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# A new dynamical approach of Emden-Fowler equations and systems

Marie Françoise BIDAUT-VERON\*

Hector GIACOMINI†

## Abstract

We give a new approach on general systems of the form

$$(G) \begin{cases} -\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u) = \varepsilon_1 |x|^a u^s v^\delta, \\ -\Delta_q v = \operatorname{div}(|\nabla v|^{q-2} \nabla v) = \varepsilon_2 |x|^b u^\mu v^m, \end{cases}$$

where  $Q, p, q, \delta, \mu, s, m, a, b$  are real parameters,  $Q, p, q \neq 1$ , and  $\varepsilon_1 = \pm 1, \varepsilon_2 = \pm 1$ . In the radial case we reduce the problem to a quadratic system of order 4, of Kolmogorov type. Then we obtain new local and global existence or nonexistence results. In the case  $\varepsilon_1 = \varepsilon_2 = 1$ , we also describe the behaviour of the ground states in two cases where the system is variational. We give an important result on existence of ground states for a nonvariational system with  $p = q = 2$  and  $s = m > 0$ . In the nonradial case we solve a conjecture of nonexistence of ground states for the system with  $p = q = 2$  and  $\delta = m + 1$  and  $\mu = s + 1$ .

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\*Laboratoire de Mathématiques et Physique Théorique, CNRS UMR 6083, Faculté des Sciences, 37200 Tours France. E-mail address:veronmf@univ-tours.fr

†Laboratoire de Mathématiques et Physique Théorique, CNRS UMR 6083, Faculté des Sciences, 37200 Tours France. E-mail address:Hector.Giacomini@lmpt.univ-tours.fr

# 1 Introduction

In this paper we consider the nonnegative solutions of Emden-Fowler equations or systems in  $\mathbb{R}^N (N \geq 1)$ ,

$$-\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u) = \varepsilon_1 |x|^a u^Q, \quad (1.1)$$

$$(G) \begin{cases} -\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u) = \varepsilon_1 |x|^a u^s v^\delta, \\ -\Delta_q v = \operatorname{div}(|\nabla v|^{q-2} \nabla v) = \varepsilon_2 |x|^b u^\mu v^m, \end{cases} \quad (1.2)$$

where  $Q, p, q, \delta, \mu, s, m, a, b$  are real parameters,  $Q, p, q \neq 1$ , and  $\varepsilon_1 = \pm 1, \varepsilon_2 = \pm 1$ .

In Section 2 we give a new way of studying the radial solutions, reducing system (G) to a quadratic autonomous system:

$$(M) \begin{cases} X_t = X \left[ X - \frac{N-p}{p-1} + \frac{Z}{p-1} \right], \\ Y_t = Y \left[ Y - \frac{N-q}{q-1} + \frac{W}{q-1} \right], \\ Z_t = Z [N + a - sX - \delta Y - Z], \\ W_t = W [N + b - \mu X - mY - W], \end{cases}$$

where  $t = \ln r$ , and

$$X(t) = -\frac{ru'}{u}, \quad Y(t) = -\frac{rv'}{v}, \quad Z(t) = -\varepsilon_1 r^{1+a} u^s v^\delta \frac{u'}{|u'|^p}, \quad W(t) = -\varepsilon_2 r^{1+b} u^\mu v^m \frac{v'}{|v'|^q}. \quad (1.3)$$

This system is of Kolmogorov type. The reduction is valid for equations and systems with source terms ( $\varepsilon_1 = \varepsilon_2 = 1$ ), absorption terms ( $\varepsilon_1 = \varepsilon_2 = -1$ ), or mixed systems ( $\varepsilon_1 = -\varepsilon_2 = 1$ ). It is remarkable that in the new system,  $p$  and  $q$  appear only as simple coefficients, which allows to treat any value of the parameters, for example  $p$  or  $q < 1$ ,  $s, m, \delta$  or  $\mu < 0$ .

In Section 3 we study the scalar case (1.1), where (G) becomes two-dimensional. We show that the phase plane of system  $(M_G)$  gives at the same time the behaviour of the two equations

$$-\Delta_p u = |x|^a u^Q \quad \text{and} \quad -\Delta_p u = -|x|^a u^Q,$$

which is a kind of unification of the two problems, with source terms or absorption terms. For the case of source term ( $\varepsilon_1 = 1$ ), we find again the results of [2], [19], showing that the new dynamical approach is simple and does not need regularity results or energy functions. Moreover it gives a model for the study of the system (G). Indeed if  $p = q$ ,  $a = b$  and  $\delta + s = \mu + m$ , the system (G) admits solutions of the form  $(u, u)$ , where  $u$  is a solution of (1.1) with  $Q = \delta + s$ .

In the sequel of the article we apply the reduction to the case of source terms:  $(G) = (S)$ , where

$$(S) \begin{cases} -\Delta_p u = |x|^a u^s v^\delta, \\ -\Delta_q v = |x|^b u^\mu v^m. \end{cases} \quad (1.4)$$

This system has been the subject of a very rich litterature, in particular the Hamiltonian problem  $s = m = 0$ , in the linear case  $p = q = 2$ , see or example [20], [31], [29], [9], [33], [14], and the potential system where  $\delta = m + 1$ ,  $\mu = s + 1$  and  $a = b$ , see [7], [34], [35]; the problem with general powers has been studied in [3], [39], [40], [41] in the linear case and [6], [12], [42] in the quasilinear case , see also [1], [10], [13].

Here we suppose that  $\delta, \mu > 0$ , so that the system is always coupled,  $s, m \geq 0$ , and we assume for simplicity

$$1 < p, q < N, \quad \min(p + a, q + b) > 0, \quad D = \delta\mu - (p - 1 - s)(q - 1 - m) > 0 \quad (1.5)$$

We say that a positive solution  $(u, v) \in (C^2((0, R)))^2$  is regular at 0 if  $u, v$  are continuous at 0; the condition  $\min(p + a, q + b) > 0$  guaranties the existence of local regular solutions; then  $u, v$  are  $C^1$  when  $a, b > -1$ , with  $u'(0) = v'(0) = 0$ . We are interested in the existence or nonexistence of ground states, called G.S., that means global positive  $(u, v)$  in  $(0, \infty)$  and regular at 0. We exclude the case of "trivial" solutions,  $(u, v) = (0, C)$  or  $(C, 0)$ , where  $C$  is a constant, which can exist when  $s > 0$  or  $m > 0$ . The assumption  $D > 0$  is a classical condition of superlinearity for the system.

In Section 4 we give a series of local existence or nonexistence results concerning the system  $(S)$ , which complete the nonexistence results found in the litterature. They are not based on the fixed point method, quite hard in general, see for example [19], [27], but on a dynamical analysis of the linearization of system  $(M_S)$  near each fixed point, which appears to be performant, even for the regular solutions. For a better exposition, the proofs are given at Section 10.

In Section 5 we study the global existence of G.S. It has been often compared with the nonexistence of positive solutions of the Dirichlet problem in a ball, see [29], [30], [12], [13]. Here we use a shooting method adapted to system  $(M_S)$ , which allows to avoid questions of regularity of system  $(S)$ . We give a new way of comparison, and improve the former results:

**Theorem 1.1** (i) Assume  $s < \frac{N(p-1)+p+pa}{N-p}$ ,  $m < \frac{N(q-1)+q+qb}{N-q}$ . If system  $(S)$  has no G.S., then

(i) there exist regular solutions such that  $X(T) = \frac{N-p}{p-1}$  and  $Y(T) = \frac{N-q}{q-1}$  for some  $T > 0$ , with  $0 < X < \frac{N-p}{p-1}$  and  $0 < Y < \frac{N-q}{q-1}$  on  $(-\infty, T)$ .

(ii) there exists a positive radial solution  $(u, v)$  of the Dirichlet problem in a ball  $B(0, R)$ .

It is a key tool in the next Sections for proving the existence of a G.S. It gives also new existence results for the Dirichlet problem, see Corollary 5.3. We also give a complementary result:

**Proposition 1.2** Assume  $s \geq \frac{N(p-1)+p+pa}{N-p}$  and  $m \geq \frac{N(q-1)+q+qb}{N-q}$ . Then all the regular solutions are G.S.

In Section 6 we study the radial solutions of the well known Hamiltonian system

$$(SH) \begin{cases} -\Delta u = |x|^a v^\delta, \\ -\Delta v = |x|^b u^\mu, \end{cases}$$

where  $p = q = 2$ ,  $s = m = 0$ , which is variational. In the case  $a = b = 0$ , a main conjecture was made:

**Conjecture 1.3** *The system (SH) with  $a = b = 0$  admits no (radial or nonradial) G.S. if and only if  $(\delta, \mu)$  is under the hyperbola of equation*

$$\frac{N}{\delta + 1} + \frac{N}{\mu + 1} = N - 2.$$

The question is still open; it was solved in the radial case in [26], [29], then partially in [31], [9], and up to the dimension  $N = 4$  in [33], see references therein. Here we find again and extend to the case  $a, b \neq 0$  some results of [20] relative to the G.S., with a shorter proof. We also give an existence result for the Dirichlet problem improving a result of [14].

**Theorem 1.4** *Let  $\mathcal{H}_0$  be the critical hyperbola in the plane  $(\delta, \mu)$  defined by*

$$\frac{N + a}{\delta + 1} + \frac{N + b}{\mu + 1} = N - 2. \quad (1.6)$$

*Then the system (SH) admits a (unique) G.S. if and only if  $(\delta, \mu)$  is above  $\mathcal{H}_0$  or on  $\mathcal{H}_0$ . The radial Dirichlet problem in a ball has a solution if and only if  $(\delta, \mu)$  is under  $\mathcal{H}_0$ . On  $\mathcal{H}_0$  the G.S. has the following behaviour at  $\infty$ : assuming for example  $\delta > \frac{N+a}{N-2}$ , then  $\lim_{r \rightarrow \infty} r^{N-2}u(r) = \alpha > 0$ , and*

$$\begin{aligned} \lim_{r \rightarrow \infty} r^{(N-2)\mu - (2+b)}v &= \beta > 0 & \text{if } \mu < \frac{N+b}{N-2}, \\ \lim_{r \rightarrow \infty} r^{N-2}v &= \beta > 0 & \text{if } \mu > \frac{N+b}{N-2}, \\ \lim_{r \rightarrow \infty} r^{N-2}|\ln r|^{-1}v &= \beta > 0 & \text{if } \mu = \frac{N+b}{N-2}. \end{aligned}$$

Observe that the G.S. can present a non-symmetric behaviour. Our proofs use a Pohozaev type function; notice that it contains a quadratic factor in  $X, Y, Z, W$ :

$$\mathcal{E}(r) = r^N \left[ u'v' + r^b \frac{|u|^{\mu+1}}{\mu+1} + r^a \frac{|v|^{\delta+1}}{\delta+1} + \frac{N+a}{\delta+1} \frac{vu'}{r} + \frac{N+b}{\mu+1} \frac{uv'}{r} \right] \quad (1.7)$$

$$= r^{N-2}uv \left[ XY - \frac{Y(N+b-W)}{\mu+1} - \frac{(N+a-Z)X}{\delta+1} \right]. \quad (1.8)$$

In Section 7 we consider the radial solutions of a nonvariational system:

$$(SN) \begin{cases} -\Delta u = |x|^a u^s v^\delta, \\ -\Delta v = |x|^a u^\mu v^s, \end{cases}$$

where  $p = 2$ ,  $a = b$  and  $m = s > 0$ . For small  $s$  it appears as a perturbation of system  $(SH)$ . Here we give a new result of *existence of G.S.* valid for any  $s$ . It is obtained by constructing a new kind of energy function by adding two terms in  $X^2, Y^2$  to the preceding one:

$$\Phi(r) = r^N \left[ u'v' + r^b \frac{|u|^{\mu+1}}{\mu+1} + r^a \frac{|v|^{\delta+1}}{\delta+1} + \frac{N+a}{\delta+1} \frac{vv'}{r} + \frac{N+b}{\mu+1} \frac{uv'}{r} + \frac{s}{2(\delta+1)} \frac{vu'^2}{u} + \frac{s}{2(\mu+1)} \frac{uv'^2}{v} \right] \quad (1.9)$$

$$= r^{N-2} uv \left[ XY - \frac{Y(N+b-W)}{\mu+1} - \frac{(N+a-Z)X}{\delta+1} + \frac{s}{2(\delta+1)} X^2 + \frac{s}{2(\mu+1)} Y^2 \right]. \quad (1.10)$$

Our main result is the following:

**Theorem 1.5** *Consider the curve  $\mathcal{C}_s$  defined by*

$$\frac{N}{\mu+1} + \frac{N}{\delta+1} = N - 2 + \frac{(N-2)s}{2} \min\left(\frac{1}{\mu+1}, \frac{1}{\delta+1}\right), \quad (1.11)$$

*located under the hyperbola defined by (1.6). Then the system (SN) admits a G.S. if  $(\delta, \mu)$  is above  $\mathcal{C}_s$ .*

In Section 8 we consider the radial solutions of the potential system

$$(SP) \begin{cases} -\Delta_p u = |x|^a u^s v^{m+1} \\ -\Delta_q v = |x|^a u^{s+1} v^m \end{cases}$$

where  $\delta = m + 1, \mu = s + 1$  and  $a = b$ , which is still variational, see [34], [35]. Here we use the following energy function, which deserves to be compared with the one of Section 6 in terms of  $(X, Y, Z, W)$ , since it also presents a quadratic factor:

$$\begin{aligned} \mathcal{E}(r) &= r^N \left[ (s+1) \left( \frac{|u'|^p}{p'} + \frac{N-p}{p} \frac{u|u'|^{p-2} u'}{r} \right) + (m+1) \left( \frac{|v'|^q}{q'} + \frac{N-q}{q} \frac{v|v'|^{q-2} v'}{r} \right) + r^a u^{s+1} v^{m+1} \right] \\ &= r^{N-2-a} \frac{|u'|^{p-1} |v'|^{q-1}}{u^s v^m} \left[ ZW - \frac{(s+1)W(N-p-(p-1)X)}{p} - \frac{(m+1)Z(N-q-(q-1)Y)}{q} \right]. \end{aligned} \quad (1.12)$$

Then we deduce new results of existence:

**Theorem 1.6** *Let  $\mathcal{D}$  be the critical line in the plane  $(m, s)$  defined by*

$$N + a = (m+1) \frac{N-q}{q} + (s+1) \frac{N-p}{p}.$$

Then the system (SP) admits a G.S. if and only if  $(m, s)$  is above or on  $\mathcal{D}$ . On  $\mathcal{D}$  the G.S. has the following behaviour: suppose for example  $q \leq p$ . Let  $\lambda^* = N + a - (s + 1)\frac{N-p}{p-1} - m\frac{N-q}{q-1}$ . Then  $\lim_{r \rightarrow \infty} r^{\frac{N-p}{p-1}} u(r) = \alpha > 0$ , and

$$\lim_{r \rightarrow \infty} r^{\frac{N-q}{q-1}} v(r) = \beta > 0 \quad \text{if } \lambda^* < 0, \quad (1.13)$$

$$\lim_{r \rightarrow \infty} r^{\frac{N-p}{p-1} \mu - (q+b)} v(r) = \beta > 0 \quad \text{if } \lambda^* > 0, \quad (1.14)$$

$$\lim_{r \rightarrow \infty} r^{\frac{N-q}{q-1}} |\ln r|^{-\frac{1}{q-1-m}} v(r) = \beta > 0 \quad \text{if } \lambda^* = 0. \quad (1.15)$$

In particular (1.13) holds if  $p = q$ , or  $q \leq m + 1$ . Also the radial Dirichlet problem in a ball has a solution if and only if  $(m, s)$  is under  $\mathcal{D}$ .

Finally in Section 9 we deduce a nonradial result for the potential system in the case of two Laplacians:

$$(SL) \begin{cases} -\Delta u = |x|^a u^s v^{m+1}, \\ -\Delta v = |x|^a u^{s+1} v^m. \end{cases}$$

Our result solves a conjecture of [7], showing that in the subcritical case there exists no G.S.:

**Theorem 1.7** *Assume  $a > -2$  and  $s, m \geq 0$ . If*

$$s + m + 1 < \min\left(\frac{N+2}{N-2}, \frac{N+2+2a}{N-2}\right),$$

*then the system (SL) admits no (radial or nonradial) G.S.*

Our proof uses a nonradial Pohozaev function extending (??), different from the energy function used in [7], coupled with the estimates of [7], which up to now are the only extensions of the results of [18] to systems.

The case of system (G) with absorption terms ( $\varepsilon_1 = \varepsilon_2 = -1$ ) or mixed terms ( $\varepsilon_1 = -\varepsilon_2 = 1$ ), studied in [4], [5], will be the subject of another article. Our approach also extends to a system with gradient terms,

$$\begin{cases} -\Delta_p u = \varepsilon_1 |x|^a u^s v^\delta |\nabla u|^\eta |\nabla v|^\ell, \\ -\Delta_q v = \varepsilon_2 |x|^b u^\mu v^m |\nabla u|^\nu |\nabla v|^\kappa, \end{cases} \quad (1.16)$$

which will be developed later.

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**Keywords** Elliptic quasilinear systems. Variational or nonvariational problems. Autonomous and quadratic systems. Stable manifolds. Heteroclinic orbits.

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## 2 Reduction to a quadratic system

Here we consider the radial positive solutions  $r \mapsto (u(r), v(r))$  of system  $(G)$  on any interval  $(R_1, R_2)$ , that means

$$\begin{cases} \left( |u'|^{p-2} u' \right)' + \frac{N-1}{r} |u'|^{p-2} u' = r^{1-N} \left( r^{N-1} |u'|^{p-2} u' \right)' = -\varepsilon_1 r^a u^s v^\delta, \\ \left( |v'|^{q-2} v' \right)' + \frac{N-1}{r} |v'|^{q-2} v' = r^{1-N} \left( r^{N-1} |v'|^{q-2} v' \right)' = -\varepsilon_2 r^b u^\mu v^m. \end{cases}$$

Near any point  $r$  where  $u(r) \neq 0$ ,  $u'(r) \neq 0$  and  $v(r) \neq 0$ ,  $v'(r) \neq 0$  we define

$$X(t) = -\frac{ru'}{u}, \quad Y(t) = -\frac{rv'}{v}, \quad Z(t) = -\varepsilon_1 r^{1+a} u^s v^\delta |u'|^{-p} u', \quad W(t) = -\varepsilon_2 r^{1+b} u^\mu v^m |v'|^{-q} v', \quad (2.1)$$

where  $t = \ln r$ . Then we find the quadratic system

$$(M) \begin{cases} X_t = X \left[ X - \frac{N-p}{p-1} + \frac{Z}{p-1} \right], \\ Y_t = Y \left[ Y - \frac{N-q}{q-1} + \frac{W}{q-1} \right], \\ Z_t = Z [N + a - sX - \delta Y - Z], \\ W_t = W [N + b - \mu X - mY - W]. \end{cases}$$

The trajectories located on its invariant hyperplanes:  $X = 0, Y = 0, Z = 0, W = 0$ , do not correspond to a solution of system  $(G)$ ; they will be called *nonadmissible*. We suppose that the discriminant of the system

$$D = \delta\mu - (p-1-s)(q-1-m) \neq 0. \quad (2.2)$$

Then one can express  $u, v$  in terms of the new variables:

$$u = r^{-\gamma} (|X|^{p-1} |Z|)^{(q-1-m)/D} (|Y|^{q-1} |W|)^{\delta/D}, \quad v = r^{-\xi} (|X|^{p-1} |Z|)^{\mu/D} (|Y|^{q-1} |W|)^{(p-1-s)/D}, \quad (2.3)$$

where  $\gamma$  and  $\xi$  are defined by

$$\gamma = \frac{(p+a)(q-1-m) + (q+b)\delta}{D}, \quad \xi = \frac{(q+b)(p-1-s) + (p+a)\mu}{D}, \quad (2.4)$$

or equivalently by

$$(p-1-s)\gamma + p + a = \delta\xi, \quad (q-1-m)\xi + q + b = \mu\gamma. \quad (2.5)$$

Since the system  $(M)$  is autonomous, each admissible trajectory  $\mathcal{T}$  in the phase space corresponds to a solution  $(u, v)$  of system  $(G)$  unique up to a scaling: if  $(u, v)$  is a solution, then for any  $\theta > 0$ ,  $r \mapsto (\theta^\gamma u(\theta r), \theta^\xi v(\theta r))$  is also a solution.



**Remark 2.1** *This system allows to study more general systems with signed solutions by reducing the study on intervals where  $u$  and  $v$  are nonzero. Consider for example the problem  $-\Delta_p u = \varepsilon_1 |x|^a |u|^s |v|^{\delta-1} v$ ,  $-\Delta_q v = \varepsilon_2 |x|^b |v|^m |u|^{\mu-1} u$ . On any interval where  $uv > 0$ , the couple  $(|u|, |v|)$  is a solution of (G). On any interval where  $u > 0 > v$ , the couple  $(u, |v|)$  satisfies (G) with  $(\varepsilon_1, \varepsilon_2)$  replaced by  $(-\varepsilon_1, -\varepsilon_2)$*

System (M) has at most 16 fixed points. The main fixed point is

$$M_0 = (X_0, Y_0, Z_0, W_0) = (\gamma, \xi, N - p - (p-1)\gamma, N - q - (q-1)\xi), \quad (2.6)$$

corresponding to the particular solutions

$$u_0(r) = Ar^{-\gamma}, v_0(r) = Br^{-\xi}, \quad A, B > 0,$$

when they exist, depending on  $\varepsilon_1, \varepsilon_2$ ; then the values of  $A$  and  $B$  are

$$\begin{aligned} A^D &= (\varepsilon_1 \gamma^{p-1} (N - p - \gamma(p-1)))^{q-1-m} (\varepsilon_2 \xi^{q-1} (N - q - (q-1)\xi))^\delta, \\ B^D &= (\varepsilon_2 \xi^{q-1} (N - q - (q-1)\xi))^{p-1-s} (\varepsilon_1 \gamma^{p-1} (N - p - (p-1)\gamma))^\mu. \end{aligned}$$

The other fixed points are

$$0 = (0, 0, 0, 0), \quad N_0 = (0, 0, N + a, N + b), \quad A_0 = \left( \frac{N-p}{p-1}, \frac{N-q}{q-1}, 0, 0 \right), \quad (2.7)$$

$$I_0 = \left( \frac{N-p}{p-1}, 0, 0, 0 \right), \quad J_0 = \left( 0, \frac{N-q}{q-1}, 0, 0 \right), \quad K_0 = (0, 0, N + a, 0), \quad L_0 = (0, 0, 0, N + b), \quad (2.8)$$

$$G_0 = \left( \frac{N-p}{p-1}, 0, 0, N + b - \frac{N-p}{p-1}\mu \right), \quad H_0 = \left( 0, \frac{N-q}{q-1}, N + a - \frac{N-q}{q-1}\delta, 0 \right), \quad (2.9)$$

and if  $m \neq q-1$ ,

$$P_0 = \left( \frac{N-p}{p-1}, \frac{\frac{N-p}{p-1}\mu - (q+b)}{q-1-m}, 0, \frac{(q-1)(N+b - \frac{N-p}{p-1}\mu) - m(N-q)}{q-1-m} \right), \quad (2.10)$$

$$C_0 = \left( 0, -\frac{q+b}{q-1-m}, 0, \frac{(N+b)(q-1) - m(N-q)}{q-1-m} \right), \quad (2.11)$$

$$R_0 = \left( 0, -\frac{q+b}{q-1-m}, N + a + \delta \frac{b+q}{q-1-m}, \frac{(N+b)(q-1) - m(N-q)}{q-1-m} \right), \quad (2.12)$$

and by symmetry, if  $s \neq p - 1$ ,

$$Q_0 = \left( \frac{\frac{N-q}{q-1}\delta - (p+a)}{p-1-s}, \frac{N-q}{q-1}, \frac{(p-1)(N+a - \frac{N-q}{q-1}\delta) - s(N-p)}{p-1-s}, 0 \right), \quad (2.13)$$

$$D_0 = \left( -\frac{p+a}{p-1-s}, 0, \frac{(N+a)(p-1) - s(N-p)}{p-1-s}, 0 \right), \quad (2.14)$$

$$S_0 = \left( -\frac{p+a}{p-1-s}, 0, \frac{(N+a)(p-1) - s(N-p)}{p-1-s}, N+b + \mu \frac{a+p}{p-1-s} \right). \quad (2.15)$$

**Remark 2.2** *There is another way for reducing the system in an autonomous form: setting*

$$U(t) = r^\gamma u, \quad V(t) = r^\xi v, \quad H(t) = -r^{(\gamma+1)(p-1)} |u'|^{p-2} u', \quad K(t) = -r^{(\xi+1)(q-1)} |v'|^{q-2} v',$$

with  $t = \ln r$ , we find

$$\begin{cases} U_t = \gamma U - |H|^{(2-p)/(p-1)} H, & V_t = \zeta U - |K|^{(2-q)/(q-1)} K, \\ H_t = (\gamma(p-1) + p - N)H + \varepsilon_1 U^s V^\delta, & K_t = (\zeta(q-1) + q - N)K + \varepsilon_2 U^\mu V^m. \end{cases}$$

*It extends the well-known transformation of Emden-Fowler in the scalar case when  $p = 2$ , used also in [2] for general  $p$ , see Section 3. When  $p = q = 2$  we obtain*

$$\begin{cases} U_{tt} + (N-2-2\gamma)U_t - \gamma(N-2-\gamma)U + \varepsilon_1 U^s V^\delta = 0, \\ V_{tt} + (N-2-2\xi)V_t - \xi(N-2-\xi)V + \varepsilon_2 U^\mu V^m = 0, \end{cases} \quad (2.16)$$

*which was extended to the nonradial case and used for Hamiltonian systems ( $s = m = 0$ ), with source terms in [9] ( $\varepsilon_1 = \varepsilon_2 = 1$ ) and absorption terms in [4] ( $\varepsilon_1 = \varepsilon_2 = -1$ ). Our system is more adequated for finding the possible behaviours, because its fixed points at infinity are not concerned, as soon as we deal with solutions  $u, v > 0$ .*

**Remark 2.3** *In the specific case  $p = q = 2$ , setting*

$$z = XZ = \varepsilon_1 r^{2+a} |u|^{s-2} u |v|^{\delta-1} v, \quad w = YW = \varepsilon_2 r^{2+b} |u|^{\mu-1} u |v|^{m-2} v$$

*we get the following system*

$$\begin{cases} X_t = X^2 - (N-2)X + z, & Y_t = Y^2 - (N-2)Y + w, \\ z_t = z [2 + a + (1-s)X - \delta Y], & w_t = w [2 + b - \mu X + (1-m)Y]. \end{cases}$$

*It has been used in [20] for studying the Hamiltonian system (SH). Even in that case we will show at Section 6 that the system ( $M_G$ ) is more performant, because it is of Kolmogorov type.*

**Remark 2.4** Assume  $p = q$  and  $a = b$ . Setting  $t = k\hat{t}$  and  $(\hat{X}, \hat{Y}, \hat{Z}, \hat{W}) = k(X, Y, Z, W)$ , we obtain a system of the same type with  $N, a$  replaced by  $\hat{N}, \hat{a}$ , with  $\frac{\hat{N}-p}{\hat{N}-p} = k = \frac{\hat{N}+\hat{a}}{N+a}$ . It corresponds to the change of unknowns

$$r = \hat{r}^k, \quad \hat{u}(\hat{r}) = C_1 u(r), \quad \hat{v}(\hat{r}) = C_2 u(r), \quad C_1 = k^{p(p-1-m+\delta)/D}, C_2 = k^{(p(p-1-s+\mu))/D}$$

And from (2.3) and (2.4), we get  $\hat{\gamma}/\gamma = \hat{\xi}/\xi = k = \frac{p+\hat{a}}{p+a}$ . There is one free parameter. In particular

- 1) we get a system without power ( $\hat{a} = 0$ ), by taking  $\hat{N} = \frac{p(N+a)}{p+a}$  and  $k = \frac{p}{p+a}$ .
- 2) We get a system in dimension  $\hat{N} = 1$ , by taking  $k = -\frac{p-1}{N-p} < 0$  and  $\hat{a} = \frac{p+a-(N+a)p}{N-p}$ .

### 3 The scalar case

We first study the signed solutions of two well-known scalar equations with source or absorption:

$$-\Delta_p u = -r^{1-N} \left( r^{N-1} |u'|^{p-2} u' \right)' = \varepsilon |x|^a |u|^{Q-1} u, \quad (3.1)$$

with  $\varepsilon = \pm 1$ ,  $1 < p < N$ ,  $Q \neq p-1$  and  $p+a > 0$ . We cannot quote all the huge litterature concerning its solutions, supersolutions or subsolutions, from the first studies of Emden and Fowler for  $p = 2$ , recalled in [16]; see for example [2] and [37], for any  $p > 1$ , and references therein. We set

$$Q_1 = \frac{(N+a)(p-1)}{N-p}, \quad Q_2 = \frac{N(p-1) + p + pa}{N-p}, \quad \gamma = \frac{p+a}{Q+1-p}.$$

From Remark 2.4 we could reduce the system to the case  $a = 0$ , in dimension  $\hat{N} = p(N+a)/(p+a)$ . However we will not do it, because we are motivated by the study of system (G), and also by the nonradial case.

#### 3.1 A common phase plane for the two equations

Near any point  $r$  where  $u(r) \neq 0$  (positive or negative), and  $u'(r) \neq 0$  setting

$$X(t) = -\frac{ru'}{u}, \quad Z(t) = -\varepsilon r^{1+a} |u|^{Q-1} u |u'|^{-p} u', \quad (3.2)$$

with  $t = \ln r$ , we get a 2-dimensional system

$$(M_{scal}) \begin{cases} X_t = X \left[ X - \frac{N-p}{p-1} + \frac{Z}{p-1} \right], \\ Z_t = Z [N+a - QX - Z]. \end{cases}$$

and then  $|u| = r^{-\gamma} (|Z| |X|^{p-1})^{1/(Q+1-p)}$ . This change of unknown was mentioned in [11] in the case  $p = 2, \varepsilon = 1$  and  $N = 3$ . It is remarkable that the system  $(M_{scal})$  is the same for the two cases  $\varepsilon = \pm 1$ , the only difference is that  $X(t)Z(t)$  has the sign of  $\varepsilon$ :

The equation with source ( $\varepsilon = 1$ ) is associated to the 1<sup>st</sup> and 3<sup>rd</sup> quadrant. It is well known that any local solution of has a unique extension on  $(0, \infty)$ . The 1<sup>st</sup> quadrant corresponds to the sets where  $|u|$  is decreasing, which can be of the following types  $(0, \infty), (0, R_2), (R_1, \infty), (R_1, R_2)$ ,  $0 < R_1 < R_2 < \infty$ ; and the 3<sup>rd</sup> quadrant corresponds to the sets  $(R_1, R_2)$  where  $|u|$  is increasing.

The equation with absorption ( $\varepsilon = -1$ ) is associated to the 2<sup>nd</sup> and 4<sup>th</sup> quadrant. It is known that the solutions have at most one zero, and their maximal interval of existence can be  $(0, R_2), (R_1, \infty), (R_1, R_2)$  or  $(0, \infty)$ . The 2<sup>nd</sup> quadrant corresponds to the sets  $(R_1, R_2)$  where  $|u|$  is increasing; and the 4<sup>th</sup> quadrant corresponds to the sets  $(0, R_2)$  or  $(R_1, \infty)$  where  $|u|$  is decreasing.

The fixed points of  $(M_{scal})$  are  $(0, 0)$ ,  $A_0 = (\frac{N-p}{p-1}, 0)$ ,  $N_0 = (0, N + a)$ ,  $M_0 = (X_0, Z_0) = (\gamma, N - p - (p - 1)\gamma)$ . In particular  $M_0$  is in the 1<sup>st</sup> quadrant whenever  $\gamma < \frac{N-p}{p-1}$ , equivalently  $Q > Q_1$ , and in the 4<sup>th</sup> quadrant whenever  $Q < Q_1$ . It corresponds to the solution  $u(r) = Ar^{-\gamma}$  for  $\varepsilon = 1$  when  $Q > Q_1$  and for  $\varepsilon = -1$  when  $Q < Q_1$ , where  $A = (\varepsilon\gamma^{p-1}(N - p - \gamma(p - 1)))^{1/(Q-p+1)}$ .

### 3.2 Local study

We examine the fixed points, where for simplicity we suppose  $Q \neq Q_1$ , and deduce local results for the two equations:

- Point  $(0, 0)$  : it is a saddle point, and the only trajectories that converge to  $(0, 0)$  are the separatrix, contained in the lines  $X = 0, Y = 0$ , they are not admissible.
- Point  $N_0$  : it is a saddle point: the eigenvalues of the linearized system are  $\frac{p}{p-1}$  and  $-N$ . the trajectories ending at  $N_0$  at  $\infty$  are located on the set  $Z = 0$ , then there exists a unique trajectory starting from  $-\infty$  at  $N_0$ ; it corresponds to the local existence and uniqueness of regular solutions, which we obtain easily.
- Point  $A_0$  : the eigenvalues of the linearized system are  $\frac{N-p}{p-1}$  and  $\frac{N-p}{p-1}(Q_1 - Q)$ . If  $Q < Q_1$ ,  $A_0$  is an unstable node. There is an infinity of trajectories starting from  $A_0$  at  $-\infty$ ; then  $X(t)$  converges exponentially to  $\frac{N-p}{p-1}$ , thus  $\lim_{r \rightarrow 0} r^{\frac{N-p}{p-1}} u = \alpha > 0$ . The corresponding solutions  $u$  satisfy the equation with a Dirac mass at 0. There exists no solution converging to  $A_0$  at  $\infty$ . If  $Q > Q_1$ ,  $A_0$  is a saddle point; the trajectories starting from  $A_0$  at  $-\infty$  are not admissible; there is a trajectory converging at  $\infty$ , and then  $\lim_{r \rightarrow \infty} r^{\frac{N-p}{p-1}} u = \alpha > 0$ .
- Point  $M_0$  : the eigenvalues  $\lambda_1, \lambda_2$  of the linearized system are the roots of equation

$$\lambda^2 + (Z_0 - X_0)\lambda + \frac{Q - p + 1}{p - 1} X_0 Z_0 = 0.$$

For  $\varepsilon = 1$ ,  $M_0$  is defined for  $Q > Q_1$ ; the eigenvalues are imaginary when  $X_0 = Z_0$ , equivalently  $\gamma = (N - p)/p$ ,  $Q = Q_2$ . When  $Q < Q_2$ ,  $M_0$  is a source, there exists an infinity of trajectories such that  $\lim_{r \rightarrow 0} r^\gamma u = A$ . When  $Q > Q_2$ ,  $M_0$  is a sink, and there exists an infinity of trajectories such that  $\lim_{r \rightarrow \infty} r^\gamma u = A$ . When  $Q = Q_2$ ,  $M_0$  is a center, see [2]. For  $\varepsilon = -1$ ,  $M_0$  is defined

for  $Q < Q_1$ , it is a saddle-point. There exist two trajectories  $\mathcal{T}_1, \mathcal{T}'_1$  converging at  $\infty$ , such that  $\lim_{r \rightarrow \infty} r^\gamma u = A$  and two trajectories  $\mathcal{T}_2, \mathcal{T}'_2$ , converging at 0, such that  $\lim_{r \rightarrow 0} r^\gamma u = A$ .

### 3.3 Global study

**Remark 3.1** System  $(M_{scal})$  has no limit cycle for  $Q \neq Q_2$ . It is evident when  $\varepsilon = -1$ . When  $\varepsilon = 1$ , as noticed in [19], it comes from the Dulac's theorem: setting  $X_t = f(X, Z)$ ,  $Z_t = g(X, Z)$ , and

$$B(X, Z) = X^{pQ/(Q+1-p)-2} Z^{p/(Q+1-p)-1}, \quad M = B_X X_t + B_Y Y_t + B(f_X + g_Y),$$

then  $M = KB$  with  $K = (Q_2 - Q)\gamma(N - p)/p$ , thus  $M$  has no zero for  $Q \neq Q_2$ .

Then from the Poincaré-Bendixson theorem, any trajectory bounded near  $\pm\infty$  converges to one of the fixed points. Thus we find again global results:

- Equation with source ( $\varepsilon = 1$ ). If  $Q < Q_2$ , there is no G.S.: the trajectories cannot converge to any fixed point.

If  $Q_1 < Q < Q_2$ , there exist solutions  $u > 0$  such that  $\lim_{r \rightarrow 0} r^\gamma u = A$  and  $\lim_{r \rightarrow 0} r^{\frac{N-p}{p-1}} u = \alpha > 0$ . Indeed the regular trajectory  $\mathcal{T}$  issued from  $N_0$  cannot converge to  $M_0$ ; if it converges to  $A_0$ , it is the unique trajectory converging to  $A_0$ ; the set delimited by  $\mathcal{T}$  and  $X = 0, Z = 0$  is invariant, thus it contains  $M_0$ ; and the trajectories issued from  $M_0$  cannot converge to any point. Then  $X$  tends to  $\infty$ , the regular solutions are changing sign. The trajectory ending at  $A_0$  converges to  $M_0$  at  $-\infty$ .

If  $Q > Q_2$ , the only singular solution at 0 is  $u_0$ , and the regular solutions are G.S., with  $\lim_{r \rightarrow \infty} r^\gamma u = A$ . Indeed  $M_0$  is a sink; the trajectory ending at  $A_0$  cannot converge to  $N_0$  at  $-\infty$ , thus  $X$  converges to 0, and  $Z$  converges to  $\infty$ , then  $u$  cannot be positive on  $(0, \infty)$ . The trajectory issued from  $N_0$  converges to  $M_0$ .

- Equation with absorption ( $\varepsilon = -1$ ). If  $Q > Q_1$ , all the solutions  $u$  defined near 0 are regular; indeed the trajectories cannot converge to a fixed point.

If  $Q < Q_1$ , we find again easily a well known result: there exists a positive solution  $u_1$ , unique up to a scaling, such that  $\lim_{r \rightarrow 0} r^{\frac{N-p}{p-1}} u_1 = \alpha > 0$ , and  $\lim_{r \rightarrow \infty} r^\gamma u = A$ . Indeed the eigenvalues at  $M_0$  satisfy  $\lambda_1 < 0 < \lambda_2$ . There are two trajectories  $\mathcal{T}_1, \mathcal{T}'_1$  associated to  $\lambda_1$ , and the eigenvector  $(X_0 + |\lambda_1|, -\frac{X_0}{p-1})$ . The trajectory  $\mathcal{T}_1$  satisfies  $X_t > 0 > Z_t$  near  $\infty$ , and  $X > \frac{N-p}{p-1}$ , since  $Z_0 < 0$ , and  $X$  cannot take the value  $\frac{N-p}{p-1}$  because at such a point  $X_t < 0$ ; then  $\frac{N-p}{p-1} < X < X_0$  and  $X_t > 0$  as long as it is defined; similarly  $Z_0 < Z < 0$  and  $Z_t < 0$ ; then  $\mathcal{T}_1$  converge to a fixed point, necessarily  $A_0$ , showing the existence of  $u_1$ . The trajectory  $\mathcal{T}'_1$  corresponds to solutions  $u$  such that  $\lim_{r \rightarrow \infty} r^\gamma u = A$  and  $\lim_{r \rightarrow R} u = \infty$  for some  $R > 0$ . There are two trajectories  $\mathcal{T}_2, \mathcal{T}'_2$ , associated to  $\lambda_2$ , defining solutions  $u$  such that  $\lim_{r \rightarrow 0} r^\gamma u = A$  and changing sign, or with a minimum point and  $\lim_{r \rightarrow R} u = \infty$  for some  $R > 0$ . The regular trajectory starts from  $N_0$  in the  $2^{nd}$  quadrant, it cannot converge to a fixed point, then  $\lim_{r \rightarrow R} u = \infty$  for some  $R > 0$ .

• Critical case  $Q = Q_2$  : it is remarkable that the system  $(M_{scal})$  admits another invariant line, namely  $A_0N_0$ , given by

$$\frac{X}{p'} + \frac{Z}{Q_2 + 1} - \frac{N - p}{p} = 0 \quad (3.3)$$

It precisely corresponds to well-known solutions of the two equations

$$u = c(K^2 + r^{(p+a)/(p-1)})^{(p-N)/(p+a)}, \quad \text{for } \varepsilon = 1; \quad u = c \left| K^2 - r^{(p+a)/(p-1)} \right|^{(p-N)/(p+a)}, \quad \text{for } \varepsilon = -1,$$

where  $K^2 = c^{Q-p+1}(N+a)^{-1}((N-p)/(p-1))^{1-p}$ .

**Remark 3.2** *The global results have been obtained without using energy functions. The study of [2] was based on a reduction of type of Remark 2.2, using an energy function linked to the new unknown. Other energy functions are well-known, of Pohozaev type:*

$$\mathcal{E}_\sigma(r) = r^N \left[ \frac{|u'|^p}{p'} + \varepsilon r^a \frac{|u|^{Q+1}}{Q+1} + \sigma \frac{u|u'|^{p-2}u'}{r} \right] = r^{N-p} |u|^p |X|^{p-2} X \left[ \frac{X}{p'} + \frac{Z}{Q+1} - \sigma \right]$$

with  $\sigma = \frac{N-p}{p}$ , satisfying  $\mathcal{E}'_\sigma(r) = r^{N-1+a} \left( \frac{N+a}{Q+1} - \frac{N-p}{p} \right) |u|^{Q+1}$ , or with  $\sigma = \frac{N+a}{Q+1}$ , leading to  $\mathcal{E}'_\sigma(r) = r^{N-1} \left( \frac{N+a}{Q+1} - \frac{N-p}{p} \right) |u|^p$ . In the critical case  $Q = Q_2$ , all these functions coincide and they are constant, in other words the system  $(M_{scal})$  has a first integral. We find again the line (3.3): the G.S. are the functions of energy 0.

## 4 Local study of system (S)

In all the sequel we study the system with source terms:  $(G) = (S)$ . Assumption (1.5) is the most interesting case for studying the existence of the G.S.

We first study the local behaviour of nonnegative solutions  $(u, v)$  defined near 0 or near  $\infty$ . It is well known that any solution  $(u, v)$  positive on some interval  $(0, R)$  satisfies  $u', v' < 0$  on  $(0, R)$ . Any solution  $(u, v)$  positive on  $(R, \infty)$ , satisfies  $u', v' < 0$  near  $\infty$ . Then we are reduced to study the system in the region  $\mathcal{R}$  where  $X, Y, Z, W > 0$ , and consider the fixed points in  $\bar{\mathcal{R}}$ . Then

$$X(t) = -\frac{ru'}{u}, \quad Y(t) = -\frac{rv'}{v}, \quad Z(t) = r^{1+a} u^s v^\delta |u'|^{1-p}, \quad W(t) = r^{1+b} v^m u^\mu |v'|^{1-q}, \quad (4.1)$$

and  $(X, Y, Z, W)$  is a solution of the system  $(M)$  in  $\mathcal{R}$  if and only if  $(u, v)$  defined by

$$u = r^{-\gamma} (ZX^{p-1})^{(q-1-m)/D} (WY^{q-1})^{\delta/D}, \quad v = r^{-\xi} (WY^{q-1})^{(p-1-s)/D} (ZX^{p-1})^{\mu/D}, \quad (4.2)$$

is a positive solution with  $u' < 0$  and  $v' < 0$ . Among the fixed points,  $M_0$  lies in  $\mathcal{R}$  if and only if

$$0 < \gamma < \frac{N-p}{p-1} \quad \text{and} \quad 0 < \xi < \frac{N-q}{q-1}. \quad (4.3)$$

The local study of the system near  $M_0$  appears to be tricky, see Remark 4.2. A main difference with the scalar case is that there always exist a trajectory converging to  $M_0$  at  $\pm\infty$  :

**Proposition 4.1** (*Point  $M_0$* ) *Assume that (4.3) holds. Then there exist trajectories converging to  $M_0$  as  $r \rightarrow \infty$ , and then solutions  $(u, v)$  being defined near  $\infty$ , such that*

$$\lim_{r \rightarrow \infty} r^\gamma u = \alpha > 0, \quad \lim_{r \rightarrow \infty} r^\xi v = \beta > 0, \quad (4.4)$$

*And there exist trajectories converging to  $M_0$  as  $r \rightarrow 0$ , and then solutions  $(u, v)$  being defined near 0 such that*

$$\lim_{r \rightarrow 0} r^\gamma u = \alpha > 0, \quad \lim_{r \rightarrow 0} r^\xi v = \beta > 0, \quad (4.5)$$

**Proof.** Here  $M_0 \in \mathcal{R}$ ; setting  $X = X_0 + \tilde{X}, Y = Y_0 + \tilde{Y}, Z = Z_0 + \tilde{Z}, W = W_0 + \tilde{W}$ , the linearized system is

$$\begin{cases} \tilde{X}_t = X_0(\tilde{X} + \frac{1}{p-1}\tilde{Z}), \\ \tilde{Y}_t = Y_0(\tilde{Y} + \frac{1}{q-1}\tilde{W}), \\ \tilde{Z}_t = Z_0(-s\tilde{X} - \delta\tilde{Y} - \tilde{Z}), \\ \tilde{W}_t = W_0(-\mu\tilde{X} - m\tilde{Y} - \tilde{W}). \end{cases}$$

The eigenvalues are the roots  $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ , of the equation

$$\left[ (\lambda - X_0)(\lambda + Z_0) + \frac{s}{p-1}X_0Z_0 \right] \left[ (\lambda - Y_0)(\lambda + W_0) + \frac{m}{q-1}Y_0W_0 \right] - \frac{\delta\mu}{(p-1)(q-1)}X_0Y_0Z_0W_0 = 0; \quad (4.6)$$

this equation is of the form

$$f(\lambda) = \lambda^4 + E\lambda^3 + F\lambda^2 + G\lambda - H = 0$$

with

$$\begin{cases} E = Z_0 - X_0 + W_0 - Y_0, \\ F = (Z_0 - X_0)(W_0 - Y_0) - \frac{s+p-1}{p-1}X_0Z_0 - \frac{m+q-1}{q-1}Y_0W_0, \\ G = -\frac{q-1-m}{q-1}Y_0W_0(Z_0 - X_0) - \frac{p-1-s}{p-1}X_0Z_0(W_0 - Y_0), \\ H = \frac{D}{(p-1)(q-1)}X_0Y_0Z_0W_0, \end{cases}$$

and  $H > 0$ , then  $\lambda_1\lambda_2\lambda_3\lambda_4 < 0$ . Hence there exist two real roots  $\lambda_3 < 0 < \lambda_4$ , and two roots  $\lambda_1, \lambda_2$ , real with  $\lambda_1\lambda_2 > 0$ , or complex. Then there exists at least one trajectory converging to  $M_0$  at  $\infty$  and another one at  $-\infty$ . Then (4.4) and (4.5) follow from (4.2). Moreover the convergence is monotone for  $X, Y, Z, W$ .  $\blacksquare$

**Remark 4.2** *There exist imaginary roots, namely  $\text{Re } \lambda_1 = \text{Re } \lambda_2 = 0$ , if and only if there exists  $c > 0$  such that  $f(ci) = 0$ , that means  $Ec^2 - G = 0$ , and  $c^4 - Fc^2 - H = 0$ , equivalently*

$$E = G = 0, \quad \text{or } EG > 0 \text{ and } G^2 - EFG - E^2H = 0.$$

Condition  $E = G = 0$  means that

(i) either  $Z_0 = X_0$  and  $W_0 = Y_0$ , equivalently in other words

$$(\gamma, \xi) = \left( \frac{N-p}{p}, \frac{N-q}{q} \right), \quad (4.7)$$

in other words  $(\delta, \mu) = \left( \frac{q(N(p-1-s)+p(1-s+a))}{p(N-q)}, \frac{p(N(q-1-m)+q(1-m+b))}{q(N-p)} \right)$ .

(ii) or  $(p-1-s)(q-1-m) > 0$  and  $(\gamma, \xi)$  satisfies

$$\begin{cases} 2N - p - q = p\gamma + q\xi, \\ \left(1 - \frac{m}{q-1}\right)\xi(N - q - (q-1)\xi) = \left(1 - \frac{s}{p-1}\right)\gamma(N - p - (p-1)\gamma). \end{cases} \quad (4.8)$$

This gives in general 0, 1 or 2 points. For example, in case  $\frac{m}{q-1} = \frac{s}{p-1} \neq 1$ , and  $(p-2)(q-2) > 0$  and  $N > \frac{pq-p-q}{p+q-2}$  we find another point, different from the one of (4.7) for  $p \neq q$ :

$$(\gamma, \xi) = \left( N \frac{q-2}{pq-p-q} - 1, N \frac{p-2}{pq-p-q} - 1 \right). \quad (4.9)$$

Moreover the computation shows that it can exist imaginary roots with  $E, G \neq 0$ .

In the case  $p = q = 2$  and  $s = m$  the situation is interesting:

**Proposition 4.3** Assume  $p = q = 2$  and  $s = m < \frac{N}{N-2}$ , and  $\delta + 1 - s > 0$ ,  $\mu + 1 - s > 0$ . Let  $\mathcal{H}_s$  be the hyperbola of equation

$$\frac{1}{\delta + 1 - s} + \frac{1}{\mu + 1 - s} = \frac{N-2}{N - (N-2)s}, \quad \text{equivalently } \gamma + \xi = N - 2 \quad (4.10)$$

It is contained in the set of points  $(\delta, \mu)$  for which the linearized system at  $M_0$  has imaginary roots, and equal when  $s \leq 1$ .

**Proof.** The assumption  $D > 0$  imply  $\delta + 1 - s > 0$  and  $\mu + 1 - s > 0$ ; condition  $E = G = 0$  implies  $s < N/(N-2)$  and reduces to condition (4.10). Moreover if  $s \leq 1$ , all the cases are covered; indeed  $2G = (s-1)E[Y_0Z_0 + X_0W_0]$ , hence  $GE \leq 0$ . ■

Next we give a summary of the local existence results obtained by linearization around the other fixed points of system (M) proved at Section 10. Recall that  $t \rightarrow -\infty \Leftrightarrow r \rightarrow 0$  and  $t \rightarrow \infty \Leftrightarrow r \rightarrow \infty$ .

**Proposition 4.4** (Point  $N_0$ ) A solution  $(u, v)$  is regular if and only if the corresponding trajectory converges to  $N_0 = (0, 0, N+a, N+b)$  as  $r \rightarrow 0$ . For any  $u_0, v_0 > 0$ , there exists a unique local regular solution  $(u, v)$  with initial data  $(u_0, v_0)$ .



**Proposition 4.5** (Point  $A_0$ ) If  $s\frac{N-p}{p-1} + \delta\frac{N-q}{q-1} > N+a$  and  $\mu\frac{N-p}{p-1} + m\frac{N-q}{q-1} > N+b$ , there exist (admissible) trajectories converging to  $A_0$  as  $r \rightarrow \infty$ . If  $s\frac{N-p}{p-1} + \delta\frac{N-q}{q-1} < N+a$  and  $\mu\frac{N-p}{p-1} + m\frac{N-q}{q-1} < N+b$ , the same happens as  $r \rightarrow 0$ . In any case

$$\lim r^{\frac{N-p}{p-1}} u = \alpha > 0, \quad \lim r^{\frac{N-q}{q-1}} v = \beta > 0. \quad (4.11)$$

If  $s\frac{N-p}{p-1} + \delta\frac{N-q}{q-1} < N+a$  or  $\mu\frac{N-p}{p-1} + m\frac{N-q}{q-1} < N+b$ , there exist no trajectory converging as  $r \rightarrow \infty$ ; if  $s\frac{N-p}{p-1} + \delta\frac{N-q}{q-1} > N+a$  or  $\mu\frac{N-p}{p-1} + m\frac{N-q}{q-1} > N+b$ , there exist no trajectory converging as  $r \rightarrow 0$ .

**Proposition 4.6** (Point  $P_0$ ) 1) Assume that  $q > m+1$  and  $q+b < \frac{N-p}{p-1}\mu < N+b - m\frac{N-q}{q-1}$ . If  $\gamma < \frac{N-p}{p-1}$  there exist there exist trajectories converging to  $P_0$  as  $r \rightarrow \infty$  (and not as  $r \rightarrow 0$ ). If  $\gamma > \frac{N-p}{p-1}$  the same happens as  $r \rightarrow 0$  (and not as  $r \rightarrow \infty$ ).

2) Assume that  $q < m+1$  and  $q+b > \frac{N-p}{p-1}\mu > N+b - m\frac{N-q}{q-1}$  and  $q\frac{N-p}{p-1}\mu + m(N-q) \neq N(q-1) + (b+1)q$ . If  $\gamma < \frac{N-p}{p-1}$  there exist trajectories converging to  $P_0$  as  $r \rightarrow 0$  (and not as  $r \rightarrow \infty$ ) if  $\gamma > \frac{N-p}{p-1}$  there exist trajectories converging as  $r \rightarrow \infty$  (and not as as  $r \rightarrow 0$ ).

In any case, setting  $\kappa = \frac{1}{q-1-m}(\frac{N-p}{p-1}\mu - (q+b))$ , there holds

$$\lim r^{\frac{N-p}{p-1}} u = \alpha > 0, \quad \lim r^\kappa v = \beta > 0. \quad (4.12)$$

**Remark 4.7** This result improves the results of existence obtained by the fixed point theorem in [27] in the case of system (RP) with  $p = q = 2, a = 0, N = 3, 2s + m \neq 3$ . The proof is quite simpler with our methods.

**Proposition 4.8** (Point  $I_0$ ) If  $\frac{N-p}{p-1}s > N+a$  and  $\frac{N-q}{q-1}\mu > N+b$ , there exist trajectories converging to  $I_0$  as  $r \rightarrow \infty$ , and then

$$\lim_{r \rightarrow \infty} r^{\frac{N-p}{p-1}} u = \beta, \quad \lim_{r \rightarrow \infty} v = \alpha > 0. \quad (4.13)$$

For any  $s, m$ , there is no trajectory converging to  $I_0$  as  $r \rightarrow 0$ .

**Proposition 4.9** (Point  $G_0$ ) Suppose  $\frac{N-p}{p-1}\mu < N+b$ . If  $q+b < \frac{N-p}{p-1}\mu$  and  $N+a < \frac{N-p}{p-1}s$ , there exist trajectories converging to  $G_0$  as  $r \rightarrow \infty$ . If  $\frac{N-p}{p-1}\mu < q+b$  and  $\frac{N-p}{p-1}s < N+a$ , the same happens as  $r \rightarrow 0$ . In any case

$$\lim r^{\frac{N-p}{p-1}} u = \beta, \quad \lim v = \alpha > 0. \quad (4.14)$$

**Proposition 4.10** (Point  $C_0$ ) Suppose  $N+b < \frac{N-q}{q-1}m$  (hence  $q < m+1$ ) and  $m \neq \frac{N(q-1)+(b+1)q}{N-q}$ , and  $\delta > \frac{(N+a)(m+1-q)}{q+b}$ , exist trajectories converging to  $G_0$  as  $r \rightarrow \infty$  (and not at 0), and then

$$\lim_{r \rightarrow \infty} u = \alpha > 0, \quad \lim_{r \rightarrow \infty} r^k v = \beta, \quad (4.15)$$

where  $k = \frac{q+b}{m+1-q}$ .

**Proposition 4.11** (Point  $R_0$ ) Assume that  $N + b < \frac{N-q}{q-1}m$  (hence  $q < m + 1$ ), and  $m \neq \frac{N(q-1)+b+bq}{N-q}$ , and  $\delta < \frac{(N+a)(m+1-q)}{q+b}$ . If  $\frac{(p+a)(m+1-q)}{q+b} < \delta$ , there exist trajectories converging to  $R_0$  as  $r \rightarrow \infty$  (and not as  $r \rightarrow 0$ ). If  $\delta < \frac{(p+a)(m+1-q)}{q+b}$ , there exist trajectories converging as  $r \rightarrow 0$  (and not as  $r \rightarrow \infty$ ), and them (4.15) holds again.

We obtain similar results of convergence to the points  $Q_0, J_0, H_0, D_0, S_0$  by exchanging  $p, \delta, s, a$  and  $q, \mu, m, b$ . There is no admissible trajectory converging to  $0, K_0, L_0$ , see Remark 10.1.

## 5 Global results for system (S)

We are concerned by the existence of global positive solutions. First we find again easily some known results by using our dynamical approach.

**Proposition 5.1** Assume that system (S) admits a positive solution  $(u, v)$  in  $(0, \infty)$ . Then  $(X, Y, Z, W)$  stay in the box

$$\mathcal{A} = \left(0, \frac{N-p}{p-1}\right) \times \left(0, \frac{N-q}{q-1}\right) \times (0, N+a) \times (0, N+b), \quad (5.1)$$

equivalently

$$ru' + \frac{N-p}{p-1}u > 0, \quad rv' + \frac{N-q}{q-1}v > 0, \quad r^{1+a}u^s v^\delta < (N+a)|u'|^{p-1}, \quad r^{1+b}u^\mu v^m < (N+b)|v'|^{q-1} \quad (5.2)$$

and then

$$u^{s-p+1}v^\delta \leq C_1 r^{-(p+a)}, \quad u^\mu v^{m-q+1} \leq C_2 r^{-(q+b)}, \quad \text{in } (0, \infty) \quad (5.3)$$

where  $C_1 = (N+a)\left(\frac{N-p}{p-1}\right)^{p-1}$ ,  $C_2 = (N+b)\left(\frac{N-q}{q-1}\right)^{q-1}$ , and

$$\lim_{r \rightarrow 0} r^{\frac{N-p}{p-1}} u = c_1 \geq 0, \quad \lim_{r \rightarrow 0} r^{\frac{N-q}{q-1}} v(r) = c_2 \geq 0, \quad \lim_{r \rightarrow \infty} \inf r^{\frac{N-p}{p-1}} u > 0, \quad \lim_{r \rightarrow \infty} \inf r^{\frac{N-q}{q-1}} v > 0. \quad (5.4)$$

As a consequence if  $s \leq p-1$  or  $m \leq q-1$ , we have

$$u \leq K_1 r^{-\gamma}, \quad v \leq K_2 r^{-\xi}, \quad \text{in } (0, \infty), \quad (5.5)$$

with  $K_1 = C_1^{(q-1-m)/D} C_2^{\delta/D}$ ,  $K_2 = C_1^{\mu/D} C_2^{(p-1-s)/D}$ .

**Proof.** Consider a solution of System (M) in  $\mathcal{R}$  defined on  $\mathbb{R}$ . On the hyperplan  $X = \frac{N-p}{p-1}$  we have  $X_t > 0$ , the field is going out. If at some point  $t_0$ ,  $X(t_0) = \frac{N-p}{p-1}$ , then  $X(t) > \frac{N-p}{p-1}$  for  $t > t_0$ ,

then, since  $Z > 0$ ,  $X_t \geq X \left[ X - \frac{N-p}{p-1} \right] > 0$ , thus  $X(t) > 2\frac{N-p}{p-1}$  for  $t > t_1 > t_0$ , then  $X_t \geq X^2/2$ , and then  $X$  blows up in finite time, contradiction, thus  $X(t) < \frac{N-p}{p-1}$  on  $\mathbb{R}$ . By integration it implies that  $r^{(N-p)/(p-1)}u(r)$  is nondecreasing near 0 or  $\infty$ , hence (5.4) holds.  $\blacksquare$

Next we prove Theorem 1.1.

**Proof of Theorem 1.1.** (i) The trajectories of the regular solutions start from  $N_0 = (0, 0, N + a, N + b)$ , from Proposition 4.4, and the unstable variety  $\mathcal{V}_u$  has dimension 2, from (10.1), (10.2). It is given locally by  $Z = \varphi(X, Y), W = \psi(X, Y)$  for  $(X, Y) \in B(0, \rho) \setminus \{0\} \subset \mathbb{R}^2$ .

To any  $(x, y) \in B(0, \rho) \setminus \{0\}$  we associate the unique trajectory  $\mathcal{T}_{x,y}$  in  $\mathcal{V}_u$  going through this point. If  $T^*$  is the maximal interval of existence of a solution on  $\mathcal{T}_{x,y}$ , then  $\lim_{t \rightarrow T^*} (X(t) + Y(t)) = \infty$ . Indeed  $Z$ , and  $W$  satisfy  $0 < Z < N + a, 0 < W < N + b$  as long as the solution exists, because at a point  $T$  where  $Z(T) = N + a$ , we have  $Z_t < 0$ . If there exists a first point  $T$  such that  $X(T) = \frac{N-p}{p-1}$  or  $Y(T) = \frac{N-q}{q-1}$ , then  $T < T^*$ . We consider the closed rectangle  $\mathcal{N}$  of submits

$$(0, 0), \quad \varpi_1 = \left( \frac{N-p}{p-1}, 0 \right), \quad \varpi_2 = \left( 0, \frac{N-q}{q-1} \right), \quad \varpi_3 = \left( \frac{N-p}{p-1}, \frac{N-q}{q-1} \right).$$

Let  $\mathcal{U} = \{(x, y) \in B(0, \rho) : x, y > 0\}$ ; then  $\mathcal{U} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 \cup \mathcal{S}$ , where

$$\left\{ \begin{array}{l} \mathcal{S}_1 = \{(x, y) \in \mathcal{U} \text{ and } \mathcal{T}_{x,y} \text{ leaves } \mathcal{N} \text{ on } (\varpi_1, \varpi)\}, \quad \mathcal{S}_2 = \{(x, y) \in \mathcal{U} : \mathcal{T}_{x,y} \text{ leaves } \mathcal{N} \text{ on } (\varpi_2, \varpi_3)\}, \\ \mathcal{S}_3 = \{(x, y) \in \mathcal{U} : \mathcal{T}_{x,y} \text{ leaves } \mathcal{N} \text{ at } \varpi_3\}, \quad \mathcal{S} = \left\{ (x, y) \in \mathcal{U} : \mathcal{T}_{x,y} \text{ stays in } \overset{\circ}{\mathcal{N}} \right\}. \end{array} \right.$$

Any element of  $\mathcal{S}$  defines a G.S. Assume  $s < \frac{N(p-1)+p+pa}{N-p}$ . Let us show that  $\mathcal{S}_1$  is nonempty. Consider the trajectory  $\mathcal{T}_{\bar{x},0}$  on  $\mathcal{V}_u$  associated to  $(\bar{x}, 0)$ , with  $\bar{x} \in (0, \rho)$ , going through  $\bar{M} = (\bar{x}, 0, \varphi(\bar{x}, 0), \psi(\bar{x}, 0))$ ; it is *nonadmissible* for our problem, in the hyperplane  $Y = 0$ : we get the system

$$\left\{ \begin{array}{l} X_t = X \left[ X - \frac{N-p}{p-1} + \frac{Z}{p-1} \right], \\ Z_t = Z [N + a - sX - Z], \\ W_t = W [N + b - \mu X - W], \end{array} \right.$$

which is *not coupled*. The two equations in  $X, Z$  corresponds to the equation

$$-\Delta_p U = r^a U^s \tag{5.6}$$

The regular solutions of (5.6) are changing sign, since  $s$  is subcritical, see Section 3. Consider the solution  $(\bar{X}, \bar{Y}, \bar{Z}, \bar{W})$  of system (M), of trajectory  $\mathcal{T}_{\bar{x},0}$ , going through  $\bar{M}$  at time 0; it satisfies  $\bar{Y} = 0$ , and  $\bar{X}(t) > 0, \bar{Z}(t) > 0$  tends to  $\infty$  in finite time  $T^*$ , then for any given  $C \geq \frac{N-p}{p-1}$ , there exist a first  $T < T^*$  such that  $\bar{X}(T) = C$ , and  $\bar{Y}(T) = 0$ . And  $\lim_{t \rightarrow -\infty} \bar{W} = N + b$ , and necessarily  $0 < \bar{W} < N + b$ , in particular  $0 < \bar{W}(T) < N + b$ ; and  $\bar{W}_t$  is bounded on  $(-\infty, T^*)$ , then  $\bar{W}$  has

a finite limit at  $T^*$ . The field at time  $T$  is transverse: we have  $\bar{X}_t \geq C \frac{Z(T)}{p-1} > 0$ , since  $\bar{Z}(T) > 0$ . From the continuous dependance of the initial data at time 0, for any  $\varepsilon > 0$ , there exist  $\eta > 0$  such that for any  $(x, y) \in B((\bar{x}, 0), \eta)$  and for any  $(X, Y, Z, W)$  on  $\mathcal{T}_{x,y}$ , there exists a first time  $T_\varepsilon$  such that  $X(T_\varepsilon) = C$ , and  $|Y(t)| \leq \varepsilon$  for any  $t \leq T_\varepsilon$ , in particular for any  $(x, y) \in B((\bar{x}, 0), \eta)$  with  $y > 0$ , and then  $0 < Y(t) \leq \varepsilon$  for any  $t \leq T_\varepsilon$ . Let us take  $C = \frac{N-p}{p-1}$ . Then  $(x, y) \in \mathcal{S}_1$ . The same arguments imply that  $\mathcal{S}_1$  is open. Similarly assuming  $m < \frac{N(q-1)+q+qb}{N-q}$  implies that  $\mathcal{S}_2$  is nonempty and open. By connexity  $\mathcal{S}$  is empty if and only if  $\mathcal{S}_3$  is nonempty.

(ii) Here the difficulty is due to the fact that the zeros of  $u, v$  correspond to infinite limits for  $X, Y$ , and then the argument of continuous dependance is no more available. We can write  $\mathcal{U} = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3 \cup \mathcal{S}$ , where

$$\begin{cases} \mathcal{M}_1 = \{(x, y) \in \mathcal{U} \text{ and } \mathcal{T}_{x,y} \text{ has an infinite branch in } X \text{ with } Y \text{ bounded}\}, \\ \mathcal{M}_2 = \{(x, y) \in \mathcal{U} : \mathcal{T}_{x,y} \text{ has an infinite branch in } Y \text{ with } X \text{ bounded}\}, \\ \mathcal{M}_3 = \{(x, y) \in \mathcal{U} : \mathcal{T}_{x,y} \text{ has an infinite branch in } (X, Y)\}, \end{cases}$$

In other words,  $\mathcal{M}_1$  is the set of  $(x, y) \in \mathcal{U}$  such that for any  $(X, Y, Z, W)$  on  $\mathcal{T}_{x,y}$ , there exists a  $T^*$  such that  $\lim_{t \rightarrow T^*} X(t) = \infty$ , and  $Y(t)$  stays bounded on  $(-\infty, T^*)$ , that means the set of  $(x, y) \in \mathcal{U}$  such that for any solution  $(u, v)$  corresponding to  $\mathcal{T}_{x,y}$ ,  $u$  vanishes before  $v$ ; similarly for  $\mathcal{M}_2$ . And  $\mathcal{M}_3$  is the set of  $(x, y) \in \mathcal{U}$  such that there exists a  $T^*$  such that  $\lim_{t \rightarrow T^*} X(t) = \lim_{t \rightarrow T^*} Y(t) = \infty$ , that means  $(u, v)$  vanish at the same  $R^* = e^{T^*}$ . In that case, from the Höpf Lemma,  $\lim_{r \rightarrow R} \frac{u'}{(r-R)u} = 1$ , then  $\lim_{t \rightarrow T^*} \frac{X}{Y} = 1$ .

We are lead to show that  $\mathcal{M}_1$  is nonempty and open for  $s < \frac{N(p-1)+p+pa}{N-p}$ . We consider again the trajectory  $\bar{T}$  and take  $C$  large enough:  $C = 2(\frac{N-p}{p-1} + \frac{N+|b|}{q-1})$ . Let  $\varepsilon \in (0, \frac{C}{2})$ . For any  $(x, y) \in B((\bar{x}, 0), \eta)$  with  $y > 0$ , and any  $(X, Y, Z, W)$  on  $\mathcal{T}_{x,y}$ , there is a first time  $T_\varepsilon$  such that  $X(T_\varepsilon) = C$ , and  $0 < Y(t) \leq \varepsilon$  for any  $t \leq T_\varepsilon$ . And  $X$  is increasing and  $X_t \geq X(X - C)$ , thus there exists  $T^*$  such that  $\lim_{t \rightarrow T^*} X(t) = \infty$ . Let  $\varphi = X/Y$ . Then

$$\frac{\varphi_t}{\varphi} = X - Y + \frac{Z}{p-1} - \frac{W}{q-1} + \frac{N-q}{q-1} - \frac{N-p}{p-1} \geq X - Y - \frac{C}{2}$$

then  $\varphi_t(T_\varepsilon) > 0$ . Let  $\theta = \sup\{t > T_\varepsilon : \varphi_t > 0\}$ ; suppose that  $\theta$  is finite; then  $\varphi(\theta) > \varphi(T_\varepsilon) = C/\varepsilon > 2$  and  $X(\theta) \leq Y(\theta) + C < X(\theta)/2 + C$ , which is contradictory. Then  $\varphi$  is increasing up to  $T^*$ ; if  $\lim_{t \rightarrow T^*} Y(t) = \infty$ , then  $\lim_{t \rightarrow T^*} \varphi = 1$ , which is impossible. Then  $(x, y)$  is in  $\mathcal{M}_1$ ; thus  $\mathcal{M}_1$  is nonempty. In the same way  $\mathcal{M}_1$  is open: let  $(\bar{x}, \bar{y})$  in  $\mathcal{M}_1$  then there exist  $M > 0$  such that  $0 < \bar{Y}(t) \leq M/2$  on  $\mathcal{T}_{\bar{x}, \bar{y}}$ ; then we argue as above, with  $(\bar{x}, 0)$  replaced by  $(\bar{x}, \bar{y})$ , and  $C$  replaced by  $C + M$ .  $\blacksquare$

**Proof of Proposition 1.2.** Assume  $s \geq \frac{N(p-1)+p+pa}{N-p}$ . Consider the Pohozaev type function

$$\mathcal{F}(r) = r^N \left[ \frac{|u'|^p}{p'} + \frac{r^a u^{s+1}}{s+1} v^\delta + \frac{N-p}{p} u \frac{|u'|^{p-2} u'}{r} \right] = r^{N-p} u^p \left[ \frac{X}{p'} + \frac{1}{s+1} Z - \frac{N-p}{p} \right]. \quad (5.7)$$

We find  $\mathcal{F}(0) = 0$  and

$$\begin{aligned}\mathcal{F}'(r) &= r^{N-1+a} \left[ \left( \frac{N+a}{s+1} - \frac{N-p}{p} \right) v^\delta u^{s+1} + \frac{\delta}{s+1} r u^{s+1} v^{\delta-1} v' \right] \\ &= r^{N-1+a} v^\delta u^{s+1} \left[ \frac{N+a}{s+1} - \frac{N-p}{p} - \frac{\delta Y}{s+1} \right]\end{aligned}\tag{5.8}$$

From our assumption,  $\mathcal{F}$  is decreasing, and  $Z > 0$ , thus  $X < \frac{N-p}{p-1}$ . Then  $\mathcal{S}_1, \mathcal{S}_3$  are empty. If moreover  $m \geq \frac{N(q-1)+q+qb}{N-q}$  then  $\mathcal{S}_2$  is empty, then  $\mathcal{S} = \mathcal{U}$ .  $\blacksquare$

**Remark 5.2** Assume only  $s \geq \frac{N(p-1)+p+pa}{N-p}$ . Then if one function has a first zero, it is  $v$ . Indeed if there exists a first point where  $u(R) = 0$ , and  $v(r) > 0$  on  $[0, R)$ , then  $\mathcal{F}(R) = \frac{R^N}{p'} |u'(R)|^p > 0$ .

As a first consequence we obtain existence results for the Dirichlet problem. It solves an open problem in case  $s > p-1$  or  $m > q-1$ , and extends some former results of [12] and [42]. Our proof, based on the shooting method differs from the proof of [12], based on degree theory and blow-up technique. Our results extend the ones of [3, Theorem 2.2] relative to the case  $p = q = 2$ , obtained by studying the equation satisfied by a suitable function of  $u, v$ .

**Corollary 5.3** System (S) admits no G.S. and then there is a radial solution of the Dirichlet problem in a ball in any of the following cases:

- (i)  $p < s+1, q < m+1$ , and  $\min(s \frac{N-p}{p-1} + \frac{N-q}{q-1} \delta - (N+a), \frac{N-p}{p-1} \mu + m \frac{N-q}{q-1} - (N+b)) \leq 0$ ;
- (ii)  $p < s+1, q > m+1$  and  $s \frac{N-p}{p-1} + \frac{N-q}{q-1} \delta - (N+a) \leq 0$  or  $\gamma - \frac{N-p}{p-1} > 0$ ;
- (iii)  $p > s+1, q > m+1$  and  $\max(\gamma - \frac{N-p}{p-1}, \xi - \frac{N-q}{q-1}) \geq 0$ ;
- (iv)  $p \geq s+1, q \geq m+1$  and  $\max(\gamma - \frac{N-p}{p-1}, \xi - \frac{N-q}{q-1}) > 0$ .

**Proof.** From Theorem 1.1, we are reduced to prove the nonexistence of G.S.

(i) Assume  $p < s+1$ , and  $s \frac{N-p}{p-1} + \frac{N-q}{q-1} \delta - (N+a) < 0$ . Also, since  $s > p-1$ , and  $-\Delta_p u \geq Cr^{a - \frac{N-q}{q-1} \delta} u^s$  for large  $r$ , then from [6, Theorem 3.1], we find  $u = O(r^{-(p+a - \frac{N-q}{q-1} \delta)/(s+1-p)})$ , and then  $s \frac{N-p}{p-1} + \frac{N-q}{q-1} \delta - (N+a) \geq 0$ , from (5.4), which contradicts our assumption. In case of equality, we find  $-\Delta_p u \geq Cr^{-N}$  for large  $r$ , which is impossible. Then there exists no G.S. This improves the result of [12] where the minimum is replaced by a maximum.

(ii) Assume  $p < s+1, q > m+1$  and  $\gamma - \frac{N-p}{p-1} > 0$ ; then  $u = O(r^{-\gamma})$ , which contradicts (5.4). If  $\gamma - \frac{N-p}{p-1} = 0$ , then  $\lim r^{\frac{N-p}{p-1}} u = \alpha > 0$ , and  $\xi > \frac{N-q}{q-1}$ ; and  $-\Delta_q v = Cr^{b - \frac{N-p}{p-1}} \mu v^m$  for large  $r$ , then  $v \geq Cr^{(q+b - \frac{N-p}{p-1} \delta)/(q-1-m)} = Cr^{-\xi}$ , then  $C_1 > r^\xi v > C_2 > 0$  for large  $r$ , from [6, Theorem 3.1] and (5.5), then  $-\Delta_p u \geq Cr^{-N}$ , which is still contradictory.

(iii) (iv) The nonexistence of G.S is obtained by extension of the proof of [12] to the case  $a, b \neq 0$ . Moreover (iii) implies the nonexistence of positive solution  $(u, v)$ , radial or not, in any exterior domain  $(R, \infty) \times (R, \infty)$ ,  $R > 0$  from [6].  $\blacksquare$

**Corollary 5.4** *Assume (4.3) with  $p = q = 2$ . If  $\delta + s \geq \frac{N+2+2a}{N-2}$  and  $\mu + m \geq \frac{N+2+2b}{N-2}$ , then system (S) admits a G.S.*

**Proof.** It was shown in [28], [41] by the moving spheres method that the Dirichlet problem has no radial or nonradial solution. Then Theorem 1.1 applies again.  $\blacksquare$

We also extend and improve a result of nonexistence of [10] relative to the case  $p = q = 2, a = 0, s > 1$ :

**Proposition 5.5** *Assume  $s + 1 > p$  or  $\gamma > \frac{N-p}{p}$ , and*

$$s + \frac{p(N-q)}{(q-1)(N-p)}\delta < \frac{N(p-1) + pa + p}{N-p} \quad (5.9)$$

*Then the system (S) admits no G.S. and then there is a solution of the Dirichlet problem. The same happens by exchanging  $p, s, \delta, a, \gamma$  with  $q, m, \mu, b, \xi$ .*

**Proof.** Consider the function  $\mathcal{F}$  defined at (5.7). Suppose that there exists a G.S. Then from (5.1) and (5.9) we find

$$\frac{N+a}{s+1} - \frac{N-p}{p} - \frac{\delta Y}{s+1} > \frac{N+a}{s+1} - \frac{N-p}{p} - \frac{\delta}{s+1} \frac{N-q}{q-1} \geq 0.$$

From (5.8),  $\mathcal{F}$  is nondecreasing. First suppose  $s + 1 > p$ . Then from (5.3) and (5.4),  $u = O(r^{-k})$  at  $\infty$ , with  $k = (p + a - \delta \frac{N-q}{q-1}) / (s - p + 1)$ , then  $r^{N-p}u^p = O(r^{(N-p)-kp}) = o(1)$  from (5.9), then  $\mathcal{F}(r) = o(1)$  near  $\infty$ . Next assume  $s + 1 \leq p$  and  $\gamma > \frac{N-p}{p}$ , then  $r^{N-p}u^p = O(r^{N-p-\gamma p})$ , hence  $\mathcal{F}(r) = o(1)$  near  $\infty$ ; in any case we get a contradiction.  $\blacksquare$

## 6 The Hamiltonian system

Here we consider the nonnegative solutions of the variational Hamiltonian problem (SH) in  $\Omega \subset \mathbb{R}^N$

$$(SH) \begin{cases} -\Delta u = |x|^a v^\delta, \\ -\Delta v = |x|^b u^\mu, \end{cases}$$

where  $p = q = 2 < N$ ,  $s = m = 0$ ,  $a > b > -2$ , and  $D = \delta\mu - 1 > 0$ . Here we find

$$\gamma = \frac{(2+a) + (2+b)\delta}{D}, \quad \xi = \frac{2+b + (2+a)\mu}{D}, \quad \gamma + 2 + a = \delta\xi, \quad \xi + 2 + b = \mu\gamma.$$

The particular solution  $(u_0(r), v_0(r)) = (Ar^{-\gamma}, Br^{-\xi})$  exists for  $0 < \gamma < N - 2$  and  $0 < \xi < N - 2$ . Here  $X, Y, Z, W$  are defined by

$$X(t) = \frac{r|u'|}{u}, \quad Y(t) = \frac{r|v'|}{v}, \quad Z(t) = \frac{r^{1+a}v^\delta}{|u'|}, \quad W(t) = \frac{r^{1+b}u^\mu}{|v'|},$$

with  $t = \ln r$ , and system  $(M)$  becomes

$$(MH) \begin{cases} X_t = X[X - (N - 2) + Z], \\ Y_t = Y[Y - (N - 2) + W], \\ Z_t = Z[N + a - \delta Y - Z], \\ W_t = W[N + b - \mu X - W] \end{cases}$$

This system has a Pohozaev type function, well known at least in the case  $a = b = 0$ , given at (1.7):

$$\begin{aligned} \mathcal{E}(r) &= r^N \left[ u'v' + r^b \frac{|u|^{\mu+1}}{\mu+1} + r^a \frac{|v|^{\delta+1}}{\delta+1} + \frac{N+a}{\delta+1} \frac{vu'}{r} + \frac{N+b}{\mu+1} \frac{wv'}{r} \right] \\ &= r^{N-2} uv \left[ XY - \frac{Y(N+b-W)}{\mu+1} - \frac{(N+a-Z)X}{\delta+1} \right] \\ &= r^{N-2-\gamma-\xi} (ZX)^{(\mu+1)/D} (WY)^{(\delta+1)/D} \left[ XY - \frac{Y(N+b-W)}{\mu+1} - \frac{(N+a-Z)X}{\delta+1} \right]. \end{aligned}$$

It can also be found by a direct computation, and satisfies

$$\mathcal{E}'(r) = r^{N-1} u'v' \left( \frac{N+a}{\delta+1} + \frac{N+b}{\mu+1} - (N-2) \right).$$

We define the critical case, when  $(\delta, \mu)$  lie on the critical hyperbola  $\mathcal{H}_0$  given by

$$\frac{N+a}{\delta+1} + \frac{N+b}{\mu+1} = N-2, \text{ equivalently } \gamma + \xi = N-2. \quad (6.1)$$

In this case  $\gamma = \frac{N+b}{\mu+1}$ ,  $\xi = \frac{N+a}{\delta+1}$ , and  $\mathcal{E}'(r) \equiv 0$ . It corresponds to the existence of a first integral of system  $(M)$ :

$$\mathcal{E}(r) = (ZX)^{(\mu+1)/D} (WY)^{(\delta+1)/D} \left[ XY - \frac{Y(N+b-W)}{\mu+1} - \frac{(N+a-Z)X}{\delta+1} \right].$$

It can also be expressed in the variables  $U = r^\gamma u, V = r^\xi v$  of Remark 2.2 under the form

$$\mathcal{E}(r) = U_t V_t - \gamma \xi UV + \frac{U^{\mu+1}}{\mu+1} + \frac{V^{\delta+1}}{\delta+1}.$$

The supercritical case is defined as the case where  $(\delta, \mu)$  is above  $\mathcal{H}$ , equivalently  $\gamma + \xi < N - 2$  and the subcritical case corresponds to  $(\delta, \mu)$  under  $\mathcal{H}$ .

**Remark 6.1** The energy  $\mathcal{E}_0$  of the particular solution associated to  $M_0$  is always negative:  $\mathcal{E}_0 = -\frac{D}{(\mu+1)(\delta+1)} r^{N-2-\gamma-\xi} X_0 Y_0 (Z_0 X_0)^{(\mu+1)/D} (W_0 Y_0)^{(\delta+1)/D}$ .

**Remark 6.2** In the case  $a = b = 0$ , it is known that there exists a solution of the Dirichlet problem in any bounded regular domain  $\Omega$  of  $\mathbb{R}^N$ , see for example [15], [20]; for general  $a, b$ , some restrictions on the coefficients appear, see [23] and [14].

Next consider the critical and supercritical case. When  $a = b = 0$ , there exists no solution if  $\Omega$  is starshaped, see [36].

Here we show the existence of G.S. for general  $a, b$ . The existence in the critical case with  $a = b = 0$  was first obtained in [22], then in the supercritical case in [29], and uniqueness was proved in [20]. The proof of [29] is quite long due to regularity problems, when  $\delta$  or  $\mu < 1$ , which play no role in our quadratic system:

**Remark 6.3** The particular case  $\delta = \mu$  and  $a = b$  is easy to treat. Indeed in that case  $u = v$  is a solution of the scalar equation  $\Delta u + |x|^\alpha |u|^{\delta-1} u = 0$ , for which the critical case is given by  $\delta = (N + 2 + 2a)/(N - 2)$ . Moreover if system (SH) admits a G.S., or a solution of the Dirichlet problem in a ball, it satisfies  $u = v$ , from [3]. Then we are completely reduced to the scalar case. In particular, in the critical case, the G.S. are given explicitly by :  $u = v = c(K + r^{(2+a)})^{(2-N)/(2+a)}$ , where  $K = c^{\delta-1}/(N + a)(N - 2)$ ; in other words they satisfy (3.3) with  $X = Y$  and  $Z = W$ , i.e.

$$\frac{X(t)}{N-2} + \frac{Z(t)}{N+a} - 1 = 0.$$

Near  $\infty$ , the G.S. is (obviously) symmetrical: it joins the points  $N_0$  and  $A_0$ .

**Remark 6.4** Consider the case  $\delta = 1$ ,  $a = b = 0$ , which is the case of the biharmonic equation

$$\Delta^2 u = u^\mu.$$

Recall that it is the only case where the conjecture (C) was completely solved by Lin in [21]. In the critical case  $\mu = (N + 4)/(N - 4)$ , the G.S. are also given explicitly, see [20]:

$$u(r) = c(K + r^2)^{(4-N)/2}, \quad K = c^{\mu-1}/(N - 4)(N - 2)N(N + 2).$$

They satisfy the relation  $XY = \frac{N-Z}{2}X + \frac{N-4}{2N}(N - W)Y$ , and moreover we find that they are on an hyperplane:

$$\frac{(N-2)X(t)}{N(N-4)} + \frac{Z(t)}{N} - 1 = 0.$$

Observe also that the G.S. is not symmetrical near  $\infty$ :  $u$  behaves like  $r^{4-N}$  and  $v$  behaves like  $r^{2-N}$ . The trajectory in the phase space joins the points  $N_0$  and  $Q_0 = (N - 4, N - 2, 2, 0)$ .



**Proof of Theorem 1.4.** 1) *Existence or nonexistence results:*

• In the supercritical or critical case we apply any of the two conditions of Theorem 1.1: Here  $\mathcal{E}(0) = 0$ ,  $\mathcal{E}$  is nonincreasing; there does not exist solutions of (M) such that at some point  $T$ ,  $X(T) = Y(T) = N - 2$ , because at the point  $T$ ,

$$XY - \frac{Y(N+b-W)}{\mu+1} - \frac{(N+a-Z)X}{\delta+1} = (N-2) \left[ N-2 - \frac{N+a}{\delta+1} - \frac{N+b}{\mu+1} + \frac{W}{\mu+1} + \frac{Z}{\delta+1} \right] > 0$$

since  $W > 0, Z > 0$ , thus  $\mathcal{E}(e^T) > 0$ , which is impossible. In the same way there exists no solution of the Dirichlet problem in a ball  $B(0, R)$ , because  $\mathcal{E}(R) = R^N u'(R)v'(R) > 0$  from the Höpf Lemma. Then there exists a G.S. The uniqueness is proved in [20].

• In the subcritical case there is no radial G.S.: it would satisfy  $\mathcal{E}(0) = 0$ ,  $\mathcal{E}$  is nondecreasing,  $\mathcal{E}(r) \leq Cr^{N-2-\gamma-\xi}$  from (5.1), and  $\gamma + \xi > (N - 2)$ , then  $\lim_{r \rightarrow \infty} \mathcal{E}(r) = 0$ . Then from Theorem 1.1, there exists a solution of the Dirichlet problem.

2) *Behaviour of the G.S. in the critical case.*

It is easy to see that condition (1.6) implies  $\mu > \frac{2+b}{N-2}$  and  $\delta > \frac{2+a}{N-2}$ , and that  $\delta \leq \frac{N+a}{N-2}$  and  $\mu \leq \frac{N+b}{N-2}$  cannot hold simultaneously. One can suppose that  $\delta > \frac{N+a}{N-2}$ . Let  $\mathcal{T}$  the unique trajectory of the G.S.; then  $\mathcal{E}(0) = 0$ , thus  $\mathcal{T}$  lies on the variety  $\mathcal{V}$  of energy 0, defined by

$$\frac{X(N+a-Z)}{\delta+1} + \frac{Y(N+b-W)}{\mu+1} = XY. \quad (6.2)$$

and from (5.2)  $\mathcal{T}$  starts from the point  $N_0$ ; from (5.1)  $\mathcal{T}$  stays in

$$\mathcal{A} = \{(X, Y, Z, W) \in \mathbb{R}^4 : 0 < X < N - 2, \quad 0 < Y < N - 2, \quad 0 < Z < N + a, \quad 0 < W < N + a\}.$$

(i) If  $\mathcal{T}$  converges to a fixed point of the system in  $\bar{\mathcal{R}}$ , the only possible points are  $A_0, P_0, Q_0$  which are effectively on  $\mathcal{V}$ . Indeed  $I_0, J_0, G_0, H_0 \notin \mathcal{V}$ . But  $Q_0 = ((N-2)\delta - (2+a), N-2, N+a - (N-2)\delta, 0) \notin \bar{\mathcal{R}}$ , since  $\delta > \frac{N+a}{N-2}$ . And  $P_0 \in \bar{\mathcal{R}}$  if  $\mu \leq \frac{N+b}{N-2}$ .

If  $\mu > \frac{N+b}{N-2}$ , then  $\mathcal{T}$  converges to  $A_0$ . If  $\mu < \frac{N+b}{N-2}$ , no trajectory converges to  $A_0$ , from Proposition 4.5, thus  $\mathcal{T}$  converges to  $P_0$ . If  $\mu \neq \frac{N+b}{N-2}$  the convergence is exponential, thus the behaviour of  $u, v$  follows. If  $\mu = \frac{N+b}{N-2}$ , then  $\mathcal{T}$  converges to  $A_0 = P_0$ ; the eigenvalues given by (10.3) satisfy  $\lambda_1 = \lambda_2 = N - 2$ ,  $\lambda_3 = N + a - \delta(N - 2) < 0$  and  $\lambda_4 = 0$ ; the projection of the trajectory on the hyperplane  $Y = N - 2$  satisfies the system

$$X_t = X [X - (N - 2) + Z], \quad Z_t = Z [N + a - \delta(N - 2) - Z]$$

which presents a saddle point at  $(N - 2, 0)$ , thus the convergence of  $X$  and  $Z$  is exponential, in particular we deduce the behaviour of  $u$ . The trajectory enters by the central variety of dimension 1, and by computation we deduce that  $Y - (N - 2) = -t^{-1} + O(t^{-2+\varepsilon})$  near  $\infty$ , and the behaviour of  $v$  follows.

(ii) Next we show that  $\mathcal{T}$  converges to a fixed point: We eliminate  $W$  from (6.2), we get a quadratic system in  $(X, Y, Z)$  :

$$\begin{cases} X_t = X [X - (N - 2) + Z], \\ Y_t = Y [Y + b + 2 - (\mu + 1)X] + \frac{\mu+1}{\delta+1}X(N + a - Z), \\ Z_t = Z [N + a - \delta Y - Z] \end{cases} \quad (6.3)$$

At the begining,  $X_t \geq 0$ , and  $Y_t \geq 0$ . Suppose that  $X$  has a maximum at  $t_0$  followed by a minimum at  $t_1$ . Then  $Z_t(t_0) < 0 < Z_t(t_1)$  then there exists  $t_2 \in (t_0, t_1)$  such that  $Z_t(t_2) = 0$ , it is a minimum, then at this point  $Z = N + a - \delta Y$ ,  $Z_{tt} = -\delta Z Y_t$ , hence

$$Y_t = Y \left[ Y + b + 2 - (\mu + 1)X + \delta \frac{\mu + 1}{\delta + 1} X \right] = Y \left[ Y + b + 2 - \frac{\mu + 1}{\delta + 1} X \right] < 0$$

and  $X_t(t_2) < 0$ , hence  $(X + Z)(t_2) < N - 2$ , At the point  $t_2$  we have

$$N - 2 - X > Z = N + a - \delta Y > N + a - \delta \left( \frac{\mu + 1}{\delta + 1} X - b - 2 \right)$$

$$(a + 2) + \delta(b + 2) < \left( \delta \frac{\mu + 1}{\delta + 1} - 1 \right) X = \frac{\delta(2 + b) + (2 + a)}{(N - 2)\delta - (2 + a)} X$$

and  $X(t_2) < X(t_0) < \delta(N - 2) - (2 + a)$ , hence we are lead to a contradiction. Then  $X$  has at most one extremum, which is a maximum, and then it has a limit in  $(0, N - 2]$  at  $\infty$ . In the same way, by symmetry,  $Y$  has at most one extremum, which is a maximum, and has a limit in  $(0, N - 2]$  at  $\infty$ . Then  $Z$  has at most one extremum, which is a minimum: indeed at the points where  $Z_t = 0$ ,  $-Z_{tt}$  has the sign of  $Y_t$ . Then  $Z$  has a limit in  $[0, N + a)$ , and similarly  $W$  has a limit in  $[0, N + b)$ .

■

**Open problems:** 1) For the case  $\delta = \mu$ , in the critical case it is well known that there exist solutions  $(u, v)$  of system  $(SH)$  of the form  $(u, u)$ , such that  $r^\gamma u$  is periodic in  $t = \ln r$ . They correspond to a periodic trajectory for the scalar system  $(M_{scal})$  with  $p = 2$ , and it admits an infinity of such trajectories. If  $\delta \neq \mu$ , does there exist solutions  $(u, v)$  such that  $(r^\gamma u, r^\xi v)$  is periodic in  $t$ , in other words a periodic trajectory for system  $(MH)$ ?

2) In the supercritical case, we cannot prove that the regular trajectory  $\mathcal{T}$  converges to  $M_0$ , that means  $\lim_{r \rightarrow \infty} r^\gamma u = A$ ,  $\lim_{r \rightarrow \infty} r^\xi v = B$ . Here  $\mathcal{E}(0) = 0$ ,  $\mathcal{E}$  is nonincreasing, then  $\mathcal{E}$  is negative. The only fixed points of negative energy are  $M_0, G_0, H_0$ , but a G.S. satisfies (5.5), then it tends to  $(0, 0)$  at  $\infty$ , hence  $\mathcal{T}$  cannot converge to  $G_0$  or  $H_0$  from Proposition 4.9; but we cannot prove that  $\mathcal{T}$  converges to some fixed point.

## 7 A nonvariational system

Here we consider the system (S) with  $p = q = 2$ ,  $a = b$  and  $s = m \neq 0$ .

$$(SN) \begin{cases} -\Delta u = |x|^a u^s v^\delta, \\ -\Delta v = |x|^a u^\mu v^s, \end{cases}$$

where  $D = \delta\mu - (1 - s)^2 > 0$ . We have seen that we can reduce the system to the case  $a = 0$ , which will be supposed in this paragraph. Here  $X = -ru'/u$ ,  $Y = -rv'/v$ ,  $Z(t) = -ru^s v^\delta / u'$ ,  $W(t) = -ru^\mu v^s / v'$ , and the system (M) becomes

$$(MN) \begin{cases} X_t = X [X - (N - 2) + Z], \\ Y_t = Y [Y - (N - 2) + W], \\ Z_t = Z [N - sX - \delta Y - Z], \\ W_t = W [N - \mu X - sY - W]. \end{cases}$$

We have chosen this system because it is not variational, and from Proposition 4.3, it can exist an hyperbola of values  $(\delta, \mu)$  for which the linearized system at  $M_0$  has two imaginary roots.

We first recall two results of [38] and [26], and complete them:

**Proposition 7.1** (i) [38] Assume  $s < \frac{N}{N-2}$ , and  $\delta + 1 - s > 0$ ,  $\mu + 1 - s > 0$ . Let  $\mathcal{H}'_s$  be the hyperbola of equation

$$\frac{1}{\delta + 1} + \frac{1}{\mu + 1} = \frac{N - 2}{N - (N - 2)s} \quad (7.1)$$

located under  $\mathcal{H}_s$ . Then under  $\mathcal{H}'_s$ , the system (SN) admits no G.S., and then there is a solution of the Dirichlet problem in a ball.

(ii) [26] Let  $\mathcal{H}_0$  be the hyperbola defined by

$$\frac{1}{\delta + 1} + \frac{1}{\mu + 1} = \frac{N - 2}{N} \quad (7.2)$$

Then above  $\mathcal{H}_0$  there exists no solution of the Dirichlet problem. Thus there exists a G.S.

**Proof.** (i) We consider an energy function with parameters  $\alpha, \beta, \sigma, \theta$  :

$$\mathcal{E}(r) = r^N \left[ u'v' + \alpha u^{\mu+1} v^s + \beta v^{\delta+1} u^s + \frac{\sigma}{r} v u' + \frac{\theta}{r} u v' \right] \quad (7.3)$$

$$= r^{N-2} u v \Psi_0 = r^{N-2-\gamma-\xi} (ZX)^{\xi/2} (WZ)^{\gamma/2} \Psi_0, \quad (7.4)$$

from (4.2), where

$$\Psi_0(X, Y, Z, W) = XY + \alpha WY + \beta ZX - \sigma X - \theta Y. \quad (7.5)$$

We get

$$\begin{aligned} r^{1-N}(uv)^{-1}\mathcal{E}'(r) &= (\sigma + \theta - (N - 2))XY + (N\alpha - \theta)YW \\ &\quad + (N\beta - \sigma)XZ - (\alpha(\mu + 1) - 1)XYW - (\beta(\delta + 1) - 1)XYZ - \alpha sY^2W - \beta sX^2Z. \end{aligned}$$

Taking  $\alpha = \frac{1}{\mu+1}, \beta = \frac{1}{\delta+1}$ , one finds

$$r^{3-N}(uv)^{-1}\mathcal{E}'(r) = (\sigma + \theta - (N - 2))XY + (N\alpha - \theta - \alpha sY)YW + (N\beta - \sigma - \beta sX)XZ. \quad (7.6)$$

If there exists a G.S., from (5.1) it satisfies  $X, Y < N - 2$ , hence

$$r^{3-N}(uv)^{-1}\mathcal{E}'(r) > (\sigma + \theta - (N - 2))XY + ((N - (N - 2)s)\alpha - \theta)YW + ((N - (N - 2)s)\beta - \sigma)XZ \quad (7.7)$$

Taking  $\theta = \frac{N - (N - 2)s}{\mu + 1}, \sigma = \frac{N - (N - 2)s}{\delta + 1}$ , then  $\mathcal{E}' > 0$  under  $\mathcal{H}'_s$ , and moreover  $\mathcal{H}'_s$  is under  $\mathcal{H}_s$ , thus  $\gamma + \xi > N - 2$ . Then  $\mathcal{E}(r) = O(r^{N-2-\gamma-\xi})$  tends to 0 at  $\infty$ , which is contradictory.

(ii) Taking  $\alpha = \frac{1}{\mu+1} = \frac{\theta}{N}, \beta = \frac{1}{\delta+1} = \frac{\sigma}{N}$ , it comes from (7.6)

$$r^{3-N}(uv)^{-1}\mathcal{E}'(r) = \left(\frac{N}{\delta + 1} + \frac{N}{\mu + 1} - (N - 2)\right)XY - \alpha sY^2W - \beta sX^2Z$$

hence  $\mathcal{E}' < 0$  when (7.2) holds. At the point  $R$  where  $u(R) = v(R) = 0$ , we find  $\mathcal{E}(R) = R^N u'(R)v'(R) > 0$ , which is a contradiction.  $\blacksquare$

**Remark 7.2** *In the case  $\delta = \mu$ , then  $u = v$  is a solution of the equation  $-\Delta U = U^{s+\delta}$ . Let  $s + \delta > 1$ . Then there exists a G.S. as soon as  $s + \delta \geq \frac{N+2}{N-2}$ . Notice that the point  $(\frac{N+2}{N-2} - s, \frac{N+2}{N-2} - s)$  lies on the hyperbola  $\mathcal{H}_s$ . Suppose moreover  $s \leq \delta$ ; if  $s + \delta \geq \frac{N+2}{N-2}$  there are no positive solutions such that  $u(R) = v(R) = 0$  for some  $R > 0$ ; if  $s + \delta < \frac{N+2}{N-2}$ , then there exists no G.S. Indeed such solutions satisfy  $u = v$ , from [3, Remark 3.3] results.*

Next we prove our main existence result of Theorem 1.5 which improves in particular Remark ???. The main idea is to introduce a new energy function by adding two terms in  $X^2$  and  $Y^2$  to the energy defined at (7.3).

**Proof of Theorem 1.5.** From Theorem 1.1, if  $s \geq \frac{N+2}{N-2}$ , all the regular solutions are G.S.. Thus we can assume  $s < \frac{N+2}{N-2}$ . Let  $j, k \in \mathbb{R}$  be parameters, and

$$\begin{aligned} \Phi(r) &= \mathcal{E}(r) + r^N \left[ k \frac{s}{2} \frac{vu'^2}{u} + j \frac{s}{2} \frac{uv'^2}{v} \right] \\ &= r^N \left[ u'v' + \alpha u^{\mu+1}v^s + \beta v^{\delta+1}u^s + \frac{\sigma}{r}vu' + \frac{\theta}{r}uv' + k \frac{s}{2} \frac{vu'^2}{u} + j \frac{s}{2} \frac{uv'^2}{v} \right] \\ &= r^{N-2}uv\Psi = r^{N-2-\gamma-\xi}(ZX)^{\xi/2}(WY)^{\gamma/2}\Psi, \end{aligned}$$

where

$$\Psi(X, Y, Z, W) = XY + \alpha WY + \beta ZX - \sigma X - \theta Y + k\frac{s}{2}X^2 + j\frac{s}{2}Y^2.$$

Then

$$\begin{aligned} r^{3-N}(uv)^{-1}\Phi'(r) &= (\sigma + \theta - (N - 2))XY + (N\alpha - \theta)YW + (N\beta - \sigma)XZ \\ &\quad - (\alpha(\mu + 1) - 1)XYW - (\beta(\delta + 1) - 1)XYZ + (j - \alpha)sY^2W + (k - \beta)sX^2Z \\ &\quad + ksX^2[X - (N - 2)] + jsY^2[Y - (N - 2)] + (N - 2 - X - Y)(k\frac{s}{2}X^2 + j\frac{s}{2}Y^2). \end{aligned}$$

We eliminate the terms in  $Z, W$  by taking  $j = \alpha = \frac{1}{\mu+1}$ ,  $k = \beta = \frac{1}{\delta+1}$ ,  $\theta = N\alpha$ ,  $\sigma = N\beta$ . We find after reduction

$$\begin{aligned} \mathcal{B}(X, Y) &:= -\frac{2}{s}r^{3-N}(uv)^{-1}\Phi'(r) \\ &= \beta X^2(N - 2 - X) + \alpha Y^2(N - 2 - Y) + XY \left[ \beta X + \alpha Y + \frac{2}{s}(N - 2 - N\alpha - N\beta) \right]. \end{aligned}$$

In particular one finds again the condition of Mitidieri if  $N(\alpha + \beta) - (N - 2) \leq 0$ .

Next we look at the case where  $N(\alpha + \beta) - (N - 2) > 0$ . We have  $\mathcal{B}(0, Y) = \alpha Y^2(N - 2 - Y) \geq 0$  and  $\mathcal{B}(X, 0) = \beta X^2(N - 2 - X) \geq 0$ . In particular  $\mathcal{B}(0, 0) = 0$ . Otherwise  $\mathcal{B}(N - 2, Y) = Y\Theta(Y)$  with

$$\Theta(Y) = \alpha Y [2(N - 2) - Y] + (N - 2)((N - 2)\beta + \frac{2}{s}(N - 2 - N\alpha - N\beta))$$

On  $[0, N - 2]$ ,  $\Theta(Y) > \Theta(0)$ . By hypothesis,  $(\delta, \mu)$  is above  $\mathcal{C}_s$ , or equivalently

$$(\alpha + \beta)\frac{N}{N - 2} - 1 \leq \frac{s}{2} \min(\alpha, \beta); \quad (7.8)$$

then  $\mathcal{B}(N - 2, Y) \geq 0$  and similarly  $\mathcal{B}(X, N - 2) \geq 0$ . Then  $\mathcal{B}$  is positive on the boundary of the square  $K = [0, N - 2] \times [0, N - 2]$  and is zero at  $(0, 0)$ ,  $(0, N - 2)$ ,  $(N - 2, 0)$ . The curve  $\mathcal{B}(X, Y) = 0$  is a cubic with a double point at  $(0, 0)$ , which is isolated under the condition (7.8):  $\mathcal{B}(X, Y) > 0$  near  $(0, 0)$ , except at this point. Then  $\mathcal{B}(X, Y) > 0$  on  $\overset{\circ}{K}$ .

Suppose that there exists a regular solution such that  $X(T) = Y(T) = N - 2$  at the same  $T$ . Indeed up to this point  $(X, Y)$  stays in  $K$ , thus the function  $\Phi$  is decreasing. There holds  $\Phi(0) = 0$ , and at the point  $R = e^T$ , we have

$$\Phi(R) = R^{N-2-\gamma-\xi}(N - 2)^{\xi+\gamma+2} \left[ \frac{\alpha W + \beta Z}{N - 2} + 1 - (\beta + \alpha)\left(\frac{N}{N - 2} - \frac{s}{2}\right) \right]$$

then  $\Phi(R) > 0$ , since  $\min(\alpha, \beta) < \alpha + \beta$ . Therefore from Theorem 1.1, there exists a G.S. ■

**Remark 7.3** We wonder if the limit curve  $\mathcal{L}_s$  for existence of G.S. would be defined by

$$\frac{1}{\delta + 1} + \frac{1}{\mu + 1} = \frac{N - 2}{N - \frac{(N-2)s}{2}}$$

which ensures that  $\Phi(R) > 0$ , and also  $\mathcal{B}(N-2, N-2) > 0$ . Like  $\mathcal{H}_s$ , this curve cuts the set  $\delta = \mu$  at the optimal point  $\delta + s = \frac{N+2}{N-2}$ . Notice that  $\mathcal{L}_s$  is under  $\mathcal{H}_s$ . Moreover we have proved the existence of G.S. when  $(\delta, \mu)$  is in the region  $\min(\delta + s, \mu + m) > \frac{N+2}{N-2}$ , which is above  $\mathcal{L}_s$ .

## 8 The radial potential system

Here we study the nonnegative radial solutions of the system (SP) :

$$(SP) \begin{cases} -\Delta_p u = |x|^a u^s v^{m+1}, \\ -\Delta_q v = |x|^a u^{s+1} v^m, \end{cases}$$

with  $a = b, \delta = m + 1, \mu = s + 1$ , and we assume (1.5). System (M) becomes

$$(MP) \begin{cases} X_t = X \left[ X - \frac{N-p}{p-1} + \frac{Z}{p-1} \right], \\ Y_t = Y \left[ Y - \frac{N-q}{q-1} + \frac{W}{q-1} \right], \\ Z_t = Z [N + a - sX - (m+1)Y - Z], \\ W_t = W [N + b - (s+1)X - mY - W]. \end{cases}$$

For this system  $D, \gamma$  and  $\xi$  are defined by

$$D = p(1+m) + q(1+s) - pq, \quad (p-1-s)\gamma + p + a = (m+1)\xi, \quad (q-1-m)\xi + q + b = (s+1)\gamma;$$

thus  $\gamma$  and  $\xi$  are linked independtly of  $s, m$  by

$$p(\gamma + 1) = q(\xi + 1) = \frac{pq(m + s + 2 + a)}{D} \quad (8.1)$$

The system is *variational*. It admits an energy function, given in (1.12); it can be given by a direct computation in terms of  $X, Y, Z, W$  :

$$\mathcal{E}(r) = \psi \left[ ZW - \frac{s+1}{p} W((N-p) - (p-1)X) - \frac{m+1}{q} Z((N-q) - (q-1)Y) \right], \quad (8.2)$$

where

$$\psi = r^{N-2-a} |u'|^{p-1} |v'|^{q-1} u^{-s} v^{-m} = r^{N-(\gamma+1)p} \left[ X^{q(s+1)(p-1)} Y^{p(m+1)(q-1)} Z^{p(q-m-1)} W^{q(p-s-1)} \right]^{1/D}.$$

Then we find

$$\mathcal{E}'(r) = (N + a - (s + 1)\frac{N - p}{p} - (m + 1)\frac{N - q}{q})r^{N-1+a}u^{s+1}v^{m+1}.$$

Thus we define a critical line  $\mathcal{D}$  as the set of  $(\delta, \mu) = (m + 1, s + 1)$  such that

$$N + a = (m + 1)\frac{N - q}{q} + (s + 1)\frac{N - p}{p}, \quad (8.3)$$

equivalent to  $pq(m + s + 2 + a) = ND$ , or  $N + a = (m + 1)\xi + (s + 1)\gamma$ , or

$$(\gamma, \xi) = \left(\frac{N - p}{p}, \frac{N - q}{q}\right)$$

The subcritical case is given by the set of points under  $\mathcal{D}$ , equivalently  $\gamma > \frac{N-p}{p}$ ,  $\xi > \frac{N-q}{q}$  or  $(s + 1)\gamma + (m + 1)\xi > N + a$ . The supercritical case is the set of points above  $\mathcal{D}$ .

**Remark 8.1** *The energy  $\mathcal{E}_0$  of the particular solution associated to  $M_0$  is still negative:  $\mathcal{E}_0 = -\frac{D}{pq}r^{N+a-(\gamma+1)p} \left[ X_0^{q(p-1)} Y_0^{p(q-1)} Z_0^{q(s+1)} W_0^{p(m+1)} \right]^{1/D}$ .*

**Remark 8.2** *When  $p = q = 2$ , another energy function can be associated to the transformation (2.2): the system (2.16) relative to  $u(r) = r^{-\gamma}U(t)$ ,  $v(r) = r^{-\xi}V(t)$  is*

$$\begin{cases} U_{tt} + (N - 2 - 2\gamma)U_t - \gamma(N - 2 - \gamma)U + U^s V^{m+1} = 0 \\ V_{tt} + (N - 2 - 2\gamma)V_t - \gamma(N - 2 - \gamma)V + U^{s+1} V^m = 0 \end{cases} \quad (8.4)$$

and the function

$$E(t) = \frac{s+1}{2}(U_t^2 - \gamma(N - 2 - \gamma)U^2) + \frac{m+1}{2}(V_t^2 - \gamma(N - 2 - \gamma)V^2) + U^{s+1}V^{m+1} \quad (8.5)$$

satisfies

$$E_t = -(N - 2 - 2\gamma) [(s + 1)U_t^2 + (m + 1)V_t^2]$$

It differs from  $\mathcal{E}$ , even in the critical case. This point is crucial for Section 9.

It has been proved in [34], [35], that in the subcritical case with  $a = 0$ , there exists a solution of the Dirichlet problem in any bounded regular domain  $\Omega$  of  $\mathbb{R}^N$ ; and in the supercritical case there exists no solution if  $\Omega$  is starshaped. Here we prove two results of existence or nonexistence of G.S. which seem to be new:

**Proof of Theorem 1.6.** 1) *Existence or nonexistence results.*

• In the supercritical or critical case there exists a G.S. Indeed from Theorem 1.1, if it were not true, then there exist regular positive solutions of  $(MP)$  such that  $X(T) = \frac{N-p}{p-1}$  and  $Y(T) = \frac{N-q}{q-1}$ .

Suppose that there exists such a solution. Then  $\mathcal{E} \leq 0$ . Then at point  $T$ ,  $\mathcal{E}(R) > 0$ , from (8.2), since  $W > 0, Z > 0$ , which is impossible.

• In the subcritical case, there exists no G.S. Indeed suppose that there exists a G.S. Here  $\mathcal{E}$  is nondecreasing, hence  $\mathcal{E} \geq 0$ . Its trajectory stays in the box  $\mathcal{A}$  defined by (5.1), thus it is bounded. If  $q \geq m + 1$  and  $p \geq s + 1$ , then from (8.2),  $\mathcal{E}(r) = O(r^{N-(\gamma+1)p})$ , then  $\mathcal{E}$  tends to 0 at  $\infty$ , which is contradictory. Next consider the general case. We have

$$\begin{aligned} \mathcal{E}(r) &\leq r^{N-(\gamma+1)p} \left[ X^{q(p-1)} Y^{p(q-1)} Z^{p(q-m-1)} W^{q(p-s-1)} \right]^{1/D} ZW \\ &= r^{N-(\gamma+1)p} \left[ X^{q(p-1)} Y^{p(q-1)} Z^{q(1+s)} W^{p(1+m)} \right]^{1/D}, \end{aligned}$$

then the same result holds. Then from Theorem 1.1 there exists a solution of the Dirichlet problem

2) *Behaviour of the G.S. in the critical case.*

Let  $\mathcal{T}$  be the trajectory of a G.S.; then  $\mathcal{E}(0) = 0$ , thus  $\mathcal{T}$  lies on the variety  $\mathcal{V}$  of energy 0, also defined by

$$qW [(s+1)((p-1)X - (N-p)) + pZ] = p(m+1)Z [(N-q) - (q-1)Y] \quad (8.6)$$

and  $Y < \frac{N-q}{q-1}$ , hence  $(s+1)((p-1)X - (N-p)) + pZ > 0$ . From (5.2),  $\mathcal{T}$  starts from  $N_0 = (0, 0, N+a, N+b)$  and stays in  $\mathcal{A}$ . Eliminating  $W$  in system (M), we find a system of three equations

$$\begin{cases} X_t = X \left[ X - \frac{N-p}{p-1} + \frac{Z}{p-1} \right], \\ Y_t = YF, \\ Z_t = Z [N+a - sX - (m+1)Y - Z], \end{cases}$$

where

$$F(X, Y, Z) = \frac{1}{q} \left[ \frac{N-q}{q-1} - Y \right] \frac{p(m+1-q)Z + q(s+1)((N-p) - (p-1)X)}{(s+1)((p-1)X - (N-p)) + pZ}.$$

(i) If  $\mathcal{T}$  converges to a fixed point of the system in  $\bar{\mathcal{R}}$ , the possible points on  $\mathcal{V}$  are  $A_0, I_0, J_0, P_0, Q_0, G_0, H_0, R_0, S_0$ . The eigenvalues of the linearized problem at  $A_0$ , given by (10.3) satisfy

$$\lambda_1, \lambda_2 > 0, \lambda_3 = N+a - s \frac{N-p}{p-1} - (m+1) \frac{N-q}{q-1} \leq \lambda_4 = N+a - (s+1) \frac{N-p}{p-1} - m \frac{N-q}{q-1} = \lambda^*,$$

since  $q \leq p$ , and  $\lambda_3 < \lambda^*$  unless  $q = p$ , and then  $\lambda_3 = \lambda^* < 0$  from (8.3). Then  $A_0$  can be attained only when  $\lambda^* \leq 0$ , from Proposition 4.5. And  $P_0$  can be attained only if  $q > m + 1$ , and

$$\lambda^* \geq 0 \text{ and } q+a < (s+1) \frac{N-p}{p-1} \quad (8.7)$$



from the proof of Proposition 4.6, because  $\gamma = \frac{N-p}{p} < \frac{N-p}{p-1}$ . We observe that the condition  $\lambda^* \geq 0$  joint to (8.3) implies  $m+1 < q < p$  and is equivalent to (8.7). Indeed it implies

$$\frac{N-p}{p-1}(s+1) \leq N+a-m\frac{N-q}{q-1} = N+a-\frac{q}{q-1}\left(N+a-\frac{N-q}{q}-(s+1)\frac{N-p}{p}\right);$$

then

$$(s+1)\frac{N-p}{p-1}\frac{q-p}{p} \leq -(a+q),$$

thus  $q < p$ . And from from (8.3)

$$(N-q)\left(\frac{m+1}{q}-1\right) = q+a-(s+1)\frac{N-p}{p} \leq (s+1)\frac{N-p}{p}\left(\frac{p-q}{p-1}-1\right) < 0,$$

hence  $m+1 < q$  and (8.7) follows. By symmetry,  $Q_0$  cannot be attained since  $q \leq p$ . Then  $A_0$  and  $P_0$  are incompatible, unless  $A_0 = P_0$ , and  $P_0$  is not attained when  $p = q$ .

(ii) Next we show that  $\mathcal{T}$  converges to  $A_0$  or  $P_0$ . If  $t$  is an extremum point of  $Y$ , then

$$\left(\frac{m+1}{q}-1\right)Z(t) + \frac{s+1}{p}\left((N-p)-(p-1)X(t)\right) = 0. \quad (8.8)$$

It implies  $q > m+1$  and

$$X_t(t) = \frac{X(t)Z(t)}{p-1} \left[1 + \frac{p(m+1-q)}{q(s+1)}\right] = \frac{DX(t)Z(t)}{(p-1)q(s+1)} > 0.$$

In the same way, if  $t$  is an extremum point of  $X$ , then  $p > s+1$  and  $Y_t(t) > 0$ . Near  $-\infty$ , there holds  $X_t, Y_t \geq 0$ , and  $Z_t, W_t \leq 0$ , from the linearization near  $N_0$ . Suppose that  $X$  has a maximum at  $t_0$  followed by a minimum at  $t_1$ . Then  $p > s+1$ , and  $Y$  is increasing on  $[t_0, t_1]$ . And at  $t_0$  we have  $(p-1)X(t_0) + Z(t_0) = N-p$  and  $X_{tt}(t_0) \leq 0$ , thus  $Z_t(t_0) \leq 0$ ; eliminating  $Z$  we deduce  $p+a+(p-1-s)X(t_0) \leq (m+1)Y(t_0)$  and similarly  $(m+1)Y(t_1) \leq p+a+(p-1-s)X(t_1)$ ; hence  $Y(t_1) < Y(t_0)$ , which is a contradiction. Then  $X$  and  $Y$  have at most one maximum, and finally no maximum point. Then  $X$  and  $Y$  are increasing, and bounded, hence  $X$  has a limit in  $\left(0, \frac{N-p}{p-1}\right]$  and  $Y$  has a limit in  $\left(0, \frac{N-q}{q-1}\right]$ . Then  $Z, W$  are decreasing; indeed at each point where  $Z_t = 0$ , we have  $Z_{tt} = Z(-sX_t - (m+1)Y_t) < 0$ , thus it is a maximum, which is impossible.

Then  $\mathcal{T}$  converges to a fixed point of the system. Moreover, since  $X$  and  $Y$  are increasing, it cannot be  $I_0, J_0, G_0, H_0, R_0, S_0$ . It is necessarily  $A_0$  or  $P_0$ . We distinguish two cases:

- Case  $q \leq m+1$ . Then  $\mathcal{T}$  converges to  $A_0$ , and  $\lambda_3, \lambda^* < 0$ , then (1.13) follows.
- Case  $q > m+1$ . Then  $\mathcal{T}$  converges to  $A_0$  (resp.  $P_0$ ) when  $\lambda^* \leq 0$  (resp.  $\lambda^* \geq 0$ ). If the inequalities are strict, we deduce the convergence of  $u$  and  $v$  from Propositions 4.5 and 4.6, and

(1.14) follows. If  $\lambda^* = 0$ , then  $P_0 = A_0$ , and  $\lambda_3 = \frac{(N-1)(q-p)}{(p-1)(q-1)} < 0$ . The projection of the trajectory  $\mathcal{T}$  in  $\mathbb{R}^3$  on the plane  $Y = \frac{N-q}{q-1}$  satisfies the system

$$X_t = X \left[ X - \frac{N-p}{p-1} + \frac{Z}{p-1} \right], \quad Z_t = Z \left[ N + a - sX - (m+1) \frac{N-q}{q-1} - Z \right]$$

which presents a saddle point at  $(\frac{N-p}{p-1}, 0)$ , thus the convergence of  $X$  and  $Z$  is exponential, in particular we deduce the behaviour of  $u$ . The trajectory enters by the central variety of dimension 1, and by computation we deduce that  $Y = \frac{N-q}{q-1} - \frac{1}{q-1-m} t^{-1} + O(t^{-2+\varepsilon})$ , then (1.15) follows.  $\blacksquare$

## 9 The nonradial potential system of Laplacians

Here we study the possibly *nonradial* solutions of the system of the preceding Section when  $p = q = 2$  :

$$(P) \begin{cases} -\Delta u = |x|^a u^s v^{m+1}, \\ -\Delta v = |x|^a u^{s+1} v^m, \end{cases}$$

with  $D = s + m$ . We solve an open problem of [7]. Recall that the main result of [7] is the obtention of apriori estimates near 0 or  $\infty$ , by using the Bernstein technique introduced in [18] and improved in [8].

In [7] it had been solved in the case  $N + a \geq 4$ . The problem was open when  $N + a < 4$ , which implies  $N < 6$ , and  $m + s + 1 > N/(N - 2)$ . In the case  $m + s + 1 \leq N/(N - 2)$ , there are no solutions of the exterior problem, see [6].

In [7] we use the transformation

$$u(r, \theta) = r^{-\gamma} U(t, \theta), \quad v(r, \theta) = r^{-\gamma} V(t, \theta), \quad t = \ln r,$$

extending the transformation of Remark 8.2 to the nonradial case (in fact here  $t$  is  $-t$  in [7] ), leading to the system

$$\begin{aligned} U_{tt} + (N - 2 - 2\gamma)U_t + \Delta_S U - \gamma(N - 2 - \gamma)U + U^s V^{m+1} &= 0 \\ V_{tt} + (N - 2 - 2\gamma)V_t + \Delta_S V - \gamma(N - 2 - \gamma)V + U^{s+1} V^m &= 0 \end{aligned}$$

where  $\Delta_S$  is the Laplace-Beltrami operator on  $S_{N-1}$ , and we define the corresponding energy extending (8.5):

$$\begin{aligned} E(t) &= \frac{s+1}{2} \int_{S^{N-1}} (U_t^2 - |\nabla_S U|^2 - \gamma(N-2-\gamma)U^2) d\theta \\ &+ \frac{m+1}{2} \int_{S^{N-1}} (V_t^2 - |\nabla_S V|^2 - \xi(N-2-\xi)V^2) d\theta + \int_{S^{N-1}} U^{s+1} V^{m+1} d\theta; \end{aligned}$$

and there holds

$$E_t = -(N-2-2\gamma) \int_{S^{N-1}} [(s+1)U_t^2 + (m+1)V_t^2] d\theta$$

Here we construct a new function, extending the Pohozaev function defined at (1.12) to the non-radial case.

**Lemma 9.1** *Consider the function  $\mathcal{E}(r)$  defined by*

$$\begin{aligned} r^{-N}\mathcal{E}(r) &= \frac{s+1}{2} \int_{S^{N-1}} \left[ u_r^2 - r^{-2} |\nabla_S u|^2 + (N-2) \frac{uu_r}{r} \right] d\theta \\ &\quad + \frac{m+1}{2} \int_{S^{N-1}} \left[ \left( \frac{\partial v}{\partial \nu} \right)^2 - r^{-2} |\nabla_S v|^2 + (N-2) \frac{vv_r}{r} \right] d\theta + r^a \int_{S^{N-1}} u^{s+1} v^{m+1} d\theta \end{aligned}$$

Then

$$r^{1-N}\mathcal{E}'(r) = (N+a-(s+1))\frac{N-2}{2} - (m+1)\frac{N-2}{2} r^a \int_{S^{N-1}} u^{s+1} v^{m+1} d\theta$$

**Proof.** In terms of  $t$ , we find

$$\begin{aligned} \mathcal{E}(t) &= \mathcal{E}_1(t) + \mathcal{E}_2(t) + \mathcal{E}_3(t), \quad \text{with } \mathcal{E}_1(t) = \frac{s+1}{2} e^{(N-2)t} \int_{S^{N-1}} \left[ u_t^2 - |\nabla_S u|^2 + (N-2)uu_t \right] d\theta, \\ \mathcal{E}_2(t) &= \frac{m+1}{2} e^{(N-2)t} \int_{S^{N-1}} \left[ v_t^2 - |\nabla_S v|^2 + (N-2)vv_t \right] d\theta, \quad \mathcal{E}_3(t) = e^{(N+a)t} \int_{S^{N-1}} u^{s+1} v^{m+1} d\theta \end{aligned} \tag{9.1}$$

and  $u, v$  satisfy

$$(e^{(N-2)t}u_t)_t + e^{(N-2)t}\Delta_S u + e^{(N+a)t}u^s v^{m+1} = 0, \quad (e^{(N-2)t}v_t)_t + e^{(N-2)t}\Delta_S v + e^{(N+a)t}u^{s+1}v^m = 0. \tag{9.2}$$

Multiplying successively by  $u$  and by  $v$ , and  $(s+1)e^{(N-2)t}u_t$  and  $(m+1)e^{(N-2)t}v_t$  and adding the equalities we obtain after computation

$$\mathcal{E}_t(t) = (N+a-(s+1))\frac{N-2}{2} - (m+1)\frac{N-2}{2} e^{(N+a)t} \int_{S^{N-1}} u^{s+1} v^{m+1} d\theta. \quad \blacksquare$$

**Proof of Theorem 1.7.** Suppose that there exists a G.S.. Since  $s+m+1 < (N+2+2a)/(N-2)$  then  $E$  and  $\mathcal{E}$  are increasing and start from 0, then they stay positive. Since  $s+m+1 < (N+$

$2)/(N-2)$  we know that either  $(u, v)$  behaves as  $M_0$ , then  $E$  has a negative limit, which is impossible, or  $(u, v)$  is regular at  $\infty$ , and then the only possibility is the following:  $m > (N + a)/(N - 2)$ ,  $\lim_{r \rightarrow \infty} u = \alpha > 0$ , and

$$\lim_{|x| \rightarrow \infty} |x|^k v = \beta > 0 \text{ or } \lim_{|x| \rightarrow \infty} |x|^k v = 0, \quad \text{with } k = (2 + a)/(m - 1), \quad (9.3)$$

where  $\beta = (\alpha^{-(s+1)} k(N-2-k))^{1/(m-1)}$ . It implies that  $N < 4 - a$  from the majorization of  $m + s + 1$  by  $\frac{N+2}{N-2}$ , and  $N < 4 + a$  from the majorization by  $\frac{N+2+2a}{N-2}$ . In [7] we find that  $\lim_{t \rightarrow \infty} E(t) = \infty$ , which gives no contradiction. Here we show that  $\lim_{t \rightarrow \infty} \mathcal{E}(t) < 0$ , which will give a contradiction. First recall the proof of (9.3). Making the substitution

$$u(r, \theta) = u(t, \theta), \quad v(r, \theta) = r^{-k} \mathbf{V}(t, \theta), \quad t = \ln r, \theta \in S_{N-1},$$

we get

$$\begin{cases} u_{tt} + (N-2)u_t + \Delta_S u + e^{-2kt} u^s \mathbf{V}^{m+1} = 0, \\ \mathbf{V}_{tt} + (N-2-2k)\mathbf{V}_t + \Delta_S \mathbf{V} - k(N-2-k)\mathbf{V} + u^{s+1} \mathbf{V}^m = 0. \end{cases} \quad (9.4)$$

Then  $u, \mathbf{V}$  are bounded near  $\infty$ , and from [7, Proposition 4.1]  $u$  converges exponentially to the constant  $\alpha$ , more precisely

$$\| |u - \alpha| + |u_t| + |\nabla_S u| \|_{C^0(S^{N-1})} = O(e^{-(N-2)t}), \quad (9.5)$$

because  $k \neq (N-2)/2$  and all the derivatives of  $\mathbf{V}$  up to the order 2 are bounded. The equation takes the form

$$\mathbf{V}_{tt} + (N-2-2k)\mathbf{V}_t + \Delta_S \mathbf{V} - k(N-2-k)\mathbf{V} + \alpha^{s+1} \mathbf{V}^m + \varphi = 0$$

where  $\varphi$  and its derivatives up to the order 2 are  $O(e^{-(N-2)t})$ ; then from [7, Theorem 4.1],  $\mathbf{V}$  converges to  $\beta$  or to 0 in  $C^2(S^{N-1})$ .

We define  $f(t) = e^{(N-2)t} \int_{S^{N-1}} u_t d\theta = r^{N-1} \int_{S^{N-1}} u_r d\theta$ . Then  $\mathcal{E}_1(t) = (N-2)\frac{s+1}{2}\alpha f(t) + O((e^{-(N-2)t}))$

from (9.5). Moreover from (9.4),

$$f_t(t) = -e^{(N-2-2k)t} \int_{S^{N-1}} u^s \mathbf{V}^{m+1} d\theta < 0$$

Since  $u$  is regular at 0,  $f(t) = 0(e^{(N-1)t})$  at  $-\infty$ , in particular  $\lim_{t \rightarrow -\infty} f(t) = 0$ . And  $f_t(t) = O(e^{(N-2-2k)t}) = O(e^{-t})$  at  $\infty$ , then  $f(t)$  has a finite negative limit  $-\ell^2$ ; and  $\mathcal{E}_1(t)$  tends to  $-(N-2)\frac{s+1}{2}\alpha\ell^2$ . Then  $\mathcal{E}_2(t) = O(e^{(N-2-2k)t}) = O(e^{-t})$  since  $v = e^{-kt}\mathbf{V}$ , and  $\mathbf{V}$  and its derivatives up to the order 2 are bounded. And  $\mathcal{E}_3(t) = O(e^{(N+a-k(m+1))t})$  and  $N + a - k(m + 1) < \frac{2-N}{m-1} < 0$ . Then  $\mathcal{E}$  has a negative finite limit at  $\infty$ , contradiction.  $\blacksquare$

## 10 Analysis of the fixed points

**Proof of Proposition 4.4.** (i) Consider a regular solution  $(u, v)$  with initial data  $(u_0, v_0)$ . Then  $(-r^{N-1}|u'|^{p-2}u')' = r^{N-1+a}u_0^s v_0^\delta(1 + o(1))$  as  $r \rightarrow 0$ , then  $-|u'|^{p-2}u' = \frac{1}{N+a}r^{1+a}u_0^s v_0^\delta(1 + o(1))$ , thus from (2.1), as  $t \rightarrow -\infty$

$$\begin{aligned} X(t) &= \left(\frac{1}{N+a}u_0^{s+1-p}v_0^\delta\right)^{1/(p-1)}e^{(p+a)t/(p-1)}(1 + o(1)), \\ Y(t) &= \left(\frac{1}{N+b}u_0^\mu v_0^{m+1-q}\right)^{1/(q-1)}e^{(q+b)t/(q-1)}(1 + o(1)), \end{aligned}$$

and  $\lim_{t \rightarrow -\infty} Z = N + a$ ,  $\lim_{t \rightarrow -\infty} W = (N + b)$ . In particular the trajectory tends to  $N_0$ .

(ii) Reciprocally, consider a trajectory converging to  $N_0$ . Setting  $Z = N + a + \tilde{Z}$ ,  $W = N + b + \tilde{W}$ , the linearized system is

$$\begin{aligned} X_t = \frac{p+a}{p-1}X, \quad Y_t = \frac{q+b}{q-1}Y, \quad \tilde{Z}_t = (N+a) \left[-sX - \delta Y - \tilde{Z}\right], \quad \tilde{W}_t = (N+b) \left[-\mu X - mY - \tilde{W}\right]. \end{aligned} \quad (10.1)$$

The eigenvalues are

$$\lambda_1 = \frac{p+a}{p-1} > 0, \quad \lambda_2 = \frac{q+b}{q-1} > 0, \quad \lambda_3 = -(N+a) < 0, \quad \lambda_4 = -(N+b) < 0. \quad (10.2)$$

The unstable variety  $\mathcal{V}_u$  and the stable one  $\mathcal{V}_s$  have dimension 2. Notice that  $\mathcal{V}_s$  is contained in the set  $X = Y = 0$ , thus no admissible trajectory converges to  $N_0$  as  $t \rightarrow \infty$ , and there exists an infinity of admissible trajectories in  $\mathcal{R}$ , converging to  $N_0$  as  $t \rightarrow -\infty$ . Since the convergence is exponential,  $X_t = \frac{p+a}{p-1}X(1 + O(e^{ct}))$  with  $c > 0$ , and then  $e^{-(p+a)/(p-1)t}X(t)$  has a positive limit  $\kappa$ , and  $e^{-(q+b)/(q-1)t}Y(t)$  has a positive limit  $\ell$ ; from (4.2), (4.1),  $(u, v)$  have a positive limit  $(u_0, v_0) = ((N+a)\kappa^{p-1})^{(q-1-m)/D}((N+b)\ell^{q-1})^{\delta/D}$ , hence  $(u, v)$  is a regular solution..

Next we show that for any  $\kappa > 0, \ell > 0$  there exists a unique local solution such that  $\lim_{t \rightarrow -\infty} e^{-(p+a)t/(p-1)}X(t) = \kappa$  and  $\lim_{t \rightarrow -\infty} e^{-(q+b)t/(q-1)}Y = \ell$ . On  $\mathcal{V}_u$ , we get a system of two equations of the form

$$X_t = X(\lambda_1 + F(X, Y)), \quad Y_t = Y(\lambda_2 + G(X, Y)),$$

where  $F = AX + BY + f(X, Y)$  and  $f$  is  $C^\infty$  and  $f_X(0, 0) = f_Y(0, 0) = 0$ , similarly for  $G$ . Setting  $X = e^{\lambda_1 t}(\kappa + x)$ ,  $Y = e^{\lambda_2 t}(\ell + y)$ , and assuming  $\lambda_2 \geq \lambda_1$  and setting  $\rho = e^{\lambda_1 t}$  we lead to

$$x_\rho = \frac{1}{\rho}(\kappa + x)F(\rho(\kappa + x), \rho^{\lambda_2/\lambda_1}(\ell + y)), \quad y_\rho = (\ell + y)G(\rho(\kappa + x), \rho^{\lambda_2/\lambda_1}(\ell + y)),$$

with  $x(0) = y(0) = 0$ . Then we get local existence and uniqueness. Hence for any  $u_0 > 0, v_0 > 0$  there exists a regular solution  $(u, v)$  with initial data  $(u_0, v_0)$ . Moreover  $u, v \in C^1([0, R))$  when  $a, b > -1$ .  $\blacksquare$

**Proof of Proposition 4.5.** The linearization at  $A_0$  at this point gives, with  $X = \frac{N-p}{p-1} + \tilde{X}$ ,  $Y = \frac{N-q}{q-1} + \tilde{Y}$ ,

$$\tilde{X}_t = \frac{N-p}{p-1} \left[ \tilde{X} + \frac{Z}{p-1} \right], \quad \tilde{Y}_t = \frac{N-q}{q-1} \left[ \tilde{Y} + \frac{W}{q-1} \right], \quad Z_t = \lambda_3 Z, \quad W_t = \lambda_4 W.$$

The eigenvalues are

$$\lambda_1 = \frac{N-p}{p-1} > 0, \quad \lambda_2 = \frac{N-q}{q-1} > 0, \quad \lambda_3 = N+a-s \frac{N-p}{p-1} - \delta \frac{N-q}{q-1}, \quad \lambda_4 = N+b-\mu \frac{N-p}{p-1} - m \frac{N-q}{q-1}. \quad (10.3)$$

• Convergence as  $r \rightarrow \infty$  : If  $\lambda_3 > 0$ , or  $\lambda_4 > 0$ , then the stable variety  $\mathcal{V}_s$  has at most dimension 1, it satisfies  $W = 0$  or  $Z = 0$ , hence there is no admissible trajectory converging to  $A_0$  at  $\infty$ . If  $\lambda_3 < 0$ , and  $\lambda_4 < 0$ , then  $\mathcal{V}_s$  has dimension 2, and contains trajectories such that  $Z > 0$  and  $W > 0$ , then in  $\mathcal{R}$ ; and  $\mathcal{V}_s \cap \{Z = 0\}$  has dimension 1 : indeed we get a system in  $X, Y, W$  with the eigenvalues  $\lambda_1, \lambda_2, \lambda_4$ ; similarly  $\mathcal{V}_s \cap \{W = 0\}$  has dimension 1; then there exist admissible trajectories. They satisfy  $\lim e^{-\lambda_3 t} Z = C_3 > 0$ ,  $\lim e^{-\lambda_4 t} W = C_4 > 0$ , then (4.11) follows from (4.2).

• Convergence as  $r \rightarrow 0$  : If  $\lambda_3 < 0$ , or  $\lambda_4 < 0$ , the unstable variety  $\mathcal{V}_u$  has at most dimension 3, and it satisfies  $W = 0$  or  $Z = 0$ , there is no admissible trajectory converging at  $-\infty$ . If  $\lambda_3 > 0$ , and  $\lambda_4 > 0$ , then  $\mathcal{V}_u$  has dimension 4, there exist admissible trajectories, and (4.11) follows as above. ■

**Proof of Proposition 4.6.** We set  $P_0 = \left( \frac{N-p}{p-1}, Y_*, 0, W_* \right)$ , for  $m+1 \neq q$ . The linearization gives, with  $X = \frac{N-p}{p-1} + \tilde{X}$ ,  $Y = Y_* + \tilde{Y}$ ,  $W = W_* + \tilde{W}$ ,

$$\tilde{X}_t = \frac{N-p}{p-1} \left[ \tilde{X} + \frac{Z}{p-1} \right], \quad \tilde{Y}_t = Y_* \left[ \tilde{Y} + \frac{\tilde{W}}{q-1} \right], \quad Z_t = \lambda_3 Z, \quad \tilde{W}_t = W_* \left[ -\mu \tilde{X} - m \tilde{Y} - \tilde{W} \right]$$

The eigenvalues are

$$\lambda_1 = \frac{N-p}{p-1} > 0, \quad \lambda_3 = N+a-s \frac{N-p}{p-1} - \delta Y_* = \frac{D}{q-1-m} \left( \gamma - \frac{N-p}{p-1} \right),$$

and the roots  $\lambda_2, \lambda_4$  of equation

$$\lambda^2 - (Y_* - W_*)\lambda + \frac{m+1-q}{q-1} Y_* W_* = 0$$

Then if  $\lambda_3 < 0$  (resp.  $\lambda_3 > 0$ ) there is no admissible trajectory converging at  $-\infty$  (resp.  $\infty$ ). Indeed  $\mathcal{V}_u = \mathcal{V}_u \cap \{Z = 0\}$  (resp.  $\mathcal{V}_s = \mathcal{V}_s \cap \{Z = 0\}$ ).

1) Suppose that  $q > m+1$ . Since  $q+b < \frac{N-p}{p-1} \mu < N+b-m \frac{N-q}{q-1}$ , we have  $P_0 \in \mathcal{R}$ , and  $\lambda_2 \lambda_4 < 0$ . First assume  $\lambda_3 < 0$ , that means  $\gamma < \frac{N-p}{p-1}$ ; then  $\mathcal{V}_s$  has dimension 2, and  $\mathcal{V}_s \cap \{Z = 0\}$

has dimension 1, there exists trajectories with  $Z > 0$ , then admissible, converging as  $t \rightarrow \infty$ . Next assume  $\lambda_3 > 0$ ; then  $\mathcal{V}_u$  has dimension 3, and  $\mathcal{V}_u \cap \{Z = 0\}$  has dimension 2, then there exist admissible trajectories converging as  $t \rightarrow -\infty$ .

2) Suppose that  $q < m + 1$ . Since  $q + b > \frac{N-p}{p-1}\mu > N + b - m\frac{N-q}{q-1}$ , we have  $P_0 \in \mathcal{R}$ , and  $\lambda_2\lambda_4 > 0$ . We assume  $q\frac{N-p}{p-1}\mu + m(N-q) \neq N(q-1) + (b+1)q$ , that means  $Y_* \neq W_*$ . First suppose  $\lambda_3 > 0$ , that means  $\gamma < \frac{N-p}{p-1}$ ; either  $\text{Ré}\lambda_2, \text{Ré}\lambda_4 > 0$  then  $\mathcal{V}_u$  has dimension 4, or  $\text{Ré}\lambda_2, \text{Ré}\lambda_4 < 0$  then  $\mathcal{V}_u$  has dimension 2 and  $\mathcal{V}_u \cap \{Z = 0\}$  has dimension 1; in any case, there exist admissible trajectories converging as  $t \rightarrow -\infty$ . Next assume  $\lambda_3 < 0$ ; either  $\text{Ré}\lambda_2, \text{Ré}\lambda_4 > 0$ ,  $\mathcal{V}_s$  has dimension 1, and  $\mathcal{V}_s \cap \{Z = 0\} = \emptyset$  or  $\text{Ré}\lambda_2, \text{Ré}\lambda_4 < 0$ , then  $\mathcal{V}_s$  has dimension 3, it contains trajectories with  $Z > 0$ , then admissible, converging as  $t \rightarrow \infty$ .

Then  $\lim e^{-\lambda_3 t} Z = C_3 > 0$ ,  $\lim X = \frac{N-p}{p-1}$ ,  $\lim Y = Y_*$  and  $\lim W = W_*$ , thus (4.12) follows from (4.2) and (2.5).  $\blacksquare$

**Proof of Proposition 4.8.** The linearization near  $I_0$  gives, with  $X = \frac{N-p}{p-1} + \tilde{X}$ ,

$$\tilde{X}_t = \frac{N-p}{p-1} \left( \tilde{X} + \frac{Z}{p-1} \right), \quad Y_t = -\frac{N-q}{q-1} Y, \quad Z_t = (N+a-s\frac{N-p}{p-1})Z, \quad W_t = (N+b-\mu\frac{N-p}{p-1})W.$$

The eigenvalues are

$$\lambda_1 = \frac{N-p}{p-1} > 0, \quad \lambda_2 = -\frac{N-q}{q-1} < 0, \quad \lambda_3 = N+a-s\frac{N-p}{p-1}, \quad \lambda_4 = N+b-\mu\frac{N-p}{p-1}.$$

- Convergence as  $r \rightarrow \infty$ : If  $\lambda_3 > 0$  or  $\lambda_4 > 0$ , then  $\mathcal{V}_s = \mathcal{V}_s \cap \{Z = 0\}$  or  $\mathcal{V}_s = \mathcal{V}_s \cap \{W = 0\}$ , there is no admissible trajectory converging at  $\infty$ . Suppose that  $\lambda_3 < 0$  and  $\lambda_4 < 0$ . Then  $\mathcal{V}_s$  has dimension 3; it contains trajectories with  $Y, Z, W > 0$ , then admissible. Then  $\lim X = \frac{N-p}{p-1}$ ,  $\lim e^{-\lambda_2 t} Y = C_2 > 0$ ,  $\lim e^{-\lambda_3 t} Z = C_3 > 0$ ,  $\lim e^{-\lambda_4 t} W = C_4 > 0$ , then (4.13) follows from (4.2) and (2.4).

- Convergence as  $r \rightarrow 0$ : Since  $\lambda_2 < 0$  we have  $\mathcal{V}_u = \mathcal{V}_u \cap \{Y = 0\}$ , hence there is no admissible trajectory converging as  $r \rightarrow 0$ .  $\blacksquare$

**Proof of Proposition 4.9.** The point  $G_0 = (\frac{N-p}{p-1}, 0, 0, N+b-\frac{N-p}{p-1}\mu) \in \mathcal{R}$  since  $\frac{N-p}{p-1}\mu < N+b$ . The linearization gives, with  $X = \frac{N-p}{p-1} + \tilde{X}$ ,  $W = N+b-\frac{N-p}{p-1}\mu + \tilde{W}$ ,

$$\begin{aligned} \tilde{X}_t &= \frac{N-p}{p-1} \left[ \tilde{X} + \frac{Z}{p-1} \right], & Y_t &= \frac{Y}{q-1} \left( q+b-\frac{N-p}{p-1}\mu \right), \\ Z_t &= (N+a-s\frac{N-p}{p-1})Z, & W_t &= (N+b-\frac{N-p}{p-1}\mu) \left[ -\mu\tilde{X} - mY - \tilde{W} \right] \end{aligned}$$

The eigenvalues are

$$\lambda_1 = \frac{N-p}{p-1} > 0, \quad \lambda_2 = \frac{1}{q-1} \left( q+b-\frac{N-p}{p-1}\mu \right), \quad \lambda_3 = N+a-s\frac{N-p}{p-1}, \quad \lambda_4 = \frac{N-p}{p-1}\mu - N-b < 0.$$

- Convergence as  $r \rightarrow \infty$  : If  $\lambda_2 > 0$ , or  $\lambda_3 > 0$ , then  $\mathcal{V}_s = \mathcal{V}_s \cap \{Y = 0\}$  or  $\mathcal{V}_s = \mathcal{V}_s \cap \{Z = 0\}$ , there is no admissible trajectory converging at  $\infty$ . Assume  $\lambda_2, \lambda_3 < 0$ , then  $\mathcal{V}_s$  has dimension 3, it contains trajectories with  $Y, Z > 0$ , then admissible.
- Convergence as  $r \rightarrow 0$  : If  $\lambda_3 < 0$ , or  $\lambda_2 < 0$  there is no admissible trajectory. If  $\lambda_2, \lambda_3 > 0$  then  $\mathcal{V}_s$  has dimension 3, it contains admissible trajectories.

In any case  $\lim X = \frac{N-p}{p-1}$ ,  $\lim e^{-\lambda_2 t} Y = C_2 > 0$ ,  $\lim e^{-\lambda_3 t} Z = C_3 > 0$ ,  $\lim W = N + b - \frac{N-p}{p-1} \mu$ , hence (4.13) still follows from (4.2) and (2.4).  $\blacksquare$

**Proof of Proposition 4.10.** We set  $C_0 = (0, \bar{Y}, 0, \bar{W})$ . It is in  $\mathcal{R}$  if  $\frac{N-q}{q-1} m > N + b$ , implying  $q < m + 1$ . The linearization at this point gives, with  $Y = \bar{Y} + \tilde{Y}$  and  $W = \bar{W} + \tilde{W}$

$$X_t = -\frac{N-p}{p-1} X, \quad \tilde{Y}_t = \bar{Y} \left[ \tilde{Y} + \frac{\tilde{W}}{q-1} \right], \quad Z_t = \lambda_3 Z, \quad W_t = \bar{W} \left[ -\mu X - m\tilde{Y} - \tilde{W} \right].$$

The eigenvalues are

$$\lambda_1 = -\frac{N-p}{p-1}, \quad \lambda_3 = N + a - \delta \bar{Y},$$

and the roots  $\lambda_2, \lambda_4$  of equation

$$\lambda^2 - (\bar{Y} - \bar{W})\lambda + \frac{m+1-q}{q-1} \bar{Y} \bar{W} = 0 \quad (10.4)$$

then  $\lambda_2 \lambda_4 > 0$ . We assume  $m \neq \frac{N(q-1)+(b+1)q}{N-q}$ , that means  $\bar{Y} \neq \bar{W}$ . ((and  $\bar{Y} - \bar{W} = \frac{(N+b)q-(m+1)(N-q)}{m+1-q}$ ; then  $\bar{Y} - \bar{W} > 0$  if  $m$  is subcritical.)).

- Convergence as  $r \rightarrow \infty$  : if  $\lambda_3 > 0$  we have  $\mathcal{V}_s = \mathcal{V}_s \cap \{Z = 0\}$ , hence there is no admissible trajectory .there is no solution near  $\infty$ . Next assume that  $\lambda_3 < 0$ , that means  $\delta > (N+a) \frac{m+1-q}{q+b}$ . If  $\text{Ré}(\lambda_2) = \text{Ré}\lambda_4 < 0$  (resp.  $> 0$ ) then  $\mathcal{V}_s$  has dimension 4 (resp. 2) and  $\mathcal{V}_s \cap \{X = 0\}$  and  $\mathcal{V}_s \cap \{Z = 0\}$  have dimension 3 (resp. 1) then there exist trajectories with  $X, Z > 0$ , then admissible.

In any case  $\lim e^{-\lambda_1 t} X = C_1 > 0$ ,  $\lim Y = \bar{Y}$ ,  $\lim e^{-\lambda_3 t} Z = C_3 > 0$ ,  $\lim W = \bar{W}$ , then (4.15) follows as above.

- Convergence as  $r \rightarrow 0$  : Since  $\lambda_1 < 0$  we have  $\mathcal{V}_u = \mathcal{V}_u \cap \{X = 0\}$ , hence there is no admissible trajectory .  $\blacksquare$

**Proof of Proposition 4.11.** We set  $R_0 = (0, \bar{Y}, \bar{Z}, \bar{W})$ . Under our assumptions it lies in  $\mathcal{R}$ . Setting  $Y = \bar{Y} + \tilde{Y}, Z = \bar{Z} + \tilde{Z}, W = \bar{W} + \tilde{W}$ , the linearization gives

$$X_t = \lambda_1 X, \quad \tilde{Y}_t = \bar{Y} \left[ \tilde{Y} + \frac{\tilde{W}}{q-1} \right], \quad Z_t = \bar{Z} \left[ -sX - \delta \tilde{Y} - \tilde{Z} \right], \quad W_t = \bar{W} \left[ -\mu X - m\tilde{Y} - \tilde{W} \right];$$

the eigenvalues are

$$\lambda_1 = \frac{1}{p-1} (p + a - \delta \frac{b+q}{m+1-q}), \quad \lambda_3 = -\bar{Z} < 0;$$



and the roots  $\lambda_2, \lambda_4$  of equation of equation (10.4).

• Convergence as  $r \rightarrow \infty$  : If  $\lambda_1 > 0$  , that means  $(p+a)\frac{m+1-q}{q+b} < \delta$ , then  $\mathcal{V}_s = \mathcal{V}_s \cap \{X = 0\}$  , hence there is no admissible trajectory . Next assume  $\lambda_1 < 0$ ; if  $\text{Ré}(\lambda_2) = \text{Ré}\lambda_4 < 0$  (resp.  $> 0$ ) then  $\mathcal{V}_s$  has dimension 4(resp. 2) and  $\mathcal{V}_s \cap \{X = 0\}$  has dimension 3 (resp. 1) then there exist admissible trajectories.

• Convergence as  $r \rightarrow 0$  : If  $\lambda_1 < 0$ , then  $\mathcal{V}_u = \mathcal{V}_u \cap \{X = 0\}$  , hence there is no admissible trajectory .Next assume  $\lambda_1 > 0$ ; if  $\text{Ré}(\lambda_2) = \text{Ré}\lambda_4 < 0$  (resp.  $> 0$ ) then  $\mathcal{V}_s$  has dimension 4 (resp. 2) and  $\mathcal{V}_s \cap \{X = 0\}$  has dimension 3 (resp. 1) then there exist admissible trajectories.

In any case  $\lim e^{-\lambda_1 t} X = C_1 > 0$ ,  $\lim Y = \bar{Y}$ ,  $\lim Z = \bar{Z}$ ,  $\lim W = \bar{W}$ , then (4.15) holds again.

■

**Remark 10.1** Finally there is no admissible trajectory converging to  $0, K_0, L_0$ . Indeed the linearization at 0 gives

$$X_t = -\frac{N-p}{p-1}X, \quad Y_t = -\frac{N-q}{q-1}Y, \quad Z_t = (N+a)Z, \quad W_t = (N+b)W$$

Then  $\mathcal{V}_s$  and  $\mathcal{V}_u$  have dimension 2; then  $\mathcal{V}_s$  is contained in  $\{Z = W = 0\}$   $\mathcal{V}_u$  in  $\{X = Y = 0\}$ . The linearization at  $K_0$  gives, with  $Z = N+a + \tilde{Z}$ ,

$$X_t = \frac{p+a}{p-1}X, \quad Y_t = -\frac{N-q}{q-1}Y, \quad Z_t = (N+a) \left[ -sX - \delta Y - \tilde{Z} \right], \quad W_t = (N+b)W.$$

The eigenvalues are  $\frac{p+a}{p-1}, -\frac{N-q}{q-1}, -(N+a), N+b$ . Then  $\mathcal{V}_s$  and  $\mathcal{V}_u$  have dimension 2;  $\mathcal{V}_s$  is contained in  $\{Z = W = 0\}$ , and  $\mathcal{V}_u$  in  $\{Y = 0\}$ . The case of  $L_0$  follows by symmetry.

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