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A going down theorem for Grothendieck Chow motives

Charles De Clercq

Abstract

Let X be a geometrically split, geometrically irreducible variety over a field F satisfying the nilpotence principle. Consider a motivic direct summand N of X and M a twisted motivic direct summand of another F -variety Y . We show that under some assumptions on a field extension E/F , if an indecomposable motive is both an outer direct summand of N_E and a direct summand of M_E , then an outer direct summand of N is also a direct summand of M .

Introduction

Let F be a field and \mathbb{F} a finite field. Throughout this note an F -variety will be understood as a smooth, projective and separated scheme of finite type over F . For any F -variety X , the Chow group of cycles on X modulo rational equivalence with coefficients in \mathbb{F} is defined by $\mathrm{CH}(X) \otimes \mathbb{F}$ and denoted $\mathrm{Ch}(X)$. An F -variety X is split (resp. geometrically split) if the motive of X (resp. the motive of X_E for some field extension E/F) in the category $\mathrm{CM}(F; \mathbb{F})$ of Grothendieck Chow motives with coefficients in \mathbb{F} is isomorphic to a finite direct sum of Tate motives.

Another important property is the following. An F -variety X satisfies the nilpotence principle if for any field extensions $L/E/F$ the kernel of the change of base field morphism $\mathrm{res}_{L/E} : \mathrm{End}(M(X_E)) \rightarrow \mathrm{End}(M(X_L))$ consists of nilpotents. The present note provides the following technical tool to study motivic decompositions of geometrically split F -varieties satisfying the nilpotence principle.

Theorem 1. Let N be a motivic direct summand of a geometrically split, geometrically irreducible F -variety X satisfying the nilpotence principle and M a twisted motivic direct summand of another F -variety Y . Assume that there is a field extension E/F such that

1. every $E(X)$ -rational cycle in $\mathrm{Ch}(\overline{X \times Y})$ is $F(X)$ -rational;
2. an indecomposable motive in $\mathrm{CM}(F; \mathbb{F})$ is both an outer direct summand of N_E and a direct summand of M_E .

Then an outer direct summand of N is a direct summand of M .

In the first section we recall briefly the definition of the category $\mathrm{CM}(F; \mathbb{F})$ while the second section is devoted to the study of direct summands of geometrically split F -varieties, introducing the notion of *upper* and *lower* direct summands. In section 3 we

prove theorem 1 and the fourth section is devoted to the already existing applications of particular cases of this result.

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1 Generalities on Grothendieck Chow motives

We refer to [3] and [8] for more details on the construction of the category $\text{CM}(F; \mathbb{F})$ of Grothendieck Chow motives over F with coefficients in \mathbb{F} .

Let X and Y be two F -varieties and $X = \coprod_{k=1}^n X_k$ be the decomposition of X as a disjoint union of irreducible components with respective dimension d_1, \dots, d_n . For any integer i the group of *correspondences* between X and Y of degree i with coefficients in \mathbb{F} is defined by $\text{Corr}_i(X, Y) = \coprod_{k=1}^n \text{Ch}_{d_k+i}(X_k \times Y)$. We now consider the category $\text{C}(F; \mathbb{F})$ whose objects are pairs $X[i]$, where X is an F -variety and i is an integer. Morphisms are defined in terms of correspondences by $\text{Hom}_{\text{C}(F; \mathbb{F})}(X[i], Y[j]) = \text{Corr}_{i-j}(X, Y)$. For any correspondences $f : X[i] \rightsquigarrow Y[j]$ and $g : Y[j] \rightsquigarrow Z[k]$ in $\text{Mor}(\text{C}(F; \mathbb{F}))$ the composite $g \circ f : X[i] \rightsquigarrow Z[k]$ is defined by

$$g \circ f = \left(\begin{smallmatrix} X & Z \\ p_Y & \end{smallmatrix} \right)_* \left(\left(\begin{smallmatrix} X \times Y \\ p_Z \end{smallmatrix} \right)^*(f) \cdot \left(\begin{smallmatrix} Y & Z \\ p_X & \end{smallmatrix} \right)^*(g) \right)$$

where $U_p V^W : U \times V \times W \rightarrow U \times W$ is the natural projection.

The category $\text{C}(F; \mathbb{F})$ is preadditive and its additive completion $\text{CR}(F; \mathbb{F})$ is the category of correspondences over F with coefficients in \mathbb{F} , which has a structure of tensor additive category given by $X[i] \otimes Y[j] = (X \times Y)[i + j]$. The category $\text{CM}(F; \mathbb{F})$ of Grothendieck Chow motives with coefficients in \mathbb{F} is the pseudo-abelian envelope of the category $\text{CR}(F; \mathbb{F})$. Its objects are finite direct sums of triples $(X, \pi)[i]$, where X is an F -variety, i is an integer and $\pi \in \text{Ch}_{\dim(X)}(X \times X)$ is a projector. Morphisms are given by $\text{Hom}_{\text{CM}(F; \mathbb{F})}((X, \pi)[i], (Y, \rho)[j]) = \rho \circ \text{Corr}_{i-j}(X, Y) \circ \pi$ and the objects of $\text{CM}(F; \mathbb{F})$ are called *motives*. For any F -variety X the motives $(X, \Gamma_{id_X})[i]$ (where Γ_{id_X} is the graph of the identity of X) will be denoted $X[i]$ and $X[0]$ is the motive of X . The motives $\mathbb{F}[i] = \text{Spec}(F)[i]$ are the *Tate motives*.

Definition 2. Let $M \in \text{CM}(F; \mathbb{F})$ be a motive and i an integer. The i -dimensional Chow group $\text{Ch}_i(M)$ of M is defined by $\text{Hom}_{\text{CM}(F; \mathbb{F})}(\mathbb{F}[i], M)$. The i -codimensional Chow group $\text{Ch}^i(M)$ of M is defined by $\text{Hom}_{\text{CM}(F; \mathbb{F})}(M, \mathbb{F}[i])$.

For any field extension E/F and any correspondence $\alpha : X[i] \rightsquigarrow Y[j]$ the pull-back of α along the natural morphism $(X \times Y)_E \rightarrow X \times Y$ will be denoted α_E . If $N = (X, \pi)[i]$ is a twisted motivic direct summand of X , the motive $(X_E, \pi_E)[i]$ will be denoted N_E .

Finally the category $\text{CM}(F; \mathbb{F})$ is endowed with the duality functor. If X and Y are two F -varieties and $\alpha \in \text{Ch}(X \times Y)$ is a correspondence, the image of α under the exchange isomorphism $X \times Y \rightarrow Y \times X$ is denoted ${}^t\alpha$. The *duality functor* is the additive functor $\dagger : \text{CM}(F; \mathbb{F})^{op} \rightarrow \text{CM}(F; \mathbb{F})$ which maps any motive $(X, \pi)[i]$ to $(X, {}^t\pi)[- \dim(X) - i]$ and any correspondence $\alpha : (X, \pi)[i] \rightsquigarrow (Y, \rho)[j]$ to ${}^t\alpha$.

2 Direct summands of geometrically split F -varieties

Throughout this section we consider a geometrically split F -variety X and E/F a splitting field of X . By [7, Remark 5.6] the pairing

$$\Psi : \begin{array}{ccc} \text{Ch}(X_E) \times \text{Ch}(X_E) & \longrightarrow & \mathbb{F} \\ (\alpha, \beta) & \longmapsto & \text{deg}(\alpha \cdot \beta) \end{array}$$

is non degenerate hence gives rise to an isomorphism of \mathbb{F} -modules between $\text{Ch}(X_E)$ and its dual space $\text{Hom}_{\mathbb{F}}(\text{Ch}(X_E), \mathbb{F})$ given by $\alpha \mapsto \Psi(\alpha, \cdot)$. The ante dual basis of an homogeneous basis $(x_k)_{k=1}^n$ of $\text{Ch}(X_E)$ with respect of Ψ is the basis $(x_k^*)_{k=1}^n$ of $\text{Ch}(X_E)$ such that for any $1 \leq i, j \leq n$, $\Psi(x_i, x_j^*) = \delta_{ij}$, where δ_{ij} is the Kronecker symbol. By definition of the composition in $\text{CM}(F; \Lambda)$, for any other F -varieties Y, Y' and any cycles $(y, y') \in \text{Ch}(Y) \times \text{Ch}(Y')$ we have the formula

$$(x_i \times y) \circ (y' \times x_j^*) = \delta_{ij}(y' \times y). \quad (1)$$

Let π be a non-zero projector in $\text{Ch}_{\dim(X)}(X \times X)$ and $N = (X, \pi)$ the associated motivic direct summand of X . The *base* of N is the set $\mathcal{B}(N) = \{i \in \mathbb{Z}, \text{Ch}_i(N_E) \text{ is not trivial}\}$. The *bottom* of N (denoted $b(N)$) is the least integer of $\mathcal{B}(N)$ and the *top* of N (denoted $t(N)$) is the greatest integer of $\mathcal{B}(N)$. We now introduce the notion of upper and lower motivic direct summands of N .

Definition 3. Let N be a direct summand of the twisted motive of a geometrically split F -variety and M a motivic direct summand of N . We say that

1. M is *upper* in N if $b(M) = b(N)$;
2. M is *lower* in N if $t(M) = t(N)$;
3. M is *outer* in N if M is both lower and upper in N .

Remark 4. Keeping the same F -variety X and any direct summand $N = (X, \pi)$, consider an homogeneous basis $(x_k)_{k=1}^n$ of $\text{Ch}(X_E)$ and its ante dual basis $(x_k^*)_{k=1}^n$. The base, bottom and top of N can be easily determined by the decomposition

$$\pi_E = \sum_{i,j=1}^n \pi_{i,j}(x_i \times x_j^*)$$

noticing that $\mathcal{B}(N) = \{\dim(x_i), \pi_{i,j} \neq 0 \text{ for some } j\}$.

Lemma 1. Let $N = (X, \pi)$ be a motivic direct summand of a geometrically split F -variety and $M = (X, \rho)$ a direct summand of N . Then M is lower in N (resp. upper in N) if and only if the dual motive M^\dagger is upper in N^\dagger (resp. M^\dagger is lower in N^\dagger).

Proof. For any motive O and for any integer i , $\text{Ch}^i(O^\dagger) = \text{Ch}_{-i}(O)$. It follows that $b(O^\dagger) = -t(O)$ and $t(O^\dagger) = -b(O)$. \square

3 Proof of the main theorem

Let \mathcal{C} be a pseudo-abelian category and \mathfrak{C} be the set of the isomorphism classes of objects of \mathcal{C} . We say that the category \mathcal{C} satisfies the *Krull-Schmidt principle* if the monoid (\mathfrak{C}, \oplus) is free. While A. Vishik has shown that the Krull-Schmidt principle holds for the motives of quadrics (even in the case of *integral coefficients*, see [8]) the Krull-Schmidt principle also holds for the motives of geometrically split F -varieties satisfying the nilpotence principle in $\text{CM}(F; \mathbb{F})$ by [5, Corollary 3.3].

Lemma 2. Let (X, π) be a direct summand of an F -variety X . A direct summand (X, ρ) of X is a direct summand of (X, π) if and only if $\pi \circ \rho \circ \pi = \rho$.

Proof. Indeed $\text{End}((X, \pi)) = \pi \circ \text{Ch}_{\dim(X)}(X \times X) \circ \pi$, thus any projector ρ in $\text{End}((X, \pi))$ satisfies $\pi \circ \rho \circ \pi = \rho$. \square

If X is an F -variety, we write $\text{Ch}(\overline{X})$ for the colimit of all $\text{Ch}(X_K)$, where K runs through all field extensions K/F . For any field extension L/F , we will say that an element lying in the image of the natural morphism of $\text{Ch}(X_L) \rightarrow \text{Ch}(\overline{X})$ is L -rational. The image of any correspondence α in $\text{Ch}(\overline{X})$ will be denoted $\overline{\alpha}$. The following lemma is the key-ingredient in the proof of theorem 1.

Lemma 3. Let N be a motivic direct summand of a geometrically split, geometrically irreducible F -variety X satisfying the nilpotence principle and M a twisted direct summand of an F -variety Y . Assume the existence of a field extension E/F such that

1. any $E(X)$ -rational cycle in $\text{Ch}(\overline{X \times Y})$ is $F(X)$ -rational;
2. there are two correspondences $h : N_E \rightsquigarrow M_E$ and $k : M_E \rightsquigarrow N_E$ such that $(X_E, k \circ h)$ is an lower direct summand of N_E .

Then there are two correspondences $f : N \rightsquigarrow M$ and $g : M_E \rightsquigarrow N_E$ such that $(X_E, g \circ f_E)$ is a direct summand of N_E which contains any lower indecomposable direct summand of $(X_E, k \circ h)$. Furthermore if k is F -rational, then g is also F -rational.

Proof. Setting $M=(Y, \rho)[i]$ and $N=(X, \pi)$, we construct explicitly the two correspondences f and g . Since $E(X)$ is a field extension of E , \overline{h} is $E(X)$ -rational, hence $F(X)$ -rational by assumption 1. The morphisms $\text{Spec}(F(X_L)) \rightarrow X_L$ for every field extension L/F induce the pull-back $\varepsilon^* : \text{Ch}(\overline{X \times Y \times X}) \rightarrow \text{Ch}(\overline{(X \times Y)_{F(X)}}$. Furthermore ε^* induces a surjection of F -rational cycles onto $F(X)$ -rational cycles by [3, Corollary 57.11] and we can consider a cycle $h_1 \in \text{Ch}(X \times Y \times X)$ such that $\varepsilon^*(\overline{h_1}) = \overline{h}$. Since ε^* maps any homogeneous cycle $\sum_i \alpha_i \times \beta_i \times 1$ to $\sum_i \alpha_i \times \beta_i$ and vanishes on homogeneous cycles whose codimension on the third factor is strictly positive, we have $\overline{h_1} = \overline{h} \times 1 + \dots$ where " \dots " is a linear combination of homogeneous cycles in $\text{Ch}(\overline{X \times Y \times X})$ with strictly positive codimension on the third factor.

We now look at h_1 as a correspondence $X \rightsquigarrow Y \times X$ and consider the cycle $h_2 = h_1 \circ \pi$. By formula 1 we have

$$\overline{h_2} = (\overline{h} \times 1) \circ \overline{\pi} + \dots,$$

where "..." is a linear combination of homogeneous cycles in $\text{Ch}(\overline{X \times Y \times X})$ with dimension lesser than $t(N)$ on the first factor and strictly positive codimension on the third factor. Finally considering the pull-back of the morphism $\Delta : X \times Y \rightarrow X \times Y \times X$ induced by the diagonal embedding X and setting $h_3 = \Delta^*(h_2)$, we have

$$\overline{h_3} = \overline{h} \circ \overline{\pi} + \dots$$

where "..." stands for a linear combination of homogeneous cycles in $\text{Ch}(\overline{X \times Y})$ with dimension strictly lesser than $t(N)$ on the first factor.

By formula 1 and since $k \circ h$ is a projector, for any positive integer n we have

$$(\overline{\pi} \circ \overline{k} \circ \overline{h_3} \circ \overline{\pi})^n = \overline{k} \circ \overline{h} + \dots$$

where "..." is a linear combination of homogeneous cycles in $\text{Ch}(\overline{X \times X})$ with dimension on the first factor strictly lesser than $t(N)$. The direct summand $k \circ h$ is lower, therefore all these correspondences are non-zero, and an appropriate power $(\overline{\pi} \circ \overline{k} \circ \overline{h_3} \circ \overline{\pi})^{n_0}$ is a non-trivial projector by [5, Corollary 3.2]. This implies (see [3, Exercise 92.6]) that for another integer n_1 , the correspondence $(\pi_E \circ k \circ (h_3)_E \circ \pi_E)^{n_1}$ is a projector. Setting $f = h_3 \circ \pi$ and $g = (\pi_E \circ k \circ (h_3)_E \circ \pi_E)^{n_1-1} \circ \pi_E \circ k$, we see that \overline{g} is F -rational if \overline{k} is F -rational. The correspondence $g \circ f_E$ is a projector which defines a direct summand of N_E by lemma 2. Furthermore any lower indecomposable direct summand of $(X_E, k \circ h)$ must lie in $(X, g \circ f_E)$, since $\overline{g} \circ \overline{f}$ contains all factors $x_i \times x_j^*$, where $\dim(x_i) = t((X_E, k \circ h))$. \square

Proof of Theorem 1. Let $O=(X_E, \kappa)$ be an outer indecomposable direct summand of N_E which is also a direct summand of M_E . We prove theorem 1 by applying lemma 3 once, then the duality functor and finally lemma 3 another time to get all our correspondences defined over the base field F .

Since O is a direct summand of M_E , there are two correspondences $h : N_E \rightsquigarrow M_E$ and $k : M_E \rightsquigarrow N_E$ such that $k \circ h = \kappa$. Moreover O is lower in N_E , so lemma 3 justifies the existence of two other correspondences $h' : N \rightsquigarrow M$ and $k' : M_E \rightsquigarrow N_E$ such that $O_2 = (X_E, k' \circ h'_E)$ is a direct summand of N_E , and the motive O_2 is outer in N_E since it contains O . The dual motive $O_2^\dagger = (X_E, {}^t h'_E \circ {}^t k')[-\dim(X)]$ is therefore outer in N_E^\dagger by lemma 1 and is a direct summand of the dual motive M_E^\dagger . Twisting these three motives by $\dim(X)$, we can apply lemma 3 again. The correspondence ${}^t \overline{h'}$ is F -rational, so lemma 3 gives two correspondences $f : N^\dagger \rightsquigarrow M^\dagger$ and $g : M^\dagger \rightsquigarrow N^\dagger$ such that the motive $(X_E, g_E \circ f_E)$ is both an outer direct summand of N^\dagger (since it contains the dual motive O^\dagger) and a direct summand of M^\dagger . Transposing again, the motive $(X, {}^t f \circ {}^t g)$ is an outer direct summand of N and a direct summand of M . \square

4 Applications

As shown by Chernousov, Gille and Merkurjev (see [1] and [2]) any projective homogeneous variety under the action of a semisimple affine algebraic group is geometrically split and satisfies the nilpotence principle. The notions of *upper* and *lower* direct summands

were introduced by Karpenko in the case of integral F -varieties (see [6, Section 2b]) to study motivic decompositions of such projective homogeneous varieties.

The theory of upper motives and the study of motivic decompositions of projective homogeneous varieties under a semisimple affine algebraic group G already have important applications. For example [6, Theorem 4.3] shows that if D is a p -primary central division F -algebra, any generalized Severi-Brauer variety $\text{SB}(p^k, D)$ has an indecomposable outer direct summand in $\text{CM}(F; \mathbb{F}_p)$. In particular those varieties are incompressible, and the computation of the canonical dimension of any variety of flags of right ideals in a central simple F -algebra follows ([6, Corollary 4.4]).

Theorem 1 allows to extract some motivic direct summands which appear on some field extension E/F to the base field. Theorem 1 generalizes [5, Proposition 4.6], replacing the whole motive of the variety X by an arbitrary direct summand. Replacing X by a direct summand requires to construct explicitly the rational cycles to get an outer direct summand defined over F , and theorem 1 thus gives a new proof of [5, Proposition 4.6]. Note that the assumption of theorem 1 on the field extension $E(X)/F(X)$ holds if the field extension is unirational.

The particular case [5, Proposition 4.6] has important applications, and is one of the main ingredients in the proof of [5, Theorem 1.1]. Finally theorem 1 was applied by Garibaldi, Petrov and Semenov in [4] to study motivic decompositions of projective homogeneous varieties under the action of a semisimple affine algebraic group G of exceptional type, and answer a question of Rost and Springer (see [4, Proposition 9.11, Proposition 9.17]).

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