \tag{4.1}$$

with

$$f^\varepsilon(s,x,y,z) := \mathbb{1}_{s<T-\varepsilon} f(s,x,y,z) + \mathbb{1}_{s\geqslant T-\varepsilon} f(s,x,y,0),$$

and g_N a Lipschitz approximation of g with Lipschitz constant N. f^ε verifies assumption (HY0) with the same constants as f. Since g_N is a Lipschitz function, Z is bounded and the BSDE (4.1) is a BSDE with a linear growth. Moreover, we can apply Theorem 3.2 to obtain:

Proposition 4.1. *There exist three constants $M_{z,1}, M_{z,2}, M_{z,3} \in \mathbb{R}^+$ that do not depend on N and ε such that*

$$\left|Z_s^{N,\varepsilon}\right| \leqslant \left(M_{z,1} + \frac{M_{z,2}}{(T-s)^{1/2}}\right) \wedge (M_{z,3}N).$$

Thanks to BMO tools we have a stability result for quadratic BSDEs (see [2] and [11]):

Proposition 4.2. *There exists a constant C that does not depend on N and ε such that*

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left|Y_t^{N,\varepsilon} - Y_t\right|^2\right] + \mathbb{E}\left[\int_0^T \left|Z_t^{N,\varepsilon} - Z_t\right|^2 dt\right] \leqslant C(e_1(N) + e_2(N,\varepsilon))$$

with

$$e_1(N) := \mathbb{E}\left[|g_N(X_T) - g(X_T)|^{2q}\right]^{1/q},^1$$

$$e_2(N,\varepsilon) := \mathbb{E}\left[\left(\int_{T-\varepsilon}^T \left|f(t, X_t, Y_t^{N,\varepsilon}, Z_t^{N,\varepsilon}) - f(t, X_t, Y_t^{N,\varepsilon}, 0)\right| dt\right)^{2q}\right]^{1/q},$$

and q defined in Theorem 2.2.

Then, in a second time, we will approximate our modified backward-forward system by a discrete-time one. We will slightly modify the classical discretization by using a non equidistant net with $2n+1$ discretization times. We define the $n+1$ first discretization times by

$$t_k = T\left(1 - \left(\frac{\varepsilon}{T}\right)^{k/n}\right),$$

and we use an equidistant net on $[T-\varepsilon, T]$ for the last n discretization times:

$$t_k = T - \left(\frac{2n-k}{n}\right)\varepsilon, \quad n \leqslant k \leqslant 2n.$$

[1] The authors of [11] obtain this result with q^2 instead of q. Nevertheless, we are able to obtain the good result by applying estimations of [2].

We denote the time step by $(h_k := t_{k+1} - t_k)_{0 \leqslant k \leqslant 2n-1}$. We consider $(X^n_{t_k})_{0 \leqslant k \leqslant 2n}$ the classical Euler scheme for X given by

$$\begin{aligned} X^n_0 &= x \\ X^n_{t_{k+1}} &= X^n_{t_k} + h_k b(t_k, X^n_{t_k}) + \sigma(t_k)(W_{t_{k+1}} - W_{t_k}), \quad 0 \leqslant k \leqslant 2n-1. \end{aligned} \tag{4.2}$$

We denote $\rho_s : \mathbb{R}^{1\times d} \to \mathbb{R}^{1\times d}$ the projection on the ball

$$B\left(0, M_{z,1} + \frac{M_{z,2}}{(T-s)^{1/2}}\right)$$

with $M_{z,1}$ and $M_{z,2}$ given by Proposition 4.1. Finally we denote $(Y^{N,\varepsilon,n}, Z^{N,\varepsilon,n})$ our time approximation of $(Y^{N,\varepsilon}, Z^{N,\varepsilon})$. This couple is obtain by a slight modification of the classical dynamic programming equation:

$$\begin{aligned} Y^{N,\varepsilon,n}_{t_{2n}} &= g_N(X^n_{t_{2n}}) \\ Z^{N,\varepsilon,n}_{t_k} &= \rho_{t_{k+1}}\left(\frac{1}{h_k}\mathbb{E}_{t_k}[Y^{N,\varepsilon,n}_{t_{k+1}}(W_{t_{k+1}} - W_{t_k})]\right), \quad 0 \leqslant k \leqslant 2n-1, \end{aligned} \tag{4.3}$$

$$Y^{N,\varepsilon,n}_{t_k} = \mathbb{E}_{t_k}[Y^{N,\varepsilon,n}_{t_{k+1}}] + h_k \mathbb{E}_{t_k}[f(t_k, X^n_{t_k}, Y^{N,\varepsilon,n}_{t_{k+1}}, Z^{N,\varepsilon,n}_{t_k})], \quad 0 \leqslant k \leqslant 2n-1. \tag{4.4}$$

The aim of our work is to study the error of discretization

$$e(N,\varepsilon,n) := \sup_{0\leqslant k\leqslant 2n} \mathbb{E}\left[\left|Y^{N,\varepsilon,n}_{t_k} - Y_{t_k}\right|^2\right] + \sum_{k=0}^{2n-1} \mathbb{E}\left[\int_{t_k}^{t_{k+1}} \left|Z^{N,\varepsilon,n}_{t_k} - Z_t\right|^2 dt\right].$$

It's easy to see that

$$e(N,\varepsilon,n) \leqslant C(e_1(N) + e_2(N,\varepsilon) + e_3(N,\varepsilon,n)),$$

with $e_1(N)$ and $e_2(N,\varepsilon)$ defined in Proposition 4.2, and

$$e_3(N,\varepsilon,n) := \sup_{0\leqslant k\leqslant 2n} \mathbb{E}\left[\left|Y^{N,\varepsilon,n}_{t_k} - Y^{N,\varepsilon}_{t_k}\right|^2\right] + \sum_{k=0}^{2n-1} \mathbb{E}\left[\int_{t_k}^{t_{k+1}} \left|Z^{N,\varepsilon,n}_{t_k} - Z^{N,\varepsilon}_t\right|^2 dt\right].$$

4.2 Studying of the time approximation error $e_3(N,\varepsilon,n)$

We need an extra assumption.

(HY1). There exists a positive constant $K_{f,t}$ such that $\forall t, t' \in [0,T]$, $\forall x \in \mathbb{R}^d$, $\forall y \in \mathbb{R}$, $\forall z \in \mathbb{R}^{1\times d}$,

$$\left|f(t,x,y,z) - f(t',x,y,z)\right| \leqslant K_{f,t}\left|t - t'\right|^{1/2}.$$

Moreover, we set $\varepsilon = Tn^{-a}$ and $N = n^b$, with $a, b \in \mathbb{R}^{+,*}$ two parameters. Before giving our error estimations, we recall two technical lemmas that we will prove in the appendix.

Lemma 4.3. *For all constant $M > 0$ there exist a constant C that depends only on T, M and a, such that*

$$\prod_{i=0}^{2n-1}(1 + Mh_i) \leqslant C, \quad \forall n \in \mathbb{N}^*.$$

Lemma 4.4. *For all constants $M_1 > 0$ and $M_2 > 0$ there exist a constant C that depends only on T, M_1, M_2 and a, such that*

$$\prod_{i=0}^{n-1}\left(1 + M_1 h_i + M_2 \frac{h_i}{T - t_{i+1}}\right) \leqslant C n^{aM_2}.$$

Firstly, we give a convergence result for the Euler scheme.

Proposition 4.5. *Assume (HX0) holds. Then there exists a constant C that does not depend on n and such that*

$$\sup_{0\leqslant k\leqslant 2n} \mathbb{E}\left[\left|X_{t_k} - X^n_{t_k}\right|^2\right] \leqslant C\frac{\ln n}{n}.$$

Proof. We just have to copy the classical proof to obtain, thanks to Lemma 4.3,

$$\sup_{0\leqslant k\leqslant 2n} \mathbb{E}\left[\left|X_{t_k} - X^n_{t_k}\right|^2\right] \leqslant C \sup_{0\leqslant i\leqslant 2n-1} h_i = Ch_0.$$

But

$$h_0 = T(1 - n^{-a/n}) \sim aT\frac{\ln n}{n},$$

and the proof is completed. □

Now, let treat the BSDE approximation. In a first time we will study the time approximation error on $[T - \varepsilon, T]$.

Proposition 4.6. *Assume that (HX0), (HY0) and (HY1) hold. Then there exist a constant C that does not depend on n and such that*

$$\sup_{n\leqslant k\leqslant 2n} \mathbb{E}\left[\left|Y^{N,\varepsilon,n}_{t_k} - Y^{N,\varepsilon}_{t_k}\right|^2\right] + \sum_{k=n}^{2n-1} \mathbb{E}\left[\int_{t_k}^{t_{k+1}} \left|Z^{N,\varepsilon,n}_{t_k} - Z^{N,\varepsilon}_t\right|^2 dt\right] \leqslant \frac{C\ln n}{n^{1-2b}}.$$

Proof. The BSDE 4.1 has a linear growth with respect to z on $[T - \varepsilon, T]$ so we are allowed to apply classical results that give us

$$\sup_{n\leqslant k\leqslant 2n} \mathbb{E}\left[\left|Y^{N,\varepsilon,n}_{t_k} - Y^{N,\varepsilon}_{t_k}\right|^2\right] + \sum_{k=n}^{2n-1} \mathbb{E}\left[\int_{t_k}^{t_{k+1}} \left|Z^{N,\varepsilon,n}_{t_k} - Z^{N,\varepsilon}_t\right|^2 dt\right] \leqslant C\left(\mathbb{E}\left[\left|g_N(X_T) - g_N(X^n_T)\right|^2\right] + \frac{\varepsilon}{n}\right).$$

Since g_N is N-Lipschitz, we obtain the result by applying Proposition 4.5. □

Now, let see what happen on $[0, T - \varepsilon]$.

Theorem 4.7. *Assume that (HX0), (HY0), (HY1) and (HX1) or (HX1') hold. Then for all $\eta > 0$, there exist a constant C that does not depend on N, ε and n, and such that*

$$\sup_{0\leqslant k\leqslant 2n} \mathbb{E}\left[\left|Y^{N,\varepsilon,n}_{t_k} - Y^{N,\varepsilon}_{t_k}\right|^2\right] + \sum_{k=0}^{2n-1} \mathbb{E}\left[\int_{t_k}^{t_{k+1}} \left|Z^{N,\varepsilon,n}_{t_k} - Z^{N,\varepsilon}_t\right|^2 dt\right] \leqslant \frac{C}{n^{1-2b-Ka}},$$

with $K = 4(1+\eta)L^2_{f,z}M^2_{z,2}$.

Proof. We will fit the classical proof to our situation. From (4.1) and (4.4) we get

$$Y_{t_k}^{N,\varepsilon}-Y_{t_k}^{N,\varepsilon,n}=\mathbb{E}_{t_k}\left[Y_{t_{k+1}}^{N,\varepsilon}-Y_{t_{k+1}}^{N,\varepsilon,n}\right]+\mathbb{E}_{t_k}\int_{t_k}^{t_{k+1}}\left(f(s,X_s,Y_s^{N,\varepsilon},Z_s^{N,\varepsilon})-f(t_k,X_{t_k}^n,Y_{t_{k+1}}^{N,\varepsilon,n},Z_{t_k}^{N,\varepsilon,n})\right)ds.$$

We introduce a parameter $\gamma_k>0$ that will be chosen later. Thank's to Proposition 4.1 and assumption (HY0), f is Lipschitz on $[t_k,t_{k+1}]$ with a Lipschitz constant $K_k:=K^1+\frac{K^2}{(T-t_{k+1})^{1/2}}$ where $K^2=2L_{f,z}M_{z,2}$. A combination of Young's inequality $(a+b)^2\leqslant(1+\gamma_k h_k)a^2+(1+\frac{1}{\gamma_k h_k})b^2$ and properties of f gives

$$\begin{aligned}\mathbb{E}\left|Y_{t_k}^{N,\varepsilon}-Y_{t_k}^{N,\varepsilon,n}\right|^2 \;\leqslant\; & (1+\gamma_k h_k)\mathbb{E}\left|\mathbb{E}_{t_k}\left[Y_{t_{k+1}}^{N,\varepsilon}-Y_{t_{k+1}}^{N,\varepsilon,n}\right]\right|^2 \\ & +(1+\eta)K_k^2(h_k+\frac{1}{\gamma_k})\mathbb{E}\int_{t_k}^{t_{k+1}}\left|Z_s^{N,\varepsilon}-Z_{t_k}^{N,\varepsilon,n}\right|^2 ds \\ & +C(h_k+\frac{1}{\gamma_k})\left(h_k^2+\int_{t_k}^{t_{k+1}}\mathbb{E}\left|X_s-X_{t_k}^n\right|^2 ds+\int_{t_k}^{t_{k+1}}\mathbb{E}\left|Y_s^{N,\varepsilon}-Y_{t_{k+1}}^{N,\varepsilon,n}\right|^2 ds\right).\end{aligned} \tag{4.5}$$

We define

$$\tilde{Z}_{t_k}^{N,\varepsilon,n}:=\frac{1}{h_k}\mathbb{E}_{t_k}[Y_{t_{k+1}}^{N,\varepsilon,n}(W_{t_{k+1}}-W_{t_k})].$$

So, $Z_{t_k}^{N,\varepsilon,n}=\rho_{t_{k+1}}(\tilde{Z}_{t_k}^{N,\varepsilon,n})$. Moreover, Proposition 4.1 implies that $Z_s^{N,\varepsilon}=\rho_{t_{k+1}}(Z_s^{N,\varepsilon})$, and, since $\rho_{t_{k+1}}$ is 1-Lipschitz, we have

$$\left|Z_s^{N,\varepsilon}-Z_{t_k}^{N,\varepsilon,n}\right|^2=\left|\rho_{t_{k+1}}(Z_s^{N,\varepsilon})-\rho_{t_{k+1}}(\tilde{Z}_{t_k}^{N,\varepsilon,n})\right|^2\leqslant\left|Z_s^{N,\varepsilon}-\tilde{Z}_{t_k}^{N,\varepsilon,n}\right|^2. \tag{4.6}$$

As in Theorem 3.4, we define $\bar{Z}_{t_k}^{N,\varepsilon}$ by

$$h_k\bar{Z}_{t_k}^{N,\varepsilon}:=\mathbb{E}_{t_k}\int_{t_k}^{t_{k+1}}Z_s^{N,\varepsilon}ds=\mathbb{E}_{t_k}\left((Y_{t_{k+1}}^{N,\varepsilon}+\int_{t_k}^{t_{k+1}}f(s,X_s,Y_s^{N,\varepsilon},Z_s^{N,\varepsilon})ds)^t(W_{t_{k+1}}-W_{t_k})\right).$$

Clearly,

$$\mathbb{E}\int_{t_k}^{t_{k+1}}\left|Z_s^{N,\varepsilon}-\tilde{Z}_{t_k}^{N,\varepsilon,n}\right|^2 ds=\mathbb{E}\int_{t_k}^{t_{k+1}}\left|Z_s^{N,\varepsilon}-\bar{Z}_{t_k}^{N,\varepsilon}\right|^2 ds+h_k\mathbb{E}\left|\bar{Z}_{t_k}^{N,\varepsilon}-\tilde{Z}_{t_k}^{N,\varepsilon,n}\right|^2. \tag{4.7}$$

The Cauchy-Schwartz inequality yields

$$\left|\mathbb{E}_{t_k}\left((Y_{t_{k+1}}^{N,\varepsilon}-Y_{t_{k+1}}^{N,\varepsilon,n})^t(W_{t_{k+1}}-W_{t_k})\right)\right|^2\leqslant h_k\{\mathbb{E}_{t_k}(\left|Y_{t_{k+1}}^{N,\varepsilon}-Y_{t_{k+1}}^{N,\varepsilon,n}\right|^2)-\left|\mathbb{E}_{t_k}(Y_{t_{k+1}}^{N,\varepsilon}-Y_{t_{k+1}}^{N,\varepsilon,n})\right|^2\},$$

and consequently

$$\begin{aligned}h_k\mathbb{E}\left|\bar{Z}_{t_k}^{N,\varepsilon}-\tilde{Z}_{t_k}^{N,\varepsilon,n}\right|^2 \;\leqslant\; & (1+\eta)\mathbb{E}\left[\mathbb{E}_{t_k}(\left|Y_{t_{k+1}}^{N,\varepsilon}-Y_{t_{k+1}}^{N,\varepsilon,n}\right|^2)-\left|\mathbb{E}_{t_k}(Y_{t_{k+1}}^{N,\varepsilon}-Y_{t_{k+1}}^{N,\varepsilon,n})\right|^2\right] \\ & +Ch_k\mathbb{E}\int_{t_k}^{t_{k+1}}f(s,X_s,Y_s^{N,\varepsilon},Z_s^{N,\varepsilon})^2 ds.\end{aligned} \tag{4.8}$$

Plugging (4.7) and (4.8) into (4.5), we get:

$$\begin{aligned}
\mathbb{E}\left|Y_{t_k}^{N,\varepsilon}-Y_{t_k}^{N,\varepsilon,n}\right|^2 &\leqslant (1+\gamma_k h_k)\mathbb{E}\left|\mathbb{E}_{t_k}\left[Y_{t_{k+1}}^{N,\varepsilon}-Y_{t_{k+1}}^{N,\varepsilon,n}\right]\right|^2 \\
&\quad +(1+\eta)K_k^2(h_k+\frac{1}{\gamma_k})\mathbb{E}\int_{t_k}^{t_{k+1}}\left|Z_s^{N,\varepsilon}-\bar{Z}_{t_k}^{N,\varepsilon}\right|^2 ds \\
&\quad +C(h_k+\frac{1}{\gamma_k})\left(h_k^2+\int_{t_k}^{t_{k+1}}\mathbb{E}\left|X_s-X_{t_k}^n\right|^2 ds+\int_{t_k}^{t_{k+1}}\mathbb{E}\left|Y_s^{N,\varepsilon}-Y_{t_{k+1}}^{N,\varepsilon,n}\right|^2 ds\right) \\
&\quad +(1+\eta)K_k^2(h_k+\frac{1}{\gamma_k})\mathbb{E}\left[\mathbb{E}_{t_k}(\left|Y_{t_{k+1}}^{N,\varepsilon}-Y_{t_{k+1}}^{N,\varepsilon,n}\right|^2)-\left|\mathbb{E}_{t_k}(Y_{t_{k+1}}^{N,\varepsilon}-Y_{t_{k+1}}^{N,\varepsilon,n})\right|^2\right] \\
&\quad +CK_k^2(h_k+\frac{1}{\gamma_k})h_k\mathbb{E}\int_{t_k}^{t_{k+1}}f(s,X_s,Y_s^{N,\varepsilon},Z_s^{N,\varepsilon})^2 ds.
\end{aligned}$$

Now write

$$\mathbb{E}\left|Y_s^{N,\varepsilon}-Y_{t_{k+1}}^{N,\varepsilon,n}\right|^2 \leqslant 2\mathbb{E}\left|Y_s^{N,\varepsilon}-Y_{t_{k+1}}^{N,\varepsilon}\right|^2+2\mathbb{E}\left|Y_{t_{k+1}}^{N,\varepsilon}-Y_{t_{k+1}}^{N,\varepsilon,n}\right|^2, \tag{4.9}$$

$$\mathbb{E}\left|X_s-X_{t_k}^n\right|^2 \leqslant 2\mathbb{E}\left|X_s-X_{t_k}\right|^2+2\mathbb{E}\left|X_{t_k}-X_{t_k}^n\right|^2. \tag{4.10}$$

We obtain

$$\begin{aligned}
\mathbb{E}\left|Y_{t_k}^{N,\varepsilon}-Y_{t_k}^{N,\varepsilon,n}\right|^2 &\leqslant (1+\gamma_k h_k)\mathbb{E}\left|\mathbb{E}_{t_k}\left[Y_{t_{k+1}}^{N,\varepsilon}-Y_{t_{k+1}}^{N,\varepsilon,n}\right]\right|^2 \\
&\quad +(1+\eta)K_k^2(h_k+\frac{1}{\gamma_k})\mathbb{E}\int_{t_k}^{t_{k+1}}\left|Z_s^{N,\varepsilon}-\bar{Z}_{t_k}^{N,\varepsilon}\right|^2 ds \\
&\quad +C(h_k+\frac{1}{\gamma_k})\left(h_k^2+\int_{t_k}^{t_{k+1}}\mathbb{E}\left|X_s-X_{t_k}\right|^2 ds+h_k\mathbb{E}\left|X_{t_k}-X_{t_k}^n\right|^2\right) \\
&\quad +C(h_k+\frac{1}{\gamma_k})\left(\int_{t_k}^{t_{k+1}}\mathbb{E}\left|Y_s^{N,\varepsilon}-Y_{t_{k+1}}^{N,\varepsilon}\right|^2 ds+h_k\mathbb{E}\left|Y_{t_{k+1}}^{N,\varepsilon}-Y_{t_{k+1}}^{N,\varepsilon,n}\right|^2\right) \\
&\quad +(1+\eta)K_k^2(h_k+\frac{1}{\gamma_k})\mathbb{E}\left[\mathbb{E}_{t_k}(\left|Y_{t_{k+1}}^{N,\varepsilon}-Y_{t_{k+1}}^{N,\varepsilon,n}\right|^2)-\left|\mathbb{E}_{t_k}(Y_{t_{k+1}}^{N,\varepsilon}-Y_{t_{k+1}}^{N,\varepsilon,n})\right|^2\right] \\
&\quad +CK_k^2(h_k+\frac{1}{\gamma_k})h_k\mathbb{E}\int_{t_k}^{t_{k+1}}f(s,X_s,Y_s^{N,\varepsilon},Z_s^{N,\varepsilon})^2 ds.
\end{aligned}$$

take $\gamma_k=(1+\eta)K_k^2$: for h_k small enough, it gives

$$\begin{aligned}
\mathbb{E}\left|Y_{t_k}^{N,\varepsilon}-Y_{t_k}^{N,\varepsilon,n}\right|^2 &\leqslant (1+(1+\eta)K_k^2h_k)\mathbb{E}\left|Y_{t_{k+1}}^{N,\varepsilon}-Y_{t_{k+1}}^{N,\varepsilon,n}\right|^2+Ch_k^2+Ch_k\max_{0\leqslant k\leqslant n}\mathbb{E}\left|X_{t_k}-X_{t_k}^n\right|^2 \\
&\quad +C\mathbb{E}\int_{t_k}^{t_{k+1}}\left|Z_s^{N,\varepsilon}-\bar{Z}_{t_k}^{N,\varepsilon}\right|^2 ds+C\int_{t_k}^{t_{k+1}}\mathbb{E}\left|X_s-X_{t_k}\right|^2 ds \\
&\quad +C\int_{t_k}^{t_{k+1}}\mathbb{E}\left|Y_s^{N,\varepsilon}-Y_{t_{k+1}}^{N,\varepsilon}\right|^2 ds+Ch_k\mathbb{E}\int_{t_k}^{t_{k+1}}f(s,X_s,Y_s^{N,\varepsilon},Z_s^{N,\varepsilon})^2 ds,
\end{aligned}$$

because $K_k^2h_k\leqslant C(h_0+h_k(T-t_{k+1})^{-1})\leqslant C\frac{\ln n}{n}$. The Gronwall's lemma gives us

$$\begin{aligned}
\mathbb{E}\left|Y_{t_k}^{N,\varepsilon}-Y_{t_k}^{N,\varepsilon,n}\right|^2 &\leqslant C\sum_{k=0}^{n-1}\left[\prod_{i=0}^{k}(1+(1+\eta)K_i^2h_i)\right]\left[h_k^2+h_k\max_{0\leqslant k\leqslant n}\mathbb{E}\left|X_{t_k}-X_{t_k}^n\right|^2\right. \\
&\quad +\mathbb{E}\int_{t_k}^{t_{k+1}}\left|Z_s^{N,\varepsilon}-\bar{Z}_{t_k}^{N,\varepsilon}\right|^2+|X_s-X_{t_k}|^2+\left|Y_s^{N,\varepsilon}-Y_{t_{k+1}}^{N,\varepsilon}\right|^2 ds \\
&\quad \left.+h_k\mathbb{E}\int_{t_k}^{t_{k+1}}f(s,X_s,Y_s^{N,\varepsilon},Z_s^{N,\varepsilon})^2 ds+\mathbb{E}\left|Y_{t_n}^{N,\varepsilon}-Y_{t_n}^{N,\varepsilon,n}\right|^2\right].
\end{aligned}$$

Then, we apply Lemma 4.4:

$$\begin{aligned}\mathbb{E}\left|Y_{t_k}^{N,\varepsilon}-Y_{t_k}^{N,\varepsilon,n}\right|^2 &\leqslant Cn^{(1+\eta)(K^2)^2a}\left[h_0+\max_{0\leqslant k\leqslant n}\mathbb{E}\left|X_{t_k}-X_{t_k}^n\right|^2\right.\\ &\quad+\sum_{k=0}^{n}\mathbb{E}\left[\int_{t_k}^{t_{k+1}}\left|Z_s^{N,\varepsilon}-\bar{Z}_{t_k}^{N,\varepsilon}\right|^2+|X_s-X_{t_k}|^2+\left|Y_s^{N,\varepsilon}-Y_{t_{k+1}}^{N,\varepsilon}\right|^2ds\right]\\ &\quad\left.+h_0\mathbb{E}\int_0^{t_n}f(s,X_s,Y_s^{N,\varepsilon},Z_s^{N,\varepsilon})^2ds+\mathbb{E}\left|Y_{t_n}^{N,\varepsilon}-Y_{t_n}^{N,\varepsilon,n}\right|^2\right].\end{aligned}$$

A classical estimation gives us $\mathbb{E}\left[|X_s-X_{t_k}|^2\right]\leqslant|s-t_k|$. Moreover, since $Z^{N,\varepsilon}$ is bounded,

$$\begin{aligned}\mathbb{E}\int_0^{t_n}f(s,X_s,Y_s^{N,\varepsilon},Z_s^{N,\varepsilon})^2ds &\leqslant CT(1+\left|Y^{N,\varepsilon}\right|_\infty)+C\mathbb{E}\int_0^{t_n}\left|Z_s^{N,\varepsilon}\right|^4ds\\ &\leqslant CT(1+\left|Y^{N,\varepsilon}\right|_\infty)+Cn^{2b}\int_0^T\left|Z_s^{N,\varepsilon}\right|^2ds.\end{aligned}$$

But we have an a priori estimate for $\int_0^T\left|Z_s^{N,\varepsilon}\right|^2ds$ that does not depend on N and ε. So

$$\mathbb{E}\int_0^{t_n}f(s,X_s,Y_s^{N,\varepsilon},Z_s^{N,\varepsilon})^2ds\leqslant Cn^{2b}.$$

With the same type of argument we also have

$$\left|Y_s^{N,\varepsilon}-Y_{t_{k+1}}^{N,\varepsilon}\right|^2\leqslant Ch_kn^{2b}. \tag{4.11}$$

If we add Zhang's path regularity theorem 3.4, Proposition 4.5 and Proposition 4.6, we finally obtain

$$\mathbb{E}\left|Y_{t_k}^{N,\varepsilon}-Y_{t_k}^{N,\varepsilon,n}\right|^2\leqslant Cn^{(1+\eta)(K^2)^2a}\frac{n^{2b}\ln n}{n}=C\frac{\ln n}{n^{1-2b-(1+\eta)(K^2)^2a}}. \tag{4.12}$$

Now, let deal with the error on Z. First of all, (4.6) gives us

$$\sum_{k=0}^{n-1}\mathbb{E}\left[\int_{t_k}^{t_{k+1}}\left|Z_{t_k}^{N,\varepsilon,n}-Z_t^{N,\varepsilon}\right|^2dt\right]\leqslant\sum_{k=0}^{n-1}\mathbb{E}\left[\int_{t_k}^{t_{k+1}}\left|\tilde{Z}_{t_k}^{N,\varepsilon,n}-Z_t^{N,\varepsilon}\right|^2dt\right].$$

For $0\leqslant k\leqslant n-1$, we can use (4.7) and (4.8) to obtain

$$\begin{aligned}\mathbb{E}\left[\int_{t_k}^{t_{k+1}}\left|\tilde{Z}_{t_k}^{N,\varepsilon,n}-Z_t^{N,\varepsilon}\right|^2dt\right] &\leqslant \mathbb{E}\left[\int_{t_k}^{t_{k+1}}\left|\bar{Z}_{t_k}^{N,\varepsilon}-Z_t^{N,\varepsilon}\right|^2dt\right]\\ &\quad+(1+\eta)\mathbb{E}\left[\mathbb{E}_{t_k}(\left|Y_{t_{k+1}}^{N,\varepsilon}-Y_{t_{k+1}}^{N,\varepsilon,n}\right|^2)-\left|\mathbb{E}_{t_k}(Y_{t_{k+1}}^{N,\varepsilon}-Y_{t_{k+1}}^{N,\varepsilon,n})\right|^2\right]\\ &\quad+Ch_k\mathbb{E}\left[\int_{t_k}^{t_{k+1}}f(s,X_s,Y_s^{N,\varepsilon},Z_s^{N,\varepsilon})^2ds\right].\end{aligned}$$

Assumptions on f and estimation for Z give us

$$\begin{aligned}\sum_{k=0}^{n-1}\mathbb{E}\left[\int_{t_k}^{t_{k+1}}\left|Z_{t_k}^{N,\varepsilon,n}-Z_t^{N,\varepsilon}\right|^2dt\right] &\leqslant \sum_{k=0}^{n-1}\mathbb{E}\left[\int_{t_k}^{t_{k+1}}\left|\bar{Z}_{t_k}^{N,\varepsilon}-Z_t^{N,\varepsilon}\right|^2dt\right]\\ &\quad+(1+\eta)\sum_{k=0}^{n-1}\mathbb{E}\left[\mathbb{E}_{t_k}(\left|Y_{t_{k+1}}^{N,\varepsilon}-Y_{t_{k+1}}^{N,\varepsilon,n}\right|^2)-\left|\mathbb{E}_{t_k}(Y_{t_{k+1}}^{N,\varepsilon}-Y_{t_{k+1}}^{N,\varepsilon,n})\right|^2\right]\\ &\quad+Ch_0\mathbb{E}\left[\int_0^T 1+|Y|_\infty+n^{2b}\left|Z_s^{N,\varepsilon}\right|ds\right]\\ &\leqslant \sum_{k=0}^{n-1}\mathbb{E}\left[\int_{t_k}^{t_{k+1}}\left|\bar{Z}_{t_k}^{N,\varepsilon}-Z_t^{N,\varepsilon}\right|^2dt\right]\\ &\quad+(1+\eta)\sum_{k=0}^{n-1}\mathbb{E}\left[\mathbb{E}_{t_k}(\left|Y_{t_k}^{N,\varepsilon}-Y_{t_k}^{N,\varepsilon,n}\right|^2)-\left|\mathbb{E}_{t_k}(Y_{t_{k+1}}^{N,\varepsilon}-Y_{t_{k+1}}^{N,\varepsilon,n})\right|^2\right]\\ &\quad+(1+\eta)\mathbb{E}\left[\left|Y_{t_n}^{N,\varepsilon}-Y_{t_n}^{N,\varepsilon,n}\right|^2\right]+Ch_0n^{2b}, \qquad (4.13)\end{aligned}$$

with an index change in the penultimate sum. Then, by using (4.5) we get

$$\begin{aligned}&(1+\eta)\sum_{k=0}^{n-1}\mathbb{E}\left[\mathbb{E}_{t_k}(\left|Y_{t_k}^{N,\varepsilon}-Y_{t_k}^{N,\varepsilon,n}\right|^2)-\left|\mathbb{E}_{t_k}(Y_{t_{k+1}}^{N,\varepsilon}-Y_{t_{k+1}}^{N,\varepsilon,n})\right|^2\right]\\ \leqslant\ &(1+\eta)\gamma_kh_k\mathbb{E}\left|\mathbb{E}_{t_k}\left[Y_{t_{k+1}}^{N,\varepsilon}-Y_{t_{k+1}}^{N,\varepsilon,n}\right]\right|^2+(1+\eta)^2K_k^2(h_k+\frac{1}{\gamma_k})\mathbb{E}\int_{t_k}^{t_{k+1}}\left|Z_s^{N,\varepsilon}-Z_{t_k}^{N,\varepsilon,n}\right|^2ds\\ &+C(h_k+\frac{1}{\gamma_k})h_k\left(h_k+\sup_{s\in[t_k,t_{k+1}]}\mathbb{E}\left[\left|X_s-X_{t_k}^n\right|^2+\left|Y_s^{N,\varepsilon}-Y_{t_{k+1}}^{N,\varepsilon,n}\right|^2\right]\right). \qquad (4.14)\end{aligned}$$

Thanks to (4.9), (4.10), (4.11) and a classical estimation on $\mathbb{E}\left[|X_s-X_{t_k}|^2\right]$ we have

$$\sup_{s\in[t_k,t_{k+1}]}\mathbb{E}\left[\left|X_s-X_{t_k}^n\right|^2+\left|Y_s^{N,\varepsilon}-Y_{t_{k+1}}^{N,\varepsilon,n}\right|^2\right]\leqslant h_kn^{2b}+\mathbb{E}\left[\left|Y_{t_{k+1}}^{N,\varepsilon}-Y_{t_{k+1}}^{N,\varepsilon,n}\right|^2\right].$$

Let set $\gamma_k=3(1+\eta)^2K_k^2$. We recall that $h_kK_k^2\leqslant\frac{C\ln n}{n}\to 0$ when $n\to 0$. So, for n big enough, (4.14) become

$$\begin{aligned}&(1+\eta)\sum_{k=0}^{n-1}\mathbb{E}\left[\mathbb{E}_{t_k}(\left|Y_{t_k}^{N,\varepsilon}-Y_{t_k}^{N,\varepsilon,n}\right|^2)-\left|\mathbb{E}_{t_k}(Y_{t_{k+1}}^{N,\varepsilon}-Y_{t_{k+1}}^{N,\varepsilon,n})\right|^2\right]\\ \leqslant\ &\frac{C\ln n}{n}\mathbb{E}\left[\left|Y_{t_{k+1}}^{N,\varepsilon}-Y_{t_{k+1}}^{N,\varepsilon,n}\right|^2\right]+\frac{1}{2}\mathbb{E}\int_{t_k}^{t_{k+1}}\left|Z_s^{N,\varepsilon}-Z_{t_k}^{N,\varepsilon,n}\right|^2ds\\ &+Ch_k^2n^{2b}.\end{aligned}$$

If we inject this last estimation in (4.13) and we use Theorem 3.4, we obtain

$$\frac{1}{2}\sum_{k=0}^{n-1}\mathbb{E}\left[\int_{t_k}^{t_{k+1}}\left|Z_{t_k}^{N,\varepsilon,n}-Z_t^{N,\varepsilon}\right|^2dt\right]\ \leqslant\ Ch_0n^{2b}+C\ln n\sup_{0\leqslant k\leqslant n-1}\mathbb{E}\left[\left|Y_{t_{k+1}}^{N,\varepsilon}-Y_{t_{k+1}}^{N,\varepsilon,n}\right|^2\right].$$

By using (4.12) and Proposition 4.6, we finally have

$$\sup_{0\leqslant k\leqslant 2n}\mathbb{E}\left[\left|Y_{t_k}^{N,\varepsilon,n}-Y_{t_k}^{N,\varepsilon}\right|^2\right]+\sum_{k=0}^{2n-1}\mathbb{E}\left[\int_{t_k}^{t_{k+1}}\left|Z_{t_k}^{N,\varepsilon,n}-Z_t^{N,\varepsilon}\right|^2dt\right]\leqslant C\frac{(\ln n)^2}{n^{1-2b-Ka}},$$

with $K=4(1+\eta)L_{f,z}^2M_{z,2}^2$. Since this estimation is true for every $\eta>0$, we have prove the result. □

4.3 Studying of the global error $e(N,\varepsilon,n)$

Let us study errors $e_1(N)$ and $e_2(N,\varepsilon)$.

Proposition 4.8. *There exists a constant $C>0$ such that*

$$e_2(N,\varepsilon)\leqslant\frac{C}{n^{2a-4b}}.$$

Proof. We just have to notice that $\left|f(t,X_t,Y_t^{N,\varepsilon},Z_t^{N,\varepsilon})-f(t,X_t,Y_t^{N,\varepsilon},0)\right|\leqslant C\left|Z_t^{N,\varepsilon}\right|^2$ and $\left|Z_t^{N,\varepsilon}\right|$ is bounded by Cn^b. □

Proposition 4.9. *We assume that g is α-Hölder. Then, there exists a constant C such that*

$$e_1(N)\leqslant\frac{C}{n^{\frac{2b\alpha}{1-\alpha}}}$$

.

Proof. We set

$$g_N(x)=\inf\left\{g(u)+N\left|x-u\right|\,|\,u\in\mathbb{R}^d\right\}.$$

Then g_N is N-Lipschitz and $g_N\to g$ when $N\to+\infty$ uniformly on $\mathbb{R}^d$. More precisely, we have

$$|g-g_N|_\infty\leqslant\frac{C}{N^{\frac{\alpha}{1-\alpha}}}.$$

□

Remark 4.10. *For some explicit examples, it is possible to have a better convergence speed. For example, let take $g(x)=(|x|^\alpha\,\mathbb{1}_{x\geqslant 0})\wedge C$ and assume that σ is invertible. Then, we can use the fact that this function is not Lipschitz only in 0, and obtain*

$$e_1(N)\leqslant\frac{C}{n^{\frac{2\alpha b}{1-\alpha}}}\mathbb{P}\left(X_T\in\left[0,N^{\frac{-1}{1-\alpha}}\right]\right)^{1/q}\leqslant\frac{C}{n^{\frac{b}{1-\alpha}\left(2\alpha+\frac{1}{q}\right)}}$$

.

Remark 4.11. *It is also possible to obtain convergence speed when g is not α-Hölder. For example, we assume that σ is invertible and we set $g(x)=\prod_{i=1}^d\mathbb{1}_{x_i>0}(x)$. Then*

$$e_1(N)\leqslant C\left[\sum_{i=1}^d\mathbb{P}((X_T)_i\in[0,1/N])\right]^{1/q}\leqslant\frac{C}{N^{1/q}}=\frac{C}{n^{b/q}}.$$

Now we are able to gather all these errors.

Theorem 4.12. *We assume that (HX0), (HY0), (HY1), and (HX1) or (HX1') hold. We assume also that g is α-Hölder. Then for all $\eta>0$, there exists a constant $C>0$ that does not depend on n such that*

$$e(n):=e(N,\varepsilon,n)\leqslant\frac{C}{n^{\frac{2\alpha}{(2-\alpha)(2+K)-2+2\alpha}}},$$

with $K=4(1+\eta)L_{f,z}^2M_{z,2}^2$.

Proof. Thanks to Theorem 4.7, Proposition 4.8 and Proposition 4.9 we have

$$e(n) \leqslant \frac{C}{n^{1-2b-Ka}} + \frac{C}{n^{2a-4b}} + \frac{C}{n^{\frac{2\alpha b}{1-\alpha}}}.$$

Then we just have to set $a := \frac{1+2b}{2+K}$ and $b := \frac{1-\alpha}{(2-\alpha)(2+K)-2+2\alpha}$ to obtain the result. □

Corollary 4.13. *We assume that assumptions of Theorem 4.12 hold. Moreover we assume that f has a sub-quadratic growth with respect to z: there exists $0 < \beta < 1$ such that, for all $t \in [0,T]$, $x \in \mathbb{R}^d$, $y \in R$, $z, z' \in \mathbb{R}^{1\times d}$,*

$$\left|f(t,x,y,z) - f(t,x,y,z')\right| \leqslant (K_{f,z} + L_{f,z}(|z|^\beta + \left|z'\right|^\beta))\left|z - z'\right|.$$

Then we are allowed to take K as small as we want. So, for all $\eta > 0$, there exists a constant $C > 0$ that does not depend on n such that

$$e(n) \leqslant \frac{C}{n^{\alpha-\eta}}.$$

Remark 4.14. *When we are allowed to take K as small as we want, then we have $\varepsilon = n^{-a} < h_0$ for K sufficiently small. So we do not need to have a discretization grid on $[T-\varepsilon, T]$: $n+2$ points of discretization are sufficient on $[0,T]$.*

Theorem 4.12 is not interesting in practice because the speed of convergence strongly depends on K. But, we just see that the global error becomes $e(n) \leqslant \frac{C}{n^{\alpha-\eta}}$ when we are allowed to choose K as small as we want. Under extra assumption we can show that we are allowed to take the constant $M_{z,2}$ as small as we want.

(HX2). b is bounded on $[0,T] \times \mathbb{R}^d$ by a constant M_b.

Theorem 4.15. *We assume that (HX0), (HY0), (HY1), (HX2) and (HX1) or (HX1') hold. We assume also that g is α-Hölder. Then for all $\eta > 0$, there exists a constant $C > 0$ that does not depend on n such that*

$$e(n) \leqslant \frac{C}{n^{\alpha-\eta}}.$$

Remark 4.16. *With assumptions of the previous theorem, it is also possible to have an estimation of the global error for examples given in remarks 4.10 and 4.11. When $g(x) = (|x|^\alpha \mathbb{1}_{x\geqslant 0}) \wedge C$, we have*

$$e(n) \leqslant \frac{C}{n^{\alpha+\frac{1-\alpha}{1+2q}-\eta}},$$

and when $g(x) = \prod_{i=1}^d \mathbb{1}_{x_i>0}(x)$, we have

$$e(n) \leqslant \frac{C}{n^{\frac{1}{1+2q}-\eta}}.$$

Proof. Firstly, we suppose that f is differentiable with respect to z. Thanks to remark 3.3 we see that it is sufficient to show that

$$\mathbb{E}^{\mathbb{Q}^{N,\varepsilon}}\left[\int_t^T \left|Z_s^{N,\varepsilon}\right|^2 ds \middle| \mathcal{F}_t\right]$$

is small uniformly on Ω, in N and ε when t is close to T.We will obtain an estimation for this quantity by applying the same computation than [2] for the BMO norm estimate of Z page 831. Thus we have

$$\mathbb{E}^{\mathbb{Q}^{N,\varepsilon}}\left[\int_t^T \left|Z_s^{N,\varepsilon}\right|^2 ds \middle| \mathcal{F}_t\right] \leqslant \mathbb{E}^{\mathbb{Q}^{N,\varepsilon}}\left[\left|\varphi(Y_T^{N,\varepsilon}) - \varphi(Y_t^{N,\varepsilon})\right| \middle| \mathcal{F}_t\right] + C(T-t),$$

with $\varphi(x) = (e^{2c(x+m)} - 2c(x+m) - 1)/(2c^2)$, $m = |Y|_\infty$ and c that depends on constants in assumption (HY0) but does not depend on $\nabla_z f$. Let notice that m, c and so φ do not depend on N and ε. Since Y is bounded, φ is a Lipschitz function, so

$$\mathbb{E}^{\mathbb{Q}^{N,\varepsilon}}\left[\int_t^T \left|Z_s^{N,\varepsilon}\right|^2 ds \middle| \mathcal{F}_t\right] \leqslant C\mathbb{E}^{\mathbb{Q}^{N,\varepsilon}}\left[\left|Y_T^{N,\varepsilon} - Y_t^{N,\varepsilon}\right| \middle| \mathcal{F}_t\right] + C(T-t).$$

We denote $(Y^{N,\varepsilon,t,x}, Z^{N,\varepsilon,t,x})$ the solution of BSDE (4.1) when $X_t^{t,x} = x$. As usual, we set $X_s^{t,x} = x$ and $Z_s^{N,\varepsilon,t,x} = 0$ for $s \leqslant t$ and we define $u^{N,\varepsilon}(t,x) := Y_t^{N,\varepsilon,t,x}$. Then we give a proposition that we will prove in the appendix.

Proposition 4.17. *We assume that (HX0), (HY0), (HY1), (HX2) and (HX1) or (HX1') hold. We assume also that g is uniformly continuous on $\mathbb{R}^d$. Then $u^{N,\varepsilon}$ is uniformly continuous on $[0,T] \times \mathbb{R}^d$ and their exists ω a concave modulus of continuity for all functions in $\left\{u^{N,\varepsilon} | N \in \mathbb{N}, \varepsilon > 0\right\}$: that is to say, ω does not depend on N and ε.*

Then

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}^{N,\varepsilon}}\left[\left|Y_T^{N,\varepsilon} - Y_t^{N,\varepsilon}\right| \middle| \mathcal{F}_t\right] &= \mathbb{E}^{\mathbb{Q}^{N,\varepsilon}}\left[\left|u^{N,\varepsilon}(T, X_T) - u^{N,\varepsilon}(t, X_t)\right| \middle| \mathcal{F}_t\right] \\
&\leqslant \mathbb{E}^{\mathbb{Q}^{N,\varepsilon}}\left[\mathbb{1}_{\left|\int_t^T \sigma(s) d\tilde{W}_s\right| \leqslant \nu} \left|u^{N,\varepsilon}(T, X_T) - u^{N,\varepsilon}(t, X_t)\right|\right. \\
&\quad \left. + 2\left|Y^{N,\varepsilon}\right|_\infty \mathbb{1}_{\left|\int_t^T \sigma(s) d\tilde{W}_s\right| > \nu} \middle| \mathcal{F}_t\right] \\
&\leqslant \mathbb{E}^{\mathbb{Q}^{N,\varepsilon}}\left[\omega\left(|T-t| + \mathbb{1}_{\left|\int_t^T \sigma(s) d\tilde{W}_s\right| \leqslant \nu} |X_T - X_t|\right)\right. \\
&\quad \left. + 2\left|Y^{N,\varepsilon}\right|_\infty \mathbb{1}_{\left|\int_t^T \sigma(s) d\tilde{W}_s\right| > \nu} \middle| \mathcal{F}_t\right],
\end{aligned}$$

with $d\tilde{W}_s = dW_s - \nabla_z f^\varepsilon(s, X_s, Y_s^{N,\varepsilon}, Z_s^{N,\varepsilon})ds$. But,

$$\begin{aligned}
& \mathbb{1}_{\left|\int_t^T \sigma(s) d\tilde{W}_s\right| \leqslant \nu} |X_T - X_t| \\
= \; & \mathbb{1}_{\left|\int_t^T \sigma(s) d\tilde{W}_s\right| \leqslant \nu} \left|\int_t^T b(s, X_s) ds + \int_t^T \nabla_z f^\varepsilon(s, X_s, Y_s^{N,\varepsilon}, Z_s^{N,\varepsilon}) ds + \int_t^T \sigma(s) d\tilde{W}_s\right| \\
\leqslant \; & M_b(T-t) + \nu + C\int_t^T (1 + \left|Z_s^{N,\varepsilon}\right|) ds \\
\leqslant \; & C(T-t) + \nu + C(T-t)^{1/2}\left(\int_t^T \left|Z_s^{N,\varepsilon}\right|^2 ds\right)^{1/2}.
\end{aligned}$$

Since ω is concave, we have by Jensen's inequality

$$\begin{aligned}
& \mathbb{E}^{\mathbb{Q}^{N,\varepsilon}}\left[\omega\left(|T-t| + \mathbb{1}_{\left|\int_t^T \sigma(s) d\tilde{W}_s\right| \leqslant \nu} |X_T - X_t|\right) \middle| \mathcal{F}_t\right] \\
\leqslant \; & \omega\left(C|T-t| + \nu + C(T-t)^{1/2}\mathbb{E}^{\mathbb{Q}^{N,\varepsilon}}\left[\left(\int_t^T \left|Z_s^{N,\varepsilon}\right|^2 ds\right)^{1/2} \middle| \mathcal{F}_t\right]\right) \\
\leqslant \; & \omega\left(C|T-t| + \nu + C(T-t)^{1/2}\mathbb{E}^{\mathbb{Q}^{N,\varepsilon}}\left[\int_t^T \left|Z_s^{N,\varepsilon}\right|^2 ds \middle| \mathcal{F}_t\right]^{1/2}\right) \\
\leqslant \; & \omega\left(C|T-t| + \nu + C(T-t)^{1/2}\left\|Z^{N,\varepsilon}\right\|_{BMO(\mathbb{Q})}\right).
\end{aligned}$$

But, $\left\|Z^{N,\varepsilon}\right\|_{BMO(\mathbb{Q})}$ only depends on constants in assumption (HY0), so it is bounded uniformly in N and ε. Moreover, $\left|\int_t^T \sigma(s)dW_s\right|$ is independent of $\mathcal{F}_t$ so we have by Bienaymé-Tchebychev inequality

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}^{N,\varepsilon}}\left[\mathbb{1}_{\left|\int_t^T \sigma(s)d\tilde{W}_s\right|>\nu}\middle|\mathcal{F}_t\right] &= \mathbb{Q}^{N,\varepsilon}\left(\left|\int_t^T \sigma(s)d\tilde{W}_s\right|>\nu\right)\\ &\leqslant \frac{\int_t^T {}^t\sigma(s)\sigma(s)ds}{\nu^2}\\ &\leqslant \frac{M_\sigma^2(T-t)}{\nu^2},\end{aligned}$$

Finally, we have

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}^{N,\varepsilon}}\left[\left|Y_T^{N,\varepsilon}-Y_t^{N,\varepsilon}\right|\middle|\mathcal{F}_t\right] &\leqslant \omega\left(C\left|T-t\right|^{1/2}+\nu\right)+C\frac{(T-t)}{\nu^2}\\ &\leqslant \omega\left(C\left|T-t\right|^{1/2}+\left|T-t\right|^{1/4}\right)+C\left|T-t\right|^{1/2},\end{aligned}$$

by setting $\nu = |T-t|^{1/4}$, and $\mathbb{E}^{\mathbb{Q}^{N,\varepsilon}}\left[\left|Y_T^{N,\varepsilon}-Y_t^{N,\varepsilon}\right|\middle|\mathcal{F}_t\right] \to 0$ uniformly on Ω, in N and ε when $t \to T$. When f is not differentiable with respect to z but is only locally Lipschitz then we can prove the result by a standard approximation. □

A Appendix

A.1 Proof of Lemma 4.3.

We have,

$$\prod_{i=0}^{2n-1}(1+Mh_i) = \left(\prod_{i=0}^{n-1}(1+Mh_i)\right)\left(\prod_{i=n}^{2n-1}(1+Mh_i)\right).$$

Firstly,

$$\prod_{i=n}^{2n-1}(1+Mh_i) \leqslant \left(1+M\frac{T}{n}\right)^n \leqslant C.$$

Moreover, for $0 \leqslant i \leqslant n-1$,

$$h_i = t_{i+1}-t_i = Tn^{-ai/n}(1-e^{-\frac{a\ln n}{n}}) \leqslant Tn^{-ai/n}a\frac{\ln n}{n},$$

thanks to the convexity of the exponential function. So

$$\begin{aligned}\prod_{i=0}^{n-1}(1+Mh_i) &\leqslant \prod_{i=0}^{n-1}\left(1+MTan^{-ai/n}\frac{\ln n}{n}\right)\\ &\leqslant \exp\left(\sum_{i=0}^{n-1}\ln\left(1+MTan^{-ai/n}\frac{\ln n}{n}\right)\right)\\ &\leqslant \exp\left(\sum_{i=0}^{n-1}MTa(n^{-a/n})^i\frac{\ln n}{n}\right)\\ &\leqslant \exp\left(MTa\frac{\ln n}{n}\left(\frac{1-(1/n^a)}{1-(1/n^{(a/n)})}\right)\right)\\ &\leqslant \exp\left(MTa\frac{\ln n}{n}\frac{n^{a/n}}{n^{a/n}-1}\right).\end{aligned}$$

But,

$$\frac{\ln n}{n}\frac{n^{a/n}}{n^{a/n}-1} \sim \frac{\ln n}{n}\frac{1}{a\frac{\ln n}{n}} \sim \frac{1}{a}.$$

Thus, we have shown the result. □

A.2 Proof of Lemma 4.4.

Thanks to Lemma 4.3, we have

$$\frac{\prod_{i=0}^{n-1}\left(1+M_1h_i+M_2\frac{h_i}{T-t_{i+1}}\right)}{\prod_{i=0}^{n-1}\left(1+M_2\frac{h_i}{T-t_{i+1}}\right)} \leqslant \prod_{i=0}^{n-1}\left(1+\frac{M_1}{1+M_2\frac{h_i}{T-t_{i+1}}}h_i\right) \leqslant \prod_{i=0}^{n-1}(1+M_1h_i) \leqslant C.$$

So, we just have to show that

$$\prod_{i=0}^{n-1}\left(1+M_2\frac{h_i}{T-t_{i+1}}\right) \leqslant Cn^{aM_2}.$$

But,

$$1+M_2\frac{h_i}{T-t_{i+1}} = 1+M_2(n^{a/n}-1).$$

So,

$$\begin{aligned}\prod_{i=0}^{n-1}\left(1+M_2\frac{h_i}{T-t_{i+1}}\right) &= \left(1+M_2(n^{a/n}-1)\right)^n \\ &= \exp\left(n\ln\left(1+aM_2\frac{\ln n}{n}+O\left(\frac{\ln^2 n}{n^2}\right)\right)\right) \\ &= \exp\left(aM_2\ln n+O\left(\frac{\ln^2 n}{n}\right)\right) \sim n^{aM_2}.\end{aligned}$$

Thus, we have shown the result. □

A.3 Proof of Proposition 4.17.

We will prove this proposition as the author of [7] do for Proposition 4.2. In all the proof we omit the superscript N,ε for u, Y and Z to be more readable. Let $x_0, x_0' \in \mathbb{R}^d$ and $t_0, t_0' \in [0,T]$. By an argument of symmetry we are allowed to suppose that $t_0 \leqslant t_0'$. We have

$$\left|u(t_0,x_0)-u(t_0',x_0')\right| \leqslant \left|u(t_0,x_0)-u(t_0,x_0')\right| + \left|u(t_0,x_0')-u(t_0',x_0')\right|.$$

Let us begin with the first term. We will use a classical argument of linearization:

$$\begin{aligned}Y_t^{t_0,x_0}-Y_t^{t_0,x_0'} &= g_N(X_T^{t_0,x_0})-g_N(X_T^{t_0,x_0'})+\int_t^T \alpha_s\left(X_s^{t_0,x_0}-X_s^{t_0,x_0'}\right)+\beta_s\left(Y_s^{t_0,x_0}-Y_s^{t_0,x_0'}\right)ds \\ &\quad -\int_t^T\left(Z_s^{t_0,x_0}-Z_s^{t_0,x_0'}\right)d\tilde{W}_s,\end{aligned}$$

with

$$\alpha_s = \begin{cases}\dfrac{f^\varepsilon(s,X_s^{t_0,x_0},Y_s^{t_0,x_0'},Z_s^{t_0,x_0'})-f^\varepsilon(s,X_s^{t_0,x_0'},Y_s^{t_0,x_0'},Z_s^{t_0,x_0'})}{X_s^{t_0,x_0}-X_s^{t_0,x_0'}} & \text{if } X_s^{t_0,x_0}-X_s^{t_0,x_0'}\neq 0, \\ 0 & \text{elsewhere,}\end{cases}$$

$$\beta_s = \begin{cases} \dfrac{f^\varepsilon(s, X_s^{t_0,x_0}, Y_s^{t_0,x_0}, Z_s^{t_0,x_0'}) - f^\varepsilon(s, X_s^{t_0,x_0}, Y_s^{t_0,x_0'}, Z_s^{t_0,x_0'})}{Y_s^{t_0,x_0} - Y_s^{t_0,x_0'}} & \text{if } Y_s^{t_0,x_0} - Y_s^{t_0,x_0'} \neq 0, \\ 0 & \text{elsewhere,} \end{cases}$$

$$\gamma_s = \begin{cases} \dfrac{f^\varepsilon(s, X_s^{t_0,x_0}, Y_s^{t_0,x_0}, Z_s^{t_0,x_0}) - f^\varepsilon(s, X_s^{t_0,x_0}, Y_s^{t_0,x_0}, Z_s^{t_0,x_0'})}{\left|Z_s^{t_0,x_0} - Z_s^{t_0,x_0'}\right|^2}{}^t (Z_s^{t_0,x_0} - Z_s^{t_0,x_0'}) & \text{if } Z_s^{t_0,x_0} - Z_s^{t_0,x_0'} \neq 0, \\ 0 & \text{elsewhere,} \end{cases}$$

and $d\tilde{W}_s := dW_s - \gamma_s ds$. By a BMO argument, there exists a probability $\mathbb{Q}$ under which $\tilde{W}$ is a brownian motion. Then we apply a classical transformation to obtain

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}\left[e^{\int_{t_0}^t \beta_s ds}\left(Y_t^{t_0,x_0} - Y_t^{t_0,x_0'}\right)\right] &= \mathbb{E}^{\mathbb{Q}}\left[e^{\int_{t_0}^T \beta_s ds}\left(g_N(X_T^{t_0,x_0}) - g_N(X_T^{t_0,x_0'})\right)\right. \\ &\quad \left. + \int_{t_0}^T \alpha_s e^{\int_{t_0}^s \beta_u du}\left(X_s^{t_0,x_0} - X_s^{t_0,x_0'}\right) ds\right], \end{aligned}$$

and

$$\left|u(t_0,x_0) - u(t_0,x_0')\right| \leqslant C\left(\mathbb{E}^{\mathbb{Q}}\left[\omega\left(\left|X_T^{t_0,x_0} - X_T^{t_0,x_0'}\right|\right)\right] + \int_{t_0}^T \mathbb{E}^{\mathbb{Q}}\left[\left|X_s^{t_0,x_0} - X_s^{t_0,x_0'}\right|\right] ds\right),$$

with ω a modulus of continuity of g that is also a modulus of continuity for g_N. We are allowed to suppose that ω is concave[2], so Jensen's inequality gives us

$$\left|u(t_0,x_0) - u(t_0,x_0')\right| \leqslant C\left(\omega\left(\mathbb{E}^{\mathbb{Q}}\left[\left|X_T^{t_0,x_0} - X_T^{t_0,x_0'}\right|\right]\right) + \int_{t_0}^T \mathbb{E}^{\mathbb{Q}}\left[\left|X_s^{t_0,x_0} - X_s^{t_0,x_0'}\right|\right] ds\right).$$

By using the fact that b is bounded we can prove the following lemma exactly as authors of [7] do for their Proposition 4.7:

Proposition A.1. *$\exists C > 0$ that does not depend on N and ε such that $\forall t, t' \in [0,T]$, $\forall x, x' \in \mathbb{R}^d$, $\forall s \in [0,T]$,*

$$\mathbb{E}^{\mathbb{Q}}\left[\left|X_s^{t,x} - X_s^{t',x'}\right|\right] \leqslant C\left(|x - x'| + |t - t'|^{1/2}\right).$$

Then,

$$\left|u(t_0,x_0) - u(t_0,x_0')\right| \leqslant C\left(\omega\left(\left|x_0 - x_0'\right|\right) + \left|x_0 - x_0'\right|\right).$$

Now we will study the second term:

$$\left|u(t_0,x_0') - u(t_0',x_0')\right| = \left|Y_{t_0}^{t_0,x_0'} - Y_{t_0'}^{t_0',x_0'}\right| \leqslant \left|Y_{t_0}^{t_0,x_0'} - Y_{t_0}^{t_0',x_0'}\right| + \left|Y_{t_0}^{t_0',x_0'} - Y_{t_0'}^{t_0',x_0'}\right|.$$

Firstly,

$$\left|Y_{t_0}^{t_0',x_0'} - Y_{t_0'}^{t_0',x_0'}\right| \leqslant \left|\int_{t_0}^{t_0'} f(s, x_0', Y_s^{t_0',x_0'}, 0) ds\right| \leqslant C\left|t_0 - t_0'\right|.$$

[2]There exist two positive constants a and b such that $\omega(x) \leqslant ax + b$. Then the concave hull of $x \mapsto \omega(x) \vee (\mathbb{1}_{x \geqslant 1}(ax + b))$ is a also a modulus of continuity of g.

Moreover, as for the first term we have

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\left[e^{\int_{t_0}^{t}\beta_s ds}\left(Y_t^{t_0,x_0'}-Y_t^{t_0',x_0'}\right)\right] &= \mathbb{E}^{\mathbb{Q}}\left[e^{\int_{t_0}^{T}\beta_s ds}\left(g_N(X_T^{t_0,x_0'})-g_N(X_T^{t_0',x_0'})\right)\right.\\ &\left.+\int_{t_0}^{T}\alpha_s e^{\int_{t_0}^{s}\beta_u du}\left(X_s^{t_0,x_0'}-X_s^{t_0',x_0'}\right)ds\right],\end{aligned}$$

and

$$\left|Y_{t_0}^{t_0,x_0'}-Y_{t_0'}^{t_0',x_0'}\right| \leqslant C\left(\omega\left(\left|t_0-t_0'\right|^{1/2}\right)+\left|t_0-t_0'\right|^{1/2}\right).$$

Finally,

$$\left|u(t_0,x_0')-u(t_0',x_0')\right| \leqslant C\left(\omega\left(\left|t_0-t_0'\right|^{1/2}\right)+\left|t_0-t_0'\right|^{1/2}\right),$$

and

$$\left|u(t_0,x_0)-u(t_0',x_0')\right| \leqslant C\left(\omega\left(\left|x_0-x_0'\right|\right)+\omega\left(\left|t_0-t_0'\right|^{1/2}\right)+\left|x_0-x_0'\right|+\left|t_0-t_0'\right|^{1/2}\right).$$

So u is uniformy continuous on $[0,T]\times\mathbb{R}^d$ and this function has a modulus of continuity that does not depend on N and ε. Moreover, we are allowed to suppose that this modulus of continuity is concave. □

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