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► **To cite this version:**

| Serguei Barannikov. Supersymmetric matrix integrals and sigma-models.. 2009. hal-00443592

HAL Id: hal-00443592

<https://hal.science/hal-00443592>

Preprint submitted on 30 Dec 2011

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SUPERSYMMETRIC MATRIX INTEGRALS AND σ -MODELS.

SERGUEI BARANNIKOV

Notations. I work in the tensor category of super vector spaces, over an algebraically closed field k , $\text{char}(k) = 0$. Let $V = V_0 \oplus V_1$ be a super vector space. I denote by $\bar{\alpha}$ the parity of an element α and by ΠV the super vector space with inversed parity $(\Pi V)_0 = V_1$, $(\Pi V)_1 = V_0$. For a finite group G acting on a vector space U , I denote via U^G the space of invariants with respect to the action of G . Element $(a_1 \otimes a_2 \otimes \dots \otimes a_n)$ of $A^{\otimes n}$ is denoted by (a_1, a_2, \dots, a_n) . Cyclic words, i.e. elements of the subspace $(V^{\otimes n})^{\mathbb{Z}/n\mathbb{Z}}$ are denoted via $(a_1 \dots a_n)^\lambda$

1. NONCOMMUTATIVE BATALIN-VILKOVISKY GEOMETRY.

1.1. Even inner product. I recall here the basics of the noncommutative Batalin-Vilkovisky geometry, introduced in [B1]. I start with the case of the even inner product. Let $g : V^{\otimes 2} \rightarrow k$ be an even symmetric inner product on the super vector space V :

$$g(x, y) = (-1)^{\bar{x}\bar{y}} g(y, x)$$

The basic space in which I look for the solutions of the noncommutative Batalin-Vilkovisky equation is the space

$$F = \text{Symm}(\oplus_{j=1}^{\infty} \Pi(\Pi V^{\otimes j})^{\mathbb{Z}/j\mathbb{Z}})$$

The space F carries the naturally defined noncommutative Batalin-Vilkovisky differential Δ . It is the operator of the second order with respect to the multiplication of the symmetric algebra and is completely determined by its action on the second power of $\oplus_{j=1}^{\infty} \Pi(\Pi V^{\otimes j})^{\mathbb{Z}/j\mathbb{Z}}$. If one chooses a basis $\{a_i\}$ in ΠV , in which the antisymmetric even inner product defined by g on ΠV has the form $(-1)^{\bar{a}_i \bar{a}_j} g(\Pi a_i, \Pi a_j) = g_{ij}$, then the operator Δ sends a product of two cyclic words $(a_{\rho_1} \dots a_{\rho_r})^c (a_{\tau_1} \dots a_{\tau_t})^c$, to

$$(1.1) \quad \sum_{p,q} (-1)^{\varepsilon_1} g_{\rho_p \tau_q} (a_{\rho_1} \dots a_{\rho_{p-1}} a_{\tau_{q+1}} \dots a_{\tau_{q-1}} a_{\rho_{p+1}} \dots a_{\rho_r})^c +$$

$$+ \sum_{p \pm 1 \neq q \text{ mod } r} (-1)^{\varepsilon_2} g_{\rho_p \rho_q} (a_{\rho_1} \dots a_{\rho_{p-1}} a_{\rho_{q+1}} \dots a_{\rho_r})^c (a_{\rho_{p+1}} \dots a_{\rho_{q-1}})^c (a_{\tau_1} \dots a_{\tau_t})^c$$

$$+ \sum_{p \pm 1 \neq q \text{ mod } r} (-1)^{\varepsilon_3} g_{\tau_p \tau_q} (a_{\rho_1} \dots a_{\rho_r})^c (a_{\tau_1} \dots a_{\tau_{p-1}} a_{\tau_{q+1}} \dots a_{\tau_t})^c (a_{\tau_{p+1}} \dots a_{\tau_{q-1}})^c$$

where ε_i are the standard Koszul signs, which take into the account that the parity of any cycle is opposite to the sum of parities of a_i : $(a_{\rho_1} \dots a_{\rho_r})^c = 1 + \sum \bar{a}_{\rho_i}$. It follows from [B1], prop. 2, that $\Delta^2 = 0$ and that Δ defines the structure of Batalin Vilkovisky algebra on F . My main object of interest in this section is *the quantum*

master equation in F

$$(1.2) \quad \begin{aligned} \hbar\Delta S + \frac{1}{2}\{S, S\} &= 0, \\ S &= \sum_{g \geq 0} \hbar^{2g-1+i} S_{g,i,n}, \quad S_{g,i,n} \in F_0^{i,n}, \end{aligned}$$

where we used the bi-grading: $F = \oplus_{i,n} F^{i,n}$ with $F^{i,n}$ the component corresponding to the products of i cycles with the total number of n elements of IV .

Remark 1. *One important feature of the operator Δ is the absence of terms in (1.1), corresponding to the neighbors $p \pm 1 = q \pmod r$. This is dictated by the relation of Δ with the cell complex of the Kontsevich compactification of moduli space of Riemann surfaces and also by the relation with equivariant Batalin-Vilkovisky geometry on matrix spaces, see propositions ?? et ??below.*

Solutions to (1.2) with $S_{0,1,1} = 0$, are in one-to one correspondence, by the loc.cit., with the structure of what I call the $\mathbb{Z}/2\mathbb{Z}$ -graded quantum A_∞ -algebra on V . The $(g = 0, i = 1)$ -part is the cyclic $\mathbb{Z}/2\mathbb{Z}$ -graded A_∞ -algebra with the even invariant inner product on $\text{Hom}(V, k) \stackrel{g}{\simeq} V$.

1.2. The homology class in the stable ribbon graph complex. I've associated in loc.cit. with any solution S to the noncommutative BV-equation (1.2), such that $S_{0,1,1} = S_{0,1,2} = 0$, the partition function $c_S(G, \text{or}(G))$ on an oriented stable ribbon graph $(G, \text{or}(G))$, where $\text{or}(G)$ is a choice of orientation on the space $\otimes_{v \in \text{Vert}(G)} k^{\text{Flag}(v) \oplus \text{Cycle}(v)}$, here $\text{Cycle}(v)$ is the set of cycles of the permutation from $\text{Aut}\{\text{Flag}(v)\}$ associated with the given vertex v of the stable ribbon graph. The value of c_S is defined by contracting the product of tensors $\otimes_{v \in \text{Vert}(G)} S_{g(v), i(v), n(v)}$ with $g^{\otimes \text{Edge}(G)}$ with appropriate signs. It is known, see loc.cit and references therein, that the complex of stable ribbon graphs with the differential, generalizing the contracting the edge differential on ribbon graphs, computes $H_*(\overline{\mathcal{M}}'_{\gamma, \nu} / \mathbb{S}_\nu)$ where $\overline{\mathcal{M}}'_{\gamma, \nu}$ is the Kontsevich compactification of the moduli space of Riemann surfaces.

Proposition 1. ([B1]) *Let The sum $c_S = \sum_{[G]} c_S(G, \text{or}(G))[(G, \text{or}(G))]$ is closed under the differential of the complex of stable ribbon graphs*

$$d \sum_{[G]} c_S(G, \text{or}(G))[G, \text{or}(G)] = 0.$$

The homology class associated with the solution S is stable under the action of the natural group of gauge transformation, whose infinitesimal action is

$$\delta S = [S, \varphi] + \hbar\Delta\varphi, \quad \varphi = \sum_{g \geq 0, i, n} \hbar^{2g-1+i} \varphi_{g,i,n}, \quad \varphi_{g,i,n} \in F_1^{i,n},$$

and the variation of the partition function is the boundary of the similar partition function associated with $S + \epsilon\varphi$, $\epsilon^2 = 0$:

$$c_{S+\epsilon\delta S} - c_S = d \sum_G c_{S+\epsilon\varphi}(G, \text{or}(G))[G, \text{or}(G)].$$

1.3. Odd inner product. In the case of the odd inner product $g : V^{\otimes 2} \rightarrow \Pi k$, the Batalin-Vilkovisky differential is defined on the space

$$(1.3) \quad \tilde{F} = \text{Symm}(\bigoplus_{j=1}^{\infty} (V^{\otimes j})^{\mathbb{Z}/j\mathbb{Z}})$$

The second order operator Δ is defined by the same formula, where now $\{a_i\}$ is a basis in V , $g_{ij} = g(a_i, a_j)$ and the Koszul signs ε_i , which now correspond to the standard parity of cycles $:(a_{\rho_1} \dots a_{\rho_r})^c = \sum \overline{a_{\rho_i}}$. Again, it follows from the loc.cit., prop. 2, that $\Delta^2 = 0$, and that Δ defines the structure of Batalin-Vilkovisky algebra on \tilde{F} .

I've associated similarly with any solution S from \tilde{F} , such that $S_{0,1,1} = S_{0,1,2} = 0$, the partition function $c_S(G, or(G))$ on oriented stable ribbon graphs $(G, or(G))$, where $or(G)$ is in this case a choice of orientation on the space $\otimes_{v \in \text{Vert}(G)} k^{\text{Edge}(G)}$. With such orientation the complex of oriented stable ribbon graphs computes $H_*(\overline{\mathcal{M}}'_{\gamma, \nu} / \mathcal{S}_{\nu}, \mathcal{L})$ the homology of $\overline{\mathcal{M}}'_{\gamma, \nu} / \mathcal{S}_{\nu}$ with coefficients in the local system $\mathcal{L} = \text{Det}(P_{\Sigma})$, where P_{Σ} is the set of marked points. Exactly the same result as the proposition 1, holds now for the solutions from \tilde{F} and the associated homology class from $H_*(\overline{\mathcal{M}}'_{\gamma, \nu} / \mathcal{S}_{\nu}, \mathcal{L})$.

2. THE SUPER MATRIX ALGEBRA WITH THE *even* TRACE.

The symmetric algebra generated by cyclic words (1.3) is familiar in the invariant theory. In particular, the n -th symmetric component of the vector space $\tilde{F}_n \subset \tilde{F}$ is canonically identified with $GL_{(N|N')}$ -invariant subspace of n -symmetric functions on $M = \text{Hom}_k(V, gl_{(N|N')})$:

$$(2.1) \quad \tilde{F}_n \simeq (S^n(\text{Hom}_k(gl_{(N|N')}, V)))^{GL_{(N|N')}},$$

where $N \geq n$ (see [B3]). The products of cyclic words correspond here to the $GL_{(N|N')}$ -invariant symmetric functions on $M = \text{Hom}_k(V, gl_{(N|N')})$ given by the products of supertraces of the corresponding matrices:

$$(a_{i_1}, \dots, a_{i_j})^c \rightarrow \text{tr}(X_{i_1} \dots X_{i_j})$$

2.1. The $GL_{(N|N)}$ -invariants and the odd noncommutative BV-equation.

The supertrace functional gives a natural extension to $\text{Hom}_k(M, k)$ of the odd symmetric inner product g . This provides M with constant odd symplectic structure and allows to define on functions on M the odd second order Batalin-Vilkovisky operator:

$$\Delta = \sum_{ij, \alpha\beta} (-1)^{\varepsilon} \frac{g_{ij}}{2} \frac{\partial^2}{\partial X_{i,\alpha}^{\beta} \partial X_{j,\beta}^{\alpha}}$$

A calculation of Δ on the $GL_{(N|N')}$ -invariant symmetric functions shows that, unless $N = N'$, the expression for Δ on such functions contains a term depending on the dimensions $(N - N')$. From now on I assume that $N = N'$ and work with the .

Proposition 2. ([B3]) *The operator Δ defined on the $GL_{(N|N)}$ -invariant subspace of $S^{\leq s}(\text{Hom}_k(gl_{(N|N)}, V))$, where N is sufficiently big ($N \geq s$), coincides with the above Batalin-Vilkovisky differential on $\bigoplus_{n \leq s} \tilde{F}_n$ defined in ([B1]).*

This implies, see ([B3]), that the cohomology of the Batalin-Vilkovisky differential Δ on \tilde{F} are trivial.

Also this implies that the Batalin-Vilkovisky algebra on \tilde{F} can be identified with the stable $GL_{(N|N)}$ -invariants of the Batalin-Vilkovisky algebra on functions on the supermatrix space $M = gl_{(N|N)} \otimes \Pi V$, $N \rightarrow \infty$.

On the level of modular operads this result can be strengthened:

Proposition 3. *The stabilisation of $GL_{(N|N)}$ -invariants of the standard tensor modular operad of vector spaces with even inner product $(GL_{(N|N)}, tr)$, as $N \rightarrow \infty$, is identified naturally with the modular operad $k[\mathbb{S}_n]$ described in ([B1]).*

3. THE SUPER MATRIX ALGEBRA WITH THE *odd* TRACE.

To describe the BV-equation in F , i.e. in the even inner product case, via invariants on matrix spaces I consider the general *queer* superalgebra q_N ([BL]) with its *odd* trace. The associative superalgebra q_N is the subalgebra of $End(U \oplus \Pi U)$,

$$q_N = \{X \in End(U \oplus \Pi U) \mid [X, \pi] = 0\}$$

where $U = k^N$ is a purely even N -dimensional vector space and $\pi : U \rightleftharpoons \Pi U$, $\pi^2 = 1$ is some fixed odd automorphism of $U \oplus \Pi U$ changing the parity. As vector space $q_N = gl_N \oplus \Pi gl_N$. The queer algebra is isomorphic to the tensor product with Clifford algebra $gl_N \otimes Cl(1)$. One of the main reasons for my interest in this algebra is that it comes equipped with the odd analog of the trace functional. The odd trace on q_N is defined as

$$otr(X) = \frac{1}{2} tr(\pi X).$$

It defines on q_N the odd symmetric inner product $otr(XY)$. The odd trace satisfies

$$otr([X, Y]) = 0$$

The analogue of (2.1) in the even inner product case can be deduced from results of A.B. Sergeev on q_N -invariants of tensor powers of the coadjoint representation of q_N , see ([B3]) and references therein.

Proposition 4. *F_n is canonically identified with Q_N -invariant subspace of n -th symmetric functions on the vector space $q_N \otimes \Pi V$:*

$$(3.1) \quad F_n \simeq (S^n Hom(q_N, \Pi V))^{Q_N},$$

where $Q_N \subset GL_{(N|N)}$, $N \geq n$, is the supergroup preserving π .

The cyclic words from $\bigoplus_j \Pi(\Pi V^{\otimes j})^{\mathbb{Z}/j\mathbb{Z}}$, with parity given by the inverse to the sum of parities of $x_i \in \Pi V$, correspond here to the Q_N -invariant symmetric functions on $M = Hom_k(\Pi V, q_N)$ given by the *odd* traces of products of the corresponding matrices:

$$(x_{i_1}, \dots, x_{i_j})^c \rightarrow otr(X_{i_1} \cdot \dots \cdot X_{i_j})$$

and the products of cyclic words corresponds to the products of odd traces.

3.1. The Q_N -invariants and the even noncommutative BV-equation. In this case the odd inner product on M comes from extension of g by the *odd* trace. This defines odd constant symplectic structure on M and the associated second order odd Batalin-Vilkovisky operator

$$\Delta = \sum_{ij, \alpha\beta} (-1)^\varepsilon \frac{g_{ij}}{2} \frac{\partial^2}{\partial Y_{i,\alpha}^\beta \partial \Xi_{j,\beta}^\alpha}$$

where $Y_i \in q_N^0$, $\Xi_i \in q_N^1$ are the corresponding components along the even and the odd parts of q_N . Then the Batalin- Vilkovisky differential on $F_{\leq s}$ is identified naturally with the matrix Batalin-Vilkovisky differential on the Q_N -invariants subspace of $S^{\leq s}(\text{Hom}_k(q_N, \Pi V))$, ($N \geq s$).

4. NONCOMMUTATIVE q_N -EQUIVARIANT σ -MODEL.

Here I explain how the supersymmetric matrix integrals from [B1] can be given the explanation as noncommutative equivariant analogue of the σ -model construction from ([?, AKSZ]).

The space $M = q_N \otimes \Pi V$ is naturally identified with the space of algebra morphisms from the free associative algebra generated by $\text{Hom}_k(\Pi V, k)$. Then F is the space of stable symmetric q_N -invariant functions on M . M has noncommutative P -structure coming from the odd constant symplectic form. The solution S corresponds to the q_N -equivariant vector field on M . Combined with the action of q_N^{odd} represented by $[\Lambda, \cdot]$, the analogue of σ -model construction from ([?, AKSZ]) leads exactly to the Lagrangian $\frac{1}{2}otr \circ g^{-1}([\Lambda, X], X) + S_q(X)$ from [B1].

5. EQUIVARIANT COHOMOLOGY OF $gl(n)$.

The Batalin-Vilkovisky algebra on symmetric functions on $q_N \otimes \Pi V$ can be identified with BV-algebra on the supermanifold $\Pi T^*gl_N \otimes \Pi V$. Using the constant volume form on gl_N this is identified with V -valued forms on gl_N . Under this identification Δ becomes the De Rham differential on forms on V -valued forms on gl_N .

Theorem 1. *The expression*

$$\frac{1}{2}otr \circ g^{-1}([\Lambda, X], X) + S_q(X)$$

defines gl_N -equivariant cohomology class

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