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# A going down theorem for Grothendieck Chow motives

Charles De Clercq

## ABSTRACT

Let  $X$  be a geometrically split, geometrically irreducible variety over a field  $F$  satisfying Rost nilpotence principle. Consider a field extension  $E/F$  and a finite field  $\mathbb{F}$ . We provide in this note a motivic tool giving sufficient conditions for so-called outer motives of direct summands of the Chow motive of  $X_E$  with coefficients in  $\mathbb{F}$  to be lifted to the base field. This going down result has been used S. Garibaldi, V. Petrov and N. Semenov to give a complete classification of the motivic decompositions of projective homogeneous varieties of inner type  $E_6$  and to answer a conjecture of Rost and Springer.

## Introduction

Throughout this note  $F$  will be the base field and by an  $F$ -variety we will mean a smooth, projective scheme over  $F$ . Given an  $F$ -variety  $X$ , denote by  $\mathrm{Ch}(X)$  the Chow group  $\mathrm{CH}(X) \otimes_{\mathbb{Z}} \mathbb{F}$  of cycles on  $X$  modulo rational equivalence with coefficients in a finite field  $\mathbb{F}$ . We write  $\mathrm{Ch}(\overline{X})$  for the colimit of all  $\mathrm{Ch}(X_K)$ , where  $K$  runs through all field extensions  $K/F$  and if  $X$  is integral we denote by  $F(X)$  its function field.

For any field extension  $L/F$ , an element lying in the image of the natural morphism of  $\mathrm{Ch}(X_L) \rightarrow \mathrm{Ch}(\overline{X})$  is called  $L$ -rational. The image of any correspondence  $\alpha \in \mathrm{Ch}(X_L)$  under the canonical morphism  $\mathrm{Ch}(X_L) \rightarrow \mathrm{Ch}(\overline{X})$  is denoted by  $\overline{\alpha}$ . An  $F$ -variety  $X$  is geometrically split, if for some field extension  $E/F$  the Grothendieck Chow motive of  $X_E = X \times_{\mathrm{Spec}(F)} \mathrm{Spec}(E)$  with coefficients in  $\mathbb{F}$  is isomorphic to a finite direct sum of Tate motives. The variety  $X$  satisfies the Rost nilpotence principle if for any field extensions  $L/E/F$  the kernel of the restriction map  $\mathrm{res}_{L/E} : \mathrm{End}(M(X_E)) \rightarrow \mathrm{End}(M(X_L))$  consists of nilpotents.

As shown in [1], any projective homogeneous  $F$ -variety under the action of a semisimple affine algebraic group is geometrically split and satisfies the Rost nilpotence principle. It follows by [2, Corollary 35] (see also [7, Corollary 2.6]) that the Grothendieck Chow motive of these varieties with coefficients in  $\mathbb{F}$  decomposes in an essentially unique way as a direct sum of indecomposable motives. The study of these decompositions have already shown to be very fruitful (see [6], [7], [9]).

The notion of *upper* motives, previously defined by Vishik in the context of quadrics in [9], was further developed by Karpenko in [7] to describe the indecomposable motives lying in the motivic decomposition of projective homogeneous varieties. If  $X$  is a homogeneous  $F$ -variety,  $E/F$  a field extension and the upper motive of  $M(X_E)$  is a direct summand of another motive  $M_E$ , [9, Theorem 4.15] and [6, Proposition 4.6] give sufficient conditions for the upper motive of

$X$  to be a direct summand of  $M$ . The purpose of the present note is to push these ideas further. We define the notions of *upper*, *lower* and *outer* direct summands of a direct summand  $N$  of the motive of a geometrically split  $F$ -variety. We then show some lifting property of outer summands of  $N_E$  to the base field with the following result.

**Theorem 0.1** *Let  $N$  be a direct summand of the motive (with coefficients in  $\mathbb{F}$ ) of a geometrically split, geometrically irreducible  $F$ -variety  $X$  satisfying the Rost nilpotence principle and  $M$  a twisted direct summand of the motive of another  $F$ -variety  $Y$ . Assume that there is a field extension  $E/F$  such that*

- (i) *every  $E(X)$ -rational cycle in  $\text{Ch}(\overline{X \times Y})$  is  $F(X)$ -rational;*
- (ii) *the motive  $N_E$  has an indecomposable outer direct summand which is also a direct summand of the motive  $M_E$ .*

*Then the motive  $N$  has an outer direct summand which is also a direct summand of  $M$ .*

Theorem 0.1 allows one to descend outer motives of direct summands projective homogeneous varieties which appear on some field extension  $E/F$  of the base field. This generalizes [6, Proposition 4.6], one of the key ingredients in the proof of [6, Theorem 1.1], replacing the whole motive of a variety  $X$  by a direct summand. To replace  $X$  by an arbitrary direct summand, one needs to construct explicitly the rational cycles to get an outer summand defined over  $F$ , and thus theorem 0.1 gives a new proof of [6, Proposition 4.6]. Note that assumption 1 of theorem 0.1 holds if the field extension  $E(X)/F(X)$  is unirational. The following particular case of theorem 0.1 was used by Garibaldi, Petrov and Semenov in [8] to both determine all the motivic decompositions of homogeneous  $F$ -varieties of inner type  $E_6$  and prove a conjecture of Rost and Springer in [8].

**Corollary 0.2** ([8, Proposition 3.2]) *Let  $X$  and  $Y$  be two projective homogeneous  $F$ -varieties for a semisimple affine algebraic group, and let  $M$  and  $N$  be direct summands of the motives of  $Y$  and  $X$  respectively with coefficients in  $\mathbb{F}$ . Assume that  $N_{F(Y)}$  is an indecomposable direct summand of  $M_{F(Y)}$  and  $Y$  has an  $F(X)$ -point. Then  $N$  is a direct summand of  $M$ .*

*Proof.* Setting  $E = F(Y)$ , the field extension  $E(X)/F(X)$  is purely transcendental, hence assumption 1) of theorem 0.1 holds.  $\square$

## 1. Grothendieck Chow motives

Our main reference for the construction of the category of Grothendieck Chow motives over  $F$  with coefficients in  $\mathbb{F}$  is [3, §63-65].

Let  $X$  and  $Y$  be two  $F$ -varieties and  $X = \coprod_{k=1}^n X_k$  be the decomposition of  $X$  into irreducible components with respective dimension  $d_1, \dots, d_n$ . The group of *correspondences* between  $X$  and  $Y$  of degree  $i$  with coefficients in  $\mathbb{F}$  is defined by  $\text{Corr}_i(X, Y) = \coprod_{k=1}^n \text{Ch}_{d_k+i}(X_k \times Y)$ . We now consider the category  $C(F; \mathbb{F})$  whose objects are pairs  $X[i]$ , where  $X$  is an  $F$ -variety and  $i$  is an integer. Morphisms are defined by  $\text{Hom}_{C(F; \mathbb{F})}(X[i], Y[j]) = \text{Corr}_{i-j}(X, Y)$ . For any correspondences  $f : X[i] \rightsquigarrow Y[j]$  and  $g : Y[j] \rightsquigarrow Z[k]$  in  $\text{Mor}(C(F; \mathbb{F}))$  the composite  $g \circ f : X[i] \rightsquigarrow Z[k]$  is given by

$$g \circ f = ({}^X p_Y^Z)_* \left( ({}^{X \times Y} p_Z)^*(f) \cdot (p_X^{Y \times Z})^*(g) \right) \quad (*)$$

where  $U p_V^W : U \times V \times W \rightarrow U \times W$  is the natural projection.

The category  $C(F; \mathbb{F})$  is preadditive and its additive completion  $CR(F; \mathbb{F})$  is the category of correspondences over  $F$  with coefficients in  $\mathbb{F}$ , which has a structure of tensor additive category given by  $X[i] \otimes Y[j] = (X \times Y)[i+j]$ . The category  $CM(F; \mathbb{F})$  of Grothendieck Chow motives with coefficients in  $\mathbb{F}$  is the pseudo-abelian envelope of the category  $CR(F; \mathbb{F})$ . Its objects are couples  $(X, \pi)$ , where  $X$  is an object of the category  $CR(F; \Lambda)$ , and  $\pi \in \text{End}(X)$  is a projector (i.e.  $\pi \circ \pi = \pi$ ). Morphisms are given by  $\text{Hom}_{CM(F; \mathbb{F})}((X, \pi)[i], (Y, \rho)[j]) = \rho \circ \text{Hom}(X, Y) \circ \pi$  and the objects of  $CM(F; \mathbb{F})$  are called *motives*. For any  $F$ -variety  $X$  the motives  $(X[i], \Gamma_{id_X})$  (where  $\Gamma_{id_X}$  is the graph of the identity of  $X$ ) will be denoted  $X[i]$  and  $X[0]$  is the motive of  $X$ . The motives  $\mathbb{F}[i] = \text{Spec}(F)[i]$  are the *Tate motives*.

**Lemma 1.1** *Let  $(X, \pi)$  be a direct summand of the motive of an  $F$ -variety  $X$ . A motivic direct summand  $(X, \rho)$  of  $X$  is a direct summand of  $(X, \pi)$  if and only if  $\pi \circ \rho \circ \pi = \rho$ .*

*Proof.* Since  $\text{End}((X, \pi)) = \pi \circ \text{Ch}_{\dim(X)}(X \times X) \circ \pi$ , any projector  $\rho$  in  $\text{End}((X, \pi))$  satisfies  $\pi \circ \rho \circ \pi = \rho$ .  $\square$

**Definition 1.2** *Let  $M \in CM(F; \mathbb{F})$  be a motive and  $i$  an integer. The  $i$ -dimensional Chow group  $\text{Ch}_i(M)$  of  $M$  is defined by  $\text{Hom}_{CM(F; \mathbb{F})}(\mathbb{F}[i], M)$ . The  $i$ -codimensional Chow group  $\text{Ch}^i(M)$  of  $M$  is defined by  $\text{Hom}_{CM(F; \mathbb{F})}(M, \mathbb{F}[i])$ .*

For any field extension  $E/F$  and any correspondence  $\alpha : X[i] \rightsquigarrow Y[j]$  the pull-back of  $\alpha$  along the natural morphism  $(X \times Y)_E \rightarrow X \times Y$  will be denoted  $\alpha_E$ . If  $N = (X, \pi)[i]$  is a twisted motivic direct summand of  $X$ , the motive  $(X_E, \pi_E)[i]$  will be denoted  $N_E$ .

Finally the category  $CM(F; \mathbb{F})$  is endowed with the duality functor. If  $X$  and  $Y$  are two  $F$ -varieties and  $\alpha \in \text{Ch}(X \times Y)$  is a correspondence, the image of  $\alpha$  under the exchange isomorphism  $X \times Y \rightarrow Y \times X$  is denoted  ${}^t\alpha$ . The *duality functor*  $\dagger : CM(F; \Lambda)^{op} \rightarrow CM(F; \Lambda)$  is the additive functor determined by the formula  $M(X)^\dagger = M(X)[- \dim(X) - i]$  and such that for any correspondence  $\alpha : X[i] \rightsquigarrow Y[j]$ ,  $\alpha^\dagger = {}^t\alpha$ .

## 2. Direct summands of geometrically split $F$ -varieties

Throughout this section we consider a geometrically split  $F$ -variety  $X$  and  $E/F$  a splitting field of  $X$ . By [5, Remark 5.6] the pairing

$$\Psi : \begin{array}{ccc} \text{Ch}(X_E) \times \text{Ch}(X_E) & \longrightarrow & \mathbb{F} \\ (\alpha, \beta) & \longmapsto & \deg(\alpha \cdot \beta) \end{array}$$

is non degenerate hence gives rise to an isomorphism of  $\mathbb{F}$ -modules between  $\text{Ch}(X_E)$  and its dual space  $\text{Hom}_{\mathbb{F}}(\text{Ch}(X_E), \mathbb{F})$  given by  $\alpha \mapsto \Psi(\alpha, \cdot)$ . The dual basis of a homogeneous basis  $(x_k)_{k=1}^n$  of  $\text{Ch}(X_E)$  with respect of  $\Psi$  is the basis  $(x_k^*)_{k=1}^n$  of  $\text{Ch}(X_E)$  such that for any  $1 \leq i, j \leq n$ ,  $\Psi(x_i, x_j^*) = \delta_{ij}$ , where  $\delta_{ij}$  is the Kronecker symbol. By definition of the composition  $(*)$  in  $CM(F; \mathbb{F})$ , if  $y$  (resp.  $y'$ ) lies in  $\text{Ch}(Y)$  (resp.  $\text{Ch}(Y')$ ) for two other  $F$ -varieties  $Y$  and  $Y'$  and if  $(i, j)$  are two integers, the composition of the correspondences  $x_i \times y \in \text{Ch}(X_E \times Y)$  and  $y' \times x_j^* \in \text{Ch}(Y' \times X_E)$  is given by

$$(x_i \times y) \circ (y' \times x_j^*) = \delta_{ij}(y' \times y) \in \text{Ch}(Y' \times Y). \quad (1)$$

Note that the Kunneth decomposition holds in  $\text{Ch}(X_E \times Y)$  and  $\text{Ch}(Y' \times X_E)$  in the view of [3, Proposition 64.3], since  $X_E$  is split, and thus the cycles of  $\text{Ch}(X_E \times Y)$  and  $\text{Ch}(Y' \times X_E)$  may always be written that way.

**Upper, lower and outer motives.** Let  $\pi \in \text{Ch}_{\dim(X)}(X \times X)$  be a non-zero projector and  $N = (X, \pi)$  the associated summand of the motive of  $X$ . The *base* of  $N$  is defined by  $\mathcal{B}(N) = \{i \in \mathbb{Z}, \text{Ch}_i(N_E) \text{ is not trivial}\}$ . The *bottom* of  $N$  (denoted by  $b(N)$ ) is the least integer of  $\mathcal{B}(N)$  and the *top* of  $N$  (denoted by  $t(N)$ ) is the greatest integer of  $\mathcal{B}(N)$ . We now introduce the notion of upper and lower direct summands of  $N$ , previously introduced by Vishik in the context of the motives of quadrics in [9, Definition 4.6].

**Definition 2.1** *Let  $N$  be a direct summand of the twisted motive of a geometrically split  $F$ -variety and  $M$  a motivic direct summand of  $N$ . We say that*

- (i)  *$M$  is upper in  $N$  if  $b(M) = b(N)$ ;*
- (ii)  *$M$  is lower in  $N$  if  $t(M) = t(N)$ ;*
- (iii)  *$M$  is outer in  $N$  if  $M$  is both lower and upper in  $N$ .*

**Remark 2.2** *Keeping the same  $F$ -variety  $X$  and any direct summand  $N = (X, \pi)$ , consider a homogeneous basis  $(x_k)_{k=1}^n$  of  $\text{Ch}(X_E)$  and its dual basis  $(x_k^*)_{k=1}^n$ . The base, bottom and top of  $N$  can be easily determined by the decomposition*

$$\pi_E = \sum_{i,j=1}^n \pi_{i,j}(x_i \times x_j^*)$$

*noticing that  $\mathcal{B}(N) = \{\dim(x_i), \pi_{i,j} \neq 0 \text{ for some } j\}$ .*

**Lemma 2.3** *Let  $N$  be a motivic direct summand of a geometrically split  $F$ -variety and  $M$  a direct summand of  $N$ . Then  $M$  is lower in  $N$  (resp. upper in  $N$ ) if and only if the dual motive  $M^\dagger$  is upper in  $N^\dagger$  (resp.  $M^\dagger$  is lower in  $N^\dagger$ ).*

*Proof.* For any motive  $O$  and for any integer  $i$ ,  $\text{Ch}^i(O^\dagger) = \text{Ch}_{-i}(O)$ . It follows that  $b(O^\dagger) = -t(O)$  and  $t(O^\dagger) = -b(O)$ .  $\square$

**The Krull-Schmidt property.** Let  $\mathcal{C}$  be a pseudo-abelian category and  $\mathfrak{C}$  be the set of the isomorphism classes of objects of  $\mathcal{C}$ . We say that the category  $\mathcal{C}$  satisfies the *Krull-Schmidt property* if the monoid  $(\mathfrak{C}, \oplus)$  is free. The Krull-Schmidt property holds for the motives of geometrically split  $F$ -varieties satisfying the Rost nilpotence principle in  $\text{CM}(F; \mathbb{F})$  by [7, Corollary 2.6].

**Proof of the main result.** In order to prove theorem 0.1, we will need the following lemma, which will allow us to construct explicitly the rational cycles lifting outer motives to the base field.

**Lemma 2.4** *Let  $N$  be a motivic direct summand of a geometrically split, geometrically irreducible  $F$ -variety  $X$  satisfying the Rost nilpotence principle and  $M$  a twisted direct summand of an  $F$ -variety  $Y$ . Assume the existence of a field extension  $E/F$  such that*

- (i) *any  $E(X)$ -rational cycle in  $\text{Ch}(\overline{X} \times \overline{Y})$  is  $F(X)$ -rational;*
- (ii) *there are two correspondences  $h : N_E \rightsquigarrow M_E$  and  $k : M_E \rightsquigarrow N_E$  such that  $k \circ h$  is a projector and  $(X_E, k \circ h)$  is a lower direct summand of  $N_E$ .*

*Then there are two correspondences  $f : N \rightsquigarrow M$  and  $g : M_E \rightsquigarrow N_E$  such that  $(X_E, g \circ f_E)$  is a direct summand of  $N_E$  which contains all lower indecomposable direct summands of  $(X_E, k \circ h)$ . Furthermore if  $\overline{k}$  is  $F$ -rational, then  $\overline{g}$  is also  $F$ -rational.*

*Proof.* Setting  $M=(Y, \rho)[i]$  and  $N=(X, \pi)$ , we construct explicitly the two correspondences  $f$  and  $g$ . Since  $E(X)$  is a field extension of  $E$ ,  $\bar{h}$  is  $E(X)$ -rational, hence  $F(X)$ -rational by assumption 1. The morphisms  $\text{Spec}(F(X_L)) \rightarrow X_L$  for every field extension  $L/F$  induce the pull-back  $\varepsilon^* : \text{Ch}(\overline{X \times Y \times X}) \rightarrow \text{Ch}((X \times Y)_{F(X)})$ . Furthermore  $\varepsilon^*$  induces a surjection of  $F$ -rational cycles onto  $F(X)$ -rational cycles by [3, Corollary 57.11] and we can consider a cycle  $h_1 \in \text{Ch}(X \times Y \times X)$  such that  $\varepsilon^*(\bar{h}_1) = \bar{h}$ . Since  $\varepsilon^*$  maps any homogeneous cycle  $\sum_i \alpha_i \times \beta_i \times 1$  to  $\sum_i \alpha_i \times \beta_i$  and vanishes on homogeneous cycles whose codimension on the third factor is strictly positive, we have  $\bar{h}_1 = \bar{h} \times 1 + \dots$  where " $\dots$ " is a linear combination of homogeneous cycles in  $\text{Ch}(\overline{X \times Y \times X})$  with strictly positive codimension on the third factor.

We now look at  $h_1$  as a correspondence  $X \rightsquigarrow Y \times X$  and consider the cycle  $h_2 = h_1 \circ \pi$ . By formula 1 we have

$$\bar{h}_2 = (\bar{h} \times 1) \circ \bar{\pi} + \dots,$$

where " $\dots$ " is a linear combination of homogeneous cycles in  $\text{Ch}(\overline{X \times Y \times X})$  with dimension at most  $t(N)$  on the first factor (since these terms come from the first factors of  $\bar{\pi}$ ) and strictly positive codimension on the third factor (since these terms come from the third factors of  $\bar{h}_1 - \bar{h} \times 1$ ). Finally considering the pull-back of the morphism  $\Delta : X \times Y \rightarrow X \times Y \times X$  induced by the diagonal embedding  $X$  and setting  $h_3 = \Delta^*(h_2)$ , we have

$$\bar{h}_3 = \bar{h} \circ \bar{\pi} + \dots$$

where " $\dots$ " stands for a linear combination of homogeneous cycles in  $\text{Ch}(\overline{X \times Y})$  with dimension strictly lesser than  $t(N)$  on the first factor.

Composing with  $\bar{\pi} \circ \bar{k}$  on the left and  $\bar{\pi}$  on the right, we get that

$$\bar{\pi} \circ \bar{k} \circ \bar{h}_3 \circ \bar{\pi} = \bar{\pi} \circ \bar{k} \circ \bar{h} \circ \bar{\pi} + z$$

where  $z$  is a linear combination of homogeneous cycles of strictly lesser dimension than  $t(N)$  on the first factor since they come from the first factors of  $\bar{h}_3 \circ \bar{\pi} - \bar{h} \circ \bar{\pi}$ . The correspondence  $k \circ h$  defines a direct summand of the motive  $N_E$  and thus by lemma 1.1

$$\bar{\pi} \circ \bar{k} \circ \bar{h}_3 \circ \bar{\pi} = \bar{k} \circ \bar{h} + z.$$

By formula 1,  $\bar{k} \circ \bar{h} \circ z$ ,  $z \circ z$  and  $z \circ \bar{k} \circ \bar{h}$  are linear combinations of homogeneous cycles of dimension strictly lesser than  $t(N)$  on the first factor. Repeating the same procedure and since  $k \circ h$  is a projector, we see that for any integer  $n$

$$(\bar{\pi} \circ \bar{k} \circ \bar{h}_3 \circ \bar{\pi})^n = \bar{k} \circ \bar{h} + \dots \quad (2)$$

where " $\dots$ " is a linear combination of homogeneous cycles in  $\text{Ch}(\overline{X \times X})$  with dimension on the first factor strictly lesser than  $t(N)$ . The direct summand  $k \circ h$  is lower, therefore all these correspondences are non-zero and by [7, Corollary 2.2] an appropriate power  $(\pi_E \circ k \circ (h_3)_E \circ \pi_E)^{n_0}$  is a projector. If we set  $f = \rho \circ h_3 \circ \pi$  and  $g = (\pi_E \circ k \circ (h_3)_E \circ \pi_E)^{n_0-1} \circ \pi_E \circ k$ , we see that  $\bar{g}$  is  $F$ -rational if  $\bar{k}$  is  $F$ -rational. The correspondence  $g \circ f_E$  is a projector which defines a direct summand of  $N_E$  by lemma 1.1.

Consider the decomposition  $\bar{k} \circ \bar{h} = \sum_{i,j=1}^s \gamma_{ij}(x_i \times x_j^*)$  of  $\bar{k} \circ \bar{h}$  with respect to a basis  $(x_i)_{i=1}^s$  of  $\text{Ch}(\overline{X})$ . By formula 2, the decomposition of  $\bar{g} \circ \bar{f}$  in  $(x_i \times x_j^*)_{i,j=1}^s$  has a non-zero coefficient for any couple  $(i, j)$  such that  $\gamma_{ij}$  is non zero and  $\dim(x_i) = t((X_E, k \circ h))$ . The Krull-Schmidt property and remark 2.2 then imply that any lower indecomposable direct summand of  $(X_E, k \circ h)$  lies in  $(X, g \circ f_E)$ .  $\square$

We now show how we can derive the proof of theorem 0.1 from the rational cycles constructed in lemma 2.4. To lift the outer motive to the base field  $F$ , we apply lemma 2.4 and the duality functor twice in order produce two correspondences which are defined on the base field.

*Proof of Theorem 1..* Let  $O=(X_E, \kappa)$  be an outer indecomposable direct summand of  $N_E$  which is also a direct summand of  $M_E$ . We prove theorem 0.1 by applying lemma 2.4 once, then the duality functor and finally lemma 2.4 another time to get all our correspondences defined over the base field  $F$ .

Since  $O$  is a direct summand of  $M_E$ , there are two correspondences  $h : N_E \rightsquigarrow M_E$  and  $k : M_E \rightsquigarrow N_E$  such that  $k \circ h = \kappa$ . Moreover  $O$  is lower in  $N_E$ , so lemma 2.4 justifies the existence of two other correspondences  $h' : N \rightsquigarrow M$  and  $k' : M_E \rightsquigarrow N_E$  such that  $O_2 = (X_E, k' \circ h'_E)$  is a direct summand of  $N_E$ , and the motive  $O_2$  is outer in  $N_E$  since it contains  $O$ . The dual motive  $O_2^\dagger = (X_E, {}^t h'_E \circ {}^t k')[-\dim(X)]$  is therefore outer in  $N_E^\dagger$  by lemma 2.3 and is a direct summand of the dual motive  $M_E^\dagger$ . Twisting these three motives by  $\dim(X)$ , we can apply lemma 2.4 again. The correspondence  ${}^t h'$  is  $F$ -rational, so lemma 2.4 gives two correspondences  $f : N^\dagger \rightsquigarrow M^\dagger$  and  $g : M^\dagger \rightsquigarrow N^\dagger$  such that the motive  $(X_E, g_E \circ f_E)$  is both an outer direct summand of  $N^\dagger$  (since it contains the dual motive  $O^\dagger$ ) and a direct summand of  $M^\dagger$ . Transposing again, the motive  $(X, {}^t f \circ {}^t g)$  is an outer direct summand of  $N$  and a direct summand of  $M$ .  $\square$

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