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UPPER SEMICONTINUOUS ATTRACTORS FOR 3D HYPERVISCOUS FLOW

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Abstract. We regularized the 3D Navier-Stokes equations by adding a high-order viscosity term. We first prove the existence of the global attractors of the Leray-Hopf weak solutions of the regularized 3D Navier-Stokes equations and then we study the upper semicontinuity, as the artificial dissipation ε goes to 0. We also consider applications of obtained results to the regularized problem by allowing the family of forcing functions to vary with ε, for $\varepsilon > 0$.

1. Introduction

In this paper, we study the robustness, or upper semicontinuity of the global attractors of the Leray-Hopf weak solutions of modified three dimensional Navier-Stokes equations. We regularized the 3D Navier-Stokes system by adding a high order artificial viscosity term to the conventional system

$$\begin{gathered} \frac{\partial u^{\varepsilon}}{\partial t}+\varepsilon(-\triangle)^{l}u^{\varepsilon}-\nu\triangle u^{\varepsilon}+\left(u^{\varepsilon}.\nabla\right)u^{\varepsilon}+\nabla p=f\left(x\right),\ \text{in }\Omega\times(0,\infty) \\ \text{div }u^{\varepsilon}=0,\ \text{in }\Omega\times\left(0,\infty\right),\ u^{\varepsilon}\left(x,0\right)=u_{0}^{\varepsilon},\ \text{in }\Omega, \\ p(x+Le_{i},t)=p(x,t),\ u^{\varepsilon}(x+Le_{i},t)=u^{\varepsilon}(x,t)\quad i=1,2\ ,3,\ t\in(0,\infty) \end{gathered} \tag{1.1}$$

where $\Omega=(0,L)^{3}$ with periodic boundary conditions and $(e_1,...,e_d)$ is the natural basis of $\mathbb{R}^d$. Here $\varepsilon>0$ is the artificial dissipation parameter, u^{ε} is the velocity vector field, p is the pressure, $\nu>0$ is the kinematic viscosity of the fluid and f is a given force field. For $\varepsilon=0$, the model is reduced to 3D Navier–Stokes system.

Mathematical model for such fluid motion has been used extensively in turbulence simulations (see e.g. [3, 4, 10]). For further discussion of theoretical results concerning (1.1), see [1, 2, 5, 12, 16, 17, 21, 23].

In the work [23], the strong convergence of the solution of this problem to the solution of the conventional system as the regularization parameter goes to zero, was established for each dimension $d\leq 4$.

For the 3D Navier–Stokes system weak solutions of problem are known to exist by a basic result by J. Leray from 1934 [11], only the uniqueness of weak solutions remains as an open problem. Then the known theory of global attractors of infinite dimensional dynamical systems is not applicable to the 3D Navier–Stokes system.

The theory of trajectory attractors for evolution partial differential equations was developed in [15, 19], which the uniqueness theorem of solutions of the corresponding initial-value problem is not proved yet, e.g. for the 3D Navier–Stokes system (see [8, 15, 18, 19]). Such trajectory attractor is a classical global attractor

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but in the space of weak solutions defined on $[0, \infty)$, with the corresponding semigroup being simply the translation in time of such solutions. A compact set $\mathfrak{A} \Subset E$ is said to be a global attractor of a semigroup $\{S(t), t > 0\}$ acting in a Banach or Hilbert space E if $\mathfrak{A}$ is strictly invariant with respect to $\{S(t)\} : S(t)\mathfrak{A} = \mathfrak{A}\ \forall t \geq 0$ and $\mathfrak{A}$ attracts any bounded set $B \subset E : dist(S(t)B, \mathfrak{A}) \to 0\ (t \to \infty)$ (see [14], [15], [18], [19], [21]).

In this article, we study the upper semicontinuity, of the global attractors of the Leray-Hopf weak solutions of a regularized 3D Navier-Stokes equations, as the artificial dissipation ε goes to 0. While there exist other examples of such robustness in the literature of the Navier-Stokes equations, the specific emphasis on the regularized problem is new for the 3D Navier-Stokes equations and is of interest. This would bean extension of the earlier work on Ou and Sritharan for the 2D Navier-Stokes equations, see references [16] and [17]. It is now known that there is a global attractor $\mathfrak{A}_0$ for the Leray-Hopf weak solutions of the 3D Navier-Stokes equations, see Sell [18, 19] and Chepyzhov [13].

The main object of this paper to show that there is a global attractor, which one might denote by $\mathfrak{A}_\varepsilon$, for the regularized problem (1.1), and that the family $\{\mathfrak{A}_\varepsilon\}$ is upper semicontinuous at $\varepsilon = 0$. Moreover, we can modify the argument described above so that the final result will have broader applicability by allowing the family of forcing functions f^ε to vary with ε, for $\varepsilon > 0$.

The family of sets $\mathfrak{A}_\varepsilon$, $0 < \varepsilon \leq 1$ is robust at $\mathfrak{A}_0$, or is upper semicontinuous with respect to ε at $\varepsilon_0 = 0$, provided that, for every $\varepsilon_0 > 0$, there is a neighborhood $O(\varepsilon_0)$ of $0 \in \mathbb{R}$ and a neighborhood $N_{\varepsilon_0}(\mathfrak{A}_0)$ of $\mathfrak{A}_0$, such that $\mathfrak{A}_\varepsilon \subset N_{\varepsilon_0}(\mathfrak{A}_0)$, for every $\varepsilon \in O(\varepsilon_0)$ with $\varepsilon > 0$, see (23.13) in [19].

The paper is organized as follows. In Section 2, we present the relevant mathematical framework for the paper. In Section 3, we recall the definition of the trajectory attractor $\mathfrak{A}_0$ of the conventional 3-D Navier-Stokes equations. In Section 4, we study the regularized problem (see equation (1.1)), then we show the existence of trajectory attractor $\mathfrak{A}_\varepsilon$. In Section 5, we present the main result of this paper, that is, a theorem on the upper semicontinuity on the attractors $\mathfrak{A}_\varepsilon$. Finally, an application of our general results to the study of the robustness of the system (1.1) with a perturbed external force.

2. Preliminary

We denote by $H^m_{per}(\Omega)$, the Sobolev space of L-periodic functions endowed with the inner product

$$(u, v) = \sum_{|\alpha| \leq m} (D^\alpha u, D^\alpha v)_{L^2(\Omega)} \text{ and the norm } \|u\|_m = \sum_{|\alpha| \leq m} (\|D^\alpha u\|^2_{L^2(\Omega)})^{\frac{1}{2}}.$$

We define the spaces V_m as completions of smooth, divergence-free, periodic, zero-average functions with respect to the H^m_{per} norms. V'_m denote the dual space of V_m and V denote the space V_0.

We present the topology to be used for generating the neighborhood of robustness. Let F any vector space. A metric $d(f, g)$ on F is said to be invariant if

$$d(f, g) = d(f - g, 0) \text{ for all } f,\ g \in F.$$

A Fréchet space is a complete topological vector space whose topology is induced by a translation invariant metric $d(f, g)$. Given a Banach space X, with norm $\|.\|_X$ and $1 \leq p < \infty$, we denote by $L^p_{loc}[0, \infty; X)$ the Fréchet space of mesurable functions $f:[0,\infty)\to X$ that are p-integrable over $[0,T]$, for each $0<T<\infty$, endow with the metric

$$d(f,g)=\sum_{n=1}^{\infty}2^{-n}\min(\|f-g\|_{L^p(0,\,n;\,X)},1).$$

We denote by $L^p_{loc}(0,\infty;X)$ the Fréchet space of mesurable functions $f:(0,\infty)\to X$ that are p-integrable over $[t_0,T]$, for each $0<t_0\leq T<\infty$ endow with the metric

$$d(f,g)=\sum_{n=2}^{\infty}2^{-n}\min(\|f-g\|_{L^p(\frac{1}{n},\,n;\,X)},1).$$

Similarly for $p=\infty$, we will let $L^\infty_{loc}(0,\infty;X)$ denote the collection of all functions $f:(0,\infty)\to X$ with the property that, for all τ and T with $0<T<\infty$, one has $\operatorname{ess}\sup_{0<s<T}\|f\|_X<\infty$. We denote by $C[0,\infty;X)$ the space of strongly continuous functions from $[0,\infty)$ to X, endowed with the topologie of the uniform convergence over compact sets and by $C_w[0,\infty;X)$ the space of weakly continuous functions from $[0,\infty)$ to X. We denote by $L^\infty C=L^\infty(\mathbb{R},X)\cap C(\mathbb{R},X)$ the Fréchet space $L^\infty C$ endow with the $L^\infty_{loc}-$topology, wich is the topology of uniform convergence on bounded sets.

Let E be a complete metric space with metric d. We write B_r for the open ball centre $0\in E$ and radius r. The following quantity is called the Hausdorff (non-symmetric) semidistance from a set X to a set Y in a Banach space E

$$dist_E(X,Y)=\sup_{x\in X}\inf_{y\in Y}\|.\|_E\,.$$

Let M be a subset of E and let $\mathbb{R}^+=[0,\infty)$. A mapping $\sigma=\sigma(u,t)$, where $\sigma:M\times[0,\infty)\to M$ is said to be a semiflow on M provided the following hold
1) $\sigma(w,0)=w$, for all $w\in M$.
2) The semigroup property holds, i. e,

$$\sigma((w,s),t)=\sigma(w,s+t)\ \text{ for all } w\in M \text{ and } s,\,t\in\mathbb{R}^+.$$

3) The mapping $\sigma:M\times(0,\infty)\to M$ is continuous.

If in addition the mapping $\sigma:M\times[0,\infty)\to M$ is continuous we will say that the semiflow is continuous at $t=0$. Here we use $t>0$ in order that the Robustness Theorem 23.14 in [19] is valid, see Sell [19] and Hale [8]. For any $u\in M$ the positive trajectory through u is defined as the set $\gamma^+(u)=\{\sigma(t)u,\ t\geq0\}$. For any set $B\subset M$ we define the positive hull $\mathcal{H}^+(B)$ and the omega limit set $\omega(B)$ as follows

$$\mathcal{H}^+(B)=Cl_M\gamma^+(B)\ \text{ and } \omega(B)=\cap_{\tau\geq0}\mathcal{H}^+(\sigma(\tau)B)\,.$$

If $\mathcal{A}\subset E$ and $\varepsilon>0$ we write

$$N_\varepsilon(\mathcal{A})=\{z\in E,\ \inf_{a\in\mathcal{A}}d(z,a)<\varepsilon\}.$$

for the open $\varepsilon-$neighbourhood of $\mathcal{A}$.

We denote by A the Stokes operator $Au=-\triangle u$ for $u\in D(A)$.We recall that the operator A is a closed positive self-adjoint unbounded operator, with
$D(A)=\{u\in V_0,\,Au\in V_0\}$. We have in fact, $D(A)=V_2$. The spectral theory of A allows us to define the powers A^l of A for $l\geq1$, A^l is an unbounded self-adjoint operator in V_0 with a domain $D(A^l)$ dense in $V_2\subset V_0$. We set here

$$A^lu=(-\triangle)^l\,u \text{ for } u\in D\left(A^l\right)=V_{2l}.$$

The space $D\left(A^l\right)$ is endowed with the scalar product and the norm

$$(u,v)_{D(A^l)} = (A^l u, A^l v), \|u\|_{D(A^l)} = \{(u,u)_{D(A^l)}\}^{\frac{1}{2}}.$$

Now define the trilinear form $b(.,.,.)$ associated with the inertia terms

$$b\left(u,v,w\right) = \sum_{i,j=1}^{3} \int_{\Omega} u_i \frac{\partial v_j}{\partial x_i} w_j dx$$

Recall that for u satisfying $\nabla . u = 0$ we have

$$b\left(u,u,u\right) = 0 \text{ and } b\left(u,v,w\right) = -b\left(u,w,v\right). \tag{2.1}$$

Hereafter, $c_i \in \mathbb{N}$,will denote a dimensionless scale invariant positive constant which might depend on the shape of the domain. The trilinear form $b\left(.,.,.\right)$ is continuous on $V_{m_1}\left(\Omega\right) \times V_{m_2+1}\left(\Omega\right) \times V_{m_3}\left(\Omega\right)$, $m_{i=1,2,3} \geq 0$

$$\left|b\left(u,v,w\right)\right| \leq c_0 \left\|u\right\|_{m_1} \left\|v\right\|_{m_2+1} \left\|w\right\|_{m_3} \quad , \ m_3 + m_2 + m_1 \geq \frac{3}{2} \tag{2.2}$$

see [6, 19]. The continuity property of the trilinear form enables us to define (using Riesz representation theorem) a bilinear continuous operator $B\left(u,v\right)$; $V_2 \times V_2 \to V_2'$ will be defined by

$$\left\langle B\left(u,v\right),w\right\rangle = b\left(u,v,w\right), \ \forall w \in V_2.$$

We recall some inequalities that we will be using in what follows.

Young's inequality

$$ab \leq \frac{\epsilon}{p} a^p + \frac{1}{q \epsilon^{\frac{q}{p}}} b^q, a, b, \epsilon > 0, p > 1, q = \frac{p}{p-1}. \tag{2.3}$$

Poincaré's inequality

$$\lambda_1 \left\|u\right\|^2 \leq \left\|u\right\|_1^2 \text{ for all } u \in V_0, \tag{2.4}$$

where λ_1 is the smallest eigenvalue of the Stokes operator A.

3. Navier-Stokes equations

The conventional Navier-Stokes system can be written in the evolution form

$$\begin{gathered} \frac{\partial u}{\partial t} + \nu A u + B\left(u,u\right) = f, \ t > 0, \\ \text{div } u = 0, \text{ in } \Omega \times (0,\infty) \text{ and } u\left(x,0\right) = u_0, \text{ in } \Omega, \end{gathered} \tag{3.1}$$

let $f \in L^\infty\left(0,\infty;V_0\right)$ be given. We will say that a function u is a weak solution of the 3D Navier-Stokes of Class LH (Leray–Hopf) on $[0,\infty)$ provided that $u\left(x,0\right) = u_0\left(x\right) \in V_0$, and the following properties hold

1) $u \in L^\infty\left(0,\infty;V_0\right) \cap L^2_{loc}\left[0,\infty;V_1\right)$.

2) $\dfrac{du}{dt} \in [L^{\frac{4}{3}}_{loc} 0,\infty; V_1')$.

Taking the inner product of (3.1) with u, and using (2.3) we have

$$\frac{d}{dt}\left\|u\left(t\right)\right\|^2 + 2\nu \left\|\nabla u\right\|^2 = 2\left\langle f,u\right\rangle. \tag{3.2}$$

by application of Young's inequality and the Poincaré's Lemma, yields

$$\frac{d}{dt}\left\|u\left(t\right)\right\|^2 + \nu \left\|\nabla u\right\|^2 \leq \frac{\left\|f\right\|^2}{\nu\lambda_1}, \tag{3.3}$$

using the Poincaré Lemma and Gronwall's inequality, to get

$$\|u(t)\|^2 \le e^{-\nu\lambda_1(t-t_0)}\|u(t_0)\|^2 + \frac{1}{\nu^2\lambda_1^2}\|f\|^2\left(1-e^{-\nu\lambda_1(t-t_0)}\right), \text{with } 0<t_0<t,$$

3) which implies that

$$\|u(t)\|^2 \le e^{-\nu\lambda_1(t-t_0)}\|u(t_0)\|^2 + \frac{1}{\nu^2\lambda_1^2}\|f\|^2. \tag{3.4}$$

Integrating (3.2) over $[t_0,t]$ we find that

$$\|u(t)\|^2 + 2\nu\int_{t_0}^{t}\|A^{\frac{1}{2}}u(s)\|^2 ds \le \|u(t_0)\|^2 + 2\int_{t_0}^{t}\langle f(s), u(s)\rangle\, ds. \tag{3.5}$$

4) The function u satisfies the following equality

$$\langle u(t)-u(t_0), v\rangle + \nu\int_{t_0}^{t}\langle A^{\frac{1}{2}}u(s), A^{\frac{1}{2}}v\rangle ds + \int_{t_0}^{t}\langle B(u(s),u(s)), v\rangle ds = \int_{t_0}^{t}\langle f, v\rangle\, ds, \tag{3.6}$$

for all $v\in V_1$ and for all $t\ge t_0\ge 0$.

The proof of the following theorem is given in [12, 19, 22].

Theorem 3.1. *Let $f\in V_1'$ and $u_0\in V_0$ be given. Then for every $T>0$, there exists a weak solution $u(t)$ of (3.1) from the space $L^2(0,T;V_1)\cap L^\infty(0,T;V_0)$, such that $u(x,0)=u_0$ and $u(t)$ satisfies the energy equality (3.6).*

Moreover (see [22]), $u(.)$ is weakly continuous from $[0,T]$ into V_0, the function $u\in C_w([0,T];\ V_0)$ and consequently $u(x,0)=u_0(x)\in V_0$. Let W is the set of all Leray–Hopf weak solutions $u(.)$ of equation (3.1) in the space $L^\infty(0,\infty;V_0)\cap L^2_{loc}[0,\infty;V_1)$ that satisfy the following properties

- $\dfrac{du}{dt}\in L^{\frac{4}{3}}_{loc}(0,\infty;V_1^{'})$;
- for almost all t and t_0, with $t>t_0>0$, inequalities (3.5 ,3.6) are valid.

Let X^0 denote the Fréchet space used to define the Leray-Hopf weak solutions. Thus

$$\varphi\in X^0 = L^\infty(0,\infty;V_0)\cap L^2_{loc}[0,\infty;V_1),$$

where $\varphi\in C_w[0,\infty;V_0)$ and we let $\mathfrak{F}^0$ denote a compact, translation invariant set of forcing functions f in

$$L^\infty C = L^\infty\left(\mathbb{R}, L^2(\Omega)\right)\cap C\left(\mathbb{R}, L^2(\Omega)\right)$$

where the topology on the Fréchet space $L^\infty C$ is the topology of uniform convergence on bounded sets in $\mathbb{R}$.

Then, we use the Leray-Hopf solutions of the 3D Navier-Stokes equations with $\varepsilon=0$ to generate a semiflow π^0 on $\mathfrak{F}^0\times X^0$, where

$$\pi^0(\tau)(f,\varphi) = \left(f_\tau, S^0(f,\tau)\varphi\right) \text{ for } \tau\ge 0,$$

$f_\tau(t)=f(\tau+t)$ and $u(t)=S^0(f,t)\varphi$ is the Leray-Hopf solution of the 3D Navier-Stokes equations that satisfies $u(0)=S^0(f,0)\varphi=\varphi(0)$. By using the theory of generalized weak solutions, as in Sell [18] or [19] , we note that π^0 has a trajectory attractor $\mathfrak{A}_0\subset\mathfrak{F}^0\times X^0$ see Theorem 65.12 in [19], and Chepyzhov [13, 14].

4. The regularized Navier-Stokes system

Using the operators defined in the previous section, we can write the modified system (1.1) in the evolution form

$$\begin{gathered}\partial_t u^\varepsilon + \varepsilon A^l u^\varepsilon + \nu A u^\varepsilon + B\left(u^\varepsilon, u^\varepsilon\right) = f\left(x\right), \text{in } \Omega\times(0,\infty) \\ \text{div } u^\varepsilon = 0, \text{ in } \Omega\times(0,\infty)\,, \; u^\varepsilon\left(x,0\right) = u_0^\varepsilon, \text{ in } \Omega.\end{gathered} \tag{4.1}$$

The existence and uniqueness results for initial value problem (1.1) can be found in J. L. Lions [12, Remark 6.11].
In three dimensions, the following theorem collects the main result in this work

Theorem 4.1. *For $l \geq \frac{5}{4}$, $\varepsilon > 0$ fixed, $f \in L^2\left(0,T;V_0'\right)$ and $u_0^\varepsilon \in V_0$ be given. There exists a unique weak solution of (4.1) which satisfies $u^\varepsilon \in L^2\left(0,T;V_l\right) \cap L^\infty\left(0,T;V_0\right), \forall T > 0$.*

We introduce the following result of the convergence of u^ε as the regularized parameter $\varepsilon \to 0$

Theorem 4.2. *For $l \geq \frac{3}{2}$, $\varepsilon > 0$ fixed, $f \in L^2\left(0,T;V_0'\right)$ and $u_0^\varepsilon \in V_0$ be given.*
i) There exists a unique weak solution of (4.1) which satisfies

$$u^\varepsilon \in L^2(0,T;V_l) \cap L^\infty\left(0,T;V_0\right), \; \forall T > 0.$$

ii) This weak solution u^ε converges strongly in $L^2(0,T;V_0)$ as $\varepsilon \to 0$ to u a weak solution of the Navier-Stokes equations.

The above theorem is established directly by using of a general result [23, Theorem 3.9.]. For $\varepsilon > 0$, we let π^ε denote the semiflow on $\mathfrak{F}^0 \times X^0$ generated by the weak solutions of regularized 3D Navier-Stokes equations of (4.1). Thus

$$\pi^\varepsilon\left(\tau\right) = \left(f_\tau, S^\varepsilon\left(f,\tau\right)\varphi\right), \tag{4.2}$$

where $u_0^\varepsilon = \varphi$ and

$$u^\varepsilon\left(t\right) = S^\varepsilon\left(f,t\right)\varphi = S^\varepsilon\left(f,t\right)u_0^\varepsilon \tag{4.3}$$

is the weak solution of (4.1) that satisfies $\varphi\left(0\right) = u_0^\varepsilon$.

Regarding the existence of the attractor $\mathfrak{A}_\varepsilon$ when $\varepsilon > 0$, we use especially the related papers of Chepyzhov and Vishik, such as [15, 13] to show that the system (4.1) possesses a global attractor. For $\varepsilon > 0$, we consider the trajectory space $\mathcal{K}_\varepsilon$ of the modified Navier-Stokes equations (4.1). $\mathcal{K}_\varepsilon$ is the union of all weak solutions $u^\varepsilon \in X^0$ that satisfy (4.1), see (6.163) in [12].

Using the described scheme in [15], we construct the spaces $\mathcal{S}_b$

$$\mathcal{S}_b = \{v\left(.\right) \in L^\infty\left(0,T;V_0\right) \cap L_b^2(0,T;V_1), \partial_t v\left(.\right) \in L_b^2(0,T;D(A^{\frac{l}{2}})')\}$$

with norm

$$\left\|v\right\|_{\mathcal{S}_b} = \left\|v\right\|_{L_b^2(0,T;V_1)} + \left\|v\right\|_{L^\infty(0,T;V_0)} + \left\|\partial_t v\right\|_{L_b^2(0,T;D(A^{\frac{l}{2}})')}$$

where

$$\left\|v\right\|_{L_b^2(0,T;V_1)} = \sup_{t\geq 0}(\int_t^{t+1}\left\|v\left(s\right)\right\|_1^2 ds)^{\frac{1}{2}}, \left\|v\right\|_{L^\infty(0,T;V_0)} = \operatorname*{ess\,sup}_{t\geq 0}\left\|v\right\|$$

and

$$\left\|\partial_t v\right\|_{L_b^2(0,T;V_l')} = \sup_{t\geq 0}(\int_t^{t+1}\left\|v\left(s\right)\right\|_{V_l'}^2 ds)^{\frac{1}{2}}.$$

We need a topology in the space $\mathcal{K}_\varepsilon$. We define on X^0 the following sequential topology which we denote Γ.

By definition, a sequence of functions $\{v_n\} \subseteq X^0$ converges to a function $v \in X^0$ in the topology Γ as $n \to \infty$ if, for any $T > 0$, $v_n \to v$ weakly in $L^2(0,T;V_1)$; $v_n \to v$ weak-$*$ in $L^\infty(0,T;V_0)$ and $v_n \to v$ strongly in $L^2(0,T;V_0)$, as $n \to \infty$.

We consider the topology Γ on $\mathcal{K}_\varepsilon$. It is easy to prove that the space $\mathcal{K}_\varepsilon$ is closed in Γ. From the definition of $\mathcal{K}_\varepsilon$, it follows that $\pi^\varepsilon \mathcal{K}_\varepsilon \subset \mathcal{K}_\varepsilon$ for all $t \geq 0$.

Corollary 4.3. *If $u^\varepsilon(t)$ is a solution of (4.1), then the following inequalities hold for all $t > 0$*

$$\|u^\varepsilon(t)\|^2 \leq e^{-\nu\lambda_1 t}\|u_0^\varepsilon\|^2 + \frac{\|f\|^2}{\nu^2\lambda_1^2}, \tag{4.4}$$

$$\int_t^{t+1}\|u^\varepsilon(s)\|^2 ds \leq \frac{e^{-\nu\lambda_1 t}}{\nu\lambda_1}\|u_0^\varepsilon\|^2 + \frac{\|f\|^2}{\nu^2\lambda_1^2}, \tag{4.5}$$

$$\nu\int_t^{t+1}\|u^\varepsilon(s)\|_1^2 ds \leq \frac{e^{-\nu\lambda_1 t}}{\nu\lambda_1}\|u_0^\varepsilon\|^2 + \frac{\|f\|^2}{\nu^2\lambda_1^2} + \frac{\|f\|^2}{\nu\lambda_1}. \tag{4.6}$$

Proof. Taking the inner product of (4.1) by u^ε, we obtain

$$\frac{d}{dt}\|u^\varepsilon\|^2 + 2\varepsilon\|A^{\frac{l}{2}}u^\varepsilon\|^2 + 2\nu\|\nabla u^\varepsilon\|^2 = 2(f, u^\varepsilon). \tag{4.7}$$

Applying Young's inequality and using the Poincaré Lemma, we obtain

$$\frac{d}{dt}\|u^\varepsilon\|^2 + \nu\|\nabla u^\varepsilon\|^2 \leq \frac{\|f\|^2}{\nu\lambda_1}. \tag{4.8}$$

Using the Gronwall's inequality over $[0,\ t]$, we obtain (4.4). Integrating (4.4) over $[t, t+1]$ we find (4.5). Integrating (4.8) over $[t, t+1]$ we find

$$\nu\int_t^{t+1}\|\nabla u^\varepsilon(s)\|^2 ds \leq \frac{\|f\|^2}{\nu\lambda_1} + \|u^\varepsilon(t)\|^2. \tag{4.9}$$

Applying inequality (4.4), we get (4.6). □

We recall the following result

Lemma 4.4. *Let $f \in L^2(0,T;V_1')$, then, for any solution $u^\varepsilon(t)$ of problem (1.1) the time derivative $\frac{du^\varepsilon}{dt}$ is uniformly bounded in $L^2(0,T;V_l')$.*

A simple consequence of Lemma 3.6 [23] is the following Corollary

Corollary 4.5. *Let $f \in L^2(0,T;V_1')$, then any solution $u^\varepsilon(t)$ of (4.1) satisfies*

$$\int_t^{t+1}\|\partial_t u^\varepsilon(s)\|^2_{D(A^{\frac{l}{2}})'} ds \leq C_3, \tag{4.10}$$

C_3 is a positive constant independent of ε .

Moreover, due to estimates (4.4) and (4.10), we also have the uniform estimate.

Proposition 4.6. *Let $f \in L^2(0,T;V_1')$, then any solution $u^\varepsilon(t)$ of (4.1) satisfies the inequality*

$$\|\pi^\varepsilon(u^\varepsilon)\|^2_{\mathcal{S}_b} \leq \frac{c_7 e^{-\nu\lambda_1 t}}{\nu\lambda_1}\|u^\varepsilon(0)\|^2 + \frac{c_7\|f\|^2}{\nu^2\lambda_1^2} + C_4 \tag{4.11}$$

where the positive constant C_4 is independent of ε.

Proposition 4.7. *For $l \geq \frac{3}{2}$ and $f \in L^\infty C$ a time independent functions, π^ε is a continuous family of semiflows on X^0.*

Proof. Since $u^\varepsilon \in L^2(0,T;V_l)$ and $\dfrac{du^\varepsilon}{dt} \in L^2(0,T;V_l')$, u^ε is almost everywhere equal to an uniform continuous function from $[0,T]$ to the space V_0. The continuity of u^ε is a direct consequence of [22, Lemma 1.4. ChIII, Sec1].

From the result of the strong convergence, there exists $\varepsilon_1 > 0$, such that

$$\|u^\varepsilon(t) - u^{\varepsilon_0}(t)\| \leq \epsilon, \forall \epsilon \geq 0, \text{ for each } \varepsilon \leq \varepsilon_1, \tag{4.12}$$

it follows from (4.12) that $\lim\limits_{\varepsilon \to \varepsilon_0} \|\pi^\varepsilon(t) - \pi^{\varepsilon_0}(t)\|$ goes to 0, for all $0 \leq t \leq T$.

This shows that π^ε is continuous semiflow on X^0 and π^ε approximates π^0 uniformly for t in compact sets in $[0,\infty)$. □

From Proposition 4.6 it follows that $\mathcal{K}_\varepsilon \subset \mathcal{S}_b$ for all $\varepsilon > 0$ and for all $\tau > 0$. Also Proposition 4.6 implies that the semigroup π^ε has absorbing set in $\mathcal{K}_\varepsilon$ for all $\varepsilon > 0$ and for all $\tau > 0$ (We note, that this absorbing set does not depend on ε, since the constant C_4 in (4.11) is independent of ε), bounded in $\mathcal{S}_b$ and inequality (4.11) implies that absorbing set is compact in Γ. The continuity of π^ε is proved. These facts are sufficient to state that π^ε has a global attractor $\mathfrak{A}_\varepsilon$. Such that $\mathfrak{A}_\varepsilon \subset \mathfrak{F}^0 \times X^0$, bounded in $\mathcal{S}_b$ and compact in Γ. For a more detailed, see [13, 14, 15].

5. Upper semicontinuity of attractors

We now prove the robustness property for the global attractor $\mathfrak{A}_\varepsilon$. We have shown in Proposition 4.7 the continuity of the family of semiflows π^ε on X^0. Having done this, We can simply invoke Theorem 23.14 in [19] to complete the proof of the robustness for the family of attractors $\mathfrak{A}_\varepsilon$ at $\varepsilon = 0$. We denote by

$$B_R = \{u^\varepsilon(x,t), 0 \leq t, 0 < \varepsilon < 1; \|\pi^\varepsilon(u^\varepsilon)\|^2_{\mathcal{S}_b} \leq R\}$$

with $R = \dfrac{c_7 e^{-\nu\lambda_1 t}}{\nu\lambda_1}\|u^\varepsilon(0)\|^2 + \dfrac{c_7\|f\|^2}{\nu^2\lambda_1^2} + C_4$, $u^\varepsilon(x,t)$ is a family of solutions of system (4.1), and the norms of $u^\varepsilon(x,t)$ in $\mathcal{S}_b$ are uniformly bounded, see Proposition 4.7. It is sufficient to show that a small $\delta-$neighbourhood of attractor $\mathfrak{A}_0$ is an absorbing set and π^ε approximates π^0 on B_R uniformly for all t in $[0,\infty)$, see [7, 19].

Theorem 5.1. *For $l \geq \frac{3}{2}$, for $\varepsilon > 0$ the family of semiflows π^ε generated by the weak solutions of the regularized 3D Navier-Stokes equations (4.1) admits a compact attractor $\{\mathfrak{A}_\varepsilon, 0 < \varepsilon \leq 1\}$ which attracts bounded sets of V_0 and is contained in the absorbing balls B_R where R is independent of ε. Moreover, $d_{X^0}(\mathfrak{A}_\varepsilon, \mathfrak{A}_0) \to 0$, as $\varepsilon \to 0$.*

Proof. Since $\mathfrak{A}_0$ is a global attractor, for any bounded set $B_{R_0} = \{u(x,0) \in V, \|u(x,0)\| \leq R_0\} \subset V$, we have

$$d_{X^0}\left(\pi^0 B_{R_0}, \mathfrak{A}_0\right) \to 0, \text{ as } t \to \infty. \tag{5.1}$$

Thus, there exists $\delta > 0$ such that

$$d_{X^0}\left(\pi^0 B_{R_0}, \mathfrak{A}_0\right) \leq \frac{\delta}{2}, \text{ for } t \geq t_\delta. \tag{5.2}$$

Consequently

$$\pi^0(t) B_{R_0} \subset N_\delta(\mathfrak{A}_0), \text{ for } t \geq t_\delta, \tag{5.3}$$

where $N_\delta(\mathfrak{A}_0)$ be the δ-neighborhood of $\mathfrak{A}_0$. This shows that $N_\delta(\mathfrak{A}_0)$ is an absorbing set. Since π^ε approximates π^0 uniformly for all $t \geq 0$, then for any $\delta > 0$, there are $\varepsilon_1 > 0$ and $t_0 \geq 0$ such that

$$\pi^\varepsilon\left(B_R \cap B_{R_0}\right) \subset N_\delta(\mathfrak{A}_0), \text{ for } 0 < \varepsilon < \varepsilon_1,\ t \geq t_0. \tag{5.4}$$

Since the attractor $\mathfrak{A}_\varepsilon$ is contained in $B_R \cap B_{R_0}$, we have

$$\pi^\varepsilon(\mathfrak{A}_\varepsilon) \subset N_\delta(\mathfrak{A}_0), \text{ for } \varepsilon \leq \varepsilon_1,\ t \geq t_0. \tag{5.5}$$

Since $\mathfrak{A}_\varepsilon$ is an invariant set, we deduce that

$$\mathfrak{A}_\varepsilon \subset N_\delta(\mathfrak{A}_0), \text{ for } 0 < \varepsilon < \varepsilon_1,\ t \geq t_0. \tag{5.6}$$

Moreover, since δ is arbitrary, we obtain the upper semicontinuity of $\mathfrak{A}_\varepsilon$, at $\varepsilon_0 = 0$

$$d_{X^0}\left(\mathfrak{A}_\varepsilon, \mathfrak{A}_0\right) \to 0, \text{ as } \varepsilon \in O\left(\varepsilon_0\right). \tag{5.7}$$

□

One can modify the argument described above so that the final result will have broader applicability by allowing the family of forcing functions to vary with ε, for $\varepsilon > 0$. Thus, we consider the regularized Navier-Stokes system (4.1) with a perturbed external force f^ε in place of f, for $\varepsilon > 0$. Then (4.1) becomes

$$\begin{aligned} \partial_t u^\varepsilon + \varepsilon A^l u^\varepsilon + \nu A u^\varepsilon + B\left(u^\varepsilon, u^\varepsilon\right) = f^\varepsilon\left(x\right), \text{in } \Omega \times (0,\infty) \\ \operatorname{div} u^\varepsilon = 0, \text{ in } \Omega \times (0,\infty)\,, u^\varepsilon\left(x,0\right) = u_0^\varepsilon, \text{ in } \Omega. \end{aligned} \tag{5.8}$$

We show that the trajectory attractor of the perturbed system (5.8) coincides with the trajectory attractor $\mathfrak{A}_\varepsilon$ of the unperturbed system (3.1). Our results rely on the work of Hale ([15]) who show that the limit behaviour is valid even through $\mathfrak{F}^\varepsilon$, where $\mathfrak{F}^\varepsilon$ denote a compact, translation invariant set of perturbed forcing functions to vary with ε, for $\varepsilon > 0$ and satisfy the condition

$$\omega(\mathcal{H}^+\left(f^\varepsilon\right)) = \omega(\mathcal{H}^+\left(f\right)). \tag{5.9}$$

Thus we would use $\mathfrak{F}^\varepsilon$ in place of $\mathfrak{F}^0$, for $\varepsilon > 0$. Moreover, by using a metric d on the $L^\infty C$-toplogy, see [19] for some samples, we can note that (5.9) is equivalent to saying that for every $\delta > 0$ there is an $\varepsilon_1 > 0$ and $T_\delta = T\left(\delta\right) \geq 0$ such that

$$d_{X^0}(f^\varepsilon, \mathfrak{F}^0) \leq \delta, \text{ for } 0 < \varepsilon \leq \varepsilon_2 \text{ and } f^\varepsilon \in \mathfrak{F}^\varepsilon \tag{5.10}$$

for any $t \geq T_\delta$, that is

$$\mathfrak{F}^\varepsilon \subset N_\delta\left(\mathfrak{F}^0\right), \text{ for } 0 < \varepsilon \leq \varepsilon_2, \tag{5.11}$$

where N_δ denotes the δ-neighborhood of $\mathfrak{F}^0$ in $L^\infty C$. The resulting argument for robustness will then depend on two parameters $\lambda = (\varepsilon, \delta)$, where $\lambda \to (0,0)$.

The following statement generalizes Theorem 5.1

Theorem 5.2. *Under the above conditions, the trajectory attractor of the perturbed 3D Navier-Stokes system (5.8) coincides with the trajectory attractor $\mathfrak{A}_\varepsilon$ of the non-perturbed system (3.1). Moreover, the perturbed attractor of (5.8) is upper semicontinuous with respect to ε at $\varepsilon = 0$.*

Proof. The existence of trajectory attractor $\mathfrak{A}_\varepsilon$ is treated above. The proof follows from formulas (5.9), (5.11) and Theorem 5.1 . □

Proposition 4.7 can be used to extend the 2D result of Ou and Sritharan[16] to l-lplacian with $l > 1$.

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