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Abdelhafid Younsi

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UPPER SEMICONTINUOUS ATTRACTORS FOR 3D HYPERVISCOUS FLOW

ABDELHAFID YOUNSI

ABSTRACT. We regularized the 3D Navier-Stokes equations by adding a high-order viscosity term. We study the upper semicontinuity, of the global attractors of the Leray-Hopf weak solutions of the regularized 3D Navier-Stokes equations, as the artificial dissipation ε goes to 0. We also consider applications of obtained results to the regularized problem by allowing the family of forcing functions to vary with ε , for $\varepsilon > 0$.

1. INTRODUCTION

In this paper, we study the robustness, or upper semicontinuity of the global attractors of the Leray-Hopf weak solutions of modified three dimensional Navier-Stokes equations. We regularized the 3D Navier-Stokes system by adding a high order artificial viscosity term to the conventional system

$$\begin{aligned} \frac{\partial u^\varepsilon}{\partial t} + \varepsilon(-\Delta)^l u^\varepsilon - \nu \Delta u^\varepsilon + (u^\varepsilon \cdot \nabla) u^\varepsilon + \nabla p &= f(x), \text{ in } \Omega \times (0, \infty) \\ \operatorname{div} u^\varepsilon &= 0, \text{ in } \Omega \times (0, \infty), u^\varepsilon(x, 0) = u_0^\varepsilon, \text{ in } \Omega, \\ p(x + Le_i, t) &= p(x, t), u^\varepsilon(x + Le_i, t) = u^\varepsilon(x, t) \quad i = 1, 2, 3, t \in (0, \infty) \end{aligned} \quad (1.1)$$

where $\Omega = (0, L)^3$ with periodic boundary conditions and (e_1, \dots, e_d) is the natural basis of \mathbb{R}^d . Here $\varepsilon > 0$ is the artificial dissipation parameter, u^ε is the velocity vector field, p is the pressure, $\nu > 0$ is the kinematic viscosity of the fluid and f is a given force field. For $\varepsilon = 0$, the model is reduced to 3D Navier–Stokes system.

Mathematical model for such fluid motion has been used extensively in turbulence simulations (see e.g. [3, 4, 7, 10]). For further discussion of theoretical results concerning (1.1), see [1, 2, 5, 12, 15, 16, 20, 23].

In the work [23], the strong convergence of the solution of this problem to the solution of the conventional system as the regularization parameter goes to zero, was established for each dimension $d \leq 4$.

For the 3D Navier–Stokes system weak solutions of problem are known to exist by a basic result by J. Leray from 1934 [11], only the uniqueness of weak solutions remains as an open problem. Then the known theory of global attractors of infinite dimensional dynamical systems is not applicable to the 3D Navier–Stokes system.

The theory of trajectory attractors for evolution partial differential equations was developed in [14, 18], which the uniqueness theorem of solutions of the corresponding initial-value problem is not proved yet, e.g. for the 3D Navier–Stokes system (see [8, 14, 17, 18]). Such trajectory attractor is a classical global attractor

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but in the space of weak solutions defined on $[0, \infty)$, with the corresponding semigroup being simply the translation in time of such solutions. A compact set $\mathfrak{A} \Subset E$ is said to be a global attractor of a semigroup $\{S(t), t > 0\}$ acting in a Banach or Hilbert space E if \mathfrak{A} is strictly invariant with respect to $\{S(t)\} : S(t)\mathfrak{A} = \mathfrak{A} \forall t \geq 0$ and \mathfrak{A} attracts any bounded set $B \subset E : \text{dist}(S(t)B, \mathfrak{A}) \rightarrow 0 (t \rightarrow \infty)$ (see [13], [14], [17], [18], [20]).

In this article, we study the upper semicontinuity, of the global attractors of the Leray-Hopf weak solutions of a regularized 3D Navier-Stokes equations, as the artificial dissipation ε goes to 0. While there exist other examples of such robustness in the literature of the Navier-Stokes equations, the specific emphasis on the regularized problem is new for the 3D Navier-Stokes equations and is of interest. This would be an extension of the earlier work on Ou and Sritharan for the 2D Navier-Stokes equations, see references [15] and [16]. It is now known that there is a global attractor \mathfrak{A}_0 for the Leray-Hopf weak solutions of the 3D Navier-Stokes equations, see Sell [17] or [18].

The main object of this paper to show that there is a global attractor, which one might denote by \mathfrak{A}_ε , for the regularized problem (1.1), and that the family $\{\mathfrak{A}_\varepsilon\}$ is upper semicontinuous at $\varepsilon = 0$. Moreover, we can modify the argument described above so that the final result will have broader applicability by allowing the family of forcing functions f^ε to vary with ε , for $\varepsilon > 0$.

The family of sets $\mathfrak{A}_\varepsilon, 0 < \varepsilon \leq 1$ is robust at \mathfrak{A}_0 , or is upper semicontinuous with respect to ε at $\varepsilon_0 = 0$, provided that, for every $\varepsilon_0 > 0$, there is a neighborhood $O(\varepsilon_0)$ of $0 \in \mathbb{R}$ and a neighborhood $N_{\varepsilon_0}(\mathfrak{A}_0)$ of \mathfrak{A}_0 , such that $\mathfrak{A}_\varepsilon \subset N_{\varepsilon_0}(\mathfrak{A}_0)$, for every $\varepsilon \in O(\varepsilon_0)$ with $\varepsilon > 0$, see (23.13) in [18].

The paper is organized as follows. In Section 2, we present the relevant mathematical framework for the paper. In Section 3, we recall the definition of the trajectory attractor \mathfrak{A}_0 of the conventional 3-D Navier-Stokes equations. In Section 4, we study the regularized problem (see equation (1.1)), then we show the existence of trajectory attractor \mathfrak{A}_ε . In Section 5, we present the main result of this paper, that is, a theorem on the upper semicontinuity on the attractors \mathfrak{A}_ε . Finally, an application of our general results to the study of the robustness of the system (1.1) with a perturbed external force.

2. PRELIMINARY

We denote by $H^m(\Omega)$, the Sobolev space of L -periodic functions endowed with the inner product

$$(u, v) = \sum_{|\beta| \leq m} (D^\beta u, D^\beta v)_{L^2(\Omega)} \text{ and the norm } \|u\|_m = \sum_{|\beta| \leq m} (\|D^\beta u\|_{L^2(\Omega)})^{\frac{1}{2}}$$

and by $H^{-m}(\Omega)$ the dual space of $H^m(\Omega)$. We denote by $\dot{H}^m(\Omega)$ the subspace of $H^m(\Omega)$ with, zero average $\dot{H}^m(\Omega) = \{u \in H^m(\Omega); \int_\Omega u(x) dx = 0\}$.

- We introduce the following solenoidal subspaces $V_s, s \in \mathbb{R}^+$ which are important to our analysis

$$V_0(\Omega) = \{u \in \dot{L}^2(\Omega), \text{div} u = 0, u.n|_{\Sigma_i} = -u.n|_{\Sigma_{i+3}}, i = 1, 2, 3\};$$

$$V_1(\Omega) = \{u \in \dot{H}^1(\Omega), \text{div} u = 0, \gamma_0 u|_{\Sigma_i} = \gamma_0 u|_{\Sigma_{i+3}}, i = 1, 2, 3\}.$$

$$V_2(\Omega) = \{u \in \dot{H}^2(\Omega), \text{div} u = 0, \gamma_0 u|_{\Sigma_i} = \gamma_0 u|_{\Sigma_{i+3}}, \gamma_1 u|_{\Sigma_i} = -\gamma_1 u|_{\Sigma_{i+3}}, i = 1, 2, 3\}.$$

see [20]. We refer the reader to Temam [21] for details on these spaces. Here the faces of Ω are numbered as

$$\Sigma_i = \partial\Omega \cap \{x_i = 0\} \text{ and } \Sigma_{i+3} = \partial\Omega \cap \{x_i = L\}, \quad i = 1, 2, 3.$$

Here γ_0, γ_1 are the trace operators and n is the unit outward normal on $\partial\Omega$.

- The space V_0 is endowed with the inner product $(u, v)_{L^2(\Omega)}$ and norm $\|u\| = (u, u)_{L^2(\Omega)}^{1/2}$.
- V_1 is the Hilbert space with the norm $\|u\|_1 = \|u\|_{V_1}$. The norm induced by $\dot{H}^1(\Omega)$ and the norm $\|\nabla u\|_{L^2(\Omega)}$ are equivalent in V_1 .
- V_2 is the Hilbert space with the norm $\|u\|_2 = \|u\|_{V_2}$. In V_2 the norm induced by $\dot{H}^2(\Omega)$ is equivalent to the norm $\|\Delta u\|_{L^2(\Omega)}$.

V'_s denote the dual space of V_s . We present the topology to be used for generating the neighborhood of robustness. Let F any vector space. A metric $d(f, g)$ on F is said to be invariant if one has

$$d(f, g) = d(f - g, 0) \text{ for all } f, g \in F.$$

A Fréchet space is a complete topological vector space whose topology is induced by a translation invariant metric $d(f, g)$. Given a Banach space X , with norm $\|\cdot\|_X$ and $1 \leq p < \infty$, we denote by $L^p_{loc}[0, \infty; X)$ the Fréchet space of measurable functions $f : [0, \infty) \rightarrow X$ that are p -integrable over $[0, T]$, for each $0 < T < \infty$, endow with the metric

$$d(f, g) = \sum_{n=1}^{\infty} 2^{-n} \min(\|f - g\|_{L^p(0, n; X)}, 1).$$

We denote by $L^p_{loc}(0, \infty; X)$ the Fréchet space of measurable functions $f : (0, \infty) \rightarrow X$ that are p -integrable over $[t_0, T]$, for each $0 < t_0 \leq T < \infty$ endow with the metric

$$d(f, g) = \sum_{n=2}^{\infty} 2^{-n} \min(\|f - g\|_{L^p(\frac{1}{n}, n; X)}, 1).$$

Similarly for $p = \infty$, we will let $L^\infty_{loc}(0, \infty; X)$ denote the collection of all functions $f : (0, \infty) \rightarrow X$ with the property that, for all τ and T with $0 < T < \infty$, one has $\text{ess sup}_{0 < s < T} \|f\|_X < \infty$. We denote by $C[0, \infty; X)$ the space of strongly continuous functions from $[0, \infty)$ to X , endowed with the topology of the uniform convergence over compact sets and by $C_w[0, \infty; X)$ the space of weakly continuous functions from $[0, \infty)$ to X . We denote by $L^\infty C = L^\infty(\mathbb{R}, X) \cap C(\mathbb{R}, X)$ the Fréchet space $L^\infty C$ endow with the L^∞_{loc} -topology, wich is the topology of uniform convergence on bounded sets.

Let E be a complete metric space with metric d . We write B_r for the open ball centre $0 \in E$ and radius r . The following quantity is called the Hausdorff (non-symmetric) semidistance from a set X to a set Y in a Banach space E

$$\text{dist}_E(X, Y) = \sup_{x \in X} \inf_{y \in Y} \|\cdot\|_E.$$

Let M be a subset of E and let $\mathbb{R}^+ = [0, \infty)$. A mapping $\sigma = \sigma(u, t)$, where $\sigma : M \times [0, \infty) \rightarrow M$ is said to be a semiflow on M provided the following hold

1) $\sigma(w, 0) = w$, for all $w \in M$.

2) The semigroup property holds, i. e,

$$\sigma((w, s), t) = \sigma(w, s + t) \text{ for all } w \in M \text{ and } s, t \in \mathbb{R}^+.$$

3) The mapping $\sigma : M \times (0, \infty) \rightarrow M$ is continuous.

If in addition the mapping $\sigma : M \times [0, \infty) \rightarrow M$ is continuous we will say that the semiflow is continuous at $t = 0$. Here we use $t > 0$ in order that the Robustness Theorem 23.14 in [18] is valid, see Sell [18] and Hale [8]. For any $u \in M$ the positive trajectory through u is defined as the set $\gamma^+(u) = \{\sigma(t)u, t \geq 0\}$. For any set $B \subset M$ we define the positive hull $\mathcal{H}^+(B)$ and the omega limit set $\omega(B)$ as follows

$$\mathcal{H}^+(B) = Cl_M \gamma^+(B) \text{ and } \omega(B) = \bigcap_{\tau \geq 0} \mathcal{H}^+(\sigma(\tau)B).$$

If $\mathcal{A} \subset E$ and $\varepsilon > 0$ we write

$$N_\varepsilon(\mathcal{A}) = \{z \in E, \inf_{a \in \mathcal{A}} d(z, a) < \varepsilon\}.$$

for the open ε -neighbourhood of \mathcal{A} .

We denote by A the Stokes operator $Au = -\Delta u$ for $u \in D(A)$. We recall that the operator A is a closed positive self-adjoint unbounded operator, with $D(A) = \{u \in V_0, Au \in V_0\}$. We have in fact, $D(A) = \dot{H}^2(\Omega) \cap V_0 = V_2$. The spectral theory of A allows us to define the powers A^l of A for $l \geq 1$, A^l is an unbounded self-adjoint operator in V_0 with a domain $D(A^l)$ dense in $V_2 \subset V_0$. We set here

$$A^l u = (-\Delta)^l u \text{ for } u \in D(A^l) = V_{2l} \cap V_0.$$

The space $D(A^l)$ is endowed with the scalar product and the norm

$$(u, v)_{D(A^l)} = (A^l u, A^l v), \|u\|_{D(A^l)} = \{(u, u)_{D(A^l)}\}^{\frac{1}{2}}.$$

Now define the trilinear form $b(., ., .)$ associated with the inertia terms

$$b(u, v, w) = \sum_{i,j=1}^3 \int_{\Omega} u_i \frac{\partial v_j}{\partial x_i} w_j dx$$

The continuity property of the trilinear form enables us to define (using Riesz representation theorem) a bilinear continuous operator $B(u, v); V_2 \times V_2 \rightarrow V_2'$ will be defined by

$$\langle B(u, v), w \rangle = b(u, v, w), \forall w \in V_2.$$

Recall that for u satisfying $\nabla \cdot u = 0$ we have

$$b(u, u, u) = 0 \text{ and } b(u, v, w) = -b(u, w, v). \quad (2.1)$$

Hereafter, $c_i \in \mathbb{N}$, will denote a dimensionless scale invariant positive constant which might depend on the shape of the domain. The trilinear form $b(., ., .)$ is continuous on $\dot{H}^{m_1}(\Omega) \times \dot{H}^{m_2+1}(\Omega) \times \dot{H}^{m_3}(\Omega)$, $m_i \geq 0$

$$|b(u, v, w)| \leq c_0 \|u\|_{m_1} \|v\|_{m_2+1} \|w\|_{m_3}, \quad m_3 + m_2 + m_1 \geq \frac{3}{2} \quad (2.2)$$

see [6, 18]. The bilinear operator satisfies the inequality

$$\|Bu\| \leq c_1 \|A^{\frac{1}{2}} u\|^{\frac{3}{2}} \|Au\|^{\frac{1}{2}}, \quad (2.3)$$

see [21, Lemma 3.8]. Similarly, we define $\hat{B}(u, v) \in V_1'$ by

$$\langle \hat{B}(u, v), w \rangle_{V_1' \times V_1} = b(u, v, w), \quad \forall w \in V_1.$$

We recall some inequalities that we will be using in what follows.

Young's inequality

$$ab \leq \frac{\epsilon}{p} a^p + \frac{1}{q\epsilon^{\frac{q}{p}}} b^q, a, b, \epsilon > 0, p > 1, q = \frac{p}{p-1}. \quad (2.4)$$

Poincaré's inequality

$$\lambda_1 \|u\|^2 \leq \|u\|_1^2 \text{ for all } u \in V_0, \quad (2.5)$$

where λ_1 is the smallest eigenvalue of the Stokes operator A .

3. NAVIER-STOKES EQUATIONS

The conventional Navier-Stokes system can be written in the evolution form

$$\begin{aligned} \frac{\partial u}{\partial t} + \nu Au + \hat{B}(u, u) &= f, \quad t > 0, \\ u_0(x) &= u_0. \end{aligned} \quad (3.1)$$

Let $f \in L^\infty(0, \infty; V_0)$ be given. We will say that a function u is a weak solution of the 3D Navier-Stokes of Class LH (Leray–Hopf) on $[0, \infty)$ provided that $u(x, 0) = u_0(x) \in V_0$, and the following properties hold

$$1) \quad u \in L^\infty(0, \infty; V_0) \cap L^2_{loc}(0, \infty; V_1).$$

$$2) \quad \frac{du}{dt} \in [L^{\frac{4}{3}}_{loc}(0, \infty; V'_1)].$$

Taking the inner product of (3.1) with u , and using (2.4) we have

$$\frac{d}{dt} \|u(t)\|^2 + 2\nu \|\nabla u\|^2 = 2\langle f, u \rangle. \quad (3.2)$$

by application of Young's inequality and the Poincaré's Lemma, yields

$$\frac{d}{dt} \|u(t)\|^2 + \nu \|\nabla u\|^2 \leq \frac{\|f\|^2}{\nu\lambda_1}, \quad (3.3)$$

using the Poincaré Lemma and Gronwall's inequality, to get

$$\|u(t)\|^2 \leq e^{-\nu\lambda_1(t-t_0)} \|u(t_0)\|^2 + \frac{1}{\nu^2\lambda_1^2} \|f\|^2 \left(1 - e^{-\nu\lambda_1(t-t_0)}\right), \text{ with } 0 < t_0 < t,$$

3) which implies that

$$\|u(t)\|^2 \leq e^{-\nu\lambda_1(t-t_0)} \|u(t_0)\|^2 + \frac{1}{\nu^2\lambda_1^2} \|f\|^2. \quad (3.4)$$

Integrating (3.2) over $[t_0, t]$ we find that

$$\|u(t)\|^2 + 2\nu \int_{t_0}^t \|A^{\frac{1}{2}}u(s)\|^2 ds \leq \|u(t_0)\|^2 + 2 \int_{t_0}^t \langle f(s), u(s) \rangle ds. \quad (3.5)$$

4) The function u satisfies the following equality

$$\langle u(t) - u(t_0), v \rangle + \nu \int_{t_0}^t \langle A^{\frac{1}{2}}u(s), A^{\frac{1}{2}}v \rangle ds + \int_{t_0}^t \langle \hat{B}(u(s), u(s)), v \rangle ds = \int_{t_0}^t \langle f, v \rangle ds, \quad (3.6)$$

for all $v \in V_1$ and for all $t \geq t_0 \geq 0$.

The proof of the following theorem is given in [12, 13, 21].

Theorem 3.1. *Let $f \in V'_1$ and $u_0 \in V_0$ be given. Then for every $T > 0$, there exists a weak solution $u(t)$ of (3.1) from the space $L^2(0, T; V_1) \cap L^\infty(0, T; V_0)$, such that $u(x, 0) = u_0$ and $u(t)$ satisfies the energy equality (3.6).*

Moreover (see [21]), $u(\cdot)$ is weakly continuous from $[0, T]$ into V_0 , the function $u \in C_w([0, T]; V_0)$ and consequently $u(x, 0) = u_0(x) \in V_0$. Let W is the set of all Leray–Hopf weak solutions $u(\cdot)$ of equation (3.1) in the space $L^\infty(0, \infty; V_0) \cap L^2_{loc}[0, \infty; V_1)$ that satisfy the following properties

- $\frac{du}{dt} \in L^{\frac{4}{3}}_{loc}(0, \infty; V'_1)$;
- for almost all t and t_0 , with $t > t_0 > 0$, inequalities (3.5 ,3.6) are valid.

Let X^0 denote the Fréchet space used to define the Leray-Hopf weak solutions. Thus

$$\varphi \in X^0 = L^\infty(0, \infty; V_0) \cap L^2_{loc}[0, \infty; V_1),$$

where $\varphi \in C_w[0, \infty; V_0)$ and we let \mathfrak{F}^0 denote a compact, translation invariant set of forcing functions f in

$$L^\infty C = L^\infty(\mathbb{R}, L^2(\Omega)) \cap C(\mathbb{R}, L^2(\Omega))$$

where the topology on the Fréchet space $L^\infty C$ is the topology of uniform convergence on bounded sets in \mathbb{R} .

Then, we use the Leray-Hopf solutions of the 3D Navier-Stokes equations with $\varepsilon = 0$ to generate a semiflow π^0 on $\mathfrak{F}^0 \times X^0$, where

$$\pi^0(\tau)(f, \varphi) = (f_\tau, S^0(f, \tau)\varphi) \text{ for } \tau \geq 0,$$

$f_\tau(t) = f(\tau + t)$ and $u(t) = S^0(f, t)\varphi$ is the Leray-Hopf solution of the 3D Navier-Stokes equations that satisfies $u(0) = S^0(f, 0)\varphi = \varphi(0)$. By using the theory of generalized weak solutions, as in Sell [17] or [18], we note that π^0 has a global attractor $\mathfrak{A}_0 \subset \mathfrak{F}^0 \times X^0$ see Theorem 65.12 in [18].

4. THE REGULARIZED NAVIER-STOKES SYSTEM

Using the operators defined in the previous section, we can write the modified system (1.1) in the evolution form

$$\begin{aligned} \partial_t u^\varepsilon + \varepsilon A^1 u^\varepsilon + \nu A u^\varepsilon + B(u^\varepsilon, u^\varepsilon) &= f(x), & \text{in } \Omega \times (0, \infty) \\ u^\varepsilon(x) &= u^\varepsilon, & \text{in } \Omega. \end{aligned} \quad (4.1)$$

For $\varepsilon > 0$, we let π^ε denote the semiflow on $\mathfrak{F}^0 \times X^0$ generated by the weak solutions of regularized 3D Navier-Stokes equations of (4.1). Thus

$$\pi^\varepsilon(\tau) = (f_\tau, S^\varepsilon(f, \tau)\varphi), \quad (4.2)$$

where $u_0^\varepsilon = \varphi$ and

$$u^\varepsilon(t) = S^\varepsilon(f, t)\varphi = S^\varepsilon(f, t)u_0^\varepsilon \quad (4.3)$$

is the weak solution of (4.1) that satisfies $u^\varepsilon(0) = \varphi(0) = u_0^\varepsilon(0)$.

The existence and uniqueness results for initial value problem (1.1) can be found in [12, Remark 6.11].

The following theorem collects the main result in this work

Theorem 4.1. *For $l \geq \frac{5}{4}$, for $\varepsilon > 0$ fixed, $f \in L^2(0, T; V'_0)$ and $u_0^\varepsilon \in V_0$ be given. There exists a unique weak solution of (4.1) which satisfies*

$$u^\varepsilon \in L^2(0, T; V_1) \cap L^\infty(0, T; V_0), \forall T > 0.$$

$u^\varepsilon \in L^\infty(0, T; V_0) \cap L^2[0, T; V_1)$ and $u^\varepsilon \in C_w([0, T]; V_0), \forall T > 0$.

We recall Lemma 3.6 [23].

Lemma 4.2. u^ε is almost everywhere equal to a continuous function from $[0, T]$ to the space V_0 .

and the following theorem

Theorem 4.3. For $l \geq \frac{3}{2}$, the weak solution u^ε of the modified Navier-Stokes equations (4.1) given by Theorem 4.1 converges strongly in $L^2(0, T; V_0)$ as $\varepsilon \rightarrow 0$ to u the weak solution of the Navier-Stokes equations.

The above theorem is established directly by using of a general result [23, Theorem 3.9].

Lemma 4.4. For fixed $\varepsilon > 0$ and for all $0 \leq t \leq T$, $u^\varepsilon \in B_{R_1} = \{u_0^\varepsilon \in V_1; \|u^\varepsilon\|_1 \leq R_1\}$ there is a constant C_1 positive independent of ε such that

$$\int_0^T \|Au^\varepsilon(t)\|^2 dt \leq C_1. \quad (4.4)$$

C_1 independent of ε is given in the proof.

Proof. Taking the inner product of (4.1) with Au^ε , we find

$$\frac{1}{2} \frac{d}{dt} \|A^{\frac{1}{2}} u^\varepsilon\|^2 + \varepsilon (A^l u^\varepsilon, Au^\varepsilon) + \nu \|Au^\varepsilon\|^2 + b(u^\varepsilon, u^\varepsilon, Au^\varepsilon) = (f, Au^\varepsilon).$$

by using (2.3), to obtain

$$|b(u^\varepsilon, u^\varepsilon, Au^\varepsilon)| \leq c_1 \|A^{\frac{1}{2}} u^\varepsilon\|^{\frac{3}{2}} \|Au^\varepsilon\|^{\frac{1}{2}} \|Au^\varepsilon\|.$$

Due to (2.4) and (2.5), we get

$$(A^l u^\varepsilon, Au^\varepsilon) = \|A^{\frac{l+1}{2}} u^\varepsilon\|^2.$$

We have the following inequality

$$\frac{1}{2} \frac{d}{dt} \|A^{\frac{1}{2}} u^\varepsilon\|^2 + \varepsilon \|A^{\frac{l+1}{2}} u^\varepsilon\|^2 + \frac{\nu}{2} \|Au^\varepsilon\|^2 \leq \frac{1}{\nu} \|f\|_\infty^2 + \frac{27c_1^4}{2\nu^3} \|A^{\frac{1}{2}} u^\varepsilon\|^6. \quad (4.5)$$

For $u^\varepsilon \in B_{R_1}$, we obtain

$$\frac{1}{2} \frac{d}{dt} \|A^{\frac{1}{2}} u^\varepsilon\|^2 + \varepsilon \|A^{\frac{l+1}{2}} u^\varepsilon\|^2 + \frac{\nu}{2} \|Au^\varepsilon\|^2 \leq \frac{1}{\nu} \|f\|_\infty^2 + \frac{27c_1^4}{2\nu^3} R_1^6.$$

We drop the positive terms $\varepsilon \|A^{\frac{l+1}{2}} u^\varepsilon\|^2$ and integrate from 0 to T to get

$$\int_0^T \|Au^\varepsilon\|^2 \leq \frac{2T}{\nu} \|f\|_\infty^2 dt + T \frac{27c_1^4}{\nu^4} R_1^6 + \frac{2}{\nu} R_1^2 = C_1, \quad (4.6)$$

this completes the proof of Lemma 4.4. \square

Theorem 4.5. For fixed $\varepsilon \geq 0$, $u^\varepsilon \in B_{R_1} = \{u^\varepsilon(0) \in V_1; \|u^\varepsilon\|_1 \leq R_1\}$ and $f \in L^\infty C$ a time independent functions, π^ε is a continuous family of semiflows on X^0 .

Proof. Let convergent sequences ε_n , φ^n and f^n , with limits $\varepsilon_n \rightarrow \varepsilon_0$ (especially with $\varepsilon_0 = 0$), $\varphi^n \rightarrow \varphi_0$ in the X^0 -topology and $f^n \rightarrow f^0$ in the $L^\infty C$ -topology as $n \rightarrow \infty$, then

$$S^{\varepsilon_n}(f^n, t) \varphi^n \rightarrow S^{\varepsilon_0}(f^0, t) \varphi^0. \quad (4.7)$$

Let

$$S^{\varepsilon_n}(f^n, t) \varphi^n - S^{\varepsilon_0}(f^0, t) \varphi^0 = u^{\varepsilon_n}(t) - u^{\varepsilon_0}(t), \quad (4.8)$$

we obtain for $w_n = u^{\varepsilon_n}(t) - u^{\varepsilon_0}(t)$ and $g_n = f^n - f^0$

$$\partial_t w_n + \varepsilon_n A^l w_n + A w_n + B(u^{\varepsilon_n}, u^{\varepsilon_n}) - B(u^{\varepsilon_0}, u^{\varepsilon_0}) = g_n. \quad (4.9)$$

By taking inner product with w_n for above equation we get

$$\frac{1}{2} \frac{d}{dt} \|w_n\|^2 + \varepsilon_n \|A^l w_n\|^2 + \nu \|A^{\frac{1}{2}} w_n\|^2 = b(w_n, w_n, u^{\varepsilon_n}) + (g_n, w_n). \quad (4.10)$$

Using Young's inequality, we obtain

$$(g_n, w_n) \leq \frac{1}{2\nu} \|g_n\|^2 + \frac{\nu}{2} \|A^{\frac{1}{2}} w_n\|^2,$$

thanks to (2.2) with $(m_3, m_2, m_1) = (0, 1, 1)$, we find

$$|b(w_n, u^{\varepsilon_n}, w_n)| \leq c_0 \|A^{\frac{1}{2}} w_n\| \|Au^{\varepsilon_n}\| \|w_n\| \quad (4.11)$$

this give

$$|b(w_n, u^{\varepsilon_n}, w_n)| \leq \frac{\nu}{2} \|A^{\frac{1}{2}} w_n\|^2 + \frac{c_4}{2} \|Au^{\varepsilon_n}\|^2 \|w_n\|^2. \quad (4.12)$$

Combining all these inequalities in (4.10), we have

$$\frac{1}{2} \frac{d}{dt} \|w_n\|^2 + \varepsilon_n \|A^l w_n\|^2 \leq \frac{c_4}{2} \|Au^{\varepsilon_n}\|^2 \|w_n\|^2 + \frac{1}{2\nu} \|g_n\|^2.$$

We drop the positive term $\varepsilon_n \|Aw_n\|^2$ to obtain the following differential inequality

$$\frac{d}{dt} \|w_n(t)\|^2 \leq c_4 \|Au^{\varepsilon_n}\|^2 \|w_n\|^2 + \frac{1}{\nu} \|g_n\|^2. \quad (4.13)$$

Applying now Gronwall's inequality to (4.13), for $t \geq 0$ we have

$$\begin{aligned} \|w_n(t)\|^2 &\leq \|w_n(0)\|^2 \exp c_4 \left(\int_0^t \|Au^{\varepsilon_n}\|^2 ds \right) \\ &\quad + \frac{1}{\nu} \int_0^t \|g_n(h)\|^2 \exp c_4 \left(\int_0^t \|Au^{\varepsilon_n}\|^2 ds \right) dh \end{aligned} \quad (4.14)$$

using estimate (4.4) we find

$$\|w_n(t)\|^2 \leq C_2 \|w_n(0)\|^2 + \frac{TC_2}{\nu} \|g_n\|^2 \quad (4.15)$$

for all t in compact sets in $[0, \infty)$, $C_2 = \exp c_4 C_1$. Since $f^n \rightarrow f^0$ in the $L^\infty C$ -topology and $\varphi^n \rightarrow \varphi_0$ in the X^0 -topology this means that $\|g_n\| \rightarrow 0$ and $\|w_n(0)\| \rightarrow 0$ as $n \rightarrow \infty$, it follows from (4.15) that

$$\|S^{\varepsilon_n}(f^n, t) \varphi^n - S^{\varepsilon_0}(f^0, t) \varphi^0\| \leq C_2 \|u_0^{\varepsilon_n} - u_0^{\varepsilon_0}\|^2 + \frac{TC_2}{\nu} \|f^n - f\|^2 \rightarrow 0, \text{ as } n \rightarrow \infty.$$

It follows that π^ε is continuous semiflows on X^0 . Hence π^ε approximates π^0 on B_{R_1} uniformly on $[0, T]$. \square

Regarding the existence of the attractor \mathfrak{A}_ε when $\varepsilon > 0$, we use especially the related papers of Chepyzhov and Vishik, such as [14] to show that the system (4.1) possesses a global attractor. For $\varepsilon > 0$, we consider the trajectory space \mathcal{K}_ε of the modified Navier-Stokes equations (4.1). \mathcal{K}_ε is the union of all weak solutions $u^\varepsilon \in X^0$ that satisfy (4.1), see (6.163) in [12]. Using the described scheme in [14], we construct the spaces \mathcal{S}_b

$$\mathcal{S}_b = \{v(\cdot) \in L^\infty(0, T; V_0) \cap L_b^2(0, T; V_1), \partial_t v(\cdot) \in L_b^2(0, T; D(A^l)')\}$$

with norm

$$\|v\|_{\mathcal{S}_b} = \|v\|_{L_b^2(0, T; V_1)} + \|v\|_{L^\infty(0, T; V_0)} + \|\partial_t v\|_{L_b^2(0, T; D(A^l)')}$$

where

$$\|v\|_{L_b^2(0,T;V_1)} = \sup_{t \geq 0} \left(\int_t^{t+1} \|v(s)\|_1^2 ds \right)^{\frac{1}{2}}, \quad \|v\|_{L^\infty(0,T;V_0)} = \operatorname{ess\,sup}_{t \geq 0} \|v\|$$

and

$$\|\partial_t v\|_{L_b^2(0,T;D(A^1)')} = \sup_{t \geq 0} \left(\int_t^{t+1} \|v(s)\|_{D(A^1)'}^2 ds \right)^{\frac{1}{2}}.$$

We need a topology in the space \mathcal{K}_ε . We define on X^0 the following sequential topology which we denote Γ .

By definition, a sequence of functions $\{v_n\} \subseteq X^0$ converges to a function $v \in X^0$ in the topology Γ as $n \rightarrow \infty$ if, for any $T > 0$, $v_n \rightarrow v$ weakly in $L^2(0, T; V_1)$; $v_n \rightarrow v$ weak-* in $L^\infty(0, T; V_0)$ and $v_n \rightarrow v$ strongly in $L^2(0, T; V_0)$, as $n \rightarrow \infty$.

We consider the topology Γ on \mathcal{K}_ε . It is easy to prove that the space \mathcal{K}_ε is closed in Γ . From the definition of \mathcal{K}_ε , it follows that $\pi^\varepsilon \mathcal{K}_\varepsilon \subset \mathcal{K}_\varepsilon$ for all $t \geq 0$.

Proposition 4.6. *If $u^\varepsilon(t)$ is a solution of (4.1), then the following inequalities hold for all $t > 0$*

$$\|u^\varepsilon(t)\|^2 \leq e^{-\nu\lambda_1 t} \|u_0^\varepsilon\|^2 + \frac{\|f\|^2}{\nu^2 \lambda_1^2}, \quad (4.16)$$

$$\int_t^{t+1} \|u^\varepsilon(s)\|^2 ds \leq \frac{e^{-\nu\lambda_1 t}}{\nu\lambda_1} \|u_0^\varepsilon\|^2 + \frac{\|f\|^2}{\nu^2 \lambda_1^2}, \quad (4.17)$$

$$\nu \int_t^{t+1} \|u^\varepsilon(s)\|_1^2 ds \leq \frac{e^{-\nu\lambda_1 t}}{\nu\lambda_1} \|u_0^\varepsilon\|^2 + \frac{\|f\|^2}{\nu^2 \lambda_1^2} + \frac{\|f\|^2}{\nu\lambda_1}. \quad (4.18)$$

Proof. Taking the inner product of (4.1) by $u^\varepsilon \in V_2$, we obtain

$$\frac{d}{dt} \|u^\varepsilon\|^2 + 2\varepsilon \|A^l u^\varepsilon\|^2 + 2\nu \|\nabla u^\varepsilon\|^2 = 2(f, u^\varepsilon). \quad (4.19)$$

Applying Young's inequality and using the Poincaré Lemma, we obtain

$$\frac{d}{dt} \|u^\varepsilon\|^2 + \nu \|\nabla u^\varepsilon\|^2 \leq \frac{\|f\|^2}{\nu\lambda_1}. \quad (4.20)$$

Using the Gronwall's inequality over $[0, t]$, we obtain (4.16). Integrating (4.16) over $[t, t+1]$ we find (4.17). Integrating (4.20) over $[t, t+1]$ we find

$$\nu \int_t^{t+1} \|\nabla u^\varepsilon(s)\|^2 ds \leq \frac{\|f\|^2}{\nu\lambda_1} + \|u^\varepsilon(t)\|^2.$$

Applying inequality (4.16), we have (4.18). \square

A simple consequence of Lemma 3.6 [23] is the following Lemma

Proposition 4.7. *Let $f \in V_0$. Then any solution $u^\varepsilon(t)$ of (4.1) satisfies*

$$\int_t^{t+1} \|\partial_t u^\varepsilon(s)\|_{D(A^1)'}^2 ds \leq C_3, \quad (4.21)$$

C_3 is a positive constant independent of ε .

Moreover, due to estimates (4.16) and (4.21), we also have the uniform estimate.

Proposition 4.8. *If $f \in V_0$, then any solution $u^\varepsilon(t)$ of problem (4.1) satisfies the inequality*

$$\|\pi^\varepsilon(u^\varepsilon)\|_{\mathcal{S}_b}^2 \leq \frac{c_7 e^{-\nu\lambda_1 t}}{\nu\lambda_1} \|u^\varepsilon(0)\|^2 + \frac{c_7 \|f\|^2}{\nu^2 \lambda_1^2} + C_4 \quad (4.22)$$

where the positive constant C_4 is independent of ε .

From Proposition 4.6 it follows that $\mathcal{K}_\varepsilon \subset \mathcal{S}_b$ for all $\varepsilon > 0$ and for all $\tau > 0$. Also Proposition 4.6 implies that the semigroup π^ε has absorbing set in \mathcal{K}_ε for all $\varepsilon > 0$ and for all $\tau > 0$ (We note, that this absorbing set does not depend on ε , since the constant C_4 in (4.22) is independent of ε), bounded in \mathcal{S}_b and inequality (4.22) implies that absorbing set is compact in Γ . The continuity of π^ε is proved. These facts are sufficient to state that π^ε has a global attractor \mathfrak{A}_ε . Such that $\mathfrak{A}_\varepsilon \subset \mathfrak{F}^0 \times X^0$, bounded in \mathcal{S}_b and compact in Γ . For a more detailed, see [14].

5. UPPER SEMICONTINUITY OF ATTRACTORS

We now prove the robustness property for the global attractor \mathfrak{A}_ε . We have shown in Theorem 4.5 the continuity of the family of semiflows π^ε on X^0 . Having done this, We can simply invoke Theorem 23.14 in [18] to complete the proof of the robustness for the family of attractors \mathfrak{A}_ε at $\varepsilon = 0$. Clearly, it is sufficient to show that the small ε_0 -neighbourhood of attractor \mathfrak{A}_0 is an absorbing set and that π^ε approximates π^0 on $B_{R_1} = \{u^\varepsilon(0) \in V_1; \|u_0^\varepsilon\|_1 \leq R_1\}$ uniformly on compact sets of $[0, \infty)$.

Theorem 5.1. *For $\varepsilon > 0$ the family of semiflows π^ε generated by the weak solutions of regularized 3D Navier-Stokes equations (1.1) admits a compact attractor $\{\mathfrak{A}_\varepsilon, 0 < \varepsilon \leq 1\}$ which attracts bounded sets of V_0 and is contained in the absorbing balls $B_{R_0} \cap B_{R_1}$ where R_0 and R_1 are independent of ε . Moreover, $d_{X^0}(\mathfrak{A}_\varepsilon, \mathfrak{A}_0) \rightarrow 0$, as $\varepsilon \rightarrow 0$.*

Proof. Let $N_{\varepsilon_0}(\mathfrak{A}_0)$ be the ε_0 -neighborhood of \mathfrak{A}_0 . Since \mathfrak{A}_0 is a global attractor, for any bounded set $B_{R_0} = \{u(0) \in V_0; \|u(0)\| \leq R_0\} \subset V_0$, we have

$$d_{X^0}(\pi^0 B_{R_0}, \mathfrak{A}_0) \rightarrow 0, \text{ as } t \rightarrow \infty. \quad (5.1)$$

Thus, there exists $\varepsilon_0 > 0$ and $t > t_{\varepsilon_0}$ such that

$$d_{X^0}(\pi^0 B_{R_0}, \mathfrak{A}_0) \leq \frac{\varepsilon_0}{2}, \text{ for } t \geq t_{\varepsilon_0}. \quad (5.2)$$

Consequently

$$\pi^0(t) B_{R_0} \subset N_{\varepsilon_0}(\mathfrak{A}_0), \text{ for } t \geq t_{\varepsilon_0}. \quad (5.3)$$

This shows that $N_{\varepsilon_0}(\mathfrak{A}_0)$ is an absorbing set. To establish the second step, we drop the positive terms $\varepsilon \|A^{\frac{l+1}{4}} u^\varepsilon\|^2$ and $\frac{\nu}{2} \|Au^\varepsilon\|^2$ in inequality (4.5) and integrate from 0 to T we get

$$\frac{1}{2} \frac{d}{dt} \|A^{\frac{1}{2}} u^\varepsilon\|^2 \leq \frac{1}{\nu} \|f\|_\infty^2 + \frac{27c_1^4}{2\nu^3} R_1^4 \|A^{\frac{1}{2}} u^\varepsilon\|^2.$$

Integrating the above inequality from 0 to t , we get

$$\|A^{\frac{1}{2}} u^\varepsilon\|^2 \leq \|A^{\frac{1}{2}} u^\varepsilon(0)\|^2 \exp T \left(\frac{27c_1^4}{\nu^3} R_1^4 \right) + \frac{2T}{\nu} \|f\|_\infty^2 = \rho_1$$

This implies that any ball $B_{R_1} = \{u_0^\varepsilon \in V_1; \|A^{\frac{1}{2}} u_0^\varepsilon(0)\| \leq R_1\}$ in V_1 with radius $R_1 > \rho_1$ will satisfy

$$\pi^\varepsilon(t) B_{R_1} \subset B_{R_1}, \text{ for } t \geq 0. \quad (5.4)$$

This means if $u_0^\varepsilon \in B_{R_1}$, then $\pi^\varepsilon(t)u_0^\varepsilon$ is defined and belongs to B_{R_1} for $t \geq 0$. The ball B_{R_1} is therefore invariant under the map π^ε . Since π^ε approximates π^0 on B_{R_1} uniformly on $[0, T]$, we have for any $\varepsilon_0 > 0$, there are $\varepsilon_1 > 0$ and $\tau_0 > 0$ such that

$$\pi^\varepsilon(B_{R_0} \cap B_{R_1}) \subset N_{\varepsilon_0}(\mathfrak{A}_0), \text{ for } 0 < \varepsilon < \varepsilon_1, t \geq \tau_0. \quad (5.5)$$

Since the attractor \mathfrak{A}_ε is contained in $B_{R_0} \cap B_{R_1}$, an open neighborhood in the X^0 Fréchet space [18, Item (2) Theorem 23.14], we have

$$\pi^\varepsilon(\mathfrak{A}_\varepsilon) \subset N_{\varepsilon_0}(\mathfrak{A}_0), \text{ for } 0 < \varepsilon < \varepsilon_1, t \geq \tau_0. \quad (5.6)$$

Since \mathfrak{A}_ε is an invariant set, we deduce that

$$\mathfrak{A}_\varepsilon \subset N_{\varepsilon_0}(\mathfrak{A}_0), \text{ for } 0 < \varepsilon < \varepsilon_1, t \geq \tau_0. \quad (5.7)$$

Moreover, since ε_0 is arbitrary, we obtain the upper semicontinuity of \mathfrak{A}_ε , at $\varepsilon = 0$

$$d_{X^0}(\mathfrak{A}_\varepsilon, \mathfrak{A}_0) \rightarrow 0, \text{ as } \varepsilon \rightarrow 0. \quad (5.8)$$

□

One can modify the argument described above so that the final result will have broader applicability by allowing the family of forcing functions to vary with ε , for $\varepsilon > 0$. Thus, we consider the regularized Navier-Stokes system (1.1) with a perturbed external force f^ε in place of f , for $\varepsilon > 0$. Then (4.1) becomes

$$\begin{aligned} \partial_t u^\varepsilon + \varepsilon A^l u^\varepsilon + \nu A u^\varepsilon + B(u^\varepsilon, u^\varepsilon) &= f^\varepsilon(x), \quad \text{in } \Omega \times (0, \infty) \\ u^\varepsilon(x) &= u_0^\varepsilon, \quad \text{in } \Omega. \end{aligned} \quad (5.9)$$

We show that the trajectory attractor of the perturbed system (5.9) coincides with the trajectory attractor \mathfrak{A}_ε of the unperturbed system (1.1). Our results rely on the work of Hale ([14]) who show that the limit behaviour is valid even through \mathfrak{F}^ε , where \mathfrak{F}^ε denote a compact, translation invariant set of perturbed forcing functions to vary with ε , for $\varepsilon > 0$ and satisfy the condition

$$\omega(\mathcal{H}^+(f^\varepsilon)) = \omega(\mathcal{H}^+(f)). \quad (5.10)$$

Thus we would use \mathfrak{F}^ε in place of \mathfrak{F}^0 , for $\varepsilon > 0$. Moreover, by using a metric d on the $L^\infty C$ -toplogy, see [18] for some samples, we can note that (5.10) is equivalent to saying that for every $\delta > 0$ there is an $\varepsilon_1 > 0$ and $T_\delta = T(\delta) \geq 0$ such that

$$d_{X^0}(f^\varepsilon, \mathfrak{F}^0) \leq \delta, \text{ for } 0 < \varepsilon \leq \varepsilon_1 \text{ and } f^\varepsilon \in \mathfrak{F}^\varepsilon$$

for any $t \geq T_\delta$, that is

$$\mathfrak{F}^\varepsilon \subset N_\delta(\mathfrak{F}^0), \text{ for } 0 < \varepsilon \leq \varepsilon_1, \quad (5.11)$$

where N_δ denotes the δ -neighborhood of \mathfrak{F}^0 in $L^\infty C$. The resulting argument for robustness will then depend on two parameters $\lambda = (\varepsilon, \delta)$, where $\lambda \rightarrow (0, 0)$.

The following statement generalizes Theorem 5.1

Theorem 5.2. *Under the above conditions, the trajectory attractor of the perturbed 3D Navier-Stokes system (5.9) coincides with the trajectory attractor \mathfrak{A}_ε of the non-perturbed system (4.1). Moreover, the perturbed attractor of (5.9) is upper semicontinuous with respect to ε at $\varepsilon = 0$.*

Proof. The existence of trajectory attractor \mathfrak{A}_ε is treated above. The proof follows from formulas (5.10), (5.11) and Theorem 5.1. □

REFERENCES

- [1] J. Avrin, The Asymptotic Finite-dimensional Character of a Spectrally-hyperviscous Model of 3D Turbulent Flow. *J. Dynam. Diff. Eqns.* 20(2008), 1-40.
- [2] J. Avrin, Singular initial data and uniform global bounds for the hyperviscous Navier–Stokes equations with periodic boundary conditions. *J. Diff. Eq.* 190(2003), 330-351.
- [3] P. Bartello, O. Metais and M. Lesieur, Coherent structures in rotating three-dimensional turbulence, *J. Fluid Mech.*, 1994, vol. 273, pp. 1-29.
- [4] C. Basdevant, B. Legras, R. Sadourny, M. Béland, A study of barotropic model flows: intermittency, waves and predictability, *J. Atmos. Sci.* 38 (1981) 2305–2326.
- [5] Marco Cannone and Grzegorz Karch, About the regularized Navier-Stokes equations, *Journal of Mathematical Fluid Mechanics* 7, No. 1 (2005), 1 - 28.
- [6] P. Constantin, C. Foias, *Navier-Stokes Equations*, The University of Chicago Press, Chicago, 1988.
- [7] C. Foias, O. Manley, R. Rosa, and R. Temam, (2001). *Navier–Stokes Equations and Turbulence*. Cambridge University Press.
- [8] J. K. Hale, Asymptotic behavior of dissipative systems, *Mathematical Surveys and Monographs* n° 25, Amer. Math. Soc, Providence, R.I., 1988.
- [9] O. A. Ladyzhenskaya, Nonstationary Navier-Stokes equations. *Amer. Math. Soc. Transl.*, Vol. 25 (1962) pp. 151-160.
- [10] Bernard Legras, G. David, Dritschel, A comparison of the contour surgery and pseudo-spectral methods, *J. Comput. Phys.* 104 (2) (1993) 287–302.
- [11] J. Leray, Sur le mouvement d'un liquide visqueux emplissant l'espace., *Acta Mathematica*, 63 (1934), pp. 193–248.
- [12] J. L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires*, Dunod Gauthier-Villars, Paris, 1969.
- [13] V.V.Chepyzhov and M.I.Vishik, *Attractors for Equations of Mathematical Physics*. AMS Colloquium Publications. V. 49. Providence: AMS, 2002.
- [14] V.V.Chepyzhov and M.I.Vishik, Trajectory attractors for evolution equations, *C. R. Acad. Sci. Paris, Series I*, 321 (1995), 1309-1314.
- [15] Yuh-Roung Ou and S.S. Sritharan, Upper Semicontinuous Global Attractors for Viscous Flow, *Journal: Dynamic Systems and Applications* 5 (1996), 59-80.
- [16] Y. U. Ou and S. S. Sritharan, Analysis Of Regularized Navier-Stokes Equations I, *Quart. Appl. Math.* 49, 651-685 (1991).
- [17] G. Sell, Global attractors for the three-dimensional Navier–Stokes equations. *J.Dyn. Diff. Eq.*,8, 1–33 (1996).
- [18] G. Sell and Y.You, *Dynamics of Evolutionary Equations*, Springer-Verlag, 68, New york, 2002.
- [19] S.S. Sritharan, Deterministic and stochastic control of Navier-Stokes equation with linear, monotone, and hyperviscosities, *Appl. Math. Optim.* 41 (2) (2000) 255–308.
- [20] Roger. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, Applied Mathematical Sciences Series, 68, New york, Springer-Verlag, 2nd ed. 1997.
- [21] R. Temam, *Navier-Stokes Equations*. North-Holland Pub. Company, Amsterdam, 1979.
- [22] M.I. Vishik, A.V. Fursikov, *Mathematical Problems of Statistical Hydromechanics*, Kluwer Academic Publishers, Dordrecht, Boston, London, 1988.
- [23] Abdelhafid Younsi; Effect of hyperviscosity on the Navier-Stokes turbulence, *Electron. J. Diff. Equ.*, Vol. 2010(2010), No. 110, pp. 1-19.

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE , UNIVERSITY OF DJELFA , ALGERIA.
E-mail address: younsihafid@gmail.com