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UPPER SEMICONTINUITY OF TRAJECTORY ATTRACTORS OF 3D HYPERVISCOUS FLOW

ABDELHAFID YOUNSI

ABSTRACT. We regularized the 3D Navier-Stokes equations by adding a high-order viscosity term. We study the upper semicontinuity, of the global attractors of the Leray-Hopf weak solutions of the regularized 3D Navier-Stokes equations, as the artificial dissipation ε goes to 0. We also consider applications of obtained results to the regularized problem by allowing the family of forcing functions to vary with ε , for $\varepsilon > 0$.

1. INTRODUCTION

In this paper, we study the robustness, or upper semicontinuity of the global attractors of the Leray-Hopf weak solutions of modified three dimensional Navier-Stokes equations. We regularized the 3D Navier-Stokes system by adding a high order artificial viscosity term to the conventional system

$$\begin{aligned} \frac{\partial u^\varepsilon}{\partial t} + \varepsilon(-\Delta)^l u^\varepsilon - \nu \Delta u^\varepsilon + (u^\varepsilon \cdot \nabla) u^\varepsilon + \nabla p &= f(x), \text{ in } \Omega \times (0, \infty) \\ \operatorname{div} u^\varepsilon &= 0, \text{ in } \Omega \times (0, \infty), u^\varepsilon(x, 0) = u_0^\varepsilon, \text{ in } \Omega, \\ p(x + Le_i, t) &= p(x, t), u^\varepsilon(x + Le_i, t) = u^\varepsilon(x, t) \quad i = 1, 2, 3, t \in (0, \infty) \end{aligned} \quad (1.1)$$

where $\Omega = (0, L)^3$ with periodic boundary conditions and (e_1, \dots, e_d) is the natural basis of \mathbb{R}^d . Here $\varepsilon > 0$ is the artificial dissipation parameter, u^ε is the velocity vector field, p is the pressure, $\nu > 0$ is the kinematic viscosity of the fluid and f is a given force field. For $\varepsilon = 0$, the model is reduced to 3D Navier-Stokes system.

Mathematical model for such fluid motion has been used extensively in turbulence simulations (see e.g. [3, 4, 7, 10]). For further discussion of theoretical results concerning (1.1), see [1, 2, 5, 12, 15, 16, 20, 23].

In the work [23], the strong convergence of the solution of this problem to the solution of the conventional system as the regularization parameter goes to zero, was established for each dimension $d \leq 4$.

For the 3D Navier-Stokes system weak solutions of problem are known to exist by a basic result by J. Leray from 1934 [11], only the uniqueness of weak solutions remains as an open problem. Then the known theory of global attractors of infinite dimensional dynamical systems is not applicable to the 3D Navier-Stokes system.

The theory of trajectory attractors for evolution partial differential equations was developed in [14, 18], which the uniqueness theorem of solutions of the corresponding initial-value problem is not proved yet, e.g. for the 3D Navier-Stokes system (see [8, 14, 17, 18]). Such trajectory attractor is a classical global attractor

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but in the space of weak solutions defined on $[0, \infty)$, with the corresponding semigroup being simply the translation in time of such solutions. A compact set $\mathfrak{A} \Subset E$ is said to be a global attractor of a semigroup $\{S(t), t > 0\}$ acting in a Banach or Hilbert space E if \mathfrak{A} is strictly invariant with respect to $\{S(t)\} : S(t)\mathfrak{A} = \mathfrak{A} \forall t \geq 0$ and \mathfrak{A} attracts any bounded set $B \subset E : \text{dist}(S(t)B, \mathfrak{A}) \rightarrow 0 (t \rightarrow \infty)$ (see [13], [14], [17], [18], [20]).

In this article, we study the upper semicontinuity, of the global attractors of the Leray-Hopf weak solutions of a regularized 3D Navier-Stokes equations, as the artificial dissipation ε goes to 0. While there exist other examples of such robustness in the literature of the Navier-Stokes equations, the specific emphasis on the regularized problem is new for the 3D Navier-Stokes equations and is of interest. This would be an extension of the earlier work on Ou and Sritharan for the 2D Navier-Stokes equations, see references [15] and [16]. It is now known that there is a global attractor \mathfrak{A}_0 for the Leray-Hopf weak solutions of the 3D Navier-Stokes equations, see Sell [17] or [18].

The main object of this paper to show that there is a global attractor, which one might denote by \mathfrak{A}_ε , for the regularized problem (1.1), and that the family $\{\mathfrak{A}_\varepsilon\}$ is upper semicontinuous at $\varepsilon = 0$. Moreover, we can modify the argument described above so that the final result will have broader applicability by allowing the family of forcing functions f^ε to vary with ε , for $\varepsilon > 0$.

The family of sets $\mathfrak{A}_\varepsilon, 0 < \varepsilon \leq 1$ is robust at \mathfrak{A}_0 , or is upper semicontinuous with respect to ε at $\varepsilon_0 = 0$, provided that, for every $\varepsilon_0 > 0$, there is a neighborhood $O(\varepsilon_0)$ of $0 \in \mathbb{R}$ and a neighborhood $N_{\varepsilon_0}(\mathfrak{A}_0)$ of \mathfrak{A}_0 , such that $\mathfrak{A}_\varepsilon \subset N_{\varepsilon_0}(\mathfrak{A}_0)$, for every $\varepsilon \in O(\varepsilon_0)$ with $\varepsilon > 0$, see (23.13) in [18].

The paper is organized as follows. In Section 2, we present the relevant mathematical framework for the paper. In Section 3, we recall the definition of the trajectory attractor \mathfrak{A}_0 of the conventional 3-D Navier-Stokes equations. In Section 4, we study the regularized problem (see equation (1.1)), then we show the existence of trajectory attractor \mathfrak{A}_ε . In Section 5, we present the main result of this paper, that is, a theorem on the upper semicontinuity on the attractors \mathfrak{A}_ε . Finally, an application of our general results to the study of the robustness of the system (1.1) with a perturbed external force.

2. PRELIMINARY

We denote by $H^m(\Omega)$, the Sobolev space of L -periodic functions endowed with the inner product

$$(u, v) = \sum_{|\beta| \leq m} (D^\beta u, D^\beta v)_{L^2(\Omega)} \text{ and the norm } \|u\|_m = \sum_{|\beta| \leq m} (\|D^\beta u\|_{L^2(\Omega)}^2)^{\frac{1}{2}}$$

and by $H^{-m}(\Omega)$ the dual space of $H^m(\Omega)$. We denote by $\dot{H}^m(\Omega)$ the subspace of $H^m(\Omega)$ with, zero average $\dot{H}^m(\Omega) = \{u \in H^m(\Omega); \int_\Omega u(x) dx = 0\}$.

- We introduce the following solenoidal subspaces $V_s, s \in \mathbb{R}^+$ which are important to our analysis

$$V_0(\Omega) = \{u \in \dot{L}^2(\Omega), \text{div} u = 0, u.n|_{\Sigma_i} = -u.n|_{\Sigma_{i+3}}, i = 1, 2, 3\};$$

$$V_1(\Omega) = \{u \in \dot{H}^1(\Omega), \text{div} u = 0, \gamma_0 u|_{\Sigma_i} = \gamma_0 u|_{\Sigma_{i+3}}, i = 1, 2, 3\}.$$

$$V_2(\Omega) = \{u \in \dot{H}^2(\Omega), \text{div} u = 0, \gamma_0 u|_{\Sigma_i} = \gamma_0 u|_{\Sigma_{i+3}}, \gamma_1 u|_{\Sigma_i} = -\gamma_1 u|_{\Sigma_{i+3}}, i = 1, 2, 3\}.$$

see [20]. We refer the reader to Temam [21] for details on these spaces. Here the faces of Ω are numbered as

$$\Sigma_i = \partial\Omega \cap \{x_i = 0\} \text{ and } \Sigma_{i+3} = \partial\Omega \cap \{x_i = L\}, \quad i = 1, 2, 3.$$

Here γ_0, γ_1 are the trace operators and n is the unit outward normal on $\partial\Omega$.

- The space V_0 is endowed with the inner product $(u, v)_{L^2(\Omega)}$ and norm $\|u\| = (u, u)_{L^2(\Omega)}^{1/2}$.
- V_1 is the Hilbert space with the norm $\|u\|_1 = \|u\|_{V_1}$. The norm induced by $\dot{H}^1(\Omega)$ and the norm $\|\nabla u\|_{L^2(\Omega)}$ are equivalent in V_1 .
- V_2 is the Hilbert space with the norm $\|u\|_2 = \|u\|_{V_2}$. In V_2 the norm induced by $\dot{H}^2(\Omega)$ is equivalent to the norm $\|\Delta u\|_{L^2(\Omega)}$.

V'_s denote the dual space of V_s . We present the topology to be used for generating the neighborhood of robustness. Let F any vector space. A metric $d(f, g)$ on F is said to be invariant if one has

$$d(f, g) = d(f - g, 0) \text{ for all } f, g \in F.$$

A Fréchet space is a complete topological vector space whose topology is induced by a translation invariant metric $d(f, g)$. Given a Banach space X , with norm $\|\cdot\|_X$ and $1 \leq p < \infty$, we denote by $L^p_{loc}[0, \infty; X)$ the Fréchet space of measurable functions $f : [0, \infty) \rightarrow X$ that are p -integrable over $[0, T]$, for each $0 < T < \infty$, endow with the metric

$$d(f, g) = \sum_{n=1}^{\infty} 2^{-n} \min(\|f - g\|_{L^p(0, n; X)}, 1).$$

We denote by $L^p_{loc}(0, \infty; X)$ the Fréchet space of measurable functions $f : (0, \infty) \rightarrow X$ that are p -integrable over $[t_0, T]$, for each $0 < t_0 \leq T < \infty$ endow with the metric

$$d(f, g) = \sum_{n=2}^{\infty} 2^{-n} \min(\|f - g\|_{L^p(\frac{1}{n}, n; X)}, 1).$$

Similarly for $p = \infty$, we will let $L^\infty_{loc}(0, \infty; X)$ denote the collection of all functions $f : (0, \infty) \rightarrow X$ with the property that, for all τ and T with $0 < T < \infty$, one has $\text{ess sup}_{0 < s < T} \|f\|_X < \infty$. We denote by $C[0, \infty; X)$ the space of strongly continuous functions from $[0, \infty)$ to X , endowed with the topology of the uniform convergence over compact sets and by $C_w[0, \infty; X)$ the space of weakly continuous functions from $[0, \infty)$ to X . We denote by $L^\infty C = L^\infty(\mathbb{R}, X) \cap C(\mathbb{R}, X)$ the Fréchet space $L^\infty C$ endow with the L^∞_{loc} -topology, which is the topology of uniform convergence on bounded sets.

Let E be a complete metric space with metric d . We write B_r for the open ball centre $0 \in E$ and radius r . The following quantity is called the Hausdorff (non-symmetric) semidistance from a set X to a set Y in a Banach space E

$$\text{dist}_E(X, Y) = \sup_{x \in X} \inf_{y \in Y} \|\cdot\|_E.$$

Let M be a subset of E and let $\mathbb{R}^+ = [0, \infty)$. A mapping $\sigma = \sigma(u, t)$, where $\sigma : M \times [0, \infty) \rightarrow M$ is said to be a semiflow on M provided the following hold

1) $\sigma(w, 0) = w$, for all $w \in M$.

2) The semigroup property holds, i. e,

$$\sigma((w, s), t) = \sigma(w, s + t) \text{ for all } w \in M \text{ and } s, t \in \mathbb{R}^+.$$

3) The mapping $\sigma : M \times (0, \infty) \rightarrow M$ is continuous.

If in addition the mapping $\sigma : M \times [0, \infty) \rightarrow M$ is continuous we will say that the semiflow is continuous at $t = 0$. Here we use $t > 0$ in order that the Robustness Theorem 23.14 in [18] is valid, see Sell [18] and Hale [8]. For any $u \in M$ the positive trajectory through u is defined as the set $\gamma^+(u) = \{\sigma(t)u, t \geq 0\}$. For any set $B \subset M$ we define the positive hull $\mathcal{H}^+(B)$ and the omega limit set $\omega(B)$ as follows

$$\mathcal{H}^+(B) = Cl_M \gamma^+(B) \text{ and } \omega(B) = \bigcap_{\tau \geq 0} \mathcal{H}^+(\sigma(\tau)B).$$

If $\mathcal{A} \subset E$ and $\varepsilon > 0$ we write

$$N_\varepsilon(\mathcal{A}) = \{z \in E, \inf_{a \in \mathcal{A}} d(z, a) < \varepsilon\}.$$

for the open ε -neighbourhood of \mathcal{A} .

We denote by A the Stokes operator $Au = -\Delta u$ for $u \in D(A)$. We recall that the operator A is a closed positive self-adjoint unbounded operator, with $D(A) = \{u \in V_0, Au \in V_0\}$. We have in fact, $D(A) = \dot{H}^2(\Omega) \cap V_0 = V_2$. The spectral theory of A allows us to define the powers A^l of A for $l \geq 1$, A^l is an unbounded self-adjoint operator in V_0 with a domain $D(A^l)$ dense in $V_2 \subset V_0$. We set here

$$A^l u = (-\Delta)^l u \text{ for } u \in D(A^l) = V_{2l} \cap V_0.$$

The space $D(A^l)$ is endowed with the scalar product and the norm

$$(u, v)_{D(A^l)} = (A^l u, A^l v), \|u\|_{D(A^l)} = \{(u, u)_{D(A^l)}\}^{\frac{1}{2}}.$$

Now define the trilinear form $b(., ., .)$ associated with the inertia terms

$$b(u, v, w) = \sum_{i,j=1}^3 \int_{\Omega} u_i \frac{\partial v_j}{\partial x_i} w_j dx$$

The continuity property of the trilinear form enables us to define (using Riesz representation theorem) a bilinear continuous operator $B(u, v); V_2 \times V_2 \rightarrow V_2'$ will be defined by

$$\langle B(u, v), w \rangle = b(u, v, w), \forall w \in V_2.$$

Recall that for u satisfying $\nabla \cdot u = 0$ we have

$$b(u, u, u) = 0 \text{ and } b(u, v, w) = -b(u, w, v). \quad (2.1)$$

Hereafter, $c_i \in \mathbb{N}$, will denote a dimensionless scale invariant positive constant which might depend on the shape of the domain. The trilinear form $b(., ., .)$ is continuous on $\dot{H}^{m_1}(\Omega) \times \dot{H}^{m_2+1}(\Omega) \times \dot{H}^{m_3}(\Omega)$, $m_i \geq 0$

$$|b(u, v, w)| \leq c_0 \|u\|_{m_1} \|v\|_{m_2+1} \|w\|_{m_3}, \quad m_3 + m_2 + m_1 \geq \frac{3}{2} \quad (2.2)$$

see [6, 18]. The bilinear operator satisfies the inequality

$$\|Bu\| \leq c_1 \|A^{\frac{1}{2}} u\|^{\frac{3}{2}} \|Au\|^{\frac{1}{2}}, \quad (2.3)$$

see [21, Lemma 3.8]. Similarly, we define $\hat{B}(u, v) \in V_1'$ by

$$\langle \hat{B}(u, v), w \rangle_{V_1' \times V_1} = b(u, v, w), \quad \forall w \in V_1.$$

We recall some inequalities that we will be using in what follows.

Young's inequality

$$ab \leq \frac{\epsilon}{p} a^p + \frac{1}{q\epsilon^{\frac{q}{p}}} b^q, a, b, \epsilon > 0, p > 1, q = \frac{p}{p-1}. \quad (2.4)$$

Poincaré's inequality

$$\lambda_1 \|u\|^2 \leq \|u\|_1^2 \text{ for all } u \in V_0, \quad (2.5)$$

where λ_1 is the smallest eigenvalue of the Stokes operator A .

3. NAVIER-STOKES EQUATIONS

The conventional Navier-Stokes system can be written in the evolution form

$$\begin{aligned} \frac{\partial u}{\partial t} + \nu Au + \hat{B}(u, u) &= f, \quad t > 0, \\ u_0(x) &= u_0. \end{aligned} \quad (3.1)$$

Let $f \in L^\infty(0, \infty; V_0)$ be given. We will say that a function u is a weak solution of the 3D Navier-Stokes of Class LH (Leray–Hopf) on $[0, \infty)$ provided that $u(x, 0) = u_0(x) \in V_0$, and the following properties hold

$$1) \quad u \in L^\infty(0, \infty; V_0) \cap L^2_{loc}(0, \infty; V_1).$$

$$2) \quad \frac{du}{dt} \in [L^{\frac{4}{3}}_{loc}(0, \infty; V'_1)].$$

Taking the inner product of (3.1) with u , and using (2.4) we have

$$\frac{d}{dt} \|u(t)\|^2 + 2\nu \|\nabla u\|^2 = 2\langle f, u \rangle. \quad (3.2)$$

by application of Young's inequality and the Poincaré's Lemma, yields

$$\frac{d}{dt} \|u(t)\|^2 + \nu \|\nabla u\|^2 \leq \frac{\|f\|^2}{\nu\lambda_1}, \quad (3.3)$$

using the Poincaré Lemma and Gronwall's inequality, to get

$$\|u(t)\|^2 \leq e^{-\nu\lambda_1(t-t_0)} \|u(t_0)\|^2 + \frac{1}{\nu^2\lambda_1^2} \|f\|^2 \left(1 - e^{-\nu\lambda_1(t-t_0)}\right), \text{ with } 0 < t_0 < t,$$

3) which implies that

$$\|u(t)\|^2 \leq e^{-\nu\lambda_1(t-t_0)} \|u(t_0)\|^2 + \frac{1}{\nu^2\lambda_1^2} \|f\|^2. \quad (3.4)$$

Integrating (3.2) over $[t_0, t]$ we find that

$$\|u(t)\|^2 + 2\nu \int_{t_0}^t \|A^{\frac{1}{2}}u(s)\|^2 ds \leq \|u(t_0)\|^2 + 2 \int_{t_0}^t \langle f(s), u(s) \rangle ds. \quad (3.5)$$

4) The function u satisfies the following equality

$$\langle u(t) - u(t_0), v \rangle + \nu \int_{t_0}^t \langle A^{\frac{1}{2}}u(s), A^{\frac{1}{2}}v \rangle ds + \int_{t_0}^t \langle \hat{B}(u(s), u(s)), v \rangle ds = \int_{t_0}^t \langle f, v \rangle ds, \quad (3.6)$$

for all $v \in V_1$ and for all $t \geq t_0 \geq 0$.

The proof of the following theorem is given in [12, 13, 21].

Theorem 3.1. *Let $f \in V'_1$ and $u_0 \in V_0$ be given. Then for every $T > 0$, there exists a weak solution $u(t)$ of (3.1) from the space $L^2(0, T; V_1) \cap L^\infty(0, T; V_0)$, such that $u(x, 0) = u_0$ and $u(t)$ satisfies the energy equality (3.6).*

Moreover (see [21]), $u(\cdot)$ is weakly continuous from $[0, T]$ into V_0 , the function $u \in C_w([0, T]; V_0)$ and consequently $u(x, 0) = u_0(x) \in V_0$. Let W is the set of all Leray–Hopf weak solutions $u(\cdot)$ of equation (3.1) in the space $L^\infty(0, \infty; V_0) \cap L^2_{loc}[0, \infty; V_1)$ that satisfy the following properties

- $\frac{du}{dt} \in L^{\frac{4}{3}}_{loc}(0, \infty; V'_1)$;
- for almost all t and t_0 , with $t > t_0 > 0$, inequalities (3.5 ,3.6) are valid.

Let X^0 denote the Fréchet space used to define the Leray-Hopf weak solutions. Thus

$$\varphi \in X^0 = L^\infty(0, \infty; V_0) \cap L^2_{loc}[0, \infty; V_1),$$

where $\varphi \in C_w[0, \infty; V_0)$ and we let \mathfrak{F}^0 denote a compact, translation invariant set of forcing functions f in

$$L^\infty C = L^\infty(\mathbb{R}, L^2(\Omega)) \cap C(\mathbb{R}, L^2(\Omega))$$

where the topology on the Fréchet space $L^\infty C$ is the topology of uniform convergence on bounded sets in \mathbb{R} .

Then, we use the Leray-Hopf solutions of the 3D Navier-Stokes equations with $\varepsilon = 0$ to generate a semiflow π^0 on $\mathfrak{F}^0 \times X^0$, where

$$\pi^0(\tau)(f, \varphi) = (f_\tau, S^0(f, \tau)\varphi) \text{ for } \tau \geq 0,$$

$f_\tau(t) = f(\tau + t)$ and $u(t) = S^0(f, t)\varphi$ is the Leray-Hopf solution of the 3D Navier-Stokes equations that satisfies $u(0) = S^0(f, 0)\varphi = \varphi(0)$. By using the theory of generalized weak solutions, as in Sell [17] or [18], we note that π^0 has a global attractor $\mathfrak{A}_0 \subset \mathfrak{F}^0 \times X^0$ see Theorem 65.12 in [18].

4. THE REGULARIZED NAVIER-STOKES SYSTEM

Using the operators defined in the previous section, we can write the modified system (1.1) in the evolution form

$$\begin{aligned} \partial_t u^\varepsilon + \varepsilon A^1 u^\varepsilon + \nu A u^\varepsilon + B(u^\varepsilon, u^\varepsilon) &= f(x), & \text{in } \Omega \times (0, \infty) \\ u^\varepsilon(x) &= u^\varepsilon, & \text{in } \Omega. \end{aligned} \quad (4.1)$$

For $\varepsilon > 0$, we let π^ε denote the semiflow on $\mathfrak{F}^0 \times X^0$ generated by the weak solutions of regularized 3D Navier-Stokes equations of (4.1). Thus

$$\pi^\varepsilon(\tau) = (f_\tau, S^\varepsilon(f, \tau)\varphi), \quad (4.2)$$

where $u_0^\varepsilon = \varphi$ and

$$u^\varepsilon(t) = S^\varepsilon(f, t)\varphi = S^\varepsilon(f, t)u_0^\varepsilon \quad (4.3)$$

is the weak solution of (4.1) that satisfies $u^\varepsilon(0) = \varphi(0) = u_0^\varepsilon(0)$.

The existence and uniqueness results for initial value problem (1.1) can be found in [12, Remark 6.11].

The following theorem collects the main result in this work

Theorem 4.1. *For $l \geq \frac{5}{4}$, for $\varepsilon > 0$ fixed, $f \in L^2(0, T; V'_0)$ and $u_0^\varepsilon \in V_0$ be given. There exists a unique weak solution of (4.1) which satisfies*

$$u^\varepsilon \in L^2(0, T; V_1) \cap L^\infty(0, T; V_0), \forall T > 0.$$

$u^\varepsilon \in L^\infty(0, T; V_0) \cap L^2[0, T; V_1)$ and $u^\varepsilon \in C_w([0, T]; V_0), \forall T > 0$.

We recall Lemma 3.6 [23].

Lemma 4.2. u^ε is almost everywhere equal to a continuous function from $[0, T]$ to the space V_0 .

and the following theorem

Theorem 4.3. For $l \geq \frac{3}{2}$, the weak solution u^ε of the modified Navier-Stokes equations (4.1) given by Theorem 4.1 converges strongly in $L^2(0, T; V_0)$ as $\varepsilon \rightarrow 0$ to u the weak solution of the Navier-Stokes equations.

The above theorem is established directly by using of a general result [23, Theorem 3.9].

Lemma 4.4. For fixed $\varepsilon > 0$ and for all $0 \leq t \leq T$, $u^\varepsilon \in B_{R_1} = \{u_0^\varepsilon \in V_1; \|u^\varepsilon\|_1 \leq R_1\}$ there is a constant C_1 positive independent of ε such that

$$\int_0^T \|Au^\varepsilon(t)\|^2 dt \leq C_1. \quad (4.4)$$

C_1 independent of ε is given in the proof.

Proof. Taking the inner product of (4.1) with Au^ε , we find

$$\frac{1}{2} \frac{d}{dt} \|A^{\frac{1}{2}} u^\varepsilon\|^2 + \varepsilon (A^l u^\varepsilon, Au^\varepsilon) + \nu \|Au^\varepsilon\|^2 + b(u^\varepsilon, u^\varepsilon, Au^\varepsilon) = (f, Au^\varepsilon).$$

by using (2.3), to obtain

$$|b(u^\varepsilon, u^\varepsilon, Au^\varepsilon)| \leq c_1 \|A^{\frac{1}{2}} u^\varepsilon\|^{\frac{3}{2}} \|Au^\varepsilon\|^{\frac{1}{2}} \|Au^\varepsilon\|.$$

Due to (2.4) and (2.5), we get

$$(A^l u^\varepsilon, Au^\varepsilon) = \|A^{\frac{l+1}{2}} u^\varepsilon\|^2.$$

We have the following inequality

$$\frac{1}{2} \frac{d}{dt} \|A^{\frac{1}{2}} u^\varepsilon\|^2 + \varepsilon \|A^{\frac{l+1}{2}} u^\varepsilon\|^2 + \frac{\nu}{2} \|Au^\varepsilon\|^2 \leq \frac{1}{\nu} \|f\|_\infty^2 + \frac{27c_1^4}{2\nu^3} \|A^{\frac{1}{2}} u^\varepsilon\|^6. \quad (4.5)$$

For $u^\varepsilon \in B_{R_1}$, we obtain

$$\frac{1}{2} \frac{d}{dt} \|A^{\frac{1}{2}} u^\varepsilon\|^2 + \varepsilon \|A^{\frac{l+1}{2}} u^\varepsilon\|^2 + \frac{\nu}{2} \|Au^\varepsilon\|^2 \leq \frac{1}{\nu} \|f\|_\infty^2 + \frac{27c_1^4}{2\nu^3} R_1^6.$$

We drop the positive terms $\varepsilon \|A^{\frac{l+1}{2}} u^\varepsilon\|^2$ and integrate from 0 to T to get

$$\int_0^T \|Au^\varepsilon\|^2 \leq \frac{2T}{\nu} \|f\|_\infty^2 dt + T \frac{27c_1^4}{\nu^4} R_1^6 + \frac{2}{\nu} R_1^2 = C_1, \quad (4.6)$$

this completes the proof of Lemma 4.4. \square

Theorem 4.5. For fixed $\varepsilon \geq 0$, $u^\varepsilon \in B_{R_1} = \{u^\varepsilon(0) \in V_1; \|u^\varepsilon\|_1 \leq R_1\}$ and $f \in L^\infty C$ a time independent functions, π^ε is a continuous family of semiflows on X^0 .

Proof. Let convergent sequences ε_n , φ^n and f^n , with limits $\varepsilon_n \rightarrow \varepsilon_0$, (especially with $\varepsilon_0 = 0$), $\varphi^n \rightarrow \varphi_0$ in the X^0 -topology and $f^n \rightarrow f^0$ in the $L^\infty C$ -topology as $n \rightarrow \infty$, then

$$S^{\varepsilon_n}(f^n, t) \varphi^n \rightarrow S^{\varepsilon_0}(f^0, t) \varphi^0. \quad (4.7)$$

Let

$$S^{\varepsilon_n}(f^n, t) \varphi^n - S^{\varepsilon_0}(f^0, t) \varphi^0 = u^{\varepsilon_n}(t) - u^{\varepsilon_0}(t), \quad (4.8)$$

we obtain for $w_n = u^{\varepsilon_n}(t) - u^{\varepsilon_0}(t)$ and $g_n = f^n - f^0$

$$\partial_t w_n + \varepsilon_n A^l w_n + A w_n + B(u^{\varepsilon_n}, u^{\varepsilon_n}) - B(u^{\varepsilon_0}, u^{\varepsilon_0}) = g_n. \quad (4.9)$$

By taking inner product with w_n for above equation we get

$$\frac{1}{2} \frac{d}{dt} \|w_n\|^2 + \varepsilon_n \|A^l w_n\|^2 + \nu \|A^{\frac{1}{2}} w_n\|^2 = b(w_n, w_n, u^{\varepsilon_n}) + (g_n, w_n). \quad (4.10)$$

Using Young's inequality, we obtain

$$(g_n, w_n) \leq \frac{1}{2\nu} \|g_n\|^2 + \frac{\nu}{2} \|A^{\frac{1}{2}} w_n\|^2,$$

thanks to (2.2) with $(m_3, m_2, m_1) = (0, 1, 1)$, we find

$$|b(w_n, u^{\varepsilon_n}, w_n)| \leq c_0 \|A^{\frac{1}{2}} w_n\| \|Au^{\varepsilon_n}\| \|w_n\| \quad (4.11)$$

this give

$$|b(w_n, u^{\varepsilon_n}, w_n)| \leq \frac{\nu}{2} \|A^{\frac{1}{2}} w_n\|^2 + \frac{c_4}{2} \|Au^{\varepsilon_n}\|^2 \|w_n\|^2. \quad (4.12)$$

Combining all these inequalities in (4.10), we have

$$\frac{1}{2} \frac{d}{dt} \|w_n\|^2 + \varepsilon_n \|A^l w_n\|^2 \leq \frac{c_4}{2} \|Au^{\varepsilon_n}\|^2 \|w_n\|^2 + \frac{1}{2\nu} \|g_n\|^2.$$

We drop the positive term $\varepsilon_n \|Aw_n\|^2$ to obtain the following differential inequality

$$\frac{d}{dt} \|w_n(t)\|^2 \leq c_4 \|Au^{\varepsilon_n}\|^2 \|w_n\|^2 + \frac{1}{\nu} \|g_n\|^2. \quad (4.13)$$

Applying now Gronwall's inequality to (4.13), for $t \geq 0$ we have

$$\begin{aligned} \|w_n(t)\|^2 &\leq \|w_n(0)\|^2 \exp c_4 \left(\int_0^t \|Au^{\varepsilon_n}\|^2 ds \right) \\ &\quad + \frac{1}{\nu} \int_0^t \|g_n(h)\|^2 \exp c_4 \left(\int_0^t \|Au^{\varepsilon_n}\|^2 ds \right) dh \end{aligned} \quad (4.14)$$

using estimate (4.4) we find

$$\|w_n(t)\|^2 \leq C_2 \|w_n(0)\|^2 + \frac{TC_2}{\nu} \|g_n\|^2 \quad (4.15)$$

for all t in compact sets in $[0, \infty)$, $C_2 = \exp c_4 C_1$. Since $f^n \rightarrow f^0$ in the $L^\infty C$ -topology and $\varphi^n \rightarrow \varphi_0$ in the X^0 -topology this means that $\|g_n\| \rightarrow 0$ and $\|w_n(0)\| \rightarrow 0$ as $n \rightarrow \infty$, it follows from (4.15) that

$$\|S^{\varepsilon_n}(f^n, t) \varphi^n - S^{\varepsilon_0}(f^0, t) \varphi^0\| \leq C_2 \|u_0^{\varepsilon_n} - u_0^{\varepsilon_0}\|^2 + \frac{TC_2}{\nu} \|f^n - f\|^2 \rightarrow 0, \text{ as } n \rightarrow \infty.$$

It follows that π^ε is continuous semiflows on X^0 . Hence π^ε approximates π^0 on B_{R_1} uniformly on $[0, T]$. \square

Regarding the existence of the attractor \mathfrak{A}_ε when $\varepsilon > 0$, we use especially the related papers of Chepyzhov and Vishik, such as [14] to show that the system (4.1) possesses a global attractor. For $\varepsilon > 0$, we consider the trajectory space \mathcal{K}_ε of the modified Navier-Stokes equations (4.1). \mathcal{K}_ε is the union of all weak solutions $u^\varepsilon \in X^0$ that satisfy (4.1), see (6.163) in [12]. Using the described scheme in [14], we construct the spaces \mathcal{S}_b

$$\mathcal{S}_b = \{v(\cdot) \in L^\infty(0, T; V_0) \cap L_b^2(0, T; V_1), \partial_t v(\cdot) \in L_b^2(0, T; D(A^l)')\}$$

with norm

$$\|v\|_{\mathcal{S}_b} = \|v\|_{L_b^2(0, T; V_1)} + \|v\|_{L^\infty(0, T; V_0)} + \|\partial_t v\|_{L_b^2(0, T; D(A^l)')}$$

where

$$\|v\|_{L_b^2(0,T;V_1)} = \sup_{t \geq 0} \left(\int_t^{t+1} \|v(s)\|_1^2 ds \right)^{\frac{1}{2}}, \quad \|v\|_{L^\infty(0,T;V_0)} = \operatorname{ess\,sup}_{t \geq 0} \|v\|$$

and

$$\|\partial_t v\|_{L_b^2(0,T;D(A^l)')} = \sup_{t \geq 0} \left(\int_t^{t+1} \|v(s)\|_{D(A^l)'}^2 ds \right)^{\frac{1}{2}}.$$

We need a topology in the space \mathcal{K}_ε . We define on X^0 the following sequential topology which we denote Γ .

By definition, a sequence of functions $\{v_n\} \subseteq X^0$ converges to a function $v \in X^0$ in the topology Γ as $n \rightarrow \infty$ if, for any $T > 0$, $v_n \rightarrow v$ weakly in $L^2(0, T; V_1)$; $v_n \rightarrow v$ weak-* in $L^\infty(0, T; V_0)$ and $v_n \rightarrow v$ strongly in $L^2(0, T; V_0)$, as $n \rightarrow \infty$.

We consider the topology Γ on \mathcal{K}_ε . It is easy to prove that the space \mathcal{K}_ε is closed in Γ . From the definition of \mathcal{K}_ε , it follows that $\pi^\varepsilon \mathcal{K}_\varepsilon \subset \mathcal{K}_\varepsilon$ for all $t \geq 0$.

Proposition 4.6. *If $u^\varepsilon(t)$ is a solution of (4.1), then the following inequalities hold for all $t > 0$*

$$\|u^\varepsilon(t)\|^2 \leq e^{-\nu\lambda_1 t} \|u_0^\varepsilon\|^2 + \frac{\|f\|^2}{\nu^2 \lambda_1^2}, \quad (4.16)$$

$$\int_t^{t+1} \|u^\varepsilon(s)\|^2 ds \leq \frac{e^{-\nu\lambda_1 t}}{\nu\lambda_1} \|u_0^\varepsilon\|^2 + \frac{\|f\|^2}{\nu^2 \lambda_1^2}, \quad (4.17)$$

$$\nu \int_t^{t+1} \|u^\varepsilon(s)\|_1^2 ds \leq \frac{e^{-\nu\lambda_1 t}}{\nu\lambda_1} \|u_0^\varepsilon\|^2 + \frac{\|f\|^2}{\nu^2 \lambda_1^2} + \frac{\|f\|^2}{\nu\lambda_1}. \quad (4.18)$$

Proof. Taking the inner product of (4.1) by $u^\varepsilon \in V_2$, we obtain

$$\frac{d}{dt} \|u^\varepsilon\|^2 + 2\varepsilon \|A^l u^\varepsilon\|^2 + 2\nu \|\nabla u^\varepsilon\|^2 = 2(f, u^\varepsilon). \quad (4.19)$$

Applying Young's inequality and using the Poincaré Lemma, we obtain

$$\frac{d}{dt} \|u^\varepsilon\|^2 + \nu \|\nabla u^\varepsilon\|^2 \leq \frac{\|f\|^2}{\nu\lambda_1}. \quad (4.20)$$

Using the Gronwall's inequality over $[0, t]$, we obtain (4.16). Integrating (4.16) over $[t, t+1]$ we find (4.17). Integrating (4.20) over $[t, t+1]$ we find

$$\nu \int_t^{t+1} \|\nabla u^\varepsilon(s)\|^2 ds \leq \frac{\|f\|^2}{\nu\lambda_1} + \|u^\varepsilon(t)\|^2.$$

Applying inequality (4.16), we have (4.18). \square

A simple consequence of Lemma 3.6 [23] is the following Lemma

Proposition 4.7. *Let $f \in V_0$. Then any solution $u^\varepsilon(t)$ of (4.1) satisfies*

$$\int_t^{t+1} \|\partial_t u^\varepsilon(s)\|_{D(A^l)'}^2 ds \leq C_3, \quad (4.21)$$

C_3 is a positive constant independent of ε .

Moreover, due to estimates (4.16) and (4.21), we also have the uniform estimate.

Proposition 4.8. *If $f \in V_0$, then any solution $u^\varepsilon(t)$ of problem (4.1) satisfies the inequality*

$$\|\pi^\varepsilon(u^\varepsilon)\|_{\mathcal{S}_b}^2 \leq \frac{c_7 e^{-\nu\lambda_1 t}}{\nu\lambda_1} \|u^\varepsilon(0)\|^2 + \frac{c_7 \|f\|^2}{\nu^2 \lambda_1^2} + C_4 \quad (4.22)$$

where the positive constant C_4 is independent of ε .

From Proposition 4.6 it follows that $\mathcal{K}_\varepsilon \subset \mathcal{S}_b$ for all $\varepsilon > 0$ and for all $\tau > 0$. Also Proposition 4.6 implies that the semigroup π^ε has absorbing set in \mathcal{K}_ε for all $\varepsilon > 0$ and for all $\tau > 0$ (We note, that this absorbing set does not depend on ε , since the constant C_4 in (4.22) is independent of ε), bounded in \mathcal{S}_b and inequality (4.22) implies that absorbing set is compact in Γ . The continuity of π^ε is proved. These facts are sufficient to state that π^ε has a global attractor \mathfrak{A}_ε . Such that $\mathfrak{A}_\varepsilon \subset \mathfrak{F}^0 \times X^0$, bounded in \mathcal{S}_b and compact in Γ . For a more detailed, see [14].

5. UPPER SEMICONTINUITY OF ATTRACTORS

We now prove the robustness property for the global attractor \mathfrak{A}_ε . We have shown in Theorem 4.5 the continuity of the family of semiflows π^ε on X^0 . Having done this, We can simply invoke Theorem 23.14 in [18] to complete the proof of the robustness for the family of attractors \mathfrak{A}_ε at $\varepsilon = 0$. Clearly, it is sufficient to show that the small ε_0 -neighbourhood of attractor \mathfrak{A}_0 is an absorbing set and that π^ε approximates π^0 on $B_{R_1} = \{u^\varepsilon(0) \in V_1; \|u_0^\varepsilon\|_1 \leq R_1\}$ uniformly on compact sets of $[0, \infty)$.

Theorem 5.1. *For $\varepsilon > 0$ the family of semiflows π^ε generated by the weak solutions of regularized 3D Navier-Stokes equations (1.1) admits a compact attractor $\{\mathfrak{A}_\varepsilon, 0 < \varepsilon \leq 1\}$ which attracts bounded sets of V_0 and is contained in the absorbing balls $B_{R_0} \cap B_{R_1}$ where R_0 and R_1 are independent of ε . Moreover, $d_{X^0}(\mathfrak{A}_\varepsilon, \mathfrak{A}_0) \rightarrow 0$, as $\varepsilon \rightarrow 0$.*

Proof. Let $N_{\varepsilon_0}(\mathfrak{A}_0)$ be the ε_0 -neighborhood of \mathfrak{A}_0 . Since \mathfrak{A}_0 is a global attractor, for any bounded set $B_{R_0} = \{u(0) \in V_0; \|u(0)\| \leq R_0\} \subset V_0$, we have

$$d_{X^0}(\pi^0 B_{R_0}, \mathfrak{A}_0) \rightarrow 0, \text{ as } t \rightarrow \infty. \quad (5.1)$$

Thus, there exists $\varepsilon_0 > 0$ and $t > t_{\varepsilon_0}$ such that

$$d_{X^0}(\pi^0 B_{R_0}, \mathfrak{A}_0) \leq \frac{\varepsilon_0}{2}, \text{ for } t \geq t_{\varepsilon_0}. \quad (5.2)$$

Consequently

$$\pi^0(t) B_{R_0} \subset N_{\varepsilon_0}(\mathfrak{A}_0), \text{ for } t \geq t_{\varepsilon_0}. \quad (5.3)$$

This shows that $N_{\varepsilon_0}(\mathfrak{A}_0)$ is an absorbing set. To establish the second step, we drop the positive terms $\varepsilon \|A^{\frac{l+1}{4}} u^\varepsilon\|^2$ and $\frac{\nu}{2} \|Au^\varepsilon\|^2$ in inequality (4.5) and integrate from 0 to T we get

$$\frac{1}{2} \frac{d}{dt} \|A^{\frac{1}{2}} u^\varepsilon\|^2 \leq \frac{1}{\nu} \|f\|_\infty^2 + \frac{27c_1^4}{2\nu^3} R_1^4 \|A^{\frac{1}{2}} u^\varepsilon\|^2.$$

Integrating the above inequality from 0 to t , we get

$$\|A^{\frac{1}{2}} u^\varepsilon\|^2 \leq \|A^{\frac{1}{2}} u^\varepsilon(0)\|^2 \exp T \left(\frac{27c_1^4}{\nu^3} R_1^4 \right) + \frac{2T}{\nu} \|f\|_\infty^2 = \rho_1$$

This implies that any ball $B_{R_1} = \{u_0^\varepsilon \in V_1; \|A^{\frac{1}{2}} u_0^\varepsilon(0)\| \leq R_1\}$ in V_1 with radius $R_1 > \rho_1$ will satisfy

$$\pi^\varepsilon(t) B_{R_1} \subset B_{R_1}, \text{ for } t \geq 0. \quad (5.4)$$

This means if $u_0^\varepsilon \in B_{R_1}$, then $\pi^\varepsilon(t)u_0^\varepsilon$ is defined and belongs to B_{R_1} for $t \geq 0$. The ball B_{R_1} is therefore invariant under the map π^ε . Since π^ε approximates π^0 on B_{R_1} uniformly on $[0, T]$, we have for any $\varepsilon_0 > 0$, there are $\varepsilon_1 > 0$ and $\tau_0 > 0$ such that

$$\pi^\varepsilon(B_{R_0} \cap B_{R_1}) \subset N_{\varepsilon_0}(\mathfrak{A}_0), \text{ for } 0 < \varepsilon < \varepsilon_1, t \geq \tau_0. \quad (5.5)$$

Since the attractor \mathfrak{A}_ε is contained in $B_{R_0} \cap B_{R_1}$, an open neighborhood in the X^0 Fréchet space [18, Item (2) Theorem 23.14], we have

$$\pi^\varepsilon(\mathfrak{A}_\varepsilon) \subset N_{\varepsilon_0}(\mathfrak{A}_0), \text{ for } 0 < \varepsilon < \varepsilon_1, t \geq \tau_0. \quad (5.6)$$

Since \mathfrak{A}_ε is an invariant set, we deduce that

$$\mathfrak{A}_\varepsilon \subset N_{\varepsilon_0}(\mathfrak{A}_0), \text{ for } 0 < \varepsilon < \varepsilon_1, t \geq \tau_0. \quad (5.7)$$

Moreover, since ε_0 is arbitrary, we obtain the upper semicontinuity of \mathfrak{A}_ε , at $\varepsilon = 0$

$$d_{X^0}(\mathfrak{A}_\varepsilon, \mathfrak{A}_0) \rightarrow 0, \text{ as } \varepsilon \rightarrow 0. \quad (5.8)$$

□

One can modify the argument described above so that the final result will have broader applicability by allowing the family of forcing functions to vary with ε , for $\varepsilon > 0$. Thus, we consider the regularized Navier-Stokes system (1.1) with a perturbed external force f^ε in place of f , for $\varepsilon > 0$. Then (4.1) becomes

$$\begin{aligned} \partial_t u^\varepsilon + \varepsilon A^l u^\varepsilon + \nu A u^\varepsilon + B(u^\varepsilon, u^\varepsilon) &= f^\varepsilon(x), & \text{in } \Omega \times (0, \infty) \\ u^\varepsilon(x) &= u_0^\varepsilon, & \text{in } \Omega. \end{aligned} \quad (5.9)$$

We show that the trajectory attractor of the perturbed system (5.9) coincides with the trajectory attractor \mathfrak{A}_ε of the unperturbed system (1.1). Our results rely on the work of Hale ([14]) who show that the limit behaviour is valid even through \mathfrak{F}^ε , where \mathfrak{F}^ε denote a compact, translation invariant set of perturbed forcing functions to vary with ε , for $\varepsilon > 0$ and satisfy the condition

$$\omega(\mathcal{H}^+(f^\varepsilon)) = \omega(\mathcal{H}^+(f)). \quad (5.10)$$

Thus we would use \mathfrak{F}^ε in place of \mathfrak{F}^0 , for $\varepsilon > 0$. Moreover, by using a metric d on the $L^\infty C$ -toplogy, see [18] for some samples, we can note that (5.10) is equivalent to saying that for every $\delta > 0$ there is an $\varepsilon_1 > 0$ and $T_\delta = T(\delta) \geq 0$ such that

$$d_{X^0}(f^\varepsilon, \mathfrak{F}^0) \leq \delta, \text{ for } 0 < \varepsilon \leq \varepsilon_1 \text{ and } f^\varepsilon \in \mathfrak{F}^\varepsilon$$

for any $t \geq T_\delta$, that is

$$\mathfrak{F}^\varepsilon \subset N_\delta(\mathfrak{F}^0), \text{ for } 0 < \varepsilon \leq \varepsilon_1, \quad (5.11)$$

where N_δ denotes the δ -neighborhood of \mathfrak{F}^0 in $L^\infty C$. The resulting argument for robustness will then depend on two parameters $\lambda = (\varepsilon, \delta)$, where $\lambda \rightarrow (0, 0)$.

The following statement generalizes Theorem 5.1

Theorem 5.2. *Under the above conditions, the trajectory attractor of the perturbed 3D Navier-Stokes system (5.9) coincides with the trajectory attractor \mathfrak{A}_ε of the non-perturbed system (4.1). Moreover, the perturbed attractor of (5.9) is upper semicontinuous with respect to ε at $\varepsilon = 0$.*

Proof. The existence of trajectory attractor \mathfrak{A}_ε is treated above. The proof follows from formulas (5.10), (5.11) and Theorem 5.1. □

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