



HAL
open science

A new dynamical approach of Emden-Fowler equations and systems

Marie-Françoise Bidaut-Véron, Hector Giacomini

► **To cite this version:**

Marie-Françoise Bidaut-Véron, Hector Giacomini. A new dynamical approach of Emden-Fowler equations and systems. 2010. hal-00443694v1

HAL Id: hal-00443694

<https://hal.science/hal-00443694v1>

Preprint submitted on 3 Jan 2010 (v1), last revised 23 Feb 2010 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A new dynamical approach of Emden-Fowler equations and systems

Marie Françoise BIDAUT-VERON*

Hector GIACOMINI†

Abstract

We give a new approach on general systems of the form

$$(G) \begin{cases} -\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u) = \varepsilon_1 |x|^a u^s v^\delta, \\ -\Delta_q v = \operatorname{div}(|\nabla v|^{q-2} \nabla v) = \varepsilon_2 |x|^b u^\mu v^m, \end{cases}$$

where $Q, p, q, \delta, \mu, s, m, a, b$ are real parameters, $Q, p, q \neq 1$, and $\varepsilon_1 = \pm 1, \varepsilon_2 = \pm 1$. In the radial case we reduce the problem to a quadratic system of order 4, of Kolmogorov type. Then we obtain new local and global existence or nonexistence results. In the case $\varepsilon_1 = \varepsilon_2 = 1$, we also describe the behaviour of the ground states in two cases where the system is variational. We give an important result on existence of ground states for a nonvariational system with $p = q = 2$ and $s = m > 0$. In the nonradial case we solve a conjecture of nonexistence of ground states for the system with $p = q = 2$ and $\delta = m + 1$ and $\mu = s + 1$.

*Laboratoire de Mathématiques et Physique Théorique, CNRS UMR 6083, Faculté des Sciences, 37200 Tours France. E-mail address:veronmf@univ-tours.fr

†Laboratoire de Mathématiques et Physique Théorique, CNRS UMR 6083, Faculté des Sciences, 37200 Tours France. E-mail address:Hector.Giacomini@lmpt.univ-tours.fr

1 Introduction

In this paper we consider the nonnegative solutions of Emden-Fowler equations or systems in $\mathbb{R}^N (N \geq 1)$,

$$-\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u) = \varepsilon_1 |x|^a u^Q, \quad (1.1)$$

$$(G) \begin{cases} -\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u) = \varepsilon_1 |x|^a u^s v^\delta, \\ -\Delta_q v = \operatorname{div}(|\nabla v|^{q-2} \nabla v) = \varepsilon_2 |x|^b u^\mu v^m, \end{cases} \quad (1.2)$$

where $Q, p, q, \delta, \mu, s, m, a, b$ are real parameters, $Q, p, q \neq 1$, and $\varepsilon_1 = \pm 1, \varepsilon_2 = \pm 1$.

In Section 2 we give a new way of studying the radial solutions, reducing system (G) to a quadratic autonomous system:

$$(M) \begin{cases} X_t = X \left[X - \frac{N-p}{p-1} + \frac{Z}{p-1} \right], \\ Y_t = Y \left[Y - \frac{N-q}{q-1} + \frac{W}{q-1} \right], \\ Z_t = Z [N + a - sX - \delta Y - Z], \\ W_t = W [N + b - \mu X - mY - W], \end{cases}$$

where $t = \ln r$, and

$$X(t) = -\frac{ru'}{u}, \quad Y(t) = -\frac{rv'}{v}, \quad Z(t) = -\varepsilon_1 r^{1+a} u^s v^\delta \frac{u'}{|u'|^p}, \quad W(t) = -\varepsilon_2 r^{1+b} u^\mu v^m \frac{v'}{|v'|^q}. \quad (1.3)$$

This system is of Kolmogorov type. The reduction is valid for equations and systems with source terms ($\varepsilon_1 = \varepsilon_2 = 1$), absorption terms ($\varepsilon_1 = \varepsilon_2 = -1$), or mixed systems ($\varepsilon_1 = -\varepsilon_2 = 1$). It is remarkable that in the new system, p and q appear only as simple coefficients, which allows to treat any value of the parameters, for example p or $q < 1$, s, m, δ or $\mu < 0$.

In Section 3 we study the scalar case (1.1), where (G) becomes two-dimensional. We show that the phase plane of system (M_G) gives at the same time the behaviour of the two equations

$$-\Delta_p u = |x|^a u^Q \quad \text{and} \quad -\Delta_p u = -|x|^a u^Q,$$

which is a kind of unification of the two problems, with source terms or absorption terms. For the case of source term ($\varepsilon_1 = 1$), we find again the results of [2], [19], showing that the new dynamical approach is simple and does not need regularity results or energy functions. Moreover it gives a model for the study of the system (G). Indeed if $p = q$, $a = b$ and $\delta + s = \mu + m$, the system (G) admits solutions of the form (u, u) , where u is a solution of (1.1) with $Q = \delta + s$.

In the sequel of the article we apply the reduction to the case of source terms: $(G) = (S)$, where

$$(S) \begin{cases} -\Delta_p u = |x|^a u^s v^\delta, \\ -\Delta_q v = |x|^b u^\mu v^m. \end{cases} \quad (1.4)$$

This system has been the subject of a very rich litterature, in particular the Hamiltonian problem $s = m = 0$, in the linear case $p = q = 2$, see or example [20], [31], [29], [9], [33], [14], and the potential system where $\delta = m + 1$, $\mu = s + 1$ and $a = b$, see [7], [34], [35]; the problem with general powers has been studied in [3], [39], [40], [41] in the linear case and [6], [12], [42] in the quasilinear case, see also [1], [10], [13].

Here we suppose that $\delta, \mu > 0$, so that the system is always coupled, $s, m \geq 0$, and we assume for simplicity

$$1 < p, q < N, \quad \min(p + a, q + b) > 0, \quad D = \delta\mu - (p - 1 - s)(q - 1 - m) > 0 \quad (1.5)$$

We say that a positive solution $(u, v) \in (C^2((0, R)))^2$ is regular at 0 if u, v are continuous at 0; the condition $\min(p + a, q + b) > 0$ guaranties the existence of local regular solutions; then u, v are C^1 when $a, b > -1$, with $u'(0) = v'(0) = 0$. We are interested in the existence or nonexistence of ground states, called G.S., that means global positive (u, v) in $(0, \infty)$ and regular at 0. We exclude the case of "trivial" solutions, $(u, v) = (0, C)$ or $(C, 0)$, where C is a constant, which can exist when $s > 0$ or $m > 0$. The assumption $D > 0$ is a classical condition of superlinearity for the system.

In Section 4 we give a series of local existence or nonexistence results concerning the system (S) , which complete the nonexistence results found in the litterature. They are not based on the fixed point method, quite hard in general, see for example [19], [27], but on a dynamical analysis of the linearization of system (M_S) near each fixed point, which appears to be performant, even for the regular solutions. For a better exposition, the proofs are given at Section 10.

In Section 5 we study the global existence of G.S. It has been often compared with the nonexistence of positive solutions of the Dirichlet problem in a ball, see [29], [30], [12], [13]. Here we use a shooting method adapted to system (M_S) , which allows to avoid questions of regularity of system (S) . We give a new way of comparison, and improve the former results:

Theorem 1.1 (i) Assume $s < \frac{N(p-1)+p+pa}{N-p}$, $m < \frac{N(q-1)+q+qb}{N-q}$. If system (S) has no G.S., then

(i) there exist regular solutions such that $X(T) = \frac{N-p}{p-1}$ and $Y(T) = \frac{N-q}{q-1}$ for some $T > 0$, with $0 < X < \frac{N-p}{p-1}$ and $0 < Y < \frac{N-q}{q-1}$ on $(-\infty, T)$.

(ii) there exists a positive radial solution (u, v) of the Dirichlet problem in a ball $B(0, R)$.

It is a key tool in the next Sections for proving the existence of a G.S. It gives also new existence results for the Dirichlet problem, see Corollary 5.3. We also give a complementary result:

Proposition 1.2 Assume $s \geq \frac{N(p-1)+p+pa}{N-p}$ and $m \geq \frac{N(q-1)+q+qb}{N-q}$. Then all the regular solutions are G.S.

In Section 6 we study the radial solutions of the well known Hamiltonian system

$$(SH) \begin{cases} -\Delta u = |x|^a v^\delta, \\ -\Delta v = |x|^b u^\mu, \end{cases}$$

where $p = q = 2$, $s = m = 0$, which is variational. In the case $a = b = 0$, a main conjecture was made:

Conjecture 1.3 *The system (SH) with $a = b = 0$ admits no (radial or nonradial) G.S. if and only if (δ, μ) is under the hyperbola of equation*

$$\frac{N}{\delta + 1} + \frac{N}{\mu + 1} = N - 2.$$

The question is still open; it was solved in the radial case in [26], [29], then partially in [31], [9], and up to the dimension $N = 4$ in [33], see references therein. Here we find again and extend to the case $a, b \neq 0$ some results of [20] relative to the G.S., with a shorter proof. We also give an existence result for the Dirichlet problem improving a result of [14].

Theorem 1.4 *Let \mathcal{H}_0 be the critical hyperbola in the plane (δ, μ) defined by*

$$\frac{N + a}{\delta + 1} + \frac{N + b}{\mu + 1} = N - 2. \quad (1.6)$$

Then the system (SH) admits a (unique) G.S. if and only if (δ, μ) is above \mathcal{H}_0 or on \mathcal{H}_0 . The radial Dirichlet problem in a ball has a solution if and only if (δ, μ) is under \mathcal{H}_0 . On \mathcal{H}_0 the G.S. has the following behaviour at ∞ : assuming for example $\delta > \frac{N+a}{N-2}$, then $\lim_{r \rightarrow \infty} r^{N-2}u(r) = \alpha > 0$, and

$$\begin{aligned} \lim_{r \rightarrow \infty} r^{(N-2)\mu - (2+b)}v &= \beta > 0 & \text{if } \mu < \frac{N+b}{N-2}, \\ \lim_{r \rightarrow \infty} r^{N-2}v &= \beta > 0 & \text{if } \mu > \frac{N+b}{N-2}, \\ \lim_{r \rightarrow \infty} r^{N-2}|\ln r|^{-1}v &= \beta > 0 & \text{if } \mu = \frac{N+b}{N-2}. \end{aligned}$$

Observe that the G.S. can present a non-symmetric behaviour. Our proofs use a Pohozaev type function; notice that it contains a quadratic factor in X, Y, Z, W :

$$\mathcal{E}(r) = r^N \left[u'v' + r^b \frac{|u|^{\mu+1}}{\mu+1} + r^a \frac{|v|^{\delta+1}}{\delta+1} + \frac{N+a}{\delta+1} \frac{vu'}{r} + \frac{N+b}{\mu+1} \frac{uv'}{r} \right] \quad (1.7)$$

$$= r^{N-2}uv \left[XY - \frac{Y(N+b-W)}{\mu+1} - \frac{(N+a-Z)X}{\delta+1} \right]. \quad (1.8)$$

In Section 7 we consider the radial solutions of a nonvariational system:

$$(SN) \begin{cases} -\Delta u = |x|^a u^s v^\delta, \\ -\Delta v = |x|^a u^\mu v^s, \end{cases}$$

where $p = 2$, $a = b$ and $m = s > 0$. For small s it appears as a perturbation of system (SH) . Here we give a new result of *existence of G.S.* valid for any s . It is obtained by constructing a new kind of energy function by adding two terms in X^2, Y^2 to the preceding one:

$$\Phi(r) = r^N \left[u'v' + r^b \frac{|u|^{\mu+1}}{\mu+1} + r^a \frac{|v|^{\delta+1}}{\delta+1} + \frac{N+a}{\delta+1} \frac{vv'}{r} + \frac{N+b}{\mu+1} \frac{uv'}{r} + \frac{s}{2(\delta+1)} \frac{vu'^2}{u} + \frac{s}{2(\mu+1)} \frac{uv'^2}{v} \right] \quad (1.9)$$

$$= r^{N-2} uv \left[XY - \frac{Y(N+b-W)}{\mu+1} - \frac{(N+a-Z)X}{\delta+1} + \frac{s}{2(\delta+1)} X^2 + \frac{s}{2(\mu+1)} Y^2 \right]. \quad (1.10)$$

Our main result is the following:

Theorem 1.5 *Consider the curve \mathcal{C}_s defined by*

$$\frac{N}{\mu+1} + \frac{N}{\delta+1} = N - 2 + \frac{(N-2)s}{2} \min\left(\frac{1}{\mu+1}, \frac{1}{\delta+1}\right), \quad (1.11)$$

located under the hyperbola defined by (1.6). Then the system (SN) admits a G.S. if (δ, μ) is above \mathcal{C}_s .

In Section 8 we consider the radial solutions of the potential system

$$(SP) \begin{cases} -\Delta_p u = |x|^a u^s v^{m+1} \\ -\Delta_q v = |x|^a u^{s+1} v^m \end{cases}$$

where $\delta = m + 1, \mu = s + 1$ and $a = b$, which is still variational, see [34], [35]. Here we use the following energy function, which deserves to be compared with the one of Section 6 in terms of (X, Y, Z, W) , since it also presents a quadratic factor:

$$\begin{aligned} \mathcal{E}(r) &= r^N \left[(s+1) \left(\frac{|u'|^p}{p'} + \frac{N-p}{p} \frac{u|u'|^{p-2} u'}{r} \right) + (m+1) \left(\frac{|v'|^q}{q'} + \frac{N-q}{q} \frac{v|v'|^{q-2} v'}{r} \right) + r^a u^{s+1} v^{m+1} \right] \\ &= r^{N-2-a} \frac{|u'|^{p-1} |v'|^{q-1}}{u^s v^m} \left[ZW - \frac{(s+1)W(N-p-(p-1)X)}{p} - \frac{(m+1)Z(N-q-(q-1)Y)}{q} \right]. \end{aligned} \quad (1.12)$$

Then we deduce new results of existence:

Theorem 1.6 *Let \mathcal{D} be the critical line in the plane (m, s) defined by*

$$N + a = (m+1) \frac{N-q}{q} + (s+1) \frac{N-p}{p}.$$

Then the system (SP) admits a G.S. if and only if (m, s) is above or on \mathcal{D} . On \mathcal{D} the G.S. has the following behaviour: suppose for example $q \leq p$. Let $\lambda^* = N + a - (s + 1)\frac{N-p}{p-1} - m\frac{N-q}{q-1}$. Then $\lim_{r \rightarrow \infty} r^{\frac{N-p}{p-1}} u(r) = \alpha > 0$, and

$$\lim_{r \rightarrow \infty} r^{\frac{N-q}{q-1}} v(r) = \beta > 0 \quad \text{if } \lambda^* < 0, \quad (1.13)$$

$$\lim_{r \rightarrow \infty} r^{\frac{N-p}{p-1} \mu - (q+b)} v(r) = \beta > 0 \quad \text{if } \lambda^* > 0, \quad (1.14)$$

$$\lim_{r \rightarrow \infty} r^{\frac{N-q}{q-1}} |\ln r|^{-\frac{1}{q-1-m}} v(r) = \beta > 0 \quad \text{if } \lambda^* = 0. \quad (1.15)$$

In particular (1.13) holds if $p = q$, or $q \leq m + 1$. Also the radial Dirichlet problem in a ball has a solution if and only if (m, s) is under \mathcal{D} .

Finally in Section 9 we deduce a nonradial result for the potential system in the case of two Laplacians:

$$(SL) \begin{cases} -\Delta u = |x|^a u^s v^{m+1}, \\ -\Delta v = |x|^a u^{s+1} v^m. \end{cases}$$

Our result solves a conjecture of [7], showing that in the subcritical case there exists no G.S.:

Theorem 1.7 *Assume $a > -2$ and $s, m \geq 0$. If*

$$s + m + 1 < \min\left(\frac{N+2}{N-2}, \frac{N+2+2a}{N-2}\right),$$

then the system (SL) admits no (radial or nonradial) G.S.

Our proof uses a nonradial Pohozaev function extending (??), different from the energy function used in [7], coupled with the estimates of [7], which up to now are the only extensions of the results of [18] to systems.

The case of system (G) with absorption terms ($\varepsilon_1 = \varepsilon_2 = -1$) or mixed terms ($\varepsilon_1 = -\varepsilon_2 = 1$), studied in [4], [5], will be the subject of another article. Our approach also extends to a system with gradient terms,

$$\begin{cases} -\Delta_p u = \varepsilon_1 |x|^a u^s v^\delta |\nabla u|^\eta |\nabla v|^\ell, \\ -\Delta_q v = \varepsilon_2 |x|^b u^\mu v^m |\nabla u|^\nu |\nabla v|^\kappa, \end{cases} \quad (1.16)$$

which will be developed later.

Acknowledgment The authors thank Raul Manasevich for having suggested us the study of system (G).

Keywords Elliptic quasilinear systems. Variational or nonvariational problems. Autonomous and quadratic systems. Stable manifolds. Heteroclinic orbits.

A.M.S. Subject Classification 34B15, 34C20, 34C37; 35J20, 35J55, 35J65, 35J70; 37J45.

2 Reduction to a quadratic system

Here we consider the radial positive solutions $r \mapsto (u(r), v(r))$ of system (G) on any interval (R_1, R_2) , that means

$$\begin{cases} \left(|u'|^{p-2} u' \right)' + \frac{N-1}{r} |u'|^{p-2} u' = r^{1-N} \left(r^{N-1} |u'|^{p-2} u' \right)' = -\varepsilon_1 r^a u^s v^\delta, \\ \left(|v'|^{q-2} v' \right)' + \frac{N-1}{r} |v'|^{q-2} v' = r^{1-N} \left(r^{N-1} |v'|^{q-2} v' \right)' = -\varepsilon_2 r^b u^\mu v^m. \end{cases}$$

Near any point r where $u(r) \neq 0$, $u'(r) \neq 0$ and $v(r) \neq 0$, $v'(r) \neq 0$ we define

$$X(t) = -\frac{ru'}{u}, \quad Y(t) = -\frac{rv'}{v}, \quad Z(t) = -\varepsilon_1 r^{1+a} u^s v^\delta |u'|^{-p} u', \quad W(t) = -\varepsilon_2 r^{1+b} u^\mu v^m |v'|^{-q} v', \quad (2.1)$$

where $t = \ln r$. Then we find the quadratic system

$$(M) \begin{cases} X_t = X \left[X - \frac{N-p}{p-1} + \frac{Z}{p-1} \right], \\ Y_t = Y \left[Y - \frac{N-q}{q-1} + \frac{W}{q-1} \right], \\ Z_t = Z [N + a - sX - \delta Y - Z], \\ W_t = W [N + b - \mu X - mY - W]. \end{cases}$$

The trajectories located on its invariant hyperplanes: $X = 0, Y = 0, Z = 0, W = 0$, do not correspond to a solution of system (G) ; they will be called *nonadmissible*. We suppose that the discriminant of the system

$$D = \delta\mu - (p-1-s)(q-1-m) \neq 0. \quad (2.2)$$

Then one can express u, v in terms of the new variables:

$$u = r^{-\gamma} (|X|^{p-1} |Z|)^{(q-1-m)/D} (|Y|^{q-1} |W|)^{\delta/D}, \quad v = r^{-\xi} (|X|^{p-1} |Z|)^{\mu/D} (|Y|^{q-1} |W|)^{(p-1-s)/D}, \quad (2.3)$$

where γ and ξ are defined by

$$\gamma = \frac{(p+a)(q-1-m) + (q+b)\delta}{D}, \quad \xi = \frac{(q+b)(p-1-s) + (p+a)\mu}{D}, \quad (2.4)$$

or equivalently by

$$(p-1-s)\gamma + p + a = \delta\xi, \quad (q-1-m)\xi + q + b = \mu\gamma. \quad (2.5)$$

Since the system (M) is autonomous, each admissible trajectory \mathcal{T} in the phase space corresponds to a solution (u, v) of system (G) unique up to a scaling: if (u, v) is a solution, then for any $\theta > 0$, $r \mapsto (\theta^\gamma u(\theta r), \theta^\xi v(\theta r))$ is also a solution.

Remark 2.1 *This system allows to study more general systems with signed solutions by reducing the study on intervals where u and v are nonzero. Consider for example the problem $-\Delta_p u = \varepsilon_1 |x|^a |u|^s |v|^{\delta-1} v$, $-\Delta_q v = \varepsilon_2 |x|^b |v|^m |u|^{\mu-1} u$. On any interval where $uv > 0$, the couple $(|u|, |v|)$ is a solution of (G). On any interval where $u > 0 > v$, the couple $(u, |v|)$ satisfies (G) with $(\varepsilon_1, \varepsilon_2)$ replaced by $(-\varepsilon_1, -\varepsilon_2)$*

System (M) has at most 16 fixed points. The main fixed point is

$$M_0 = (X_0, Y_0, Z_0, W_0) = (\gamma, \xi, N - p - (p-1)\gamma, N - q - (q-1)\xi), \quad (2.6)$$

corresponding to the particular solutions

$$u_0(r) = Ar^{-\gamma}, v_0(r) = Br^{-\xi}, \quad A, B > 0,$$

when they exist, depending on $\varepsilon_1, \varepsilon_2$; then the values of A and B are

$$\begin{aligned} A^D &= (\varepsilon_1 \gamma^{p-1} (N - p - \gamma(p-1)))^{q-1-m} (\varepsilon_2 \xi^{q-1} (N - q - (q-1)\xi))^\delta, \\ B^D &= (\varepsilon_2 \xi^{q-1} (N - q - (q-1)\xi))^{p-1-s} (\varepsilon_1 \gamma^{p-1} (N - p - (p-1)\gamma))^\mu. \end{aligned}$$

The other fixed points are

$$0 = (0, 0, 0, 0), \quad N_0 = (0, 0, N + a, N + b), \quad A_0 = \left(\frac{N-p}{p-1}, \frac{N-q}{q-1}, 0, 0 \right), \quad (2.7)$$

$$I_0 = \left(\frac{N-p}{p-1}, 0, 0, 0 \right), \quad J_0 = \left(0, \frac{N-q}{q-1}, 0, 0 \right), \quad K_0 = (0, 0, N + a, 0), \quad L_0 = (0, 0, 0, N + b), \quad (2.8)$$

$$G_0 = \left(\frac{N-p}{p-1}, 0, 0, N + b - \frac{N-p}{p-1}\mu \right), \quad H_0 = \left(0, \frac{N-q}{q-1}, N + a - \frac{N-q}{q-1}\delta, 0 \right), \quad (2.9)$$

and if $m \neq q-1$,

$$P_0 = \left(\frac{N-p}{p-1}, \frac{\frac{N-p}{p-1}\mu - (q+b)}{q-1-m}, 0, \frac{(q-1)(N+b - \frac{N-p}{p-1}\mu) - m(N-q)}{q-1-m} \right), \quad (2.10)$$

$$C_0 = \left(0, -\frac{q+b}{q-1-m}, 0, \frac{(N+b)(q-1) - m(N-q)}{q-1-m} \right), \quad (2.11)$$

$$R_0 = \left(0, -\frac{q+b}{q-1-m}, N + a + \delta \frac{b+q}{q-1-m}, \frac{(N+b)(q-1) - m(N-q)}{q-1-m} \right), \quad (2.12)$$

and by symmetry, if $s \neq p - 1$,

$$Q_0 = \left(\frac{\frac{N-q}{q-1}\delta - (p+a)}{p-1-s}, \frac{N-q}{q-1}, \frac{(p-1)(N+a - \frac{N-q}{q-1}\delta) - s(N-p)}{p-1-s}, 0 \right), \quad (2.13)$$

$$D_0 = \left(-\frac{p+a}{p-1-s}, 0, \frac{(N+a)(p-1) - s(N-p)}{p-1-s}, 0 \right), \quad (2.14)$$

$$S_0 = \left(-\frac{p+a}{p-1-s}, 0, \frac{(N+a)(p-1) - s(N-p)}{p-1-s}, N+b + \mu \frac{a+p}{p-1-s} \right). \quad (2.15)$$

Remark 2.2 *There is another way for reducing the system in an autonomous form: setting*

$$U(t) = r^\gamma u, \quad V(t) = r^\xi v, \quad H(t) = -r^{(\gamma+1)(p-1)} |u'|^{p-2} u', \quad K(t) = -r^{(\xi+1)(q-1)} |v'|^{q-2} v',$$

with $t = \ln r$, we find

$$\begin{cases} U_t = \gamma U - |H|^{(2-p)/(p-1)} H, & V_t = \zeta U - |K|^{(2-q)/(q-1)} K, \\ H_t = (\gamma(p-1) + p - N)H + \varepsilon_1 U^s V^\delta, & K_t = (\zeta(q-1) + q - N)K + \varepsilon_2 U^\mu V^m. \end{cases}$$

It extends the well-known transformation of Emden-Fowler in the scalar case when $p = 2$, used also in [2] for general p , see Section 3. When $p = q = 2$ we obtain

$$\begin{cases} U_{tt} + (N-2-2\gamma)U_t - \gamma(N-2-\gamma)U + \varepsilon_1 U^s V^\delta = 0, \\ V_{tt} + (N-2-2\xi)V_t - \xi(N-2-\xi)V + \varepsilon_2 U^\mu V^m = 0, \end{cases} \quad (2.16)$$

which was extended to the nonradial case and used for Hamiltonian systems ($s = m = 0$), with source terms in [9] ($\varepsilon_1 = \varepsilon_2 = 1$) and absorption terms in [4] ($\varepsilon_1 = \varepsilon_2 = -1$). Our system is more adequated for finding the possible behaviours, because its fixed points at infinity are not concerned, as soon as we deal with solutions $u, v > 0$.

Remark 2.3 *In the specific case $p = q = 2$, setting*

$$z = XZ = \varepsilon_1 r^{2+a} |u|^{s-2} u |v|^{\delta-1} v, \quad w = YW = \varepsilon_2 r^{2+b} |u|^{\mu-1} u |v|^{m-2} v$$

we get the following system

$$\begin{cases} X_t = X^2 - (N-2)X + z, & Y_t = Y^2 - (N-2)Y + w, \\ z_t = z [2 + a + (1-s)X - \delta Y], & w_t = w [2 + b - \mu X + (1-m)Y]. \end{cases}$$

It has been used in [20] for studying the Hamiltonian system (SH). Even in that case we will show at Section 6 that the system (M_G) is more performant, because it is of Kolmogorov type.

Remark 2.4 Assume $p = q$ and $a = b$. Setting $t = k\hat{t}$ and $(\hat{X}, \hat{Y}, \hat{Z}, \hat{W}) = k(X, Y, Z, W)$, we obtain a system of the same type with N, a replaced by \hat{N}, \hat{a} , with $\frac{\hat{N}-p}{\hat{N}-p} = k = \frac{\hat{N}+\hat{a}}{\hat{N}+\hat{a}}$. It corresponds to the change of unknowns

$$r = \hat{r}^k, \quad \hat{u}(\hat{r}) = C_1 u(r), \quad \hat{v}(\hat{r}) = C_2 u(r), \quad C_1 = k^{p(p-1-m+\delta)/D}, C_2 = k^{(p(p-1-s+\mu))/D}$$

And from (2.3) and (2.4), we get $\hat{\gamma}/\gamma = \hat{\xi}/\xi = k = \frac{p+\hat{a}}{p+a}$. There is one free parameter. In particular

- 1) we get a system without power ($\hat{a} = 0$), by taking $\hat{N} = \frac{p(N+a)}{p+a}$ and $k = \frac{p}{p+a}$.
- 2) We get a system in dimension $\hat{N} = 1$, by taking $k = -\frac{p-1}{N-p} < 0$ and $\hat{a} = \frac{p+a-(N+a)p}{N-p}$.

3 The scalar case

We first study the signed solutions of two well-known scalar equations with source or absorption:

$$-\Delta_p u = -r^{1-N} \left(r^{N-1} |u'|^{p-2} u' \right)' = \varepsilon |x|^a |u|^{Q-1} u, \quad (3.1)$$

with $\varepsilon = \pm 1$, $1 < p < N$, $Q \neq p-1$ and $p+a > 0$. We cannot quote all the huge litterature concerning its solutions, supersolutions or subsolutions, from the first studies of Emden and Fowler for $p = 2$, recalled in [16]; see for example [2] and [37], for any $p > 1$, and references therein. We set

$$Q_1 = \frac{(N+a)(p-1)}{N-p}, \quad Q_2 = \frac{N(p-1) + p + pa}{N-p}, \quad \gamma = \frac{p+a}{Q+1-p}.$$

From Remark 2.4 we could reduce the system to the case $a = 0$, in dimension $\hat{N} = p(N+a)/(p+a)$. However we will not do it, because we are motivated by the study of system (G), and also by the nonradial case.

3.1 A common phase plane for the two equations

Near any point r where $u(r) \neq 0$ (positive or negative), and $u'(r) \neq 0$ setting

$$X(t) = -\frac{ru'}{u}, \quad Z(t) = -\varepsilon r^{1+a} |u|^{Q-1} u |u'|^{-p} u', \quad (3.2)$$

with $t = \ln r$, we get a 2-dimensional system

$$(M_{scal}) \begin{cases} X_t = X \left[X - \frac{N-p}{p-1} + \frac{Z}{p-1} \right], \\ Z_t = Z [N+a - QX - Z]. \end{cases}$$

and then $|u| = r^{-\gamma} (|Z| |X|^{p-1})^{1/(Q+1-p)}$. This change of unknown was mentioned in [11] in the case $p = 2, \varepsilon = 1$ and $N = 3$. It is remarkable that the system (M_{scal}) is the same for the *two cases* $\varepsilon = \pm 1$, the only difference is that $X(t)Z(t)$ has the sign of ε :

The equation with source ($\varepsilon = 1$) is associated to the 1st and 3rd quadrant. It is well known that any local solution of has a unique extension on $(0, \infty)$. The 1st quadrant corresponds to the sets where $|u|$ is decreasing, which can be of the following types $(0, \infty), (0, R_2), (R_1, \infty), (R_1, R_2)$, $0 < R_1 < R_2 < \infty$; and the 3rd quadrant corresponds to the sets (R_1, R_2) where $|u|$ is increasing.

The equation with absorption ($\varepsilon = -1$) is associated to the 2nd and 4th quadrant. It is known that the solutions have at most one zero, and their maximal interval of existence can be $(0, R_2), (R_1, \infty), (R_1, R_2)$ or $(0, \infty)$. The 2nd quadrant corresponds to the sets (R_1, R_2) where $|u|$ is increasing; and the 4th quadrant corresponds to the sets $(0, R_2)$ or (R_1, ∞) where $|u|$ is decreasing.

The fixed points of (M_{scal}) are $(0, 0)$, $A_0 = (\frac{N-p}{p-1}, 0)$, $N_0 = (0, N + a)$, $M_0 = (X_0, Z_0) = (\gamma, N - p - (p - 1)\gamma)$. In particular M_0 is in the 1st quadrant whenever $\gamma < \frac{N-p}{p-1}$, equivalently $Q > Q_1$, and in the 4th quadrant whenever $Q < Q_1$. It corresponds to the solution $u(r) = Ar^{-\gamma}$ for $\varepsilon = 1$ when $Q > Q_1$ and for $\varepsilon = -1$ when $Q < Q_1$, where $A = (\varepsilon\gamma^{p-1}(N - p - \gamma(p - 1)))^{1/(Q-p+1)}$.

3.2 Local study

We examine the fixed points, where for simplicity we suppose $Q \neq Q_1$, and deduce local results for the two equations:

- Point $(0, 0)$: it is a saddle point, and the only trajectories that converge to $(0, 0)$ are the separatrix, contained in the lines $X = 0, Y = 0$, they are not admissible.
- Point N_0 : it is a saddle point: the eigenvalues of the linearized system are $\frac{p}{p-1}$ and $-N$. the trajectories ending at N_0 at ∞ are located on the set $Z = 0$, then there exists a unique trajectory starting from $-\infty$ at N_0 ; it corresponds to the local existence and uniqueness of regular solutions, which we obtain easily.
- Point A_0 : the eigenvalues of the linearized system are $\frac{N-p}{p-1}$ and $\frac{N-p}{p-1}(Q_1 - Q)$. If $Q < Q_1$, A_0 is an unstable node. There is an infinity of trajectories starting from A_0 at $-\infty$; then $X(t)$ converges exponentially to $\frac{N-p}{p-1}$, thus $\lim_{r \rightarrow 0} r^{\frac{N-p}{p-1}} u = \alpha > 0$. The corresponding solutions u satisfy the equation with a Dirac mass at 0. There exists no solution converging to A_0 at ∞ . If $Q > Q_1$, A_0 is a saddle point; the trajectories starting from A_0 at $-\infty$ are not admissible; there is a trajectory converging at ∞ , and then $\lim_{r \rightarrow \infty} r^{\frac{N-p}{p-1}} u = \alpha > 0$.
- Point M_0 : the eigenvalues λ_1, λ_2 of the linearized system are the roots of equation

$$\lambda^2 + (Z_0 - X_0)\lambda + \frac{Q - p + 1}{p - 1} X_0 Z_0 = 0.$$

For $\varepsilon = 1$, M_0 is defined for $Q > Q_1$; the eigenvalues are imaginary when $X_0 = Z_0$, equivalently $\gamma = (N - p)/p$, $Q = Q_2$. When $Q < Q_2$, M_0 is a source, there exists an infinity of trajectories such that $\lim_{r \rightarrow 0} r^\gamma u = A$. When $Q > Q_2$, M_0 is a sink, and there exists an infinity of trajectories such that $\lim_{r \rightarrow \infty} r^\gamma u = A$. When $Q = Q_2$, M_0 is a center, see [2]. For $\varepsilon = -1$, M_0 is defined

for $Q < Q_1$, it is a saddle-point. There exist two trajectories $\mathcal{T}_1, \mathcal{T}'_1$ converging at ∞ , such that $\lim_{r \rightarrow \infty} r^\gamma u = A$ and two trajectories $\mathcal{T}_2, \mathcal{T}'_2$, converging at 0, such that $\lim_{r \rightarrow 0} r^\gamma u = A$.

3.3 Global study

Remark 3.1 System (M_{scal}) has no limit cycle for $Q \neq Q_2$. It is evident when $\varepsilon = -1$. When $\varepsilon = 1$, as noticed in [19], it comes from the Dulac's theorem: setting $X_t = f(X, Z)$, $Z_t = g(X, Z)$, and

$$B(X, Z) = X^{pQ/(Q+1-p)-2} Z^{p/(Q+1-p)-1}, \quad M = B_X X_t + B_Y Y_t + B(f_X + g_Y),$$

then $M = KB$ with $K = (Q_2 - Q)\gamma(N - p)/p$, thus M has no zero for $Q \neq Q_2$.

Then from the Poincaré-Bendixson theorem, any trajectory bounded near $\pm\infty$ converges to one of the fixed points. Thus we find again global results:

- Equation with source ($\varepsilon = 1$). If $Q < Q_2$, there is no G.S.: the trajectories cannot converge to any fixed point.

If $Q_1 < Q < Q_2$, there exist solutions $u > 0$ such that $\lim_{r \rightarrow 0} r^\gamma u = A$ and $\lim_{r \rightarrow 0} r^{\frac{N-p}{p-1}} u = \alpha > 0$. Indeed the regular trajectory \mathcal{T} issued from N_0 cannot converge to M_0 ; if it converges to A_0 , it is the unique trajectory converging to A_0 ; the set delimited by \mathcal{T} and $X = 0, Z = 0$ is invariant, thus it contains M_0 ; and the trajectories issued from M_0 cannot converge to any point. Then X tends to ∞ , the regular solutions are changing sign. The trajectory ending at A_0 converges to M_0 at $-\infty$.

If $Q > Q_2$, the only singular solution at 0 is u_0 , and the regular solutions are G.S., with $\lim_{r \rightarrow \infty} r^\gamma u = A$. Indeed M_0 is a sink; the trajectory ending at A_0 cannot converge to N_0 at $-\infty$, thus X converges to 0, and Z converges to ∞ , then u cannot be positive on $(0, \infty)$. The trajectory issued from N_0 converges to M_0 .

- Equation with absorption ($\varepsilon = -1$). If $Q > Q_1$, all the solutions u defined near 0 are regular; indeed the trajectories cannot converge to a fixed point.

If $Q < Q_1$, we find again easily a well known result: there exists a positive solution u_1 , unique up to a scaling, such that $\lim_{r \rightarrow 0} r^{\frac{N-p}{p-1}} u_1 = \alpha > 0$, and $\lim_{r \rightarrow \infty} r^\gamma u = A$. Indeed the eigenvalues at M_0 satisfy $\lambda_1 < 0 < \lambda_2$. There are two trajectories $\mathcal{T}_1, \mathcal{T}'_1$ associated to λ_1 , and the eigenvector $(X_0 + |\lambda_1|, -\frac{X_0}{p-1})$. The trajectory \mathcal{T}_1 satisfies $X_t > 0 > Z_t$ near ∞ , and $X > \frac{N-p}{p-1}$, since $Z_0 < 0$, and X cannot take the value $\frac{N-p}{p-1}$ because at such a point $X_t < 0$; then $\frac{N-p}{p-1} < X < X_0$ and $X_t > 0$ as long as it is defined; similarly $Z_0 < Z < 0$ and $Z_t < 0$; then \mathcal{T}_1 converge to a fixed point, necessarily A_0 , showing the existence of u_1 . The trajectory \mathcal{T}'_1 corresponds to solutions u such that $\lim_{r \rightarrow \infty} r^\gamma u = A$ and $\lim_{r \rightarrow R} u = \infty$ for some $R > 0$. There are two trajectories $\mathcal{T}_2, \mathcal{T}'_2$, associated to λ_2 , defining solutions u such that $\lim_{r \rightarrow 0} r^\gamma u = A$ and changing sign, or with a minimum point and $\lim_{r \rightarrow R} u = \infty$ for some $R > 0$. The regular trajectory starts from N_0 in the 2^{nd} quadrant, it cannot converge to a fixed point, then $\lim_{r \rightarrow R} u = \infty$ for some $R > 0$.

• Critical case $Q = Q_2$: it is remarkable that the system (M_{scal}) admits another invariant line, namely A_0N_0 , given by

$$\frac{X}{p'} + \frac{Z}{Q_2 + 1} - \frac{N - p}{p} = 0 \quad (3.3)$$

It precisely corresponds to well-known solutions of the two equations

$$u = c(K^2 + r^{(p+a)/(p-1)})^{(p-N)/(p+a)}, \quad \text{for } \varepsilon = 1; \quad u = c \left| K^2 - r^{(p+a)/(p-1)} \right|^{(p-N)/(p+a)}, \quad \text{for } \varepsilon = -1,$$

where $K^2 = c^{Q-p+1}(N+a)^{-1}((N-p)/(p-1))^{1-p}$.

Remark 3.2 *The global results have been obtained without using energy functions. The study of [2] was based on a reduction of type of Remark 2.2, using an energy function linked to the new unknown. Other energy functions are well-known, of Pohozaev type:*

$$\mathcal{E}_\sigma(r) = r^N \left[\frac{|u'|^p}{p'} + \varepsilon r^a \frac{|u|^{Q+1}}{Q+1} + \sigma \frac{u |u'|^{p-2} u'}{r} \right] = r^{N-p} |u|^p |X|^{p-2} X \left[\frac{X}{p'} + \frac{Z}{Q+1} - \sigma \right]$$

with $\sigma = \frac{N-p}{p}$, satisfying $\mathcal{E}'_\sigma(r) = r^{N-1+a} \left(\frac{N+a}{Q+1} - \frac{N-p}{p} \right) |u|^{Q+1}$, or with $\sigma = \frac{N+a}{Q+1}$, leading to $\mathcal{E}'_\sigma(r) = r^{N-1} \left(\frac{N+a}{Q+1} - \frac{N-p}{p} \right) |u|^p$. In the critical case $Q = Q_2$, all these functions coincide and they are constant, in other words the system (M_{scal}) has a first integral. We find again the line (3.3): the G.S. are the functions of energy 0.

4 Local study of system (S)

In all the sequel we study the system with source terms: $(G) = (S)$. Assumption (1.5) is the most interesting case for studying the existence of the G.S.

We first study the local behaviour of nonnegative solutions (u, v) defined near 0 or near ∞ . It is well known that any solution (u, v) positive on some interval $(0, R)$ satisfies $u', v' < 0$ on $(0, R)$. Any solution (u, v) positive on (R, ∞) , satisfies $u', v' < 0$ near ∞ . Then we are reduced to study the system in the region \mathcal{R} where $X, Y, Z, W > 0$, and consider the fixed points in $\bar{\mathcal{R}}$. Then

$$X(t) = -\frac{ru'}{u}, \quad Y(t) = -\frac{rv'}{v}, \quad Z(t) = r^{1+a} u^s v^\delta |u'|^{1-p}, \quad W(t) = r^{1+b} v^m u^\mu |v'|^{1-q}, \quad (4.1)$$

and (X, Y, Z, W) is a solution of the system (M) in \mathcal{R} if and only if (u, v) defined by

$$u = r^{-\gamma} (ZX^{p-1})^{(q-1-m)/D} (WY^{q-1})^{\delta/D}, \quad v = r^{-\xi} (WY^{q-1})^{(p-1-s)/D} (ZX^{p-1})^{\mu/D}, \quad (4.2)$$

is a positive solution with $u' < 0$ and $v' < 0$. Among the fixed points, M_0 lies in \mathcal{R} if and only if

$$0 < \gamma < \frac{N-p}{p-1} \quad \text{and} \quad 0 < \xi < \frac{N-q}{q-1}. \quad (4.3)$$

The local study of the system near M_0 appears to be tricky, see Remark 4.2. A main difference with the scalar case is that there always exist a trajectory converging to M_0 at $\pm\infty$:

Proposition 4.1 (*Point M_0*) *Assume that (4.3) holds. Then there exist trajectories converging to M_0 as $r \rightarrow \infty$, and then solutions (u, v) being defined near ∞ , such that*

$$\lim_{r \rightarrow \infty} r^\gamma u = \alpha > 0, \quad \lim_{r \rightarrow \infty} r^\xi v = \beta > 0, \quad (4.4)$$

And there exist trajectories converging to M_0 as $r \rightarrow 0$, and then solutions (u, v) being defined near 0 such that

$$\lim_{r \rightarrow 0} r^\gamma u = \alpha > 0, \quad \lim_{r \rightarrow 0} r^\xi v = \beta > 0, \quad (4.5)$$

Proof. Here $M_0 \in \mathcal{R}$; setting $X = X_0 + \tilde{X}, Y = Y_0 + \tilde{Y}, Z = Z_0 + \tilde{Z}, W = W_0 + \tilde{W}$, the linearized system is

$$\begin{cases} \tilde{X}_t = X_0(\tilde{X} + \frac{1}{p-1}\tilde{Z}), \\ \tilde{Y}_t = Y_0(\tilde{Y} + \frac{1}{q-1}\tilde{W}), \\ \tilde{Z}_t = Z_0(-s\tilde{X} - \delta\tilde{Y} - \tilde{Z}), \\ \tilde{W}_t = W_0(-\mu\tilde{X} - m\tilde{Y} - \tilde{W}). \end{cases}$$

The eigenvalues are the roots $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, of the equation

$$\left[(\lambda - X_0)(\lambda + Z_0) + \frac{s}{p-1}X_0Z_0 \right] \left[(\lambda - Y_0)(\lambda + W_0) + \frac{m}{q-1}Y_0W_0 \right] - \frac{\delta\mu}{(p-1)(q-1)}X_0Y_0Z_0W_0 = 0; \quad (4.6)$$

this equation is of the form

$$f(\lambda) = \lambda^4 + E\lambda^3 + F\lambda^2 + G\lambda - H = 0$$

with

$$\begin{cases} E = Z_0 - X_0 + W_0 - Y_0, \\ F = (Z_0 - X_0)(W_0 - Y_0) - \frac{s+p-1}{p-1}X_0Z_0 - \frac{m+q-1}{q-1}Y_0W_0, \\ G = -\frac{q-1-m}{q-1}Y_0W_0(Z_0 - X_0) - \frac{p-1-s}{p-1}X_0Z_0(W_0 - Y_0), \\ H = \frac{D}{(p-1)(q-1)}X_0Y_0Z_0W_0, \end{cases}$$

and $H > 0$, then $\lambda_1\lambda_2\lambda_3\lambda_4 < 0$. Hence there exist two real roots $\lambda_3 < 0 < \lambda_4$, and two roots λ_1, λ_2 , real with $\lambda_1\lambda_2 > 0$, or complex. Then there exists at least one trajectory converging to M_0 at ∞ and another one at $-\infty$. Then (4.4) and (4.5) follow from (4.2). Moreover the convergence is monotone for X, Y, Z, W . \blacksquare

Remark 4.2 *There exist imaginary roots, namely $\text{Re } \lambda_1 = \text{Re } \lambda_2 = 0$, if and only if there exists $c > 0$ such that $f(ci) = 0$, that means $Ec^2 - G = 0$, and $c^4 - Fc^2 - H = 0$, equivalently*

$$E = G = 0, \quad \text{or } EG > 0 \text{ and } G^2 - EFG - E^2H = 0.$$

Condition $E = G = 0$ means that

(i) either $Z_0 = X_0$ and $W_0 = Y_0$, equivalently in other words

$$(\gamma, \xi) = \left(\frac{N-p}{p}, \frac{N-q}{q} \right), \quad (4.7)$$

in other words $(\delta, \mu) = \left(\frac{q(N(p-1-s)+p(1-s+a))}{p(N-q)}, \frac{p(N(q-1-m)+q(1-m+b))}{q(N-p)} \right)$.

(ii) or $(p-1-s)(q-1-m) > 0$ and (γ, ξ) satisfies

$$\begin{cases} 2N - p - q = p\gamma + q\xi, \\ \left(1 - \frac{m}{q-1}\right)\xi(N - q - (q-1)\xi) = \left(1 - \frac{s}{p-1}\right)\gamma(N - p - (p-1)\gamma). \end{cases} \quad (4.8)$$

This gives in general 0, 1 or 2 points. For example, in case $\frac{m}{q-1} = \frac{s}{p-1} \neq 1$, and $(p-2)(q-2) > 0$ and $N > \frac{pq-p-q}{p+q-2}$ we find another point, different from the one of (4.7) for $p \neq q$:

$$(\gamma, \xi) = \left(N \frac{q-2}{pq-p-q} - 1, N \frac{p-2}{pq-p-q} - 1 \right). \quad (4.9)$$

Moreover the computation shows that it can exist imaginary roots with $E, G \neq 0$.

In the case $p = q = 2$ and $s = m$ the situation is interesting:

Proposition 4.3 Assume $p = q = 2$ and $s = m < \frac{N}{N-2}$, and $\delta + 1 - s > 0$, $\mu + 1 - s > 0$. Let \mathcal{H}_s be the hyperbola of equation

$$\frac{1}{\delta + 1 - s} + \frac{1}{\mu + 1 - s} = \frac{N-2}{N - (N-2)s}, \quad \text{equivalently } \gamma + \xi = N - 2 \quad (4.10)$$

It is contained in the set of points (δ, μ) for which the linearized system at M_0 has imaginary roots, and equal when $s \leq 1$.

Proof. The assumption $D > 0$ imply $\delta + 1 - s > 0$ and $\mu + 1 - s > 0$; condition $E = G = 0$ implies $s < N/(N-2)$ and reduces to condition (4.10). Moreover if $s \leq 1$, all the cases are covered; indeed $2G = (s-1)E[Y_0Z_0 + X_0W_0]$, hence $GE \leq 0$. ■

Next we give a summary of the local existence results obtained by linearization around the other fixed points of system (M) proved at Section 10. Recall that $t \rightarrow -\infty \Leftrightarrow r \rightarrow 0$ and $t \rightarrow \infty \Leftrightarrow r \rightarrow \infty$.

Proposition 4.4 (Point N_0) A solution (u, v) is regular if and only if the corresponding trajectory converges to $N_0 = (0, 0, N+a, N+b)$ as $r \rightarrow 0$. For any $u_0, v_0 > 0$, there exists a unique local regular solution (u, v) with initial data (u_0, v_0) .

Proposition 4.5 (Point A_0) If $s\frac{N-p}{p-1} + \delta\frac{N-q}{q-1} > N+a$ and $\mu\frac{N-p}{p-1} + m\frac{N-q}{q-1} > N+b$, there exist (admissible) trajectories converging to A_0 as $r \rightarrow \infty$. If $s\frac{N-p}{p-1} + \delta\frac{N-q}{q-1} < N+a$ and $\mu\frac{N-p}{p-1} + m\frac{N-q}{q-1} < N+b$, the same happens as $r \rightarrow 0$. In any case

$$\lim r^{\frac{N-p}{p-1}} u = \alpha > 0, \quad \lim r^{\frac{N-q}{q-1}} v = \beta > 0. \quad (4.11)$$

If $s\frac{N-p}{p-1} + \delta\frac{N-q}{q-1} < N+a$ or $\mu\frac{N-p}{p-1} + m\frac{N-q}{q-1} < N+b$, there exist no trajectory converging as $r \rightarrow \infty$; if $s\frac{N-p}{p-1} + \delta\frac{N-q}{q-1} > N+a$ or $\mu\frac{N-p}{p-1} + m\frac{N-q}{q-1} > N+b$, there exist no trajectory converging as $r \rightarrow 0$.

Proposition 4.6 (Point P_0) 1) Assume that $q > m+1$ and $q+b < \frac{N-p}{p-1}\mu < N+b - m\frac{N-q}{q-1}$. If $\gamma < \frac{N-p}{p-1}$ there exist there exist trajectories converging to P_0 as $r \rightarrow \infty$ (and not as $r \rightarrow 0$). If $\gamma > \frac{N-p}{p-1}$ the same happens as $r \rightarrow 0$ (and not as $r \rightarrow \infty$).

2) Assume that $q < m+1$ and $q+b > \frac{N-p}{p-1}\mu > N+b - m\frac{N-q}{q-1}$ and $q\frac{N-p}{p-1}\mu + m(N-q) \neq N(q-1) + (b+1)q$. If $\gamma < \frac{N-p}{p-1}$ there exist trajectories converging to P_0 as $r \rightarrow 0$ (and not as $r \rightarrow \infty$) if $\gamma > \frac{N-p}{p-1}$ there exist trajectories converging as $r \rightarrow \infty$ (and not as as $r \rightarrow 0$).

In any case, setting $\kappa = \frac{1}{q-1-m}(\frac{N-p}{p-1}\mu - (q+b))$, there holds

$$\lim r^{\frac{N-p}{p-1}} u = \alpha > 0, \quad \lim r^\kappa v = \beta > 0. \quad (4.12)$$

Remark 4.7 This result improves the results of existence obtained by the fixed point theorem in [27] in the case of system (RP) with $p = q = 2, a = 0, N = 3, 2s + m \neq 3$. The proof is quite simpler with our methods.

Proposition 4.8 (Point I_0) If $\frac{N-p}{p-1}s > N+a$ and $\frac{N-q}{q-1}\mu > N+b$, there exist trajectories converging to I_0 as $r \rightarrow \infty$, and then

$$\lim_{r \rightarrow \infty} r^{\frac{N-p}{p-1}} u = \beta, \quad \lim_{r \rightarrow \infty} v = \alpha > 0. \quad (4.13)$$

For any s, m , there is no trajectory converging to I_0 as $r \rightarrow 0$.

Proposition 4.9 (Point G_0) Suppose $\frac{N-p}{p-1}\mu < N+b$. If $q+b < \frac{N-p}{p-1}\mu$ and $N+a < \frac{N-p}{p-1}s$, there exist trajectories converging to G_0 as $r \rightarrow \infty$. If $\frac{N-p}{p-1}\mu < q+b$ and $\frac{N-p}{p-1}s < N+a$, the same happens as $r \rightarrow 0$. In any case

$$\lim r^{\frac{N-p}{p-1}} u = \beta, \quad \lim v = \alpha > 0. \quad (4.14)$$

Proposition 4.10 (Point C_0) Suppose $N+b < \frac{N-q}{q-1}m$ (hence $q < m+1$) and $m \neq \frac{N(q-1)+(b+1)q}{N-q}$, and $\delta > \frac{(N+a)(m+1-q)}{q+b}$, exist trajectories converging to G_0 as $r \rightarrow \infty$ (and not at 0), and then

$$\lim_{r \rightarrow \infty} u = \alpha > 0, \quad \lim_{r \rightarrow \infty} r^k v = \beta, \quad (4.15)$$

where $k = \frac{q+b}{m+1-q}$.

Proposition 4.11 (Point R_0) Assume that $N + b < \frac{N-q}{q-1}m$ (hence $q < m + 1$), and $m \neq \frac{N(q-1)+b+bq}{N-q}$, and $\delta < \frac{(N+a)(m+1-q)}{q+b}$. If $\frac{(p+a)(m+1-q)}{q+b} < \delta$, there exist trajectories converging to R_0 as $r \rightarrow \infty$ (and not as $r \rightarrow 0$). If $\delta < \frac{(p+a)(m+1-q)}{q+b}$, there exist trajectories converging as $r \rightarrow 0$ (and not as $r \rightarrow \infty$), and them (4.15) holds again.

We obtain similar results of convergence to the points Q_0, J_0, H_0, D_0, S_0 by exchanging p, δ, s, a and q, μ, m, b . There is no admissible trajectory converging to $0, K_0, L_0$, see Remark 10.1.

5 Global results for system (S)

We are concerned by the existence of global positive solutions. First we find again easily some known results by using our dynamical approach.

Proposition 5.1 Assume that system (S) admits a positive solution (u, v) in $(0, \infty)$. Then (X, Y, Z, W) stay in the box

$$\mathcal{A} = \left(0, \frac{N-p}{p-1}\right) \times \left(0, \frac{N-q}{q-1}\right) \times (0, N+a) \times (0, N+b), \quad (5.1)$$

equivalently

$$ru' + \frac{N-p}{p-1}u > 0, \quad rv' + \frac{N-q}{q-1}v > 0, \quad r^{1+a}u^s v^\delta < (N+a)|u'|^{p-1}, \quad r^{1+b}u^\mu v^m < (N+b)|v'|^{q-1} \quad (5.2)$$

and then

$$u^{s-p+1}v^\delta \leq C_1 r^{-(p+a)}, \quad u^\mu v^{m-q+1} \leq C_2 r^{-(q+b)}, \quad \text{in } (0, \infty) \quad (5.3)$$

where $C_1 = (N+a)\left(\frac{N-p}{p-1}\right)^{p-1}$, $C_2 = (N+b)\left(\frac{N-q}{q-1}\right)^{q-1}$, and

$$\lim_{r \rightarrow 0} r^{\frac{N-p}{p-1}} u = c_1 \geq 0, \quad \lim_{r \rightarrow 0} r^{\frac{N-q}{q-1}} v(r) = c_2 \geq 0, \quad \liminf_{r \rightarrow \infty} r^{\frac{N-p}{p-1}} u > 0, \quad \liminf_{r \rightarrow \infty} r^{\frac{N-q}{q-1}} v > 0. \quad (5.4)$$

As a consequence if $s \leq p-1$ or $m \leq q-1$, we have

$$u \leq K_1 r^{-\gamma}, \quad v \leq K_2 r^{-\xi}, \quad \text{in } (0, \infty), \quad (5.5)$$

with $K_1 = C_1^{(q-1-m)/D} C_2^{\delta/D}$, $K_2 = C_1^{\mu/D} C_2^{(p-1-s)/D}$.

Proof. Consider a solution of System (M) in \mathcal{R} defined on \mathbb{R} . On the hyperplan $X = \frac{N-p}{p-1}$ we have $X_t > 0$, the field is going out. If at some point t_0 , $X(t_0) = \frac{N-p}{p-1}$, then $X(t) > \frac{N-p}{p-1}$ for $t > t_0$,

then, since $Z > 0$, $X_t \geq X \left[X - \frac{N-p}{p-1} \right] > 0$, thus $X(t) > 2\frac{N-p}{p-1}$ for $t > t_1 > t_0$, then $X_t \geq X^2/2$, and then X blows up in finite time, contradiction, thus $X(t) < \frac{N-p}{p-1}$ on \mathbb{R} . By integration it implies that $r^{(N-p)/(p-1)}u(r)$ is nondecreasing near 0 or ∞ , hence (5.4) holds. \blacksquare

Next we prove Theorem 1.1.

Proof of Theorem 1.1. (i) The trajectories of the regular solutions start from $N_0 = (0, 0, N + a, N + b)$, from Proposition 4.4, and the unstable variety \mathcal{V}_u has dimension 2, from (10.1), (10.2). It is given locally by $Z = \varphi(X, Y), W = \psi(X, Y)$ for $(X, Y) \in B(0, \rho) \setminus \{0\} \subset \mathbb{R}^2$.

To any $(x, y) \in B(0, \rho) \setminus \{0\}$ we associate the unique trajectory $\mathcal{T}_{x,y}$ in \mathcal{V}_u going through this point. If T^* is the maximal interval of existence of a solution on $\mathcal{T}_{x,y}$, then $\lim_{t \rightarrow T^*} (X(t) + Y(t)) = \infty$. Indeed Z , and W satisfy $0 < Z < N + a, 0 < W < N + b$ as long as the solution exists, because at a point T where $Z(T) = N + a$, we have $Z_t < 0$. If there exists a first point T such that $X(T) = \frac{N-p}{p-1}$ or $Y(T) = \frac{N-q}{q-1}$, then $T < T^*$. We consider the closed rectangle \mathcal{N} of submits

$$(0, 0), \quad \varpi_1 = \left(\frac{N-p}{p-1}, 0 \right), \quad \varpi_2 = \left(0, \frac{N-q}{q-1} \right), \quad \varpi_3 = \left(\frac{N-p}{p-1}, \frac{N-q}{q-1} \right).$$

Let $\mathcal{U} = \{(x, y) \in B(0, \rho) : x, y > 0\}$; then $\mathcal{U} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 \cup \mathcal{S}$, where

$$\left\{ \begin{array}{l} \mathcal{S}_1 = \{(x, y) \in \mathcal{U} \text{ and } \mathcal{T}_{x,y} \text{ leaves } \mathcal{N} \text{ on } (\varpi_1, \varpi)\}, \quad \mathcal{S}_2 = \{(x, y) \in \mathcal{U} : \mathcal{T}_{x,y} \text{ leaves } \mathcal{N} \text{ on } (\varpi_2, \varpi_3)\}, \\ \mathcal{S}_3 = \{(x, y) \in \mathcal{U} : \mathcal{T}_{x,y} \text{ leaves } \mathcal{N} \text{ at } \varpi_3\}, \quad \mathcal{S} = \left\{ (x, y) \in \mathcal{U} : \mathcal{T}_{x,y} \text{ stays in } \overset{\circ}{\mathcal{N}} \right\}. \end{array} \right.$$

Any element of \mathcal{S} defines a G.S. Assume $s < \frac{N(p-1)+p+pa}{N-p}$. Let us show that \mathcal{S}_1 is nonempty. Consider the trajectory $\mathcal{T}_{\bar{x},0}$ on \mathcal{V}_u associated to $(\bar{x}, 0)$, with $\bar{x} \in (0, \rho)$, going through $\bar{M} = (\bar{x}, 0, \varphi(\bar{x}, 0), \psi(\bar{x}, 0))$; it is *nonadmissible* for our problem, in the hyperplane $Y = 0$: we get the system

$$\left\{ \begin{array}{l} X_t = X \left[X - \frac{N-p}{p-1} + \frac{Z}{p-1} \right], \\ Z_t = Z [N + a - sX - Z], \\ W_t = W [N + b - \mu X - W], \end{array} \right.$$

which is *not coupled*. The two equations in X, Z corresponds to the equation

$$-\Delta_p U = r^a U^s \tag{5.6}$$

The regular solutions of (5.6) are changing sign, since s is subcritical, see Section 3. Consider the solution $(\bar{X}, \bar{Y}, \bar{Z}, \bar{W})$ of system (M), of trajectory $\mathcal{T}_{\bar{x},0}$, going through \bar{M} at time 0; it satisfies $\bar{Y} = 0$, and $\bar{X}(t) > 0, \bar{Z}(t) > 0$ tends to ∞ in finite time T^* , then for any given $C \geq \frac{N-p}{p-1}$, there exist a first $T < T^*$ such that $\bar{X}(T) = C$, and $\bar{Y}(T) = 0$. And $\lim_{t \rightarrow -\infty} \bar{W} = N + b$, and necessarily $0 < \bar{W} < N + b$, in particular $0 < \bar{W}(T) < N + b$; and \bar{W}_t is bounded on $(-\infty, T^*)$, then \bar{W} has

a finite limit at T^* . The field at time T is transverse: we have $\bar{X}_t \geq C \frac{Z(T)}{p-1} > 0$, since $\bar{Z}(T) > 0$. From the continuous dependance of the initial data at time 0, for any $\varepsilon > 0$, there exist $\eta > 0$ such that for any $(x, y) \in B((\bar{x}, 0), \eta)$ and for any (X, Y, Z, W) on $\mathcal{T}_{x,y}$, there exists a first time T_ε such that $X(T_\varepsilon) = C$, and $|Y(t)| \leq \varepsilon$ for any $t \leq T_\varepsilon$, in particular for any $(x, y) \in B((\bar{x}, 0), \eta)$ with $y > 0$, and then $0 < Y(t) \leq \varepsilon$ for any $t \leq T_\varepsilon$. Let us take $C = \frac{N-p}{p-1}$. Then $(x, y) \in \mathcal{S}_1$. The same arguments imply that \mathcal{S}_1 is open. Similarly assuming $m < \frac{N(q-1)+q+qb}{N-q}$ implies that \mathcal{S}_2 is nonempty and open. By connexity \mathcal{S} is empty if and only if \mathcal{S}_3 is nonempty.

(ii) Here the difficulty is due to the fact that the zeros of u, v correspond to infinite limits for X, Y , and then the argument of continuous dependance is no more available. We can write $\mathcal{U} = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3 \cup \mathcal{S}$, where

$$\begin{cases} \mathcal{M}_1 = \{(x, y) \in \mathcal{U} \text{ and } \mathcal{T}_{x,y} \text{ has an infinite branch in } X \text{ with } Y \text{ bounded}\}, \\ \mathcal{M}_2 = \{(x, y) \in \mathcal{U} : \mathcal{T}_{x,y} \text{ has an infinite branch in } Y \text{ with } X \text{ bounded}\}, \\ \mathcal{M}_3 = \{(x, y) \in \mathcal{U} : \mathcal{T}_{x,y} \text{ has an infinite branch in } (X, Y)\}, \end{cases}$$

In other words, \mathcal{M}_1 is the set of $(x, y) \in \mathcal{U}$ such that for any (X, Y, Z, W) on $\mathcal{T}_{x,y}$, there exists a T^* such that $\lim_{t \rightarrow T^*} X(t) = \infty$, and $Y(t)$ stays bounded on $(-\infty, T^*)$, that means the set of $(x, y) \in \mathcal{U}$ such that for any solution (u, v) corresponding to $\mathcal{T}_{x,y}$, u vanishes before v ; similarly for \mathcal{M}_2 . And \mathcal{M}_3 is the set of $(x, y) \in \mathcal{U}$ such that there exists a T^* such that $\lim_{t \rightarrow T^*} X(t) = \lim_{t \rightarrow T^*} Y(t) = \infty$, that means (u, v) vanish at the same $R^* = e^{T^*}$. In that case, from the Höpf Lemma, $\lim_{r \rightarrow R} \frac{u'}{(r-R)u} = 1$, then $\lim_{t \rightarrow T^*} \frac{X}{Y} = 1$.

We are lead to show that \mathcal{M}_1 is nonempty and open for $s < \frac{N(p-1)+p+pa}{N-p}$. We consider again the trajectory \bar{T} and take C large enough: $C = 2(\frac{N-p}{p-1} + \frac{N+|b|}{q-1})$. Let $\varepsilon \in (0, \frac{C}{2})$. For any $(x, y) \in B((\bar{x}, 0), \eta)$ with $y > 0$, and any (X, Y, Z, W) on $\mathcal{T}_{x,y}$, there is a first time T_ε such that $X(T_\varepsilon) = C$, and $0 < Y(t) \leq \varepsilon$ for any $t \leq T_\varepsilon$. And X is increasing and $X_t \geq X(X - C)$, thus there exists T^* such that $\lim_{t \rightarrow T^*} X(t) = \infty$. Let $\varphi = X/Y$. Then

$$\frac{\varphi_t}{\varphi} = X - Y + \frac{Z}{p-1} - \frac{W}{q-1} + \frac{N-q}{q-1} - \frac{N-p}{p-1} \geq X - Y - \frac{C}{2}$$

then $\varphi_t(T_\varepsilon) > 0$. Let $\theta = \sup\{t > T_\varepsilon : \varphi_t > 0\}$; suppose that θ is finite; then $\varphi(\theta) > \varphi(T_\varepsilon) = C/\varepsilon > 2$ and $X(\theta) \leq Y(\theta) + C < X(\theta)/2 + C$, which is contradictory. Then φ is increasing up to T^* ; if $\lim_{t \rightarrow T^*} Y(t) = \infty$, then $\lim_{t \rightarrow T^*} \varphi = 1$, which is impossible. Then (x, y) is in \mathcal{M}_1 ; thus \mathcal{M}_1 is nonempty. In the same way \mathcal{M}_1 is open: let (\bar{x}, \bar{y}) in \mathcal{M}_1 then there exist $M > 0$ such that $0 < \bar{Y}(t) \leq M/2$ on $\mathcal{T}_{\bar{x}, \bar{y}}$; then we argue as above, with $(\bar{x}, 0)$ replaced by (\bar{x}, \bar{y}) , and C replaced by $C + M$. \blacksquare

Proof of Proposition 1.2. Assume $s \geq \frac{N(p-1)+p+pa}{N-p}$. Consider the Pohozaev type function

$$\mathcal{F}(r) = r^N \left[\frac{|u'|^p}{p'} + \frac{r^a u^{s+1}}{s+1} v^\delta + \frac{N-p}{p} u \frac{|u'|^{p-2} u'}{r} \right] = r^{N-p} u^p \left[\frac{X}{p'} + \frac{1}{s+1} Z - \frac{N-p}{p} \right]. \quad (5.7)$$

We find $\mathcal{F}(0) = 0$ and

$$\begin{aligned}\mathcal{F}'(r) &= r^{N-1+a} \left[\left(\frac{N+a}{s+1} - \frac{N-p}{p} \right) v^\delta u^{s+1} + \frac{\delta}{s+1} r u^{s+1} v^{\delta-1} v' \right] \\ &= r^{N-1+a} v^\delta u^{s+1} \left[\frac{N+a}{s+1} - \frac{N-p}{p} - \frac{\delta Y}{s+1} \right]\end{aligned}\tag{5.8}$$

From our assumption, \mathcal{F} is decreasing, and $Z > 0$, thus $X < \frac{N-p}{p-1}$. Then $\mathcal{S}_1, \mathcal{S}_3$ are empty. If moreover $m \geq \frac{N(q-1)+q+qb}{N-q}$ then \mathcal{S}_2 is empty, then $\mathcal{S} = \mathcal{U}$. \blacksquare

Remark 5.2 Assume only $s \geq \frac{N(p-1)+p+pa}{N-p}$. Then if one function has a first zero, it is v . Indeed if there exists a first point where $u(R) = 0$, and $v(r) > 0$ on $[0, R)$, then $\mathcal{F}(R) = \frac{R^N}{p'} |u'(R)|^p > 0$.

As a first consequence we obtain existence results for the Dirichlet problem. It solves an open problem in case $s > p-1$ or $m > q-1$, and extends some former results of [12] and [42]. Our proof, based on the shooting method differs from the proof of [12], based on degree theory and blow-up technique. Our results extend the ones of [3, Theorem 2.2] relative to the case $p = q = 2$, obtained by studying the equation satisfied by a suitable function of u, v .

Corollary 5.3 System (S) admits no G.S. and then there is a radial solution of the Dirichlet problem in a ball in any of the following cases:

- (i) $p < s+1, q < m+1$, and $\min(s \frac{N-p}{p-1} + \frac{N-q}{q-1} \delta - (N+a), \frac{N-p}{p-1} \mu + m \frac{N-q}{q-1} - (N+b)) \leq 0$;
- (ii) $p < s+1, q > m+1$ and $s \frac{N-p}{p-1} + \frac{N-q}{q-1} \delta - (N+a) \leq 0$ or $\gamma - \frac{N-p}{p-1} > 0$;
- (iii) $p > s+1, q > m+1$ and $\max(\gamma - \frac{N-p}{p-1}, \xi - \frac{N-q}{q-1}) \geq 0$;
- (iv) $p \geq s+1, q \geq m+1$ and $\max(\gamma - \frac{N-p}{p-1}, \xi - \frac{N-q}{q-1}) > 0$.

Proof. From Theorem 1.1, we are reduced to prove the nonexistence of G.S.

(i) Assume $p < s+1$, and $s \frac{N-p}{p-1} + \frac{N-q}{q-1} \delta - (N+a) < 0$. Also, since $s > p-1$, and $-\Delta_p u \geq Cr^{a - \frac{N-q}{q-1} \delta} u^s$ for large r , then from [6, Theorem 3.1], we find $u = O(r^{-(p+a - \frac{N-q}{q-1} \delta)/(s+1-p)})$, and then $s \frac{N-p}{p-1} + \frac{N-q}{q-1} \delta - (N+a) \geq 0$, from (5.4), which contradicts our assumption. In case of equality, we find $-\Delta_p u \geq Cr^{-N}$ for large r , which is impossible. Then there exists no G.S. This improves the result of [12] where the minimum is replaced by a maximum.

(ii) Assume $p < s+1, q > m+1$ and $\gamma - \frac{N-p}{p-1} > 0$; then $u = O(r^{-\gamma})$, which contradicts (5.4). If $\gamma - \frac{N-p}{p-1} = 0$, then $\lim r^{\frac{N-p}{p-1}} u = \alpha > 0$, and $\xi > \frac{N-q}{q-1}$; and $-\Delta_q v = Cr^{b - \frac{N-p}{p-1}} \mu v^m$ for large r , then $v \geq Cr^{(q+b - \frac{N-p}{p-1} \delta)/(q-1-m)} = Cr^{-\xi}$, then $C_1 > r^\xi v > C_2 > 0$ for large r , from [6, Theorem 3.1] and (5.5), then $-\Delta_p u \geq Cr^{-N}$, which is still contradictory.

(iii) (iv) The nonexistence of G.S is obtained by extension of the proof of [12] to the case $a, b \neq 0$. Moreover (iii) implies the nonexistence of positive solution (u, v) , radial or not, in any exterior domain $(R, \infty) \times (R, \infty)$, $R > 0$ from [6]. \blacksquare

Corollary 5.4 *Assume (4.3) with $p = q = 2$. If $\delta + s \geq \frac{N+2+2a}{N-2}$ and $\mu + m \geq \frac{N+2+2b}{N-2}$, then system (S) admits a G.S.*

Proof. It was shown in [28], [41] by the moving spheres method that the Dirichlet problem has no radial or nonradial solution. Then Theorem 1.1 applies again. \blacksquare

We also extend and improve a result of nonexistence of [10] relative to the case $p = q = 2, a = 0, s > 1$:

Proposition 5.5 *Assume $s + 1 > p$ or $\gamma > \frac{N-p}{p}$, and*

$$s + \frac{p(N-q)}{(q-1)(N-p)}\delta < \frac{N(p-1) + pa + p}{N-p} \quad (5.9)$$

Then the system (S) admits no G.S. and then there is a solution of the Dirichlet problem. The same happens by exchanging p, s, δ, a, γ with q, m, μ, b, ξ .

Proof. Consider the function \mathcal{F} defined at (5.7). Suppose that there exists a G.S. Then from (5.1) and (5.9) we find

$$\frac{N+a}{s+1} - \frac{N-p}{p} - \frac{\delta Y}{s+1} > \frac{N+a}{s+1} - \frac{N-p}{p} - \frac{\delta}{s+1} \frac{N-q}{q-1} \geq 0.$$

From (5.8), \mathcal{F} is nondecreasing. First suppose $s + 1 > p$. Then from (5.3) and (5.4), $u = O(r^{-k})$ at ∞ , with $k = (p + a - \delta \frac{N-q}{q-1}) / (s - p + 1)$, then $r^{N-p}u^p = O(r^{(N-p)-kp}) = o(1)$ from (5.9), then $\mathcal{F}(r) = o(1)$ near ∞ . Next assume $s + 1 \leq p$ and $\gamma > \frac{N-p}{p}$, then $r^{N-p}u^p = O(r^{N-p-\gamma p})$, hence $\mathcal{F}(r) = o(1)$ near ∞ ; in any case we get a contradiction. \blacksquare

6 The Hamiltonian system

Here we consider the nonnegative solutions of the variational Hamiltonian problem (SH) in $\Omega \subset \mathbb{R}^N$

$$(SH) \begin{cases} -\Delta u = |x|^a v^\delta, \\ -\Delta v = |x|^b u^\mu, \end{cases}$$

where $p = q = 2 < N$, $s = m = 0$, $a > b > -2$, and $D = \delta\mu - 1 > 0$. Here we find

$$\gamma = \frac{(2+a) + (2+b)\delta}{D}, \quad \xi = \frac{2+b + (2+a)\mu}{D}, \quad \gamma + 2 + a = \delta\xi, \quad \xi + 2 + b = \mu\gamma.$$

The particular solution $(u_0(r), v_0(r)) = (Ar^{-\gamma}, Br^{-\xi})$ exists for $0 < \gamma < N - 2$ and $0 < \xi < N - 2$. Here X, Y, Z, W are defined by

$$X(t) = \frac{r|u'|}{u}, \quad Y(t) = \frac{r|v'|}{v}, \quad Z(t) = \frac{r^{1+a}v^\delta}{|u'|}, \quad W(t) = \frac{r^{1+b}u^\mu}{|v'|},$$

with $t = \ln r$, and system (M) becomes

$$(MH) \begin{cases} X_t = X[X - (N - 2) + Z], \\ Y_t = Y[Y - (N - 2) + W], \\ Z_t = Z[N + a - \delta Y - Z], \\ W_t = W[N + b - \mu X - W] \end{cases}$$

This system has a Pohozaev type function, well known at least in the case $a = b = 0$, given at (1.7):

$$\begin{aligned} \mathcal{E}(r) &= r^N \left[u'v' + r^b \frac{|u|^{\mu+1}}{\mu+1} + r^a \frac{|v|^{\delta+1}}{\delta+1} + \frac{N+a}{\delta+1} \frac{vu'}{r} + \frac{N+b}{\mu+1} \frac{wv'}{r} \right] \\ &= r^{N-2} uv \left[XY - \frac{Y(N+b-W)}{\mu+1} - \frac{(N+a-Z)X}{\delta+1} \right] \\ &= r^{N-2-\gamma-\xi} (ZX)^{(\mu+1)/D} (WY)^{(\delta+1)/D} \left[XY - \frac{Y(N+b-W)}{\mu+1} - \frac{(N+a-Z)X}{\delta+1} \right]. \end{aligned}$$

It can also be found by a direct computation, and satisfies

$$\mathcal{E}'(r) = r^{N-1} u'v' \left(\frac{N+a}{\delta+1} + \frac{N+b}{\mu+1} - (N-2) \right).$$

We define the critical case, when (δ, μ) lie on the critical hyperbola \mathcal{H}_0 given by

$$\frac{N+a}{\delta+1} + \frac{N+b}{\mu+1} = N-2, \text{ equivalently } \gamma + \xi = N-2. \quad (6.1)$$

In this case $\gamma = \frac{N+b}{\mu+1}$, $\xi = \frac{N+a}{\delta+1}$, and $\mathcal{E}'(r) \equiv 0$. It corresponds to the existence of a first integral of system (M) :

$$\mathcal{E}(r) = (ZX)^{(\mu+1)/D} (WY)^{(\delta+1)/D} \left[XY - \frac{Y(N+b-W)}{\mu+1} - \frac{(N+a-Z)X}{\delta+1} \right].$$

It can also be expressed in the variables $U = r^\gamma u, V = r^\xi v$ of Remark 2.2 under the form

$$\mathcal{E}(r) = U_t V_t - \gamma \xi UV + \frac{U^{\mu+1}}{\mu+1} + \frac{V^{\delta+1}}{\delta+1}.$$

The supercritical case is defined as the case where (δ, μ) is above \mathcal{H} , equivalently $\gamma + \xi < N - 2$ and the subcritical case corresponds to (δ, μ) under \mathcal{H} .

Remark 6.1 The energy \mathcal{E}_0 of the particular solution associated to M_0 is always negative: $\mathcal{E}_0 = -\frac{D}{(\mu+1)(\delta+1)} r^{N-2-\gamma-\xi} X_0 Y_0 (Z_0 X_0)^{(\mu+1)/D} (W_0 Y_0)^{(\delta+1)/D}$.

Remark 6.2 In the case $a = b = 0$, it is known that there exists a solution of the Dirichlet problem in any bounded regular domain Ω of \mathbb{R}^N , see for example [15], [20]; for general a, b , some restrictions on the coefficients appear, see [23] and [14].

Next consider the critical and supercritical case. When $a = b = 0$, there exists no solution if Ω is starshaped, see [36].

Here we show the existence of G.S. for general a, b . The existence in the critical case with $a = b = 0$ was first obtained in [22], then in the supercritical case in [29], and uniqueness was proved in [20]. The proof of [29] is quite long due to regularity problems, when δ or $\mu < 1$, which play no role in our quadratic system:

Remark 6.3 The particular case $\delta = \mu$ and $a = b$ is easy to treat. Indeed in that case $u = v$ is a solution of the scalar equation $\Delta u + |x|^\alpha |u|^{\delta-1} u = 0$, for which the critical case is given by $\delta = (N + 2 + 2a)/(N - 2)$. Moreover if system (SH) admits a G.S., or a solution of the Dirichlet problem in a ball, it satisfies $u = v$, from [3]. Then we are completely reduced to the scalar case. In particular, in the critical case, the G.S. are given explicitly by : $u = v = c(K + r^{(2+a)})^{(2-N)/(2+a)}$, where $K = c^{\delta-1}/(N + a)(N - 2)$; in other words they satisfy (3.3) with $X = Y$ and $Z = W$, i.e.

$$\frac{X(t)}{N-2} + \frac{Z(t)}{N+a} - 1 = 0.$$

Near ∞ , the G.S. is (obviously) symmetrical: it joins the points N_0 and A_0 .

Remark 6.4 Consider the case $\delta = 1$, $a = b = 0$, which is the case of the biharmonic equation

$$\Delta^2 u = u^\mu.$$

Recall that it is the only case where the conjecture (C) was completely solved by Lin in [21]. In the critical case $\mu = (N + 4)/(N - 4)$, the G.S. are also given explicitly, see [20]:

$$u(r) = c(K + r^2)^{(4-N)/2}, \quad K = c^{\mu-1}/(N - 4)(N - 2)N(N + 2).$$

They satisfy the relation $XY = \frac{N-Z}{2}X + \frac{N-4}{2N}(N - W)Y$, and moreover we find that they are on an hyperplane:

$$\frac{(N-2)X(t)}{N(N-4)} + \frac{Z(t)}{N} - 1 = 0.$$

Observe also that the G.S. is not symmetrical near ∞ : u behaves like r^{4-N} and v behaves like r^{2-N} . The trajectory in the phase space joins the points N_0 and $Q_0 = (N - 4, N - 2, 2, 0)$.

Proof of Theorem 1.4. 1) *Existence or nonexistence results:*

• In the supercritical or critical case we apply any of the two conditions of Theorem 1.1: Here $\mathcal{E}(0) = 0$, \mathcal{E} is nonincreasing; there does not exist solutions of (M) such that at some point T , $X(T) = Y(T) = N - 2$, because at the point T ,

$$XY - \frac{Y(N+b-W)}{\mu+1} - \frac{(N+a-Z)X}{\delta+1} = (N-2) \left[N-2 - \frac{N+a}{\delta+1} - \frac{N+b}{\mu+1} + \frac{W}{\mu+1} + \frac{Z}{\delta+1} \right] > 0$$

since $W > 0, Z > 0$, thus $\mathcal{E}(e^T) > 0$, which is impossible. In the same way there exists no solution of the Dirichlet problem in a ball $B(0, R)$, because $\mathcal{E}(R) = R^N u'(R)v'(R) > 0$ from the Höpf Lemma. Then there exists a G.S. The uniqueness is proved in [20].

• In the subcritical case there is no radial G.S.: it would satisfy $\mathcal{E}(0) = 0$, \mathcal{E} is nondecreasing, $\mathcal{E}(r) \leq Cr^{N-2-\gamma-\xi}$ from (5.1), and $\gamma + \xi > (N - 2)$, then $\lim_{r \rightarrow \infty} \mathcal{E}(r) = 0$. Then from Theorem 1.1, there exists a solution of the Dirichlet problem.

2) *Behaviour of the G.S. in the critical case.*

It is easy to see that condition (1.6) implies $\mu > \frac{2+b}{N-2}$ and $\delta > \frac{2+a}{N-2}$, and that $\delta \leq \frac{N+a}{N-2}$ and $\mu \leq \frac{N+b}{N-2}$ cannot hold simultaneously. One can suppose that $\delta > \frac{N+a}{N-2}$. Let \mathcal{T} the unique trajectory of the G.S.; then $\mathcal{E}(0) = 0$, thus \mathcal{T} lies on the variety \mathcal{V} of energy 0, defined by

$$\frac{X(N+a-Z)}{\delta+1} + \frac{Y(N+b-W)}{\mu+1} = XY. \quad (6.2)$$

and from (5.2) \mathcal{T} starts from the point N_0 ; from (5.1) \mathcal{T} stays in

$$\mathcal{A} = \{(X, Y, Z, W) \in \mathbb{R}^4 : 0 < X < N - 2, \quad 0 < Y < N - 2, \quad 0 < Z < N + a, \quad 0 < W < N + a\}.$$

(i) If \mathcal{T} converges to a fixed point of the system in $\bar{\mathcal{R}}$, the only possible points are A_0, P_0, Q_0 which are effectively on \mathcal{V} . Indeed $I_0, J_0, G_0, H_0 \notin \mathcal{V}$. But $Q_0 = ((N-2)\delta - (2+a), N-2, N+a - (N-2)\delta, 0) \notin \bar{\mathcal{R}}$, since $\delta > \frac{N+a}{N-2}$. And $P_0 \in \bar{\mathcal{R}}$ if $\mu \leq \frac{N+b}{N-2}$.

If $\mu > \frac{N+b}{N-2}$, then \mathcal{T} converges to A_0 . If $\mu < \frac{N+b}{N-2}$, no trajectory converges to A_0 , from Proposition 4.5, thus \mathcal{T} converges to P_0 . If $\mu \neq \frac{N+b}{N-2}$ the convergence is exponential, thus the behaviour of u, v follows. If $\mu = \frac{N+b}{N-2}$, then \mathcal{T} converges to $A_0 = P_0$; the eigenvalues given by (10.3) satisfy $\lambda_1 = \lambda_2 = N - 2$, $\lambda_3 = N + a - \delta(N - 2) < 0$ and $\lambda_4 = 0$; the projection of the trajectory on the hyperplane $Y = N - 2$ satisfies the system

$$X_t = X [X - (N - 2) + Z], \quad Z_t = Z [N + a - \delta(N - 2) - Z]$$

which presents a saddle point at $(N - 2, 0)$, thus the convergence of X and Z is exponential, in particular we deduce the behaviour of u . The trajectory enters by the central variety of dimension 1, and by computation we deduce that $Y - (N - 2) = -t^{-1} + O(t^{-2+\varepsilon})$ near ∞ , and the behaviour of v follows.

(ii) Next we show that \mathcal{T} converges to a fixed point: We eliminate W from (6.2), we get a quadratic system in (X, Y, Z) :

$$\begin{cases} X_t = X [X - (N - 2) + Z], \\ Y_t = Y [Y + b + 2 - (\mu + 1)X] + \frac{\mu+1}{\delta+1}X(N + a - Z), \\ Z_t = Z [N + a - \delta Y - Z] \end{cases} \quad (6.3)$$

At the beginning, $X_t \geq 0$, and $Y_t \geq 0$. Suppose that X has a maximum at t_0 followed by a minimum at t_1 . Then $Z_t(t_0) < 0 < Z_t(t_1)$ then there exists $t_2 \in (t_0, t_1)$ such that $Z_t(t_2) = 0$, it is a minimum, then at this point $Z = N + a - \delta Y$, $Z_{tt} = -\delta Z Y_t$, hence

$$Y_t = Y \left[Y + b + 2 - (\mu + 1)X + \delta \frac{\mu + 1}{\delta + 1} X \right] = Y \left[Y + b + 2 - \frac{\mu + 1}{\delta + 1} X \right] < 0$$

and $X_t(t_2) < 0$, hence $(X + Z)(t_2) < N - 2$, At the point t_2 we have

$$N - 2 - X > Z = N + a - \delta Y > N + a - \delta \left(\frac{\mu + 1}{\delta + 1} X - b - 2 \right)$$

$$(a + 2) + \delta(b + 2) < \left(\delta \frac{\mu + 1}{\delta + 1} - 1 \right) X = \frac{\delta(2 + b) + (2 + a)}{(N - 2)\delta - (2 + a)} X$$

and $X(t_2) < X(t_0) < \delta(N - 2) - (2 + a)$, hence we are lead to a contradiction. Then X has at most one extremum, which is a maximum, and then it has a limit in $(0, N - 2]$ at ∞ . In the same way, by symmetry, Y has at most one extremum, which is a maximum, and has a limit in $(0, N - 2]$ at ∞ . Then Z has at most one extremum, which is a minimum: indeed at the points where $Z_t = 0$, $-Z_{tt}$ has the sign of Y_t . Then Z has a limit in $[0, N + a)$, and similarly W has a limit in $[0, N + b)$.

■

Open problems: 1) For the case $\delta = \mu$, in the critical case it is well known that there exist solutions (u, v) of system (SH) of the form (u, u) , such that $r^\gamma u$ is periodic in $t = \ln r$. They correspond to a periodic trajectory for the scalar system (M_{scal}) with $p = 2$, and it admits an infinity of such trajectories. If $\delta \neq \mu$, does there exist solutions (u, v) such that $(r^\gamma u, r^\xi v)$ is periodic in t , in other words a periodic trajectory for system (MH) ?

2) In the supercritical case, we cannot prove that the regular trajectory \mathcal{T} converges to M_0 , that means $\lim_{r \rightarrow \infty} r^\gamma u = A$, $\lim_{r \rightarrow \infty} r^\xi v = B$. Here $\mathcal{E}(0) = 0$, \mathcal{E} is nonincreasing, then \mathcal{E} is negative. The only fixed points of negative energy are M_0, G_0, H_0 , but a G.S. satisfies (5.5), then it tends to $(0, 0)$ at ∞ , hence \mathcal{T} cannot converge to G_0 or H_0 from Proposition 4.9; but we cannot prove that \mathcal{T} converges to some fixed point.

7 A nonvariational system

Here we consider the system (S) with $p = q = 2$, $a = b$ and $s = m \neq 0$.

$$(SN) \begin{cases} -\Delta u = |x|^a u^s v^\delta, \\ -\Delta v = |x|^a u^\mu v^s, \end{cases}$$

where $D = \delta\mu - (1 - s)^2 > 0$. We have seen that we can reduce the system to the case $a = 0$, which will be supposed in this paragraph. Here $X = -ru'/u$, $Y = -rv'/v$, $Z(t) = -ru^s v^\delta / u'$, $W(t) = -ru^\mu v^s / v'$, and the system (M) becomes

$$(MN) \begin{cases} X_t = X [X - (N - 2) + Z], \\ Y_t = Y [Y - (N - 2) + W], \\ Z_t = Z [N - sX - \delta Y - Z], \\ W_t = W [N - \mu X - sY - W]. \end{cases}$$

We have chosen this system because it is not variational, and from Proposition 4.3, it can exist an hyperbola of values (δ, μ) for which the linearized system at M_0 has two imaginary roots.

We first recall two results of [38] and [26], and complete them:

Proposition 7.1 (i) [38] Assume $s < \frac{N}{N-2}$, and $\delta + 1 - s > 0$, $\mu + 1 - s > 0$. Let \mathcal{H}'_s be the hyperbola of equation

$$\frac{1}{\delta + 1} + \frac{1}{\mu + 1} = \frac{N - 2}{N - (N - 2)s} \quad (7.1)$$

located under \mathcal{H}_s . Then under \mathcal{H}'_s , the system (SN) admits no G.S., and then there is a solution of the Dirichlet problem in a ball.

(ii) [26] Let \mathcal{H}_0 be the hyperbola defined by

$$\frac{1}{\delta + 1} + \frac{1}{\mu + 1} = \frac{N - 2}{N} \quad (7.2)$$

Then above \mathcal{H}_0 there exists no solution of the Dirichlet problem. Thus there exists a G.S.

Proof. (i) We consider an energy function with parameters $\alpha, \beta, \sigma, \theta$:

$$\mathcal{E}(r) = r^N \left[u'v' + \alpha u^{\mu+1} v^s + \beta v^{\delta+1} u^s + \frac{\sigma}{r} v u' + \frac{\theta}{r} u v' \right] \quad (7.3)$$

$$= r^{N-2} u v \Psi_0 = r^{N-2-\gamma-\xi} (ZX)^{\xi/2} (WZ)^{\gamma/2} \Psi_0, \quad (7.4)$$

from (4.2), where

$$\Psi_0(X, Y, Z, W) = XY + \alpha WY + \beta ZX - \sigma X - \theta Y. \quad (7.5)$$

We get

$$\begin{aligned} r^{1-N}(uv)^{-1}\mathcal{E}'(r) &= (\sigma + \theta - (N - 2))XY + (N\alpha - \theta)YW \\ &\quad + (N\beta - \sigma)XZ - (\alpha(\mu + 1) - 1)XYW - (\beta(\delta + 1) - 1)XYZ - \alpha sY^2W - \beta sX^2Z. \end{aligned}$$

Taking $\alpha = \frac{1}{\mu+1}, \beta = \frac{1}{\delta+1}$, one finds

$$r^{3-N}(uv)^{-1}\mathcal{E}'(r) = (\sigma + \theta - (N - 2))XY + (N\alpha - \theta - \alpha sY)YW + (N\beta - \sigma - \beta sX)XZ. \quad (7.6)$$

If there exists a G.S., from (5.1) it satisfies $X, Y < N - 2$, hence

$$r^{3-N}(uv)^{-1}\mathcal{E}'(r) > (\sigma + \theta - (N - 2))XY + ((N - (N - 2)s)\alpha - \theta)YW + ((N - (N - 2)s)\beta - \sigma)XZ \quad (7.7)$$

Taking $\theta = \frac{N-(N-2)s}{\mu+1}, \sigma = \frac{N-(N-2)s}{\delta+1}$, then $\mathcal{E}' > 0$ under \mathcal{H}'_s , and moreover \mathcal{H}'_s is under \mathcal{H}_s , thus $\gamma + \xi > N - 2$. Then $\mathcal{E}(r) = O(r^{N-2-\gamma-\xi})$ tends to 0 at ∞ , which is contradictory.

(ii) Taking $\alpha = \frac{1}{\mu+1} = \frac{\theta}{N}, \beta = \frac{1}{\delta+1} = \frac{\sigma}{N}$, it comes from (7.6)

$$r^{3-N}(uv)^{-1}\mathcal{E}'(r) = \left(\frac{N}{\delta+1} + \frac{N}{\mu+1} - (N-2)\right)XY - \alpha sY^2W - \beta sX^2Z$$

hence $\mathcal{E}' < 0$ when (7.2) holds. At the point R where $u(R) = v(R) = 0$, we find $\mathcal{E}(R) = R^N u'(R)v'(R) > 0$, which is a contradiction. \blacksquare

Remark 7.2 *In the case $\delta = \mu$, then $u = v$ is a solution of the equation $-\Delta U = U^{s+\delta}$. Let $s + \delta > 1$. Then there exists a G.S. as soon as $s + \delta \geq \frac{N+2}{N-2}$. Notice that the point $(\frac{N+2}{N-2} - s, \frac{N+2}{N-2} - s)$ lies on the hyperbola \mathcal{H}_s . Suppose moreover $s \leq \delta$; if $s + \delta \geq \frac{N+2}{N-2}$ there are no positive solutions such that $u(R) = v(R) = 0$ for some $R > 0$; if $s + \delta < \frac{N+2}{N-2}$, then there exists no G.S. Indeed such solutions satisfy $u = v$, from [3, Remark 3.3] results.*

Next we prove our main existence result of Theorem 1.5 which improves in particular Remark ???. The main idea is to introduce a new energy function by adding two terms in X^2 and Y^2 to the energy defined at (7.3).

Proof of Theorem 1.5. From Theorem 1.1, if $s \geq \frac{N+2}{N-2}$, all the regular solutions are G.S.. Thus we can assume $s < \frac{N+2}{N-2}$. Let $j, k \in \mathbb{R}$ be parameters, and

$$\begin{aligned} \Phi(r) &= \mathcal{E}(r) + r^N \left[k \frac{s}{2} \frac{vu'^2}{u} + j \frac{s}{2} \frac{uv'^2}{v} \right] \\ &= r^N \left[u'v' + \alpha u^{\mu+1}v^s + \beta v^{\delta+1}u^s + \frac{\sigma}{r}vu' + \frac{\theta}{r}uv' + k \frac{s}{2} \frac{vu'^2}{u} + j \frac{s}{2} \frac{uv'^2}{v} \right] \\ &= r^{N-2}uv\Psi = r^{N-2-\gamma-\xi}(ZX)^{\xi/2}(WY)^{\gamma/2}\Psi, \end{aligned}$$

where

$$\Psi(X, Y, Z, W) = XY + \alpha WY + \beta ZX - \sigma X - \theta Y + k\frac{s}{2}X^2 + j\frac{s}{2}Y^2.$$

Then

$$\begin{aligned} r^{3-N}(uv)^{-1}\Phi'(r) &= (\sigma + \theta - (N - 2))XY + (N\alpha - \theta)YW + (N\beta - \sigma)XZ \\ &\quad - (\alpha(\mu + 1) - 1)XYW - (\beta(\delta + 1) - 1)XYZ + (j - \alpha)sY^2W + (k - \beta)sX^2Z \\ &\quad + ksX^2[X - (N - 2)] + jsY^2[Y - (N - 2)] + (N - 2 - X - Y)(k\frac{s}{2}X^2 + j\frac{s}{2}Y^2). \end{aligned}$$

We eliminate the terms in Z, W by taking $j = \alpha = \frac{1}{\mu+1}$, $k = \beta = \frac{1}{\delta+1}$, $\theta = N\alpha$, $\sigma = N\beta$. We find after reduction

$$\begin{aligned} \mathcal{B}(X, Y) &:= -\frac{2}{s}r^{3-N}(uv)^{-1}\Phi'(r) \\ &= \beta X^2(N - 2 - X) + \alpha Y^2(N - 2 - Y) + XY \left[\beta X + \alpha Y + \frac{2}{s}(N - 2 - N\alpha - N\beta) \right]. \end{aligned}$$

In particular one finds again the condition of Mitidieri if $N(\alpha + \beta) - (N - 2) \leq 0$.

Next we look at the case where $N(\alpha + \beta) - (N - 2) > 0$. We have $\mathcal{B}(0, Y) = \alpha Y^2(N - 2 - Y) \geq 0$ and $\mathcal{B}(X, 0) = \beta X^2(N - 2 - X) \geq 0$. In particular $\mathcal{B}(0, 0) = 0$. Otherwise $\mathcal{B}(N - 2, Y) = Y\Theta(Y)$ with

$$\Theta(Y) = \alpha Y [2(N - 2) - Y] + (N - 2)((N - 2)\beta + \frac{2}{s}(N - 2 - N\alpha - N\beta))$$

On $[0, N - 2]$, $\Theta(Y) > \Theta(0)$. By hypothesis, (δ, μ) is above \mathcal{C}_s , or equivalently

$$(\alpha + \beta)\frac{N}{N - 2} - 1 \leq \frac{s}{2} \min(\alpha, \beta); \quad (7.8)$$

then $\mathcal{B}(N - 2, Y) \geq 0$ and similarly $\mathcal{B}(X, N - 2) \geq 0$. Then \mathcal{B} is positive on the boundary of the square $K = [0, N - 2] \times [0, N - 2]$ and is zero at $(0, 0)$, $(0, N - 2)$, $(N - 2, 0)$. The curve $\mathcal{B}(X, Y) = 0$ is a cubic with a double point at $(0, 0)$, which is isolated under the condition (7.8): $\mathcal{B}(X, Y) > 0$ near $(0, 0)$, except at this point. Then $\mathcal{B}(X, Y) > 0$ on $\overset{\circ}{K}$.

Suppose that there exists a regular solution such that $X(T) = Y(T) = N - 2$ at the same T . Indeed up to this point (X, Y) stays in K , thus the function Φ is decreasing. There holds $\Phi(0) = 0$, and at the point $R = e^T$, we have

$$\Phi(R) = R^{N-2-\gamma-\xi}(N - 2)^{\xi+\gamma+2} \left[\frac{\alpha W + \beta Z}{N - 2} + 1 - (\beta + \alpha)\left(\frac{N}{N - 2} - \frac{s}{2}\right) \right]$$

then $\Phi(R) > 0$, since $\min(\alpha, \beta) < \alpha + \beta$. Therefore from Theorem 1.1, there exists a G.S. \blacksquare

Remark 7.3 We wonder if the limit curve \mathcal{L}_s for existence of G.S. would be defined by

$$\frac{1}{\delta + 1} + \frac{1}{\mu + 1} = \frac{N - 2}{N - \frac{(N-2)s}{2}}$$

which ensures that $\Phi(R) > 0$, and also $\mathcal{B}(N-2, N-2) > 0$. Like \mathcal{H}_s , this curve cuts the set $\delta = \mu$ at the optimal point $\delta + s = \frac{N+2}{N-2}$. Notice that \mathcal{L}_s is under \mathcal{H}_s . Moreover we have proved the existence of G.S. when (δ, μ) is in the region $\min(\delta + s, \mu + m) > \frac{N+2}{N-2}$, which is above \mathcal{L}_s .

8 The radial potential system

Here we study the nonnegative radial solutions of the system (SP) :

$$(SP) \begin{cases} -\Delta_p u = |x|^a u^s v^{m+1}, \\ -\Delta_q v = |x|^a u^{s+1} v^m, \end{cases}$$

with $a = b, \delta = m + 1, \mu = s + 1$, and we assume (1.5). System (M) becomes

$$(MP) \begin{cases} X_t = X \left[X - \frac{N-p}{p-1} + \frac{Z}{p-1} \right], \\ Y_t = Y \left[Y - \frac{N-q}{q-1} + \frac{W}{q-1} \right], \\ Z_t = Z [N + a - sX - (m+1)Y - Z], \\ W_t = W [N + b - (s+1)X - mY - W]. \end{cases}$$

For this system D, γ and ξ are defined by

$$D = p(1+m) + q(1+s) - pq, \quad (p-1-s)\gamma + p + a = (m+1)\xi, \quad (q-1-m)\xi + q + b = (s+1)\gamma;$$

thus γ and ξ are linked independtly of s, m by

$$p(\gamma + 1) = q(\xi + 1) = \frac{pq(m + s + 2 + a)}{D} \quad (8.1)$$

The system is *variational*. It admits an energy function, given in (1.12); it can be given by a direct computation in terms of X, Y, Z, W :

$$\mathcal{E}(r) = \psi \left[ZW - \frac{s+1}{p} W((N-p) - (p-1)X) - \frac{m+1}{q} Z((N-q) - (q-1)Y) \right], \quad (8.2)$$

where

$$\psi = r^{N-2-a} |u'|^{p-1} |v'|^{q-1} u^{-s} v^{-m} = r^{N-(\gamma+1)p} \left[X^{q(s+1)(p-1)} Y^{p(m+1)(q-1)} Z^{p(q-m-1)} W^{q(p-s-1)} \right]^{1/D}.$$

Then we find

$$\mathcal{E}'(r) = (N + a - (s + 1)\frac{N - p}{p} - (m + 1)\frac{N - q}{q})r^{N-1+a}u^{s+1}v^{m+1}.$$

Thus we define a critical line \mathcal{D} as the set of $(\delta, \mu) = (m + 1, s + 1)$ such that

$$N + a = (m + 1)\frac{N - q}{q} + (s + 1)\frac{N - p}{p}, \quad (8.3)$$

equivalent to $pq(m + s + 2 + a) = ND$, or $N + a = (m + 1)\xi + (s + 1)\gamma$, or

$$(\gamma, \xi) = \left(\frac{N - p}{p}, \frac{N - q}{q}\right)$$

The subcritical case is given by the set of points under \mathcal{D} , equivalently $\gamma > \frac{N-p}{p}$, $\xi > \frac{N-q}{q}$ or $(s + 1)\gamma + (m + 1)\xi > N + a$. The supercritical case is the set of points above \mathcal{D} .

Remark 8.1 *The energy \mathcal{E}_0 of the particular solution associated to M_0 is still negative: $\mathcal{E}_0 = -\frac{D}{pq}r^{N+a-(\gamma+1)p} \left[X_0^{q(p-1)} Y_0^{p(q-1)} Z_0^{q(s+1)} W_0^{p(m+1)} \right]^{1/D}$.*

Remark 8.2 *When $p = q = 2$, another energy function can be associated to the transformation (2.2): the system (2.16) relative to $u(r) = r^{-\gamma}U(t)$, $v(r) = r^{-\xi}V(t)$ is*

$$\begin{cases} U_{tt} + (N - 2 - 2\gamma)U_t - \gamma(N - 2 - \gamma)U + U^s V^{m+1} = 0 \\ V_{tt} + (N - 2 - 2\gamma)V_t - \gamma(N - 2 - \gamma)V + U^{s+1} V^m = 0 \end{cases} \quad (8.4)$$

and the function

$$E(t) = \frac{s+1}{2}(U_t^2 - \gamma(N - 2 - \gamma)U^2) + \frac{m+1}{2}(V_t^2 - \gamma(N - 2 - \gamma)V^2) + U^{s+1}V^{m+1} \quad (8.5)$$

satisfies

$$E_t = -(N - 2 - 2\gamma) [(s + 1)U_t^2 + (m + 1)V_t^2]$$

It differs from \mathcal{E} , even in the critical case. This point is crucial for Section 9.

It has been proved in [34], [35], that in the subcritical case with $a = 0$, there exists a solution of the Dirichlet problem in any bounded regular domain Ω of \mathbb{R}^N ; and in the supercritical case there exists no solution if Ω is starshaped. Here we prove two results of existence or nonexistence of G.S. which seem to be new:

Proof of Theorem 1.6. 1) *Existence or nonexistence results.*

• In the supercritical or critical case there exists a G.S. Indeed from Theorem 1.1, if it were not true, then there exist regular positive solutions of (MP) such that $X(T) = \frac{N-p}{p-1}$ and $Y(T) = \frac{N-q}{q-1}$.

Suppose that there exists such a solution. Then $\mathcal{E} \leq 0$. Then at point T , $\mathcal{E}(R) > 0$, from (8.2), since $W > 0, Z > 0$, which is impossible.

• In the subcritical case, there exists no G.S. Indeed suppose that there exists a G.S. Here \mathcal{E} is nondecreasing, hence $\mathcal{E} \geq 0$. Its trajectory stays in the box \mathcal{A} defined by (5.1), thus it is bounded. If $q \geq m + 1$ and $p \geq s + 1$, then from (8.2), $\mathcal{E}(r) = O(r^{N-(\gamma+1)p})$, then \mathcal{E} tends to 0 at ∞ , which is contradictory. Next consider the general case. We have

$$\begin{aligned} \mathcal{E}(r) &\leq r^{N-(\gamma+1)p} \left[X^{q(p-1)} Y^{p(q-1)} Z^{p(q-m-1)} W^{q(p-s-1)} \right]^{1/D} ZW \\ &= r^{N-(\gamma+1)p} \left[X^{q(p-1)} Y^{p(q-1)} Z^{q(1+s)} W^{p(1+m)} \right]^{1/D}, \end{aligned}$$

then the same result holds. Then from Theorem 1.1 there exists a solution of the Dirichlet problem

2) *Behaviour of the G.S. in the critical case.*

Let \mathcal{T} be the trajectory of a G.S.; then $\mathcal{E}(0) = 0$, thus \mathcal{T} lies on the variety \mathcal{V} of energy 0, also defined by

$$qW [(s+1)((p-1)X - (N-p)) + pZ] = p(m+1)Z [(N-q) - (q-1)Y] \quad (8.6)$$

and $Y < \frac{N-q}{q-1}$, hence $(s+1)((p-1)X - (N-p)) + pZ > 0$. From (5.2), \mathcal{T} starts from $N_0 = (0, 0, N+a, N+b)$ and stays in \mathcal{A} . Eliminating W in system (M), we find a system of three equations

$$\begin{cases} X_t = X \left[X - \frac{N-p}{p-1} + \frac{Z}{p-1} \right], \\ Y_t = YF, \\ Z_t = Z [N+a - sX - (m+1)Y - Z], \end{cases}$$

where

$$F(X, Y, Z) = \frac{1}{q} \left[\frac{N-q}{q-1} - Y \right] \frac{p(m+1-q)Z + q(s+1)((N-p) - (p-1)X)}{(s+1)((p-1)X - (N-p)) + pZ}.$$

(i) If \mathcal{T} converges to a fixed point of the system in $\bar{\mathcal{R}}$, the possible points on \mathcal{V} are $A_0, I_0, J_0, P_0, Q_0, G_0, H_0, R_0, S_0$. The eigenvalues of the linearized problem at A_0 , given by (10.3) satisfy

$$\lambda_1, \lambda_2 > 0, \lambda_3 = N+a - s \frac{N-p}{p-1} - (m+1) \frac{N-q}{q-1} \leq \lambda_4 = N+a - (s+1) \frac{N-p}{p-1} - m \frac{N-q}{q-1} = \lambda^*,$$

since $q \leq p$, and $\lambda_3 < \lambda^*$ unless $q = p$, and then $\lambda_3 = \lambda^* < 0$ from (8.3). Then A_0 can be attained only when $\lambda^* \leq 0$, from Proposition 4.5. And P_0 can be attained only if $q > m + 1$, and

$$\lambda^* \geq 0 \text{ and } q+a < (s+1) \frac{N-p}{p-1} \quad (8.7)$$

from the proof of Proposition 4.6, because $\gamma = \frac{N-p}{p} < \frac{N-p}{p-1}$. We observe that the condition $\lambda^* \geq 0$ joint to (8.3) implies $m+1 < q < p$ and is equivalent to (8.7). Indeed it implies

$$\frac{N-p}{p-1}(s+1) \leq N+a-m\frac{N-q}{q-1} = N+a-\frac{q}{q-1}\left(N+a-\frac{N-q}{q}-(s+1)\frac{N-p}{p}\right);$$

then

$$(s+1)\frac{N-p}{p-1}\frac{q-p}{p} \leq -(a+q),$$

thus $q < p$. And from from (8.3)

$$(N-q)\left(\frac{m+1}{q}-1\right) = q+a-(s+1)\frac{N-p}{p} \leq (s+1)\frac{N-p}{p}\left(\frac{p-q}{p-1}-1\right) < 0,$$

hence $m+1 < q$ and (8.7) follows. By symmetry, Q_0 cannot be attained since $q \leq p$. Then A_0 and P_0 are incompatible, unless $A_0 = P_0$, and P_0 is not attained when $p = q$.

(ii) Next we show that \mathcal{T} converges to A_0 or P_0 . If t is an extremum point of Y , then

$$\left(\frac{m+1}{q}-1\right)Z(t) + \frac{s+1}{p}((N-p)-(p-1)X(t)) = 0. \quad (8.8)$$

It implies $q > m+1$ and

$$X_t(t) = \frac{X(t)Z(t)}{p-1} \left[1 + \frac{p(m+1-q)}{q(s+1)}\right] = \frac{DX(t)Z(t)}{(p-1)q(s+1)} > 0.$$

In the same way, if t is an extremum point of X , then $p > s+1$ and $Y_t(t) > 0$. Near $-\infty$, there holds $X_t, Y_t \geq 0$, and $Z_t, W_t \leq 0$, from the linearization near N_0 . Suppose that X has a maximum at t_0 followed by a minimum at t_1 . Then $p > s+1$, and Y is increasing on $[t_0, t_1]$. And at t_0 we have $(p-1)X(t_0) + Z(t_0) = N-p$ and $X_{tt}(t_0) \leq 0$, thus $Z_t(t_0) \leq 0$; eliminating Z we deduce $p+a+(p-1-s)X(t_0) \leq (m+1)Y(t_0)$ and similarly $(m+1)Y(t_1) \leq p+a+(p-1-s)X(t_1)$; hence $Y(t_1) < Y(t_0)$, which is a contradiction. Then X and Y have at most one maximum, and finally no maximum point. Then X and Y are increasing, and bounded, hence X has a limit in $\left(0, \frac{N-p}{p-1}\right]$ and Y has a limit in $\left(0, \frac{N-q}{q-1}\right]$. Then Z, W are decreasing; indeed at each point where $Z_t = 0$, we have $Z_{tt} = Z(-sX_t - (m+1)Y_t) < 0$, thus it is a maximum, which is impossible.

Then \mathcal{T} converges to a fixed point of the system. Moreover, since X and Y are increasing, it cannot be $I_0, J_0, G_0, H_0, R_0, S_0$. It is necessarily A_0 or P_0 . We distinguish two cases:

- Case $q \leq m+1$. Then \mathcal{T} converges to A_0 , and $\lambda_3, \lambda^* < 0$, then (1.13) follows.
- Case $q > m+1$. Then \mathcal{T} converges to A_0 (resp. P_0) when $\lambda^* \leq 0$ (resp. $\lambda^* \geq 0$). If the inequalities are strict, we deduce the convergence of u and v from Propositions 4.5 and 4.6, and

(1.14) follows. If $\lambda^* = 0$, then $P_0 = A_0$, and $\lambda_3 = \frac{(N-1)(q-p)}{(p-1)(q-1)} < 0$. The projection of the trajectory \mathcal{T} in \mathbb{R}^3 on the plane $Y = \frac{N-q}{q-1}$ satisfies the system

$$X_t = X \left[X - \frac{N-p}{p-1} + \frac{Z}{p-1} \right], \quad Z_t = Z \left[N + a - sX - (m+1) \frac{N-q}{q-1} - Z \right]$$

which presents a saddle point at $(\frac{N-p}{p-1}, 0)$, thus the convergence of X and Z is exponential, in particular we deduce the behaviour of u . The trajectory enters by the central variety of dimension 1, and by computation we deduce that $Y = \frac{N-q}{q-1} - \frac{1}{q-1-m} t^{-1} + O(t^{-2+\varepsilon})$, then (1.15) follows. \blacksquare

9 The nonradial potential system of Laplacians

Here we study the possibly *nonradial* solutions of the system of the preceding Section when $p = q = 2$:

$$(P) \begin{cases} -\Delta u = |x|^a u^s v^{m+1}, \\ -\Delta v = |x|^a u^{s+1} v^m, \end{cases}$$

with $D = s + m$. We solve an open problem of [7]. Recall that the main result of [7] is the obtention of apriori estimates near 0 or ∞ , by using the Bernstein technique introduced in [18] and improved in [8].

In [7] it had been solved in the case $N + a \geq 4$. The problem was open when $N + a < 4$, which implies $N < 6$, and $m + s + 1 > N/(N - 2)$. In the case $m + s + 1 \leq N/(N - 2)$, there are no solutions of the exterior problem, see [6].

In [7] we use the transformation

$$u(r, \theta) = r^{-\gamma} U(t, \theta), \quad v(r, \theta) = r^{-\gamma} V(t, \theta), \quad t = \ln r,$$

extending the transformation of Remark 8.2 to the nonradial case (in fact here t is $-t$ in [7]), leading to the system

$$\begin{aligned} U_{tt} + (N - 2 - 2\gamma)U_t + \Delta_S U - \gamma(N - 2 - \gamma)U + U^s V^{m+1} &= 0 \\ V_{tt} + (N - 2 - 2\gamma)V_t + \Delta_S V - \gamma(N - 2 - \gamma)V + U^{s+1} V^m &= 0 \end{aligned}$$

where Δ_S is the Laplace-Beltrami operator on S_{N-1} , and we define the corresponding energy extending (8.5):

$$\begin{aligned} E(t) &= \frac{s+1}{2} \int_{S^{N-1}} (U_t^2 - |\nabla_S U|^2 - \gamma(N-2-\gamma)U^2) d\theta \\ &+ \frac{m+1}{2} \int_{S^{N-1}} (V_t^2 - |\nabla_S V|^2 - \xi(N-2-\xi)V^2) d\theta + \int_{S^{N-1}} U^{s+1} V^{m+1} d\theta; \end{aligned}$$

and there holds

$$E_t = -(N-2-2\gamma) \int_{S^{N-1}} [(s+1)U_t^2 + (m+1)V_t^2] d\theta$$

Here we construct a new function, extending the Pohozaev function defined at (1.12) to the non-radial case.

Lemma 9.1 *Consider the function $\mathcal{E}(r)$ defined by*

$$\begin{aligned} r^{-N}\mathcal{E}(r) &= \frac{s+1}{2} \int_{S^{N-1}} \left[u_r^2 - r^{-2} |\nabla_S u|^2 + (N-2) \frac{uu_r}{r} \right] d\theta \\ &\quad + \frac{m+1}{2} \int_{S^{N-1}} \left[\left(\frac{\partial v}{\partial \nu} \right)^2 - r^{-2} |\nabla_S v|^2 + (N-2) \frac{vv_r}{r} \right] d\theta + r^a \int_{S^{N-1}} u^{s+1} v^{m+1} d\theta \end{aligned}$$

Then

$$r^{1-N}\mathcal{E}'(r) = (N+a-(s+1))\frac{N-2}{2} - (m+1)\frac{N-2}{2} r^a \int_{S^{N-1}} u^{s+1} v^{m+1} d\theta$$

Proof. In terms of t , we find

$$\begin{aligned} \mathcal{E}(t) &= \mathcal{E}_1(t) + \mathcal{E}_2(t) + \mathcal{E}_3(t), \quad \text{with } \mathcal{E}_1(t) = \frac{s+1}{2} e^{(N-2)t} \int_{S^{N-1}} \left[u_t^2 - |\nabla_S u|^2 + (N-2)uu_t \right] d\theta, \\ \mathcal{E}_2(t) &= \frac{m+1}{2} e^{(N-2)t} \int_{S^{N-1}} \left[v_t^2 - |\nabla_S v|^2 + (N-2)vv_t \right] d\theta, \quad \mathcal{E}_3(t) = e^{(N+a)t} \int_{S^{N-1}} u^{s+1} v^{m+1} d\theta \end{aligned} \tag{9.1}$$

and u, v satisfy

$$(e^{(N-2)t}u_t)_t + e^{(N-2)t}\Delta_S u + e^{(N+a)t}u^s v^{m+1} = 0, \quad (e^{(N-2)t}v_t)_t + e^{(N-2)t}\Delta_S v + e^{(N+a)t}u^{s+1}v^m = 0. \tag{9.2}$$

Multiplying successively by u and by v , and $(s+1)e^{(N-2)t}u_t$ and $(m+1)e^{(N-2)t}v_t$ and adding the equalities we obtain after computation

$$\mathcal{E}_t(t) = (N+a-(s+1))\frac{N-2}{2} - (m+1)\frac{N-2}{2} e^{(N+a)t} \int_{S^{N-1}} u^{s+1} v^{m+1} d\theta. \quad \blacksquare$$

Proof of Theorem 1.7. Suppose that there exists a G.S.. Since $s+m+1 < (N+2+2a)/(N-2)$ then E and \mathcal{E} are increasing and start from 0, then they stay positive. Since $s+m+1 < (N+$

$2)/(N-2)$ we know that either (u, v) behaves as M_0 , then E has a negative limit, which is impossible, or (u, v) is regular at ∞ , and then the only possibility is the following: $m > (N + a)/(N - 2)$, $\lim_{r \rightarrow \infty} u = \alpha > 0$, and

$$\lim_{|x| \rightarrow \infty} |x|^k v = \beta > 0 \text{ or } \lim_{|x| \rightarrow \infty} |x|^k v = 0, \quad \text{with } k = (2 + a)/(m - 1), \quad (9.3)$$

where $\beta = (\alpha^{-(s+1)} k(N-2-k))^{1/(m-1)}$. It implies that $N < 4 - a$ from the majorization of $m + s + 1$ by $\frac{N+2}{N-2}$, and $N < 4 + a$ from the majorization by $\frac{N+2+2a}{N-2}$. In [7] we find that $\lim_{t \rightarrow \infty} E(t) = \infty$, which gives no contradiction. Here we show that $\lim_{t \rightarrow \infty} \mathcal{E}(t) < 0$, which will give a contradiction. First recall the proof of (9.3). Making the substitution

$$u(r, \theta) = u(t, \theta), \quad v(r, \theta) = r^{-k} \mathbf{V}(t, \theta), \quad t = \ln r, \theta \in S_{N-1},$$

we get

$$\begin{cases} u_{tt} + (N-2)u_t + \Delta_S u + e^{-2kt} u^s \mathbf{V}^{m+1} = 0, \\ \mathbf{V}_{tt} + (N-2-2k)\mathbf{V}_t + \Delta_S \mathbf{V} - k(N-2-k)\mathbf{V} + u^{s+1} \mathbf{V}^m = 0. \end{cases} \quad (9.4)$$

Then u, \mathbf{V} are bounded near ∞ , and from [7, Proposition 4.1] u converges exponentially to the constant α , more precisely

$$\| |u - \alpha| + |u_t| + |\nabla_S u| \|_{C^0(S^{N-1})} = O(e^{-(N-2)t}), \quad (9.5)$$

because $k \neq (N-2)/2$ and all the derivatives of \mathbf{V} up to the order 2 are bounded. The equation takes the form

$$\mathbf{V}_{tt} + (N-2-2k)\mathbf{V}_t + \Delta_S \mathbf{V} - k(N-2-k)\mathbf{V} + \alpha^{s+1} \mathbf{V}^m + \varphi = 0$$

where φ and its derivatives up to the order 2 are $O(e^{-(N-2)t})$; then from [7, Theorem 4.1], \mathbf{V} converges to β or to 0 in $C^2(S^{N-1})$.

We define $f(t) = e^{(N-2)t} \int_{S^{N-1}} u_t d\theta = r^{N-1} \int_{S^{N-1}} u_r d\theta$. Then $\mathcal{E}_1(t) = (N-2)\frac{s+1}{2}\alpha f(t) + O((e^{-(N-2)t}))$

from (9.5). Moreover from (9.4),

$$f_t(t) = -e^{(N-2-2k)t} \int_{S^{N-1}} u^s \mathbf{V}^{m+1} d\theta < 0$$

Since u is regular at 0, $f(t) = 0(e^{(N-1)t})$ at $-\infty$, in particular $\lim_{t \rightarrow -\infty} f(t) = 0$. And $f_t(t) = O(e^{(N-2-2k)t}) = O(e^{-t})$ at ∞ , then $f(t)$ has a finite negative limit $-\ell^2$; and $\mathcal{E}_1(t)$ tends to $-(N-2)\frac{s+1}{2}\alpha\ell^2$. Then $\mathcal{E}_2(t) = O(e^{(N-2-2k)t}) = O(e^{-t})$ since $v = e^{-kt}\mathbf{V}$, and \mathbf{V} and its derivatives up to the order 2 are bounded. And $\mathcal{E}_3(t) = O(e^{(N+a-k(m+1))t})$ and $N + a - k(m + 1) < \frac{2-N}{m-1} < 0$. Then \mathcal{E} has a negative finite limit at ∞ , contradiction. \blacksquare

10 Analysis of the fixed points

Proof of Proposition 4.4. (i) Consider a regular solution (u, v) with initial data (u_0, v_0) . Then $(-r^{N-1}|u'|^{p-2}u')' = r^{N-1+a}u_0^s v_0^\delta(1 + o(1))$ as $r \rightarrow 0$, then $-|u'|^{p-2}u' = \frac{1}{N+a}r^{1+a}u_0^s v_0^\delta(1 + o(1))$, thus from (2.1), as $t \rightarrow -\infty$

$$\begin{aligned} X(t) &= \left(\frac{1}{N+a}u_0^{s+1-p}v_0^\delta\right)^{1/(p-1)}e^{(p+a)t/(p-1)}(1 + o(1)), \\ Y(t) &= \left(\frac{1}{N+b}u_0^\mu v_0^{m+1-q}\right)^{1/(q-1)}e^{(q+b)t/(q-1)}(1 + o(1)), \end{aligned}$$

and $\lim_{t \rightarrow -\infty} Z = N + a$, $\lim_{t \rightarrow -\infty} W = (N + b)$. In particular the trajectory tends to N_0 .

(ii) Reciprocally, consider a trajectory converging to N_0 . Setting $Z = N + a + \tilde{Z}$, $W = N + b + \tilde{W}$, the linearized system is

$$X_t = \frac{p+a}{p-1}X, \quad Y_t = \frac{q+b}{q-1}Y, \quad \tilde{Z}_t = (N+a) \left[-sX - \delta Y - \tilde{Z}\right], \quad \tilde{W}_t = (N+b) \left[-\mu X - mY - \tilde{W}\right]. \quad (10.1)$$

The eigenvalues are

$$\lambda_1 = \frac{p+a}{p-1} > 0, \quad \lambda_2 = \frac{q+b}{q-1} > 0, \quad \lambda_3 = -(N+a) < 0, \quad \lambda_4 = -(N+b) < 0. \quad (10.2)$$

The unstable variety \mathcal{V}_u and the stable one \mathcal{V}_s have dimension 2. Notice that \mathcal{V}_s is contained in the set $X = Y = 0$, thus no admissible trajectory converges to N_0 as $t \rightarrow \infty$, and there exists an infinity of admissible trajectories in \mathcal{R} , converging to N_0 as $t \rightarrow -\infty$. Since the convergence is exponential, $X_t = \frac{p+a}{p-1}X(1 + O(e^{ct}))$ with $c > 0$, and then $e^{-(p+a)/(p-1)t}X(t)$ has a positive limit κ , and $e^{-(q+b)/(q-1)t}Y(t)$ has a positive limit ℓ ; from (4.2), (4.1), (u, v) have a positive limit $(u_0, v_0) = ((N+a)\kappa^{p-1})^{(q-1-m)/D}((N+b)\ell^{q-1})^{\delta/D}$, hence (u, v) is a regular solution..

Next we show that for any $\kappa > 0, \ell > 0$ there exists a unique local solution such that $\lim_{t \rightarrow -\infty} e^{-(p+a)t/(p-1)}X(t) = \kappa$ and $\lim_{t \rightarrow -\infty} e^{-(q+b)t/(q-1)}Y = \ell$. On \mathcal{V}_u , we get a system of two equations of the form

$$X_t = X(\lambda_1 + F(X, Y)), \quad Y_t = Y(\lambda_2 + G(X, Y)),$$

where $F = AX + BY + f(X, Y)$ and f is C^∞ and $f_X(0, 0) = f_Y(0, 0) = 0$, similarly for G . Setting $X = e^{\lambda_1 t}(\kappa + x)$, $Y = e^{\lambda_2 t}(\ell + y)$, and assuming $\lambda_2 \geq \lambda_1$ and setting $\rho = e^{\lambda_1 t}$ we lead to

$$x_\rho = \frac{1}{\rho}(\kappa + x)F(\rho(\kappa + x), \rho^{\lambda_2/\lambda_1}(\ell + y)), \quad y_\rho = (\ell + y)G(\rho(\kappa + x), \rho^{\lambda_2/\lambda_1}(\ell + y)),$$

with $x(0) = y(0) = 0$. Then we get local existence and uniqueness. Hence for any $u_0 > 0, v_0 > 0$ there exists a regular solution (u, v) with initial data (u_0, v_0) . Moreover $u, v \in C^1([0, R))$ when $a, b > -1$. \blacksquare

Proof of Proposition 4.5. The linearization at A_0 at this point gives, with $X = \frac{N-p}{p-1} + \tilde{X}$, $Y = \frac{N-q}{q-1} + \tilde{Y}$,

$$\tilde{X}_t = \frac{N-p}{p-1} \left[\tilde{X} + \frac{Z}{p-1} \right], \quad \tilde{Y}_t = \frac{N-q}{q-1} \left[\tilde{Y} + \frac{W}{q-1} \right], \quad Z_t = \lambda_3 Z, \quad W_t = \lambda_4 W.$$

The eigenvalues are

$$\lambda_1 = \frac{N-p}{p-1} > 0, \quad \lambda_2 = \frac{N-q}{q-1} > 0, \quad \lambda_3 = N+a-s \frac{N-p}{p-1} - \delta \frac{N-q}{q-1}, \quad \lambda_4 = N+b-\mu \frac{N-p}{p-1} - m \frac{N-q}{q-1}. \quad (10.3)$$

• Convergence as $r \rightarrow \infty$: If $\lambda_3 > 0$, or $\lambda_4 > 0$, then the stable variety \mathcal{V}_s has at most dimension 1, it satisfies $W = 0$ or $Z = 0$, hence there is no admissible trajectory converging to A_0 at ∞ . If $\lambda_3 < 0$, and $\lambda_4 < 0$, then \mathcal{V}_s has dimension 2, and contains trajectories such that $Z > 0$ and $W > 0$, then in \mathcal{R} ; and $\mathcal{V}_s \cap \{Z = 0\}$ has dimension 1 : indeed we get a system in X, Y, W with the eigenvalues $\lambda_1, \lambda_2, \lambda_4$; similarly $\mathcal{V}_s \cap \{W = 0\}$ has dimension 1; then there exist admissible trajectories. They satisfy $\lim e^{-\lambda_3 t} Z = C_3 > 0, \lim e^{-\lambda_4 t} W = C_4 > 0$, then (4.11) follows from (4.2).

• Convergence as $r \rightarrow 0$: If $\lambda_3 < 0$, or $\lambda_4 < 0$, the unstable variety \mathcal{V}_u has at most dimension 3, and it satisfies $W = 0$ or $Z = 0$, there is no admissible trajectory converging at $-\infty$. If $\lambda_3 > 0$, and $\lambda_4 > 0$, then \mathcal{V}_u has dimension 4, there exist admissible trajectories, and (4.11) follows as above. ■

Proof of Proposition 4.6. We set $P_0 = \left(\frac{N-p}{p-1}, Y_*, 0, W_* \right)$, for $m+1 \neq q$. The linearization gives, with $X = \frac{N-p}{p-1} + \tilde{X}$, $Y = Y_* + \tilde{Y}$, $W = W_* + \tilde{W}$,

$$\tilde{X}_t = \frac{N-p}{p-1} \left[\tilde{X} + \frac{Z}{p-1} \right], \quad \tilde{Y}_t = Y_* \left[\tilde{Y} + \frac{\tilde{W}}{q-1} \right], \quad Z_t = \lambda_3 Z, \quad \tilde{W}_t = W_* \left[-\mu \tilde{X} - m \tilde{Y} - \tilde{W} \right]$$

The eigenvalues are

$$\lambda_1 = \frac{N-p}{p-1} > 0, \quad \lambda_3 = N+a-s \frac{N-p}{p-1} - \delta Y_* = \frac{D}{q-1-m} \left(\gamma - \frac{N-p}{p-1} \right),$$

and the roots λ_2, λ_4 of equation

$$\lambda^2 - (Y_* - W_*)\lambda + \frac{m+1-q}{q-1} Y_* W_* = 0$$

Then if $\lambda_3 < 0$ (resp. $\lambda_3 > 0$) there is no admissible trajectory converging at $-\infty$ (resp. ∞). Indeed $\mathcal{V}_u = \mathcal{V}_u \cap \{Z = 0\}$ (resp. $\mathcal{V}_s = \mathcal{V}_s \cap \{Z = 0\}$).

1) Suppose that $q > m+1$. Since $q+b < \frac{N-p}{p-1} \mu < N+b-m \frac{N-q}{q-1}$, we have $P_0 \in \mathcal{R}$, and $\lambda_2 \lambda_4 < 0$. First assume $\lambda_3 < 0$, that means $\gamma < \frac{N-p}{p-1}$; then \mathcal{V}_s has dimension 2, and $\mathcal{V}_s \cap \{Z = 0\}$

has dimension 1, there exists trajectories with $Z > 0$, then admissible, converging as $t \rightarrow \infty$. Next assume $\lambda_3 > 0$; then \mathcal{V}_u has dimension 3, and $\mathcal{V}_u \cap \{Z = 0\}$ has dimension 2, then there exist admissible trajectories converging as $t \rightarrow -\infty$.

2) Suppose that $q < m + 1$. Since $q + b > \frac{N-p}{p-1}\mu > N + b - m\frac{N-q}{q-1}$, we have $P_0 \in \mathcal{R}$, and $\lambda_2\lambda_4 > 0$. We assume $q\frac{N-p}{p-1}\mu + m(N-q) \neq N(q-1) + (b+1)q$, that means $Y_* \neq W_*$. First suppose $\lambda_3 > 0$, that means $\gamma < \frac{N-p}{p-1}$; either $\text{Ré}\lambda_2, \text{Ré}\lambda_4 > 0$ then \mathcal{V}_u has dimension 4, or $\text{Ré}\lambda_2, \text{Ré}\lambda_4 < 0$ then \mathcal{V}_u has dimension 2 and $\mathcal{V}_u \cap \{Z = 0\}$ has dimension 1; in any case, there exist admissible trajectories converging as $t \rightarrow -\infty$. Next assume $\lambda_3 < 0$; either $\text{Ré}\lambda_2, \text{Ré}\lambda_4 > 0$, \mathcal{V}_s has dimension 1, and $\mathcal{V}_s \cap \{Z = 0\} = \emptyset$ or $\text{Ré}\lambda_2, \text{Ré}\lambda_4 < 0$, then \mathcal{V}_s has dimension 3, it contains trajectories with $Z > 0$, then admissible, converging as $t \rightarrow \infty$.

Then $\lim e^{-\lambda_3 t} Z = C_3 > 0$, $\lim X = \frac{N-p}{p-1}$, $\lim Y = Y_*$ and $\lim W = W_*$, thus (4.12) follows from (4.2) and (2.5). \blacksquare

Proof of Proposition 4.8. The linearization near I_0 gives, with $X = \frac{N-p}{p-1} + \tilde{X}$,

$$\tilde{X}_t = \frac{N-p}{p-1} \left(\tilde{X} + \frac{Z}{p-1} \right), \quad Y_t = -\frac{N-q}{q-1} Y, \quad Z_t = (N+a-s\frac{N-p}{p-1})Z, \quad W_t = (N+b-\mu\frac{N-p}{p-1})W.$$

The eigenvalues are

$$\lambda_1 = \frac{N-p}{p-1} > 0, \quad \lambda_2 = -\frac{N-q}{q-1} < 0, \quad \lambda_3 = N+a-s\frac{N-p}{p-1}, \quad \lambda_4 = N+b-\mu\frac{N-p}{p-1}.$$

- Convergence as $r \rightarrow \infty$: If $\lambda_3 > 0$ or $\lambda_4 > 0$, then $\mathcal{V}_s = \mathcal{V}_s \cap \{Z = 0\}$ or $\mathcal{V}_s = \mathcal{V}_s \cap \{W = 0\}$, there is no admissible trajectory converging at ∞ . Suppose that $\lambda_3 < 0$ and $\lambda_4 < 0$. Then \mathcal{V}_s has dimension 3; it contains trajectories with $Y, Z, W > 0$, then admissible. Then $\lim X = \frac{N-p}{p-1}$, $\lim e^{-\lambda_2 t} Y = C_2 > 0$, $\lim e^{-\lambda_3 t} Z = C_3 > 0$, $\lim e^{-\lambda_4 t} W = C_4 > 0$, then (4.13) follows from (4.2) and (2.4).

- Convergence as $r \rightarrow 0$: Since $\lambda_2 < 0$ we have $\mathcal{V}_u = \mathcal{V}_u \cap \{Y = 0\}$, hence there is no admissible trajectory converging as $r \rightarrow 0$. \blacksquare

Proof of Proposition 4.9. The point $G_0 = (\frac{N-p}{p-1}, 0, 0, N+b-\frac{N-p}{p-1}\mu) \in \mathcal{R}$ since $\frac{N-p}{p-1}\mu < N+b$. The linearization gives, with $X = \frac{N-p}{p-1} + \tilde{X}$, $W = N+b-\frac{N-p}{p-1}\mu + \tilde{W}$,

$$\begin{aligned} \tilde{X}_t &= \frac{N-p}{p-1} \left[\tilde{X} + \frac{Z}{p-1} \right], & Y_t &= \frac{Y}{q-1} \left(q+b-\frac{N-p}{p-1}\mu \right), \\ Z_t &= (N+a-s\frac{N-p}{p-1})Z, & W_t &= (N+b-\frac{N-p}{p-1}\mu) \left[-\mu\tilde{X} - mY - \tilde{W} \right] \end{aligned}$$

The eigenvalues are

$$\lambda_1 = \frac{N-p}{p-1} > 0, \quad \lambda_2 = \frac{1}{q-1} \left(q+b-\frac{N-p}{p-1}\mu \right), \quad \lambda_3 = N+a-s\frac{N-p}{p-1}, \quad \lambda_4 = \frac{N-p}{p-1}\mu - N-b < 0.$$

- Convergence as $r \rightarrow \infty$: If $\lambda_2 > 0$, or $\lambda_3 > 0$, then $\mathcal{V}_s = \mathcal{V}_s \cap \{Y = 0\}$ or $\mathcal{V}_s = \mathcal{V}_s \cap \{Z = 0\}$, there is no admissible trajectory converging at ∞ . Assume $\lambda_2, \lambda_3 < 0$, then \mathcal{V}_s has dimension 3, it contains trajectories with $Y, Z > 0$, then admissible.
- Convergence as $r \rightarrow 0$: If $\lambda_3 < 0$, or $\lambda_2 < 0$ there is no admissible trajectory. If $\lambda_2, \lambda_3 > 0$ then \mathcal{V}_s has dimension 3, it contains admissible trajectories.

In any case $\lim X = \frac{N-p}{p-1}$, $\lim e^{-\lambda_2 t} Y = C_2 > 0$, $\lim e^{-\lambda_3 t} Z = C_3 > 0$, $\lim W = N + b - \frac{N-p}{p-1} \mu$, hence (4.13) still follows from (4.2) and (2.4). \blacksquare

Proof of Proposition 4.10. We set $C_0 = (0, \bar{Y}, 0, \bar{W})$. It is in \mathcal{R} if $\frac{N-q}{q-1} m > N + b$, implying $q < m + 1$. The linearization at this point gives, with $Y = \bar{Y} + \tilde{Y}$ and $W = \bar{W} + \tilde{W}$

$$X_t = -\frac{N-p}{p-1} X, \quad \tilde{Y}_t = \bar{Y} \left[\tilde{Y} + \frac{\tilde{W}}{q-1} \right], \quad Z_t = \lambda_3 Z, \quad W_t = \bar{W} \left[-\mu X - m\tilde{Y} - \tilde{W} \right].$$

The eigenvalues are

$$\lambda_1 = -\frac{N-p}{p-1}, \quad \lambda_3 = N + a - \delta \bar{Y},$$

and the roots λ_2, λ_4 of equation

$$\lambda^2 - (\bar{Y} - \bar{W})\lambda + \frac{m+1-q}{q-1} \bar{Y} \bar{W} = 0 \quad (10.4)$$

then $\lambda_2 \lambda_4 > 0$. We assume $m \neq \frac{N(q-1)+(b+1)q}{N-q}$, that means $\bar{Y} \neq \bar{W}$. ((and $\bar{Y} - \bar{W} = \frac{(N+b)q-(m+1)(N-q)}{m+1-q}$; then $\bar{Y} - \bar{W} > 0$ if m is subcritical.)).

- Convergence as $r \rightarrow \infty$: if $\lambda_3 > 0$ we have $\mathcal{V}_s = \mathcal{V}_s \cap \{Z = 0\}$, hence there is no admissible trajectory .there is no solution near ∞ . Next assume that $\lambda_3 < 0$, that means $\delta > (N+a) \frac{m+1-q}{q+b}$. If $\text{Ré}(\lambda_2) = \text{Ré}\lambda_4 < 0$ (resp. > 0) then \mathcal{V}_s has dimension 4 (resp. 2) and $\mathcal{V}_s \cap \{X = 0\}$ and $\mathcal{V}_s \cap \{Z = 0\}$ have dimension 3 (resp. 1) then there exist trajectories with $X, Z > 0$, then admissible.

In any case $\lim e^{-\lambda_1 t} X = C_1 > 0$, $\lim Y = \bar{Y}$, $\lim e^{-\lambda_3 t} Z = C_3 > 0$, $\lim W = \bar{W}$, then (4.15) follows as above.

- Convergence as $r \rightarrow 0$: Since $\lambda_1 < 0$ we have $\mathcal{V}_u = \mathcal{V}_u \cap \{X = 0\}$, hence there is no admissible trajectory . \blacksquare

Proof of Proposition 4.11. We set $R_0 = (0, \bar{Y}, \bar{Z}, \bar{W})$. Under our assumptions it lies in \mathcal{R} . Setting $Y = \bar{Y} + \tilde{Y}, Z = \bar{Z} + \tilde{Z}, W = \bar{W} + \tilde{W}$, the linearization gives

$$X_t = \lambda_1 X, \quad \tilde{Y}_t = \bar{Y} \left[\tilde{Y} + \frac{\tilde{W}}{q-1} \right], \quad Z_t = \bar{Z} \left[-sX - \delta \tilde{Y} - \tilde{Z} \right], \quad W_t = \bar{W} \left[-\mu X - m\tilde{Y} - \tilde{W} \right];$$

the eigenvalues are

$$\lambda_1 = \frac{1}{p-1} (p + a - \delta \frac{b+q}{m+1-q}), \quad \lambda_3 = -\bar{Z} < 0;$$

and the roots λ_2, λ_4 of equation of equation (10.4).

• Convergence as $r \rightarrow \infty$: If $\lambda_1 > 0$, that means $(p+a)\frac{m+1-q}{q+b} < \delta$, then $\mathcal{V}_s = \mathcal{V}_s \cap \{X = 0\}$, hence there is no admissible trajectory . Next assume $\lambda_1 < 0$; if $\text{Ré}(\lambda_2) = \text{Ré}\lambda_4 < 0$ (resp. > 0) then \mathcal{V}_s has dimension 4(resp. 2) and $\mathcal{V}_s \cap \{X = 0\}$ has dimension 3 (resp. 1) then there exist admissible trajectories.

• Convergence as $r \rightarrow 0$: If $\lambda_1 < 0$, then $\mathcal{V}_u = \mathcal{V}_u \cap \{X = 0\}$, hence there is no admissible trajectory .Next assume $\lambda_1 > 0$; if $\text{Ré}(\lambda_2) = \text{Ré}\lambda_4 < 0$ (resp. > 0) then \mathcal{V}_s has dimension 4 (resp. 2) and $\mathcal{V}_s \cap \{X = 0\}$ has dimension 3 (resp. 1) then there exist admissible trajectories.

In any case $\lim e^{-\lambda_1 t} X = C_1 > 0$, $\lim Y = \bar{Y}$, $\lim Z = \bar{Z}$, $\lim W = \bar{W}$, then (4.15) holds again.

■

Remark 10.1 Finally there is no admissible trajectory converging to $0, K_0, L_0$. Indeed the linearization at 0 gives

$$X_t = -\frac{N-p}{p-1}X, \quad Y_t = -\frac{N-q}{q-1}Y, \quad Z_t = (N+a)Z, \quad W_t = (N+b)W$$

Then \mathcal{V}_s and \mathcal{V}_u have dimension 2; then \mathcal{V}_s is contained in $\{Z = W = 0\}$ \mathcal{V}_u in $\{X = Y = 0\}$. The linearization at K_0 gives, with $Z = N+a + \tilde{Z}$,

$$X_t = \frac{p+a}{p-1}X, \quad Y_t = -\frac{N-q}{q-1}Y, \quad Z_t = (N+a) \left[-sX - \delta Y - \tilde{Z} \right], \quad W_t = (N+b)W.$$

The eigenvalues are $\frac{p+a}{p-1}, -\frac{N-q}{q-1}, -(N+a), N+b$. Then \mathcal{V}_s and \mathcal{V}_u have dimension 2; \mathcal{V}_s is contained in $\{Z = W = 0\}$, and \mathcal{V}_u in $\{Y = 0\}$. The case of L_0 follows by symmetry.

References

- [1] Azizier C., Clément P. and Mitidieri E., *Existence and a Priori Estimates for Positive Solutions of p-Laplace Systems*, Journal of Differential Equations 184 (2002), 422–442.
- [2] Bidaut-Véron M.F., *Local and global behavior of solutions of quasilinear equations of Emden-Fowler type*, Arc. Rat. Mech. and Anal., 107 (1989), 293-324.
- [3] Bidaut-Véron M.F., *Local behaviour of solutions of a class of nonlinear elliptic systems*, Adv. in Diff. Equ. 5 (2000), 147-192.
- [4] Bidaut-Véron M.F. and Grillo P., *Singularities in Elliptic systems with absorption terms*, Ann. Scuola Norm. Sup. Pisa, 28 (1999), 229-271.
- [5] Bidaut-Véron M.F. and Grillo P., *Asymptotic behaviour of elliptic systems with mixed absorption and source terms*, Asymptotic Anal., 19 (1999), 117-147.

- [6] Bidaut-Véron M.F. and Pohozaev S., *Nonexistence results and estimates for some nonlinear elliptic problems*, J. Anal. Mathématique, 84 (2001), 1-49.
- [7] Bidaut-Véron M.F. and Raoux T., *Asymptotics of solutions of some nonlinear elliptic systems*, Comm. Part. Diff. Equ., 21 (1996), 1035-1086.
- [8] Bidaut-Véron M.F. and Véron L., *Nonlinear elliptic equations on compact Riemannian manifolds and asymptotics of Emden equations*, Invent. Math. 106 (1991), 489-539.
- [9] Busca J., and Manasevitch R., *A Liouville-type theorem for Emden system*, Indiana Univ. Math. J. 51 (2002), 37-51.
- [10] Chen S., and Lu G., *Existence and nonexistence of positive radial solutions for a class of semilinear elliptic system*, Nonlinear Analysis 38 (1999) 919-932.
- [11] Chicone, C. and Tian J., *On general properties of quadratic systems*, Proc. Amer. Math. Soc. 89 (1982), 167–178.
- [12] Clément P., Fleckinger J., Mitidieri E. and de Thélin F., *Existence of positive solutions for a nonvariational quasilinear elliptic system*, J. Differential Equations 166 (2000), 455–477.
- [13] Clément P., Garcia-Huidobro M., Guerra I. and Manasevich R., *On regions of existence and nonexistence for a system of p - q Laplacians*, Asymptot. Anal. 48 (2006), 1–18.
- [14] de Figueiredo D., Peral I. and Rossi J., *The critical hyperbola for a Hamiltonian elliptic system with weights*, Annali di Matematica 187(2008), 531–545
- [15] de Figueiredo D. and Felmer P., *A Liouville type theorem for elliptic systems*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 2& (1994), 387-397.
- [16] Fowler R.H., *Further studies of Emden's and similar differential equations*, Quart. J. Math 2 (1931), 259-288.
- [17] Garcia M., Manasevitch R., Mitidieri E., and Yarur C., *Existence and nonexistence of singular positive solutions for a class of semilinear elliptic systems*, Arch. Rat. Mech. Anal. 140 (1997), 253-284.
- [18] Gidas and Spruck J., *Global and local behavior of positive solutions of nonlinear elliptic equations*, Comm. Pure Appl. Math. 34 (1981), 525–598.
- [19] Guedda M. and Véron L, *Local and global properties of solutions of nonlinear elliptic equations*, J. Diff. Equ., 76 (1988), 159–189.
- [20] Hülshof J. and vander Vorst R., *Asymptotic behaviour of ground states*, Proc. Amer. Math. Soc. 124 (1996), 2423-2431.

- [21] Lin C.S., *A classification of solutions of a conformally invariant fourth order equation in \mathbb{R}^N* , Comment. Math. Helv. 73 (1998), 206–231.
- [22] Lions P.L., *The concentration-compactness principle in the calculus of variations, part 1*, Rev. Mat. Iberoam. 1 (1985), 145-201.
- [23] Liu F. and Yang J., *Nontrivial solutions of Hardy-Henon type elliptic systems*, preprint.
- [24] Lu Q. and Yang Z., *Blow-up estimates for a quasi-linear reaction–diffusion system*, Math. Meth. Appl. Sci. 26 (2003), 1005–1023.
- [25] Mitidieri E., *A Rellich type identity and applications*, Comm. P.D.E. 18 (1993), no. 1-2, 125–151.
- [26] Mitidieri E., *Nonexistence of positive solutions of semilinear elliptic systems in \mathbb{R}^N* , Diff. Int. Equ. 9 (1996), 465-479.
- [27] Raoux T., Thèse de Doctorat, Université de Tours (1995).
- [28] Reichel W. and Zou H., *Nonexistence results for semilinear cooperative systems via moving spheres*, J. Diff. Equ., 161 (2000), 219-243.
- [29] Serrin J. and Zou H., *Existence of positive solutions of the Lane-Emden system*, Atti Sem. Mat. Fis. Univ. Modena, 46 (1998), suppl., 369-380.
- [30] Serrin J. and Zou H., *Existence of positive entire solutions of elliptic Hamiltonian systems*, Comm. Partial Differential Equations 23 (1998), 577–599.
- [31] Serrin J. and Zou H., *Non-existence of positive solutions of Lane-Emden systems*, Diff. Int. Equ., 9 (1996), 635-653.
- [32] Serrin J. and Zou H., *Non-existence of positive solutions of semilinear elliptic systems*, A tribute to Ilya Bakelman (College Station, TX, 1993), 55–68, Discourses Math. Appl., 3, Texas A & M Univ., College Station, TX, 1994.
- [33] Souplet P., *The proof of the Lane-Emden conjecture in 4 space dimensions*, Adv. Math. 221 (2009), 1409–1427.
- [34] De Thélin F. and Vélin J., *Existence et non-existence de solutions non triviales pour des systèmes elliptiques non linéaires*, C.R. Acad. Sci. Paris 313 (1991), 589 592.
- [35] De Thélin F. and Vélin J., *Existence and nonexistence of nontrivial solutions for some non-linear elliptic systems*, Rev. Mat. Univ. Comput. Madrid 6 (1993), 153 194.
- [36] vanderVorst R., *Variational identities and applications to differential systems*, Arch. Rat. Mech. Anal. 116 (1991), 375-398.

- [37] Véron L., *Singularities of solutions of second order quasilinear equations*, Pitman Research Notes in Math., Longman (1995).
- [38] Zheng S., *Nonexistence of positive solutions to a semilinear elliptic system and blow-up estimates for a reaction–diffusion system*, J. Math. Anal. Applic. 232 (1999), 293–311.
- [39] Zou H., *A priori estimates and existence on strongly coupled cooperative elliptic systems*, Glasg. Math. J. 48 (2006), 437-457.
- [40] Zou H., *A priori estimates and existence for strongly coupled cooperative elliptic systems*, Comm. Part. Diff. Equ. 31 (2006), 735-773
- [41] Zou H., *A note on non-existence results for semi-linear cooperative elliptic systems via moving spheres*, Proc. Amer. Math. Soc. 134 (2006), 1635-1646.
- [42] Zou H., *Existence and non-existence for strongly coupled quasi-linear cooperative elliptic systems*, J. Math. Soc. Japan 59 (2007), 393-421.