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A going-down theorem for Chow-Grothendieck motives

Charles De Clercq

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Abstract

Let $(M(X), p)$ be a direct summand of the motive associated with a geometrically split, geometrically irreducible variety over a field F satisfying the nilpotence principle. We show that under some conditions, if $(M(X_E), p_E)$ is a direct summand of another motive M_E over a field extension E , then $(M(X), p)$ is a direct summand of M over F .

I Introduction

Throughout this note, Λ will always be a finite field. Given a field F , an F -variety will be understood as a separated scheme of finite type over F . Given such Λ and an F -variety X , we can consider $CH_i(X; \Lambda)$, the *Chow group* of i -dimensional cycles on X with coefficients in Λ , defined by $CH_i(X) \otimes_{\mathbb{Z}} \Lambda$. These groups are the first step in the construction of the category $CM(F; \Lambda)$ of *Chow-Grothendieck motives* with coefficients in Λ . This category is constructed as the *pseudo-abelian envelope* of the category $CR(F; \Lambda)$ of *correspondences* with coefficients in Λ . Our main reference for the construction and the main properties of these categories is [2].

Definition I.1. *Let X be an F -variety. A field extension E/F is a splitting field of X if the E -motive X_E is isomorphic to a finite direct sum of twisted Tate motives.*

Following [2], we will write $CH(\overline{X}; \Lambda)$ for the colimit of all $CH(X_E; \Lambda)$, where E runs through all field extensions E/F . We will say that an F -variety X is *geometrically split* if X splits over the algebraic closure of F .

For any field extension E/F , we will say that any element of $CH(\overline{X}; \Lambda)$ lying in the image of the restriction morphism $CH(X_E; \Lambda) \rightarrow CH(\overline{X}; \Lambda)$ is E -rational.

The purpose of this note is to generalize the following result, proved by N. Karpenko ([3, Proposition 4.6]).

Theorem I.2. *Let X be a geometrically irreducible, geometrically split variety satisfying the nilpotence principle and M be a motive. Assume that there exists a field extension E/F such that*

1. *the field extension $E(X)/F(X)$ is purely transcendental;*
2. *the upper indecomposable summand of $M(X)_E$ is also lower and is a summand of M_E .*

Then the upper indecomposable summand of $M(X)$ is a summand of M .

The class of geometrically split, geometrically irreducible F -varieties satisfying the nilpotence principle is quite large (see [2], [1], [4]). N. Karpenko uses this result for instance to prove several motivic decompositions in $CM(F; \Lambda)$ ([3, lemma 7.1]). Here we generalize theorem I.2 by replacing the motive $M(X)$ by the direct summand $(M(X), p)$ associated with a projector $p \in CH_{\dim(X)}(X \times X; \Lambda)$.

I would like to thank Nikita Karpenko, my Ph.D. thesis adviser, for raising this question and guiding me during this work.

II The Chow group of a geometrically split F -variety

Let X be a geometrically split F -variety. Let us consider the motivic decomposition

$$M(X_E) \simeq \bigoplus_{i=1}^m \Lambda(a_i)$$

where m and a_i are integers, over a splitting field E of X .

By definition of morphisms in $CM(E; \Lambda)$, $Hom_{CM(E; \Lambda)}(\Lambda(i), \Lambda(j)) \simeq \delta_{ij} \Lambda$, where δ_{ij} stands for the Kronecker symbol. Therefore there is an integer n called the *rank* of X such that

$$CH(X_E; \Lambda) \simeq \bigoplus_{i=1}^n \Lambda.$$

Moreover, the isomorphism between $M(X_E)$ and the previous direct sum of twisted Tate motives defines an homogeneous basis $(x_k)_{k=1}^n$ of the Λ -module $CH(X_E; \Lambda)$.

Proposition II.1. *Let X be a geometrically split F -variety. then the pairing*

$$\Psi : \begin{array}{ccc} CH(\overline{X}; \Lambda) \times CH(\overline{X}; \Lambda) & \longrightarrow & \Lambda \\ (\alpha, \beta) & \longmapsto & \deg(\alpha \cdot \beta) \end{array}$$

is bilinear, symmetric and non-degenerate.

Proof. c.f. [5, Remark 5.6] □

Since the bilinear form Ψ is non-degenerate, it gives rise to an isomorphism of Λ -modules between $CH(\overline{X}; \Lambda)$ and its dual space $Hom_{\Lambda}(CH(\overline{X}; \Lambda), \Lambda)$ given by

$$f : \begin{array}{ccc} CH(\overline{X}; \Lambda) & \longrightarrow & Hom_{\Lambda}(CH(\overline{X}; \Lambda), \Lambda) \\ u & \longmapsto & \Psi(u, \cdot) \end{array}$$

Now consider the *dual basis* $(x'_i)_{i=0}^n$ of the basis $(x_i)_{i=0}^n$. We define the *antedual* basis of $(x_i)_{i=0}^n$ by $(x_i^*)_{i=0}^n$, where $x_i^* := f^{-1}(x'_i)$. By definition of the antedual basis, we have $\Psi(x_i, x_j^*) = \delta_{ij}$, where δ_{ij} is the Kronecker symbol.

Proposition II.2. *Let M and N be two motives in $CM(F; \Lambda)$, with M split. Then there is an isomorphism of Λ -modules*

$$CH^*(M; \Lambda) \otimes CH^*(N; \Lambda) \longrightarrow CH^*(M \otimes N; \Lambda)$$

Proof. c.f. [2, Proposition 64.3]. □

Computations become much simpler on split varieties.

Lemma II.3. *Let X be a split variety of rank n and two smooth complete varieties Y and Y' . Consider the homogeneous basis $(x_k)_{k=1}^n$ defined previously and its antedual basis $(x_k^*)_{k=1}^n$. Then for all cycles $y \in CH(Y; \Lambda)$, $y' \in CH(Y'; \Lambda)$ and for all $1 \leq i, j \leq n$,*

$$(x_i \times y) \circ (y' \times x_j^*) = \delta_{ij}(y' \times y).$$

Proof. We set 1 for the identity class, either in $CH(Y; \Lambda)$ or in $CH(Y'; \Lambda)$.

$$\begin{aligned} (x_i \times y) \circ (y' \times x_j^*) &= (p_X^Y)_* \left((p_Y^{X \times Y})^*(y' \times x_j^*) \cdot (p_{Y'}^{X \times Y})^*(x_i \times y) \right) \\ &= (p_X^Y)_* \left((y' \times x_j^* \times 1) \cdot (1 \times x_i \times y) \right) \\ &= (p_X^Y)_* (y' \times (x_j^* \cdot x_i) \times y) \\ &= \deg(x_j^* \cdot x_i)(y' \times y) \\ &= \delta_{ij}(y' \times y) \end{aligned}$$

□

III Direct summands of geometrically split F -varieties

Let X be an F -variety and $p \in CH_{\dim(X)}(X \times X; \Lambda)$ be the projector defining the direct summand (X, p) . A direct summand of the motive (X, p) is given by a projector in $End_{CM(F; \Lambda)}((X, p), (X, p))$, by definition of the category $CM(F; \Lambda)$.

Lemma III.1. *Let X be an F -variety, and $\bar{p} \in CH_{\dim(X)}(X \times X; \Lambda)$ the projector defining the direct summand (X, p) . Then every direct summand of (X, p) is a summand (X, q) associated with a projector $q \in CH_{\dim(X)}(X \times X; \Lambda)$ satisfying $p \circ q \circ p = q$.*

Proof. Indeed we have $End_{CM(F; \Lambda)}((X, p), (X, p)) = p \circ CH_{\dim(X)}(X \times X; \Lambda) \circ p$, thus for any projector $q \in End_{CM(F; \Lambda)}((X, p), (X, p))$, there is a cycle $s \in CH_{\dim(X)}(X \times X; \Lambda)$ such that $q = p \circ s \circ p$. But p is itself a projector, therefore $p \circ q \circ p = p^2 \circ s \circ p^2 = p \circ s \circ p = q$. \square

We now study the notion of *upper* and *lower* direct summands of a direct summand $(M(X), p)$. From now on, we consider a geometrically split F -variety X of rank n . We keep the notation $(x_k)_{k=1}^n$ for the homogeneous basis of $CH(\bar{X}; \Lambda)$ and $(x_k^*)_{k=1}^n$ for the *antedual* basis of $(x_k)_{k=1}^n$ associated with the pairing Ψ .

Definition III.2. *Let X a geometrically split F -variety and $p \in CH_{\dim(X)}(X \times X; \Lambda)$ be a non-zero projector. We denote by \bar{p} the image of p under the restriction map to a splitting field E of X . Considering the bases $(x_i)_{i=1}^n$ and $(x_i^*)_{i=1}^n$ of $CH(X_E; \Lambda)$, we have*

$$\bar{p} = \sum_{i=1}^n \sum_{j=1}^n p_{ij} (x_i \times x_j^*).$$

We define the lowest codimension of p by

$$cdmin(p) := \min_{p_{ij} \neq 0} (codim(x_i))$$

where $codim(x_i)$ stands for the codimension in $CH(X; \Lambda)$.

As X is geometrically split, the summand (X, p) associated with the projector $p \in CH_{\dim(X)}(X \times X; \Lambda)$ splits over a field extension E/F :

$$(X_E, p_E) \simeq \bigoplus_{i=1}^m (\Lambda(a_i))^{m_i} \tag{III.1}$$

where a_i, m_i and m are integers, and we set $i < j \Rightarrow a_i < a_j$.

Definition III.3. *Let X be a geometrically split F -variety, and (X, p) be a summand of X . Let also (X, q) be a direct summand of the motive (X, p) . Consider the decomposition III.1 of (X, p) in a splitting field E of X . The summand (X, q) is :*

1. upper in (X, p) if $CH^{a_1}((X_E, q_E)) \neq 0$.
2. lower in (X, p) if $CH_{a_m}((X_E, q_E)) \neq 0$.

By definition, a direct summand (X, q) is upper in (X, p) if and only if $(X(-\dim(X)), {}^t q)$ is lower in the dual motive $(X(-\dim(X)), {}^t p)$. We can now study the link between the *lowest codimension* of a summand and the position of its motivic decomposition.

Proposition III.4. *Let X be a geometrically split F -variety, and $p \in CH_{\dim(X)}(X \times X; \Lambda)$ the projector defining the summand (X, p) . Consider the motivic decomposition III.1 given above of the motive (X_E, p_E) , where E is a splitting field of X . Then*

$$a_1 = cdmin(p)$$

Proof. Let us show first that $cdmin(p) \geq a_1$. Indeed we consider the motivic decomposition III.1, thus

$$CH^{a_1}((X_E, p_E); \Lambda) \simeq \bigoplus_{i=1}^m Hom_{CM(E; \Lambda)}(\Lambda(a_i), \Lambda(a_1))^{m_i} \simeq \Lambda^{m_1} \neq 0.$$

Consequently $CH^{a_1}(X_E, \Lambda) \circ p_E \neq 0$ by definition of Chow groups in $CM(E; \Lambda)$.

Let $x \in CH^{a_1}(X_E; \Lambda)$ such that $x \circ p_E \neq 0$. Since $(x_i)_{i=1}^n$ is homogeneous, we can consider the base $(x_k)_{k \in K}$ of the Λ -module $CH^{a_1}(X_E; \Lambda)$ and write $x = \sum_{k \in K} \lambda_k x_k$. Then

$$\begin{aligned} x \circ p_E \neq 0 &\Rightarrow \left(\sum_{k \in K} \lambda_k x_k \right) \circ \left(\sum_{i=1}^n \sum_{j=1}^n p_{ij}(x_i \times x_j^*) \right) \neq 0 \\ &\Rightarrow \sum_{i=1}^n \sum_{k \in K} \lambda_k p_{ik} x_i \neq 0 \\ &\Rightarrow \exists (i_0, k_0) \in \{1, \dots, n\} \times K, \quad p_{i_0 k_0} \neq 0 \end{aligned}$$

Then since $codim(x_{i_0}) = \dim(X_E) - codim(x_{k_0}^*) = codim(x_{k_0}) = a_1$ we have proven that $cdmin(p) \leq a_1$.

Suppose on the other hand that $cdmin(p) = s < a_1$. Then there are two index i_0, j_0 in $\{1, \dots, n\}$ such that $p_{i_0 j_0} \neq 0$, and $codim(x_{i_0}) = s$. Then $x_{j_0} \circ p_E = \sum_{i=0}^n p_{ij_0} x_i \neq 0$, therefore $CH^s(X_E; \Lambda) \circ p_E \neq 0$, as $codim(x_{i_0}) = codim(x_{j_0})$. But this contradicts the fact that

$$\begin{aligned} CH^s(X_E; \Lambda) \circ p_E &= Hom_{CM(E; \Lambda)}((X_E, p_E); \Lambda(s)) \\ &= \bigoplus_{i=1}^m Hom_{CM(E; \Lambda)}(\Lambda(a_i); \Lambda(s))^{m_i} \\ &= 0 \end{aligned}$$

Therefore $cdmin(p) = a_1$. □

Proposition III.5. *Let X be a geometrically split F -variety, and p the projector defining the summand $(M(X), p)$. Let also be $(M(X), q)$ a direct summand of $(M(X), p)$. Then*

$$cdmin(p) = cdmin(q) \Leftrightarrow (X, q) \text{ is upper in } (X, p).$$

Proof. We keep the same notations as in proposition III.4.

As q defines a direct summand of (X, p) we have $q = p \circ q \circ p$ by lemma III.1, thus for any k

$$\begin{aligned} CH^k((X_E, q_E); \Lambda) &= CH^k(X_E; \Lambda) \circ q_E \\ &= CH^k(X_E; \Lambda) \circ (p_E \circ q_E \circ p_E) \end{aligned}$$

But $q_E \circ p_E = (p_E \circ q_E \circ p_E) \circ p_E = p_E \circ q_E \circ p_E = q_E$, thus

$$CH^k((X_E, q_E); \Lambda) = (CH^k(X_E; \Lambda) \circ p_E) \circ q_E = CH^k((X_E, p_E); \Lambda) \circ q_E. (*)$$

Therefore for any k such that $CH^k((X_E, p_E); \Lambda) = 0$, we have shown that $CH^k((X_E, q_E); \Lambda) = 0$.

Now write the decomposition of (X_E, q_E) :

$$(X_E, q_E) \simeq \bigoplus_{j=1}^s (\Lambda(b_j))^{s_j}$$

with $i < j \Rightarrow b_i < b_j$. Equality (*) implies that for all $j \in \{1, \dots, s\}$, $b_j \geq a_1$. But proposition III.4 implies that $b_1 = cdmin(q)$, thus

$$\begin{aligned} cdmin(p) = cdmin(q) &\Leftrightarrow b_1 = a_1 \\ &\Leftrightarrow CH^{a_1}((X_E, q_E); \Lambda) \neq 0 \end{aligned}$$

□

IV Proof of the main theorem

Following [3], we recall some results on the category generated by geometrically split F -varieties satisfying the nilpotence principle in $CM(F; \Lambda)$. We will say that a pseudo-abelian category satisfies the *Krull-Schmidt principle* if every object of this category decomposes in a unique way as a direct sum of indecomposable objects (up to permutation of these indecomposable summands).

Lemma IV.1. ([3, Corollary 3.2]) *Let X be a geometrically split F -variety satisfying the nilpotence principle. Then an appropriate power of any endomorphism of the motive X is a projector.*

Proposition IV.2. ([3, Corollary 3.3]) *The Krull-Schmidt principle holds for the pseudo-abelian Tate subcategory of $CM(F, \Lambda)$ generated by the motives of geometrically split F -varieties satisfying the nilpotence principle.*

Remark IV.3. *More generally, proposition IV.2 is true when Λ is any finite commutative ring (see [3, Corollary 3.2]). The Krull-Schmidt principle also holds for the category generated by motives of quadrics when the ring of coefficients Λ is \mathbb{Z} . Nevertheless, the Krull-Schmidt principle does not hold in the category generated by the motives of projective homogeneous varieties when $\Lambda = \mathbb{Z}$ (see [6]) and this is one of the reasons why we suppose that Λ is a finite field.*

Let X be a geometrically split F -variety. Let $p \in CH_{\dim(X)}(X \times X; \Lambda)$ be the projector defining the direct summand (X, p) of X . As seen before, the homogeneous base $(x_k)_{k=1}^n$ and its antidual base $(x_k^*)_{k=1}^n$ relatively to the pairing Ψ allow us to decompose p in a splitting field E of the motive X , that is to say $p_E = \sum_{i=1}^n \sum_{j=1}^n p_{ij}(x_i \times x_j^*)$.

Definition IV.4. *Let X be a geometrically split F -variety and E a splitting field of the motive X . Let p be the projector defining the direct summand (X, p) . We say that couple $(i, j) \in \{1, \dots, n\}^2$ is contained in p_E if $p_{ij} \neq 0$. We will also say that an element $x_i \times x_j^*$ of the base of $CH_{\dim(X_E)}(X_E \times X_E; \Lambda)$ is contained in p_E if the couple (i, j) is contained in p_E .*

Notation IV.5. *The set of all elements $x_i \times x_j^*$ of $CH_{\dim(X_E)}(X_E \times X_E; \Lambda)$ which are contained in the projector p_E will be called the base of p_E , and denoted $\mathcal{B}(p_E)$.*

Definition IV.6. *Let X be a geometrically split F -variety and two projectors p and q defining the two direct summand (X, p) and (X, q) . Let E be a splitting field of X . We say that p contains q if $\mathcal{B}(q_E) \subset \mathcal{B}(p_E)$. We will say that q intersects p if $\mathcal{B}(q_E) \cap \mathcal{B}(p_E) \neq \emptyset$.*

Remark IV.7. *Given a geometrically split F -variety X and two direct summands (X, p) and (X, q) of the motive X defined by the two projectors p and q , we will also say that the summand (X, p) contains the summand (X, q) if the projector p contains the projector q .*

Since Λ is a finite field, we have seen that the category generated by motives of geometrically split F -varieties satisfying the nilpotence principle satisfies the Krull-Schmidt principle (proposition IV.2). By definition of the category $CM(F; \Lambda)$, direct summands of the motive of an F -variety correspond to projectors in $End_{CM(F; \Lambda)}(X)$, thus in a splitting field E of the motive X , every F -rational cycle in $CH_{\dim(X)}(\overline{X} \times \overline{X}; \Lambda)$ is a linear combination of minimal F -rational cycles, corresponding to indecomposable direct summands of X . We now prove the key-lemma.

Lemma IV.8. *Let X be a geometrically split, geometrically irreducible F -variety satisfying the nilpotence principle. Let p be the projector defining the direct summand $N := (X, p)$ of X . Let also Y be a smooth complete F -variety, and $M := (Y, q)$ the direct summand of Y induced by the projector q . Assume that there is a field extension E/F and two E -rational correspondences $h : X_E \rightarrow M_E$ and $k : M_E \rightarrow X_E$ such that*

1. *every $E(X)$ -rational cycle in $CH(\overline{X} \times \overline{Y}; \Lambda)$ is also $F(X)$ -rational;*
2. *$k \circ h \in CH_{\dim(X_E)}(X_E \times X_E; \Lambda)$ defines an upper direct summand $(X_E, k \circ h)$ of N_E .*

Then there exists an F -rational cycle h' and a E -rational cycle k' such that

- $k' \circ h'_E$ is a non-zero projector in $CH_{\dim(X_E)}(X_E \times X_E; \Lambda)$;
- the E -motive $(X_E, k' \circ h'_E)$ contains every upper indecomposable summand of $(X_E, k \circ h)$;
- the cycle k' is F -rational if k is F -rational.

Proof. We denote by s the composite $k \circ h$. By assumption, s is a projector in $CH_{\dim(X_E)}(X_E \times X_E; \Lambda)$ satisfying $s = p_E \circ s \circ p_E$ by lemma III.1. Moreover we suppose that $(M(X), s)$ is upper in $(M(X), p_E)$, that is to say $cdmin(s) = cdmin(p)$ by proposition III.5.

Let us decompose the images of the cycles h , k , s and p in a splitting field of the F -variety X of rank n :

1. $\bar{h} = \sum_{i=1}^n h_i(x_i \times y_i)$
2. $\bar{k} = \sum_{j=1}^n k_j(y'_j \times x_j^*)$
3. $\bar{s} = \sum_{i=1}^n \sum_{j=1}^n s_{ij}(x_i \times x_j^*)$
4. $\bar{p} = \sum_{i=1}^n \sum_{j=1}^n p_{ij}(x_i \times x_j^*)$

with $s_{ij} = h_i k_j \deg(y'_j \cdot y_i)$ by definition of s .

Since $E(X)$ is a field extension of E , the cycle \bar{h} is $E(X)$ -rational, hence $F(X)$ -rational by assumption 1).

Now consider the morphism $Spec(F(X)) \longrightarrow \bar{X}$ induced by the generic point of the geometrically irreducible variety X . It induces a morphism

$$\epsilon : (\bar{X} \times \bar{Y})_{F(X)} \longrightarrow \bar{X} \times \bar{Y} \times \bar{X}$$

whose pull-back $\epsilon^* : CH_{\dim(X)}(\bar{X} \times \bar{Y} \times \bar{X}; \Lambda) \longrightarrow CH_{\dim(X)}(\bar{X} \times \bar{Y}; \Lambda)$ maps every cycles of the form $\sum_i \lambda_i(\alpha_i \times \beta_i \times 1)$ on $\sum_i \lambda_i(\alpha_i \times \beta_i)$ and vanishes on elements of the form $\sum_i \lambda_i(\alpha_i \times \beta_i \times \gamma_i)$, where $codim(\gamma_i) > 0$.

Moreover, the pull-back ϵ^* induces a surjection of F -rational cycles onto $F(X)$ -rational cycles ([2, corollary 57.11]). Therefore we can choose an F -rational cycle $\bar{h}_1 \in CH_{\dim(X)}(\bar{X} \times \bar{Y} \times \bar{X}; \Lambda)$ such that $\epsilon^*(\bar{h}_1) = \bar{h}$.

By definition of ϵ^* we have

$$\bar{h}_1 = \sum_{i=1}^n h_i(x_i \times y_i \times 1) + \sum(\alpha \times \beta \times \gamma)$$

where cycles γ are of strictly positive codimension.

Now let us look at the cycle \bar{h}_1 as a correspondence $\bar{h}_1 : \bar{X} \longrightarrow \bar{Y} \times \bar{X}$. Setting $\bar{h}_2 := \bar{h}_1 \circ \bar{p}$, \bar{h}_2 is F -rational and its image in a splitting field of X is

$$\begin{aligned} \bar{h}_2 &= \left(\sum_{i=1}^n h_i(x_i \times y_i \times 1) \right) \circ p + \left(\sum(\alpha \times \beta \times \gamma) \right) \circ p \\ &= \sum_{i=1}^n \sum_{j=1}^n h_j p_{ij}(x_i \times y_j \times 1) + \sum \tilde{\alpha} \times \tilde{\beta} \times \tilde{\gamma} \end{aligned}$$

where cycles $\tilde{\gamma}$ are of strictly positive codimension and cycles $\tilde{\alpha}$ satisfies $codim(\tilde{\alpha}) \geq cdmin(p)$.

Consider now the diagonal embedding

$$\Delta : \begin{array}{ccc} \bar{X} \times \bar{Y} & \longrightarrow & \bar{X} \times \bar{Y} \times \bar{X} \\ (x, y) & \longmapsto & (x, y, x) \end{array} .$$

The morphism Δ induces a pull-back $\Delta^* : CH_{\dim(X)}(\bar{X} \times \bar{Y} \times \bar{X}; \Lambda) \longrightarrow CH_{\dim(X)}(\bar{X} \times \bar{Y}; \Lambda)$. Setting $\bar{h}_3 := \Delta^*(\bar{h}_2)$, \bar{h}_3 is an F -rational cycle and

$$\bar{h}_3 = \sum_{i=1}^n \sum_{j=1}^n h_j p_{ij}(x_i \times y_j) + \sum(\tilde{\alpha} \cdot \tilde{\gamma}) \times \tilde{\beta}$$

where $\text{codim}(\tilde{\alpha} \cdot \tilde{\gamma}) > \text{cdmin}(p)$ since $\text{codim}(\tilde{\alpha}) \geq \text{cdmin}(p)$ and $\text{codim}(\tilde{\gamma}) > 0$.

The next step consists of computing the composite $\bar{k} \circ \bar{h}_3$:

$$\begin{aligned} \bar{k} \circ \bar{h}_3 &= \left(\sum_{l=1}^n k_l (y'_l \times x_l^*) \right) \circ \left(\sum_{i=1}^n \sum_{j=1}^n h_j p_{ij} (x_i \times y_j) \right) + \left(\sum_{l=1}^n k_l (y'_l \times x_l^*) \right) \circ \left(\sum (\tilde{\alpha} \cdot \tilde{\gamma}) \times \tilde{\beta} \right) \\ &= \sum_{i=1}^n \sum_{l=1}^n \left(\sum_{j=1}^n k_l h_j p_{ij} (y'_l \times x_l^*) \circ (x_i \times y_j) \right) + \sum \bar{\alpha} \times \bar{\beta} \\ &= \sum_{i=1}^n \sum_{l=1}^n \left(\sum_{j=1}^n k_l h_j p_{ij} \deg(y'_l \cdot y_j) \right) (x_i \times x_l^*) + \sum \bar{\alpha} \times \bar{\beta} \end{aligned}$$

Where cycles $\bar{\alpha}$ satisfy $\text{codim}(\bar{\alpha}) > \text{cdmin}(p) = \text{cdmin}(s)$.

Let us show that for any $(i, l) \in \{1, \dots, n\}^2$, $\sum_{j=1}^n k_l h_j p_{ij} \deg(y'_l \cdot y_j) = s_{il}$. First s defines a direct summand of (X, p_E) , thus by lemma III.1 $s \circ p_E = p_E \circ s \circ p_E = p_E \circ s \circ p_E = s$. Now compute $\bar{s} \circ \bar{p}$:

$$\begin{aligned} \bar{s} \circ \bar{p} &= \left(\sum_{i=1}^n \sum_{j=1}^n s_{ij} (x_i \times x_j^*) \right) \circ \left(\sum_{l=1}^n \sum_{t=1}^n p_{lt} (x_l \times x_t^*) \right) \\ &= \sum_{j=1}^n \sum_{l=1}^n \sum_{i=1}^n \sum_{t=1}^n s_{ij} p_{lt} (x_i \times x_j^*) \circ (x_l \times x_t^*) \\ &= \sum_{j=1}^n \sum_{l=1}^n \left(\sum_{i=1}^n s_{ij} p_{li} \right) (x_l \times x_j^*) \end{aligned}$$

Therefore for any (i, l) with $\text{codim}(x_i) = \text{cdmin}(s)$, the component of $\bar{k} \circ \bar{h}_3$ over $x_i \times x_l^*$ is equal to $\sum_{j=1}^n k_l h_j p_{ij} \deg(y'_l \cdot y_j)$ and we have

$$\begin{aligned} \sum_{j=1}^n k_l h_j p_{ij} \deg(y'_l \cdot y_j) &= \sum_{j=1}^n s_{jl} p_{ij} \\ &= s_{il}. \end{aligned}$$

We thus have obtained

$$(\bar{k} \circ \bar{h}_3) = \sum_{\text{codim}(x_i) = \text{cdmin}(p)} \sum_{j=1}^n s_{ij} (x_i \times x_j^*) + \sum \bar{\alpha} \times \bar{\beta}$$

where cycles $\bar{\alpha}$ satisfy $\text{codim}(\bar{\alpha}) > \text{cdmin}(p) = \text{cdmin}(s)$.

As Λ is finite, there exists an integer n_0 such that $(g \circ (f_3)_E)^{n_0}$ is a projector by lemma IV.1. Setting $h' = \bar{h}_3$ and $k' = (\bar{k} \circ \bar{h}_3)^{n_0-1} \circ \bar{k}$, we see that

$$(\bar{k}' \circ \bar{h}') = \sum_{\text{codim}(x_i) = \text{cdmin}(p)} \sum_{j=1}^n s_{ij} (x_i \times x_j^*) + \sum \hat{\alpha} \times \hat{\beta}$$

where $\text{codim}(\hat{\alpha}) > \text{cdmin}(p)$, therefore $k' \circ h'_E$ is non-zero.

If k is F -rational, then $k' = (\bar{k} \circ \bar{h}_3)^{n_0-1} \circ k$ is also F -rational. Moreover, the cycle $k' \circ h'$ intersects every projector $w \in CH(\bar{X} \times \bar{X}; \Lambda)$ satisfying $\text{cdmin}(w) = \text{cdmin}(k \circ h)$, and $\mathcal{B}(w) \subset \mathcal{B}(k \circ h)$. In particular $k' \circ h'$ intersects every upper direct summand of the motive $(X_E, k \circ h)$, and therefore contains every indecomposable direct summand of the motive $(X_E, k \circ h)$. \square

The main theorem is a direct consequence of lemma IV.8.

Theorem IV.9. *Let X be a geometrically irreducible, geometrically split F -variety satisfying the nilpotence principle. Consider the projector p defining the indecomposable direct summand $N := (X, p)$ of the motive X . Let also Y be a smooth complete F -variety and $M := (Y, q)$ the direct summand induced by a projector q . Assume the existence of a field extension E/F such that*

1. every $E(X)$ -rational cycle in $CH(\overline{X} \times \overline{Y}; \Lambda)$ is also $F(X)$ -rational;
2. the E -motive N_E contains an indecomposable direct summand (X_E, s) both lower and upper and is a direct summand of M_E .

Then the motive N is a direct summand of the motive M .

Proof. By assumption, the E -motive N_E is a direct summand of M_E , that is to say there are two correspondences $f \in CH_{\dim(X_E)}(X_E \times Y_E; \Lambda)$ and $g \in CH_{\dim(Y_E)}(Y_E \times X_E; \Lambda)$ such that $g \circ f = p_E$. Let us apply lemma IV.8.

Lemma IV.8 implies that there are two correspondences $g' \in CH_{\dim(Y_E)}(Y_E \times X_E; \Lambda)$ and $f' \in CH_{\dim(X_E)}(X_E \times Y_E; \Lambda)$, with $\overline{f'}$ F -rational, such that $g' \circ f'_E$ is a non-zero projector and the motive $(X_E, g' \circ f'_E)$ contains every upper indecomposable summand of (X_E, p_E) .

Now consider the cycle $p_E \circ g' \circ f'_E \circ p_E$. Lemma IV.1 justifies the existence of an integer n_0 such that $(p_E \circ g' \circ f'_E \circ p_E)^{n_0}$ is a projector. Moreover we have the decomposition

$$\overline{g'} \circ \overline{f'} = \sum_{\text{codim}(x_i)=\text{cdmin}(p)} \sum_{j=1}^n p_{ij}(x_i \times x_j^*) + \sum \overline{\alpha} \times \overline{\beta}$$

with $\text{codim}(\overline{\alpha}) > \text{cdmin}(p)$.

Set $\overline{g} := (p_E \circ g' \circ f'_E \circ p_E)^{n_0-1} \circ p_E \circ g'$ and $\overline{f} := f' \circ p$. The cycle $\overline{g} \circ \overline{f}_E$ is a projector satisfying $p_E \circ \overline{g} \circ \overline{f}_E \circ p_E = \overline{g} \circ \overline{f}_E$. Besides in a splitting field of X

$$\overline{g} \circ \overline{f}_E = \sum_{\text{codim}(x_i)=\text{cdmin}(p)} \sum_{j=1}^n p_{ij}(x_i \times x_j^*) + \sum \hat{\alpha} \times \hat{\beta}$$

with $\text{codim}(\hat{\alpha}) > \text{cdmin}(p)$, therefore the motive $(X_E, \overline{g} \circ \overline{f}_E)$ is an upper direct summand of the motive N_E . Besides $(X_E, \overline{g} \circ \overline{f}_E)$ intersects any upper direct summand of N_E , thus contains the indecomposable direct summand (X_E, s) . It follows that the summand $(X_E, \overline{g} \circ \overline{f}_E)$ is both upper and lower in N_E .

Transposing, the summand $(X_E(-\dim(X_E)), {}^t \overline{f}_E \circ {}^t \overline{g})$ is a direct summand of the dual motive N_E^* which is upper, as $(X_E, \overline{g} \circ \overline{f}_E)$ is lower in N_E , therefore we can apply lemma IV.8 again.

Lemma IV.8 justifies the existence of an F -rational cycle \tilde{g} and another cycle \tilde{f} which is also F -rational, such that ${}^t \tilde{f}_E \circ {}^t \tilde{g}_E$ is a non-zero projector in $\text{End}_{CM(E; \Lambda)}(N_E^*)$. Transposing again, the cycle $\tilde{g}_E \circ \tilde{f}_E$ is a non-zero projector in $CH_{\dim(X_E)}(X_E \times X_E; \Lambda)$.

By nilpotence principle, an appropriate power $(\tilde{g} \circ \tilde{f})^{\tilde{n}}$ is a non-zero projector in $CH_{\dim(X)}(X \times X; \Lambda)$. Setting $\mathfrak{g} := (\tilde{g} \circ \tilde{f})^{\tilde{n}-1} \circ \tilde{g}$ and $\mathfrak{f} := \tilde{f}$, the cycle $\mathfrak{g} \circ \mathfrak{f}$ is a non zero-projector. But the motive N is indecomposable, therefore we can conclude that $\mathfrak{g} \circ \mathfrak{f} = p$, that is to say N is a direct summand of M . \square

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