



**HAL**  
open science

# A going down theorem for Grothendieck Chow motives

Charles de Clercq

► **To cite this version:**

| Charles de Clercq. A going down theorem for Grothendieck Chow motives. 2009. hal-00443691v3

**HAL Id: hal-00443691**

**<https://hal.science/hal-00443691v3>**

Preprint submitted on 6 Jan 2012 (v3), last revised 13 Sep 2012 (v4)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# A going down theorem for Grothendieck Chow motives

Charles De Clercq

## ABSTRACT

Let  $X$  be a geometrically split, geometrically irreducible variety over a field  $F$  satisfying Rost nilpotence principle. Consider a field extension  $E/F$  and a finite field  $\mathbb{F}$ . We provide in this note a motivic tool giving sufficient conditions for so-called outer motives of direct summands of the Chow motive of  $X_E$  with coefficients in  $\mathbb{F}$  to be lifted to the base field. This going down result has been used S. Garibaldi, V. Petrov and N. Semenov to give a complete classification of the motivic decompositions of projective homogeneous varieties of inner type  $E_6$  and to answer a conjecture of Rost and Springer.

## Introduction

Throughout this note  $F$  will be the base field and by an  $F$ -variety we will mean a smooth, projective scheme over  $F$ . Given an  $F$ -variety  $X$ , denote by  $\mathrm{Ch}(X)$  the Chow group  $\mathrm{CH}(X) \otimes_{\mathbb{Z}} \mathbb{F}$  of cycles on  $X$  modulo rational equivalence with coefficients in a finite field  $\mathbb{F}$ . We write  $\mathrm{Ch}(\overline{X})$  for the colimit of all  $\mathrm{Ch}(X_K)$ , where  $K$  runs through all field extensions  $K/F$  and if  $X$  is integral we denote by  $F(X)$  its function field.

For any field extension  $L/F$ , an element lying in the image of the natural morphism of  $\mathrm{Ch}(X_L) \rightarrow \mathrm{Ch}(\overline{X})$  is called  $L$ -rational. The image of any correspondence  $\alpha \in \mathrm{Ch}(X_L)$  under the canonical morphism  $\mathrm{Ch}(X_L) \rightarrow \mathrm{Ch}(\overline{X})$  is denoted by  $\overline{\alpha}$ . An  $F$ -variety  $X$  is geometrically split, if for some field extension  $E/F$  the Grothendieck Chow motive of  $X_E = X \times_{\mathrm{Spec}(F)} \mathrm{Spec}(E)$  with coefficients in  $\mathbb{F}$  is isomorphic to a finite direct sum of Tate motives. The variety  $X$  satisfies the Rost nilpotence principle if for any field extensions  $L/E/F$  the kernel of the restriction map  $\mathrm{res}_{L/E} : \mathrm{End}(M(X_E)) \rightarrow \mathrm{End}(M(X_L))$  consists of nilpotents.

As shown in [1], any projective homogeneous  $F$ -variety under the action of a semisimple affine algebraic group is geometrically split and satisfies the Rost nilpotence principle. It follows by [2, Corollary 35] (see also [7, Corollary 2.6]) that the Grothendieck Chow motive of these varieties with coefficients in  $\mathbb{F}$  decomposes in an essentially unique way as a direct sum of indecomposable motives. The study of these decompositions have already shown to be very fruitful (see [6], [7], [9]).

The notion of *upper* motives, previously defined by Vishik in the context of quadrics in [9], was further developed by Karpenko in [7] to describe the indecomposable motives lying in the motivic decomposition of projective homogeneous varieties. If  $X$  is a homogeneous  $F$ -variety,  $E/F$  a field extension and the upper motive of  $M(X_E)$  is a direct summand of another motive  $M_E$ , [9, Theorem 4.15] and [6, Proposition 4.6] give sufficient conditions for the upper motive of

$X$  to be a direct summand of  $M$ . The purpose of the present note is to push these ideas further. We define the notions of *upper*, *lower* and *outer* direct summands of a direct summand  $N$  of the motive of a geometrically split  $F$ -variety. We then show some lifting property of outer summands of  $N_E$  to the base field with the following result.

**Theorem 0.1** *Let  $N$  be a direct summand of the motive (with coefficients in  $\mathbb{F}$ ) of a geometrically split, geometrically irreducible  $F$ -variety  $X$  satisfying the Rost nilpotence principle and  $M$  a twisted direct summand of the motive of another  $F$ -variety  $Y$ . Assume that there is a field extension  $E/F$  such that*

- (i) *every  $E(X)$ -rational cycle in  $\text{Ch}(\overline{X \times Y})$  is  $F(X)$ -rational;*
- (ii) *the motive  $N_E$  has an indecomposable outer direct summand which is also a direct summand of the motive  $M_E$ .*

*Then the motive  $N$  has an outer direct summand which is also a direct summand of  $M$ .*

Theorem 0.1 allows one to descend outer motives of direct summands projective homogeneous varieties which appear on some field extension  $E/F$  of the base field. This generalizes [6, Proposition 4.6], one of the key ingredients in the proof of [6, Theorem 1.1], replacing the whole motive of a variety  $X$  by a direct summand. To replace  $X$  by an arbitrary direct summand, one needs to construct explicitly the rational cycles to get an outer summand defined over  $F$ , and thus theorem 0.1 gives a new proof of [6, Proposition 4.6]. Note that assumption 1 of theorem 0.1 holds if the field extension  $E(X)/F(X)$  is unirational. The following particular case of theorem 0.1 was used by Garibaldi, Petrov and Semenov in [8] to both determine all the motivic decompositions of homogeneous  $F$ -varieties of inner type  $E_6$  and prove a conjecture of Rost and Springer in [8].

**Corollary 0.2** *([8, Proposition 3.2]) Let  $X$  and  $Y$  be two projective homogeneous  $F$ -varieties for a semisimple affine algebraic group, and let  $M$  and  $N$  be direct summands of the motives of  $Y$  and  $X$  respectively with coefficients in  $\mathbb{F}$ . Assume that  $N_{F(Y)}$  is an indecomposable direct summand of  $M_{F(Y)}$  and  $Y$  has an  $F(X)$ -point. Then  $N$  is a direct summand of  $M$ .*

*Proof.* Setting  $E = F(Y)$ , the field extension  $E(X)/F(X)$  is purely transcendental, hence assumption 1) of theorem 0.1 holds.  $\square$

## 1. Grothendieck Chow motives

Our main reference for the construction of the category of Grothendieck Chow motives over  $F$  with coefficients in  $\mathbb{F}$  is [3, §63-65].

Let  $X$  and  $Y$  be two  $F$ -varieties and  $X = \coprod_{k=1}^n X_k$  be the decomposition of  $X$  into irreducible components with respective dimension  $d_1, \dots, d_n$ . The group of *correspondences* between  $X$  and  $Y$  of degree  $i$  with coefficients in  $\mathbb{F}$  is defined by  $\text{Corr}_i(X, Y) = \coprod_{k=1}^n \text{Ch}_{d_k+i}(X_k \times Y)$ . We now consider the category  $\mathcal{C}(F; \mathbb{F})$  whose objects are pairs  $X[i]$ , where  $X$  is an  $F$ -variety and  $i$  is an integer. Morphisms are defined by  $\text{Hom}_{\mathcal{C}(F; \mathbb{F})}(X[i], Y[j]) = \text{Corr}_{i-j}(X, Y)$ . For any correspondences  $f : X[i] \rightsquigarrow Y[j]$  and  $g : Y[j] \rightsquigarrow Z[k]$  in  $\text{Mor}(\mathcal{C}(F; \mathbb{F}))$  the composite  $g \circ f : X[i] \rightsquigarrow Z[k]$  is given by

$$g \circ f = ({}^X p_Y^Z)_* \left( ({}^{X \times Y} p_Z)^*(f) \cdot ({}^Y p_X^{Y \times Z})^*(g) \right) \quad (*)$$

where  $U p_V^W : U \times V \times W \rightarrow U \times W$  is the natural projection.

The category  $\mathbf{C}(F; \mathbb{F})$  is preadditive and its additive completion  $\mathbf{CR}(F; \mathbb{F})$  is the category of correspondences over  $F$  with coefficients in  $\mathbb{F}$ , which has a structure of tensor additive category given by  $X[i] \otimes Y[j] = (X \times Y)[i + j]$ . The category  $\mathbf{CM}(F; \mathbb{F})$  of Grothendieck Chow motives with coefficients in  $\mathbb{F}$  is the pseudo-abelian envelope of the category  $\mathbf{CR}(F; \mathbb{F})$ . Its objects are couples  $(X, \pi)$ , where  $X$  is an object of the category  $\mathbf{CR}(F; \mathbb{F})$ , and  $\pi \in \text{End}(X)$  is a projector (i.e.  $\pi \circ \pi = \pi$ ). Morphisms are given by  $\text{Hom}_{\mathbf{CM}(F; \mathbb{F})}((X, \pi)[i], (Y, \rho)[j]) = \rho \circ \text{Hom}(X, Y) \circ \pi$  and the objects of  $\mathbf{CM}(F; \mathbb{F})$  are called *motives*. For any  $F$ -variety  $X$  the motives  $(X[i], \Gamma_{id_X})$  (where  $\Gamma_{id_X}$  is the graph of the identity of  $X$ ) will be denoted  $X[i]$  and  $X[0]$  is the motive of  $X$ . The motives  $\mathbb{F}[i] = \text{Spec}(F)[i]$  are the *Tate motives*.

**Lemma 1.1** *Let  $(X, \pi)$  be a direct summand of the motive of an  $F$ -variety  $X$ . A motivic direct summand  $(X, \rho)$  of  $X$  is a direct summand of  $(X, \pi)$  if and only if  $\pi \circ \rho \circ \pi = \rho$ .*

*Proof.* Since  $\text{End}((X, \pi)) = \pi \circ \text{Ch}_{\dim(X)}(X \times X) \circ \pi$ , any projector  $\rho$  in  $\text{End}((X, \pi))$  satisfies  $\pi \circ \rho \circ \pi = \rho$ .  $\square$

**Definition 1.2** *Let  $M \in \mathbf{CM}(F; \mathbb{F})$  be a motive and  $i$  an integer. The  $i$ -dimensional Chow group  $\text{Ch}_i(M)$  of  $M$  is defined by  $\text{Hom}_{\mathbf{CM}(F; \mathbb{F})}(\mathbb{F}[i], M)$ . The  $i$ -codimensional Chow group  $\text{Ch}^i(M)$  of  $M$  is defined by  $\text{Hom}_{\mathbf{CM}(F; \mathbb{F})}(M, \mathbb{F}[i])$ .*

For any field extension  $E/F$  and any correspondence  $\alpha : X[i] \rightsquigarrow Y[j]$  the pull-back of  $\alpha$  along the natural morphism  $(X \times Y)_E \rightarrow X \times Y$  will be denoted  $\alpha_E$ . If  $N = (X, \pi)[i]$  is a twisted motivic direct summand of  $X$ , the motive  $(X_E, \pi_E)[i]$  will be denoted  $N_E$ .

Finally the category  $\mathbf{CM}(F; \mathbb{F})$  is endowed with the duality functor. If  $X$  and  $Y$  are two  $F$ -varieties and  $\alpha \in \text{Ch}(X \times Y)$  is a correspondence, the image of  $\alpha$  under the exchange isomorphism  $X \times Y \rightarrow Y \times X$  is denoted  ${}^t\alpha$ . The *duality functor*  $\dagger : \mathbf{CM}(F; \mathbb{F})^{op} \rightarrow \mathbf{CM}(F; \mathbb{F})$  is the additive functor determined by the formula  $M(X)^\dagger = M(X)[- \dim(X) - i]$  and such that for any correspondence  $\alpha : X[i] \rightsquigarrow Y[j]$ ,  $\alpha^\dagger = {}^t\alpha$ .

## 2. Direct summands of geometrically split $F$ -varieties

Throughout this section we consider a geometrically split  $F$ -variety  $X$  and  $E/F$  a splitting field of  $X$ . By [5, Remark 5.6] the pairing

$$\Psi : \begin{array}{ccc} \text{Ch}(X_E) \times \text{Ch}(X_E) & \longrightarrow & \mathbb{F} \\ (\alpha, \beta) & \longmapsto & \text{deg}(\alpha \cdot \beta) \end{array}$$

is non degenerate hence gives rise to an isomorphism of  $\mathbb{F}$ -modules between  $\text{Ch}(X_E)$  and its dual space  $\text{Hom}_{\mathbb{F}}(\text{Ch}(X_E), \mathbb{F})$  given by  $\alpha \mapsto \Psi(\alpha, \cdot)$ . The dual basis of a homogeneous basis  $(x_k)_{k=1}^n$  of  $\text{Ch}(X_E)$  with respect of  $\Psi$  is the basis  $(x_k^*)_{k=1}^n$  of  $\text{Ch}(X_E)$  such that for any  $1 \leq i, j \leq n$ ,  $\Psi(x_i, x_j^*) = \delta_{ij}$ , where  $\delta_{ij}$  is the Kronecker symbol. By definition of the composition  $(*)$  in  $\mathbf{CM}(F; \mathbb{F})$ , if  $y$  (resp.  $y'$ ) lies in  $\text{Ch}(Y)$  (resp.  $\text{Ch}(Y')$ ) for two other  $F$ -varieties  $Y$  and  $Y'$  and if  $(i, j)$  are two integers, the composition of the correspondences  $x_i \times y \in \text{Ch}(X_E \times Y)$  and  $y' \times x_j^* \in \text{Ch}(Y' \times X_E)$  is given by

$$(x_i \times y) \circ (y' \times x_j^*) = \delta_{ij}(y' \times y) \in \text{Ch}(Y' \times Y). \quad (1)$$

Note that the Kunnet decomposition holds in  $\text{Ch}(X_E \times Y)$  and  $\text{Ch}(Y' \times X_E)$  in the view of [3, Proposition 64.3], since  $X_E$  is split, and thus the cycles of  $\text{Ch}(X_E \times Y)$  and  $\text{Ch}(Y' \times X_E)$  may always be written that way.

**Upper, lower and outer motives.** Let  $\pi \in \mathrm{Ch}_{\dim(X)}(X \times X)$  be a non-zero projector and  $N = (X, \pi)$  the associated summand of the motive of  $X$ . The *base* of  $N$  is defined by  $\mathcal{B}(N) = \{i \in \mathbb{Z}, \mathrm{Ch}_i(N_E) \text{ is not trivial}\}$ . The *bottom* of  $N$  (denoted by  $b(N)$ ) is the least integer of  $\mathcal{B}(N)$  and the *top* of  $N$  (denoted by  $t(N)$ ) is the greatest integer of  $\mathcal{B}(N)$ . We now introduce the notion of upper and lower direct summands of  $N$ , previously introduced by Vishik in the context of the motives of quadrics in [9, Definition 4.6].

**Definition 2.1** *Let  $N$  be a direct summand of the twisted motive of a geometrically split  $F$ -variety and  $M$  a motivic direct summand of  $N$ . We say that*

- (i)  $M$  is upper in  $N$  if  $b(M) = b(N)$ ;
- (ii)  $M$  is lower in  $N$  if  $t(M) = t(N)$ ;
- (iii)  $M$  is outer in  $N$  if  $M$  is both lower and upper in  $N$ .

**Remark 2.2** *Keeping the same  $F$ -variety  $X$  and any direct summand  $N = (X, \pi)$ , consider a homogeneous basis  $(x_k)_{k=1}^n$  of  $\mathrm{Ch}(X_E)$  and its dual basis  $(x_k^*)_{k=1}^n$ . The base, bottom and top of  $N$  can be easily determined by the decomposition*

$$\pi_E = \sum_{i,j=1}^n \pi_{i,j}(x_i \times x_j^*)$$

*noticing that  $\mathcal{B}(N) = \{\dim(x_i), \pi_{i,j} \neq 0 \text{ for some } j\}$ .*

**Lemma 2.3** *Let  $N$  be a motivic direct summand of a geometrically split  $F$ -variety and  $M$  a direct summand of  $N$ . Then  $M$  is lower in  $N$  (resp. upper in  $N$ ) if and only if the dual motive  $M^\dagger$  is upper in  $N^\dagger$  (resp.  $M^\dagger$  is lower in  $N^\dagger$ ).*

*Proof.* For any motive  $O$  and for any integer  $i$ ,  $\mathrm{Ch}^i(O^\dagger) = \mathrm{Ch}_{-i}(O)$ . It follows that  $b(O^\dagger) = -t(O)$  and  $t(O^\dagger) = -b(O)$ .  $\square$

**The Krull-Schmidt property.** Let  $\mathcal{C}$  be a pseudo-abelian category and  $\mathfrak{C}$  be the set of the isomorphism classes of objects of  $\mathcal{C}$ . We say that the category  $\mathcal{C}$  satisfies the *Krull-Schmidt property* if the monoid  $(\mathfrak{C}, \oplus)$  is free. The Krull-Schmidt property holds for the motives of geometrically split  $F$ -varieties satisfying the Rost nilpotence principle in  $\mathrm{CM}(F; \mathbb{F})$  by [7, Corollary 2.6].

**Proof of the main result.** In order to prove theorem 0.1, we will need the following lemma, which will allow us to construct explicitly the rational cycles lifting outer motives to the base field.

**Lemma 2.4** *Let  $N$  be a motivic direct summand of a geometrically split, geometrically irreducible  $F$ -variety  $X$  satisfying the Rost nilpotence principle and  $M$  a twisted direct summand of an  $F$ -variety  $Y$ . Assume the existence of a field extension  $E/F$  such that*

- (i) *any  $E(X)$ -rational cycle in  $\mathrm{Ch}(\overline{X \times Y})$  is  $F(X)$ -rational;*
- (ii) *there are two correspondences  $h : N_E \rightsquigarrow M_E$  and  $k : M_E \rightsquigarrow N_E$  such that  $k \circ h$  is a projector and  $(X_E, k \circ h)$  is a lower direct summand of  $N_E$ .*

*Then there are two correspondences  $f : N \rightsquigarrow M$  and  $g : M_E \rightsquigarrow N_E$  such that  $(X_E, g \circ f_E)$  is a direct summand of  $N_E$  which contains all lower indecomposable direct summands of  $(X_E, k \circ h)$ . Furthermore if  $\overline{k}$  is  $F$ -rational, then  $\overline{g}$  is also  $F$ -rational.*

*Proof.* Setting  $M=(Y, \rho)[i]$  and  $N=(X, \pi)$ , we construct explicitly the two correspondences  $f$  and  $g$ . Since  $E(X)$  is a field extension of  $E$ ,  $\bar{h}$  is  $E(X)$ -rational, hence  $F(X)$ -rational by assumption 1. The morphisms  $\text{Spec}(F(X_L)) \rightarrow X_L$  for every field extension  $L/F$  induce the pull-back  $\varepsilon^* : \text{Ch}(\overline{X \times Y \times X}) \rightarrow \text{Ch}((X \times Y)_{F(X)})$ . Furthermore  $\varepsilon^*$  induces a surjection of  $F$ -rational cycles onto  $F(X)$ -rational cycles by [3, Corollary 57.11] and we can consider a cycle  $h_1 \in \text{Ch}(X \times Y \times X)$  such that  $\varepsilon^*(\bar{h}_1) = \bar{h}$ . Since  $\varepsilon^*$  maps any homogeneous cycle  $\sum_i \alpha_i \times \beta_i \times 1$  to  $\sum_i \alpha_i \times \beta_i$  and vanishes on homogeneous cycles whose codimension on the third factor is strictly positive, we have  $\bar{h}_1 = \bar{h} \times 1 + \dots$  where " $\dots$ " is a linear combination of homogeneous cycles in  $\text{Ch}(\overline{X \times Y \times X})$  with strictly positive codimension on the third factor.

We now look at  $h_1$  as a correspondence  $X \rightsquigarrow Y \times X$  and consider the cycle  $h_2 = h_1 \circ \pi$ . By formula 1 we have

$$\bar{h}_2 = (\bar{h} \times 1) \circ \bar{\pi} + \dots,$$

where " $\dots$ " is a linear combination of homogeneous cycles in  $\text{Ch}(\overline{X \times Y \times X})$  with dimension at most  $t(N)$  on the first factor (since these terms come from the first factors of  $\bar{\pi}$ ) and strictly positive codimension on the third factor (since these terms come from the third factors of  $\bar{h}_1 - \bar{h} \times 1$ ). Finally considering the pull-back of the morphism  $\Delta : X \times Y \rightarrow X \times Y \times X$  induced by the diagonal embedding  $X$  and setting  $h_3 = \Delta^*(h_2)$ , we have

$$\bar{h}_3 = \bar{h} \circ \bar{\pi} + \dots$$

where " $\dots$ " stands for a linear combination of homogeneous cycles in  $\text{Ch}(\overline{X \times Y})$  with dimension strictly lesser than  $t(N)$  on the first factor.

Composing with  $\bar{\pi} \circ \bar{k}$  on the left and  $\bar{\pi}$  on the right, we get that

$$\bar{\pi} \circ \bar{k} \circ \bar{h}_3 \circ \bar{\pi} = \bar{\pi} \circ \bar{k} \circ \bar{h} \circ \bar{\pi} + z$$

where  $z$  is a linear combination of homogeneous cycles of strictly lesser dimension than  $t(N)$  on the first factor since they come from the first factors of  $\bar{h}_3 \circ \bar{\pi} - \bar{h} \circ \bar{\pi}$ . The correspondence  $k \circ h$  defines a direct summand of the motive  $N_E$  and thus by lemma 1.1

$$\bar{\pi} \circ \bar{k} \circ \bar{h}_3 \circ \bar{\pi} = \bar{k} \circ \bar{h} + z.$$

By formula 1,  $\bar{k} \circ \bar{h} \circ z$ ,  $z \circ z$  and  $z \circ \bar{k} \circ \bar{h}$  are linear combinations of homogeneous cycles of dimension strictly lesser than  $t(N)$  on the first factor. Repeating the same procedure and since  $k \circ h$  is a projector, we see that for any integer  $n$

$$(\bar{\pi} \circ \bar{k} \circ \bar{h}_3 \circ \bar{\pi})^n = \bar{k} \circ \bar{h} + \dots \quad (2)$$

where " $\dots$ " is a linear combination of homogeneous cycles in  $\text{Ch}(\overline{X \times X})$  with dimension on the first factor strictly lesser than  $t(N)$ . The direct summand  $k \circ h$  is lower, therefore all these correspondences are non-zero and by [7, Corollary 2.2] an appropriate power  $(\pi_E \circ k \circ (h_3)_E \circ \pi_E)^{n_0}$  is a projector. If we set  $f = \rho \circ h_3 \circ \pi$  and  $g = (\pi_E \circ k \circ (h_3)_E \circ \pi_E)^{n_0-1} \circ \pi_E \circ k$ , we see that  $\bar{g}$  is  $F$ -rational if  $\bar{k}$  is  $F$ -rational. The correspondence  $g \circ f_E$  is a projector which defines a direct summand of  $N_E$  by lemma 1.1.

Consider the decomposition  $\bar{k} \circ \bar{h} = \sum_{i,j=1}^s \gamma_{ij} (x_i \times x_j^*)$  of  $\bar{k} \circ \bar{h}$  with respect to a basis  $(x_i)_{i=1}^s$  of  $\text{Ch}(\overline{X})$ . By formula 2, the decomposition of  $\bar{g} \circ \bar{f}$  in  $(x_i \times x_j^*)_{i,j=1}^s$  has a non-zero coefficient for any couple  $(i, j)$  such that  $\gamma_{ij}$  is non zero and  $\dim(x_i) = t((X_E, k \circ h))$ . The Krull-Schmidt property and remark 2.2 then imply that any lower indecomposable direct summand of  $(X_E, k \circ h)$  lies in  $(X, g \circ f_E)$ .  $\square$

We now show how we can derive the proof of theorem 0.1 from the rational cycles constructed in lemma 2.4. To lift the outer motive to the base field  $F$ , we apply lemma 2.4 and the duality functor twice in order to produce two correspondences which are defined on the base field.

*Proof of Theorem 1.* Let  $O=(X_E, \kappa)$  be an outer indecomposable direct summand of  $N_E$  which is also a direct summand of  $M_E$ . We prove theorem 0.1 by applying lemma 2.4 once, then the duality functor and finally lemma 2.4 another time to get all our correspondences defined over the base field  $F$ .

Since  $O$  is a direct summand of  $M_E$ , there are two correspondences  $h : N_E \rightsquigarrow M_E$  and  $k : M_E \rightsquigarrow N_E$  such that  $k \circ h = \kappa$ . Moreover  $O$  is lower in  $N_E$ , so lemma 2.4 justifies the existence of two other correspondences  $h' : N \rightsquigarrow M$  and  $k' : M_E \rightsquigarrow N_E$  such that  $O_2 = (X_E, k' \circ h'_E)$  is a direct summand of  $N_E$ , and the motive  $O_2$  is outer in  $N_E$  since it contains  $O$ . The dual motive  $O_2^\dagger = (X_E, {}^t h'_E \circ {}^t k')$   $[-\dim(X)]$  is therefore outer in  $N_E^\dagger$  by lemma 2.3 and is a direct summand of the dual motive  $M_E^\dagger$ . Twisting these three motives by  $\dim(X)$ , we can apply lemma 2.4 again. The correspondence  ${}^t h'$  is  $F$ -rational, so lemma 2.4 gives two correspondences  $f : N^\dagger \rightsquigarrow M^\dagger$  and  $g : M^\dagger \rightsquigarrow N^\dagger$  such that the motive  $(X_E, g_E \circ f_E)$  is both an outer direct summand of  $N^\dagger$  (since it contains the dual motive  $O^\dagger$ ) and a direct summand of  $M^\dagger$ . Transposing again, the motive  $(X, {}^t f \circ {}^t g)$  is an outer direct summand of  $N$  and a direct summand of  $M$ .  $\square$

**Acknowledgements :** I would like to thank N. Karpenko for raising this question and for his suggestions.

#### REFERENCES

- 1 CHERNOUSOV, V., GILLE, S., MERKURJEV, A. *Motivic decomposition of isotropic projective homogeneous varieties*, Duke Math. J., 126(1) :137-159, 2005.
- 2 CHERNOUSOV, V., MERKURJEV, A. *Motivic decomposition of projective homogeneous varieties and the Krull-Schmidt theorem*, Transformation Groups 11, 371-386, 2006.
- 3 ELMAN, R., KARPENKO, N., MERKURJEV, A. *The Algebraic and Geometric Theory of Quadratic Forms*, AMS Colloquium Publications, Vol. 56, 2008.
- 4 FULTON, W. *Intersection theory*, Springer-Verlag, 1998.
- 5 KARPENKO, N., MERKURJEV, A. *Canonical  $p$ -dimension of algebraic groups*, Adv. Math. 205 (2006), 410–433.
- 6 KARPENKO, N. *Hyperbolicity of orthogonal involutions*, Doc. Math. Extra Volume: Andrei A. Suslin's Sixtieth Birthday (2010), 371–392, with an appendix of J.-P. Tignol
- 7 KARPENKO, N. *Upper motives of algebraic groups and incompressibility of Severi-Brauer varieties*, to appear in J. Reine Angew. Math.
- 8 GARIBALDI, S., PETROV, V., SEMENOV, N. *Shells of twisted flag varieties and non-decomposibility of the Rost invariant*, preprint (2010), available on the webpage of the authors.
- 9 VISHIK, A. *Motives of quadrics with applications to the theory of quadratic forms*, Lect. Notes in Math. 1835, Proceedings of the Summer School "Geometric Methods in the Algebraic Theory of Quadratic Forms, Lens 2000", 25-101, 2004.

Charles De Clercq    declercq@math.jussieu.fr  
 Université Paris VI, 4 place Jussieu, 75252 Paris CEDEX 5