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# UPPER SEMICONTINUITY OF TRAJECTORY ATTRACTORS OF 3D NAVIER-STOKES EQUATIONS WITH HYPERVISCOSITY

ABDELHAFID YOUNSI

ABSTRACT. We regularized the 3D Navier-Stokes equations by adding a fourth-order viscosity term. We study the upper semicontinuity, of the global attractors of the Leray-Hopf weak solutions of the regularized 3D Navier-Stokes equations, as the artificial dissipation  $\varepsilon$  goes to 0, and the regularized problem (see equation (1.1)) limits to the standard 3D Navier-Stokes equations. We also consider applications of obtained results to the regularized problem by allowing the family of forcing functions to vary with  $\varepsilon$ , for  $\varepsilon > 0$ .

## 1. INTRODUCTION

In this paper, we study the robustness, or upper semicontinuity of the global attractors of the Leray-Hopf weak solutions of modified three dimensional Navier-Stokes equations. We regularized the 3D Navier-Stokes system by adding a fourth order artificial viscosity term (Laplacian square) to the conventional system

$$\begin{aligned} \frac{\partial u^\varepsilon}{\partial t} + \varepsilon \Delta^2 u^\varepsilon - \nu \Delta u^\varepsilon + (u^\varepsilon \cdot \nabla) u^\varepsilon + \nabla p &= f(x), \text{ in } \Omega \times (0, \infty) \\ \operatorname{div} u^\varepsilon &= 0, \text{ in } \Omega \times (0, \infty), u^\varepsilon(x, 0) = u_0^\varepsilon, \text{ in } \Omega, \\ p(x + Le_i, t) &= p(x, t), u^\varepsilon(x + Le_i, t) = u^\varepsilon(x, t) \quad i = 1, 2, 3, t \in (0, \infty) \end{aligned} \quad (1.1)$$

where  $\Omega = (0, L)^3$  with periodic boundary conditions and  $(e_1, \dots, e_d)$  is the natural basis of  $\mathbb{R}^d$ . Here  $\varepsilon > 0$  is the artificial dissipation parameter,  $u^\varepsilon$  is the velocity vector field,  $p$  is the pressure,  $\nu > 0$  is the kinematic viscosity of the fluid and  $f$  is a given force field. For  $\varepsilon = 0$ , the model is reduced to 3D Navier-Stokes system.

Mathematical model for such fluid motion has been used extensively in turbulence simulations (see e.g. [3, 4, 7, 10]). For further discussion of theoretical results concerning (1.1), see [1, 2, 5, 12, 15, 16, 20].

For the 3D Navier-Stokes system weak solutions of problem are known to exist by a basic result by J. Leray from 1934 [11], only the uniqueness of weak solutions remains as an open problem. Then the known theory of global attractors of infinite dimensional dynamical systems is not applicable to the 3D Navier-Stokes system.

The theory of trajectory attractors for evolution partial differential equations was developed in [14, 18], which the uniqueness theorem of solutions of the corresponding initial-value problem is not proved yet, e.g. for the 3D Navier-Stokes system (see [8, 14, 17, 18]). Such trajectory attractor is a classical global attractor but in the space of weak solutions defined on  $[0, \infty)$ , with the corresponding semigroup being simply the translation in time of such solutions. A compact set  $\mathfrak{A} \Subset E$  is said to be a global attractor of a semigroup  $\{S(t), t > 0\}$  acting in a Banach or

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Hilbert space  $E$  if  $\mathfrak{A}$  is strictly invariant with respect to  $\{S(t)\} : S(t)\mathfrak{A} = \mathfrak{A} \forall t \geq 0$  and  $\mathfrak{A}$  attracts any bounded set  $B \subset E : \text{dist}(S(t)B, \mathfrak{A}) \rightarrow 0 (t \rightarrow \infty)$  (see [13], [14], [17], [18], [20]).

In this article, we study the upper semicontinuity, of the global attractors of the Leray-Hopf weak solutions of a regularized 3D. Navier-Stokes equations, as the artificial dissipation  $\varepsilon$  goes to 0, and the regularized problem (1.1 ) limits to the standard 3D Navier-Stokes equations. While there exist other examples of such robustness in the literature of the Navier-Stokes equations, the specific emphasis on the regularized problem is new for the 3D Navier-Stokes equations and is of interest. This would be an extension of the earlier work on Ou and Sritharan for the 2D Navier-Stokes equations, see references [15] and [16]. It is now known that there is a global attractor  $\mathfrak{A}_0$  for the Leray-Hopf weak solutions of the 3D Navier-Stokes equations, see Sell [17] or [18].

The main object of this paper to show that there is a global attractor, which one might denote by  $\mathfrak{A}_\varepsilon$ , for the regularized problem (1.1 ), and that the family  $\{\mathfrak{A}_\varepsilon\}$  is upper semicontinuous at  $\varepsilon = 0$ . Moreover, we can modify the argument described above so that the final result will have broader applicability by allowing the family of forcing functions  $f^\varepsilon$  to vary with  $\varepsilon$ , for  $\varepsilon > 0$ .

The family of sets  $\mathfrak{A}_\varepsilon, 0 < \varepsilon \leq 1$  is robust at  $\mathfrak{A}_0$ , or is upper semicontinuous with respect to  $\varepsilon$  at  $\varepsilon_0 = 0$ , provided that, for every  $\varepsilon_0 > 0$ , there is a neighborhood  $O(\varepsilon_0)$  of  $0 \in \mathbb{R}$  and a neighborhood  $N_{\varepsilon_0}(\mathfrak{A}_0)$  of  $\mathfrak{A}_0$ , such that  $\mathfrak{A}_\varepsilon \subset N_{\varepsilon_0}(\mathfrak{A}_0)$ , for every  $\varepsilon \in O(\varepsilon_0)$  with  $\varepsilon > 0$ , see (23.13) in [18].

The paper is organized as follows. In Section 2, we present the relevant mathematical framework for the paper. In Section 3, we recall the definition of the trajectory attractor  $\mathfrak{A}_0$  of the conventional 3-D Navier-Stokes equations. In Section 4, we study the regularized problem (see equation (1.1 )) limits to the standard 3D Navier-Stokes equations, as the artificial dissipation  $\varepsilon$  goes to 0, then we show the existence of trajectory attractor  $\mathfrak{A}_\varepsilon$ . In Section 5, we present the main result of this paper, that is, a theorem on the upper semicontinuity on the attractors  $\mathfrak{A}_\varepsilon$ . Finally, an application of our general results to the study of the robustness of the system (1.1 ) with a perturbed external force.

## 2. PRELIMINARY

We denote by  $H^m(\Omega)$ , the Sobolev space of  $L$ -periodic functions endowed with the inner product

$$(u, v) = \sum_{|\beta| \leq m} (D^\beta u, D^\beta v)_{L^2(\Omega)} \text{ and the norm } \|u\|_m = \sum_{|\beta| \leq m} (\|D^\beta u\|_{L^2(\Omega)}^2)^{\frac{1}{2}}$$

and by  $H^{-m}(\Omega)$  the dual space of  $H^m(\Omega)$ . We denote by  $\dot{H}^m(\Omega)$  the subspace of  $H^m(\Omega)$  with, zero average  $\dot{H}^m(\Omega) = \{u \in H^m(\Omega) ; \int_{\Omega} u(x) dx = 0\}$ .

- We introduce the following solenoidal subspaces  $V_s, s \in \mathbb{R}^+$  which are important to our analysis

$$V_0(\Omega) = \{u \in \dot{L}^2(\Omega), \text{div} u = 0, u.n|_{\Sigma_i} = -u.n|_{\Sigma_{i+3}}, i = 1, 2, 3\};$$

$$V_1(\Omega) = \{u \in \dot{H}^1(\Omega), \text{div} u = 0, \gamma_0 u|_{\Sigma_i} = \gamma_0 u|_{\Sigma_{i+3}}, i = 1, 2, 3\}.$$

$$V_2(\Omega) = \{u \in \dot{H}^2(\Omega), \text{div} u = 0, \gamma_0 u|_{\Sigma_i} = \gamma_0 u|_{\Sigma_{i+3}}, \gamma_1 u|_{\Sigma_i} = -\gamma_1 u|_{\Sigma_{i+3}}, i = 1, 2, 3\}.$$

see [20]. We refer the reader to Temam [21] for details on these spaces. Here the faces of  $\Omega$  are numbered as

$$\Sigma_i = \partial\Omega \cap \{x_i = 0\} \text{ and } \Sigma_{i+3} = \partial\Omega \cap \{x_i = L\}, \quad i = 1, 2, 3.$$

Here  $\gamma_0, \gamma_1$  are the trace operators and  $n$  is the unit outward normal on  $\partial\Omega$ .

- The space  $V_0$  is endowed with the inner product  $(u, v)_{L^2(\Omega)}$  and norm  $\|u\| = (u, u)_{L^2(\Omega)}^{1/2}$ .
- $V_1$  is the Hilbert space with the norm  $\|u\|_1 = \|u\|_{V_1}$ . The norm induced by  $\dot{H}^1(\Omega)$  and the norm  $\|\nabla u\|_{L^2(\Omega)}$  are equivalent in  $V_1$ .
- $V_2$  is the Hilbert space with the norm  $\|u\|_2 = \|u\|_{V_2}$ . In  $V_2$  the norm induced by  $\dot{H}^2(\Omega)$  is equivalent to the norm  $\|\Delta u\|_{L^2(\Omega)}$ .

$V'_s$  denote the dual space of  $V_s$ . We present the topology to be used for generating the neighborhood of robustness. Let  $F$  any vector space. A metric  $d(f, g)$  on  $F$  is said to be invariant if one has

$$d(f, g) = d(f - g, 0) \text{ for all } f, g \in F.$$

A Fréchet space is a complete topological vector space whose topology is induced by a translation invariant metric  $d(f, g)$ . Given a Banach space  $X$ , with norm  $\|\cdot\|_X$  and  $1 \leq p < \infty$ , we denote by  $L^p_{loc}[0, \infty; X)$  the Fréchet space of measurable functions  $f : [0, \infty) \rightarrow X$  that are  $p$ -integrable over  $[0, T]$ , for each  $0 < T < \infty$ , endowed with the metric

$$d(f, g) = \sum_{n=1}^{\infty} 2^{-n} \min(\|f - g\|_{L^p(0, n; X)}, 1).$$

We denote by  $L^p_{loc}(0, \infty; X)$  the Fréchet space of measurable functions  $f : (0, \infty) \rightarrow X$  that are  $p$ -integrable over  $[t_0, T]$ , for each  $0 < t_0 \leq T < \infty$  endowed with the metric

$$d(f, g) = \sum_{n=2}^{\infty} 2^{-n} \min(\|f - g\|_{L^p(\frac{1}{n}, n; X)}, 1).$$

Similarly for  $p = \infty$ , we will let  $L^\infty_{loc}(0, \infty; X)$  denote the collection of all functions  $f : (0, \infty) \rightarrow X$  with the property that, for all  $\tau$  and  $T$  with  $0 < T < \infty$ , one has  $\text{ess sup}_{0 < s < T} \|f\|_X < \infty$ . We denote by  $C[0, \infty; X)$  the space of strongly continuous functions from  $[0, \infty)$  to  $X$ , endowed with the topologie of the uniform convergence over compact sets and by  $C_w[0, \infty; X)$  the space of weakly continuous functions from  $[0, \infty)$  to  $X$ . We denote by  $L^\infty C = L^\infty(\mathbb{R}, X) \cap C(\mathbb{R}, X)$  the Fréchet space  $L^\infty C$  endowed with the  $L^\infty_{loc}$ -topology, which is the topology of uniform convergence on bounded sets.

Let  $E$  be a complete metric space with metric  $d$ . We write  $B_r$  for the open ball centre  $0 \in E$  and radius  $r$ . The following quantity is called the Hausdorff (non-symmetric) semidistance from a set  $X$  to a set  $Y$  in a Banach space  $E$

$$\text{dist}_E(X, Y) = \sup_{x \in X} \inf_{y \in Y} \|x - y\|_E.$$

Let  $M$  be a subset of  $E$  and let  $\mathbb{R}^+ = [0, \infty)$ . A mapping  $\sigma = \sigma(u, t)$ , where  $\sigma : M \times [0, \infty) \rightarrow M$  is said to be a semiflow on  $M$  provided the following hold

- 1)  $\sigma(w, 0) = w$ , for all  $w \in M$ .
- 2) The semigroup property holds, i. e.,

$$\sigma((w, s), t) = \sigma(w, s + t) \text{ for all } w \in M \text{ and } s, t \in \mathbb{R}^+.$$

- 3) The mapping  $\sigma : M \times (0, \infty) \rightarrow M$  is continuous.

If in addition the mapping  $\sigma : M \times [0, \infty) \rightarrow M$  is continuous we will say that the semiflow is continuous at  $t = 0$ . Here we use  $t > 0$  in order that the Robustness Theorem 23.14 in [18] is valid, see Sell [18] and Hale [8]. For any  $u \in M$  the positive trajectory through  $u$  is defined as the set  $\gamma^+(u) = \{\sigma(t)u, t \geq 0\}$ . For any set  $B \subset M$  we define the positive hull  $\mathcal{H}^+(B)$  and the omega limit set  $\omega(B)$  as follows

$$\mathcal{H}^+(B) = Cl_M \gamma^+(B) \text{ and } \omega(B) = \bigcap_{\tau \geq 0} \mathcal{H}^+(\sigma(\tau)B).$$

If  $\mathcal{A} \subset E$  and  $\varepsilon > 0$  we write

$$N_\varepsilon(\mathcal{A}) = \{z \in E, \inf_{a \in \mathcal{A}} d(z, a) < \varepsilon\}.$$

for the open  $\varepsilon$ -neighbourhood of  $\mathcal{A}$ .

We denote by  $A$  the Stokes operator  $Au = -\Delta u$  for  $u \in D(A)$ . We recall that the operator  $A$  is a closed positive self-adjoint unbounded operator, with  $D(A) = \{u \in V_0, Au \in V_0\}$ . We have in fact,  $D(A) = \dot{H}^2(\Omega) \cap V_0 = V_2$ . The spectral theory of  $A$  allows us to define the powers  $A^l$  of  $A$  for  $l \geq 1$ ,  $A^l$  is an unbounded self-adjoint operator in  $V_0$  with a domain  $D(A^l)$  dense in  $V_2 \subset V_0$ . We set here

$$A^l u = (-\Delta)^l u \text{ for } u \in D(A^l) = V_{2l} \cap V_0.$$

The space  $D(A^l)$  is endowed with the scalar product and the norm

$$(u, v)_{D(A^l)} = (A^l u, A^l v), \|u\|_{D(A^l)} = \{(u, v)_{D(A^l)}\}^{\frac{1}{2}}.$$

Now define the trilinear form  $b(., ., .)$  associated with the inertia terms

$$b(u, v, w) = \sum_{i,j=1}^3 \int_{\Omega} u_i \frac{\partial v_j}{\partial x_i} w_j dx$$

The continuity property of the trilinear form enables us to define (using Riesz representation theorem) a bilinear continuous operator  $B(u, v); V_2 \times V_2 \rightarrow V_2'$  will be defined by

$$\langle B(u, v), w \rangle = b(u, v, w), \forall w \in V_2.$$

Recall that for  $u$  satisfying  $\nabla \cdot u = 0$  we have

$$b(u, u, u) = 0 \text{ and } b(u, v, w) = -b(u, w, v). \quad (2.1)$$

Hereafter,  $c_i \in \mathbb{N}$ , will denote a dimensionless scale invariant positive constant which might depend on the shape of the domain. The trilinear form  $b(., ., .)$  is continuous on  $\dot{H}^{m_1}(\Omega) \times \dot{H}^{m_2+1}(\Omega) \times \dot{H}^{m_3}(\Omega)$ ,  $m_i \geq 0$

$$|b(u, v, w)| \leq c_0 \|u\|_{m_1} \|v\|_{m_2+1} \|w\|_{m_3}, \quad m_3 + m_2 + m_1 \geq \frac{3}{2} \quad (2.2)$$

see [6, 18]. The bilinear operator satisfies the inequality

$$\|Bu\| \leq c_1 \|A^{\frac{1}{2}}u\|^{\frac{3}{2}} \|Au\|^{\frac{1}{2}}, \quad (2.3)$$

see [21, Lemma 3.8]. Similarly, we define  $\hat{B}(u, v) \in V_1'$  by

$$\left\langle \hat{B}(u, v), w \right\rangle_{V_1' \times V_1} = b(u, v, w), \quad \forall w \in V_1.$$

We recall some inequalities that we will be using in what follows.

Young's inequality

$$ab \leq \frac{\epsilon}{p} a^p + \frac{1}{q\epsilon^{\frac{q}{p}}} b^q, a, b, \epsilon > 0, p > 1, q = \frac{p}{p-1}. \quad (2.4)$$

Poincaré's inequality

$$\lambda_1 \|u\|^2 \leq \|u\|_1^2 \text{ for all } u \in V_0, \quad (2.5)$$

where  $\lambda_1$  is the smallest eigenvalue of the Stokes operator  $A$ .

### 3. NAVIER-STOKES EQUATIONS

The conventional Navier-Stokes system can be written in the evolution form

$$\begin{aligned} \frac{\partial u}{\partial t} + \nu Au + \hat{B}(u, u) &= f, \quad t > 0, \\ u_0(x) &= u_0. \end{aligned} \quad (3.1)$$

Let  $f \in L^\infty(0, \infty; V_0)$  be given. We will say that a function  $u$  is a weak solution of the 3D Navier-Stokes of Class  $LH$  (Leray–Hopf) on  $[0, \infty)$  provided that  $u(x, 0) = u_0(x) \in V_0$ , and the following properties hold

1)  $u \in L^\infty(0, \infty; V_0) \cap L^2_{loc}[0, \infty; V_1)$ .

2)  $\frac{du}{dt} \in [L^{\frac{4}{3}}_{loc} 0, \infty; V'_1)$ .

Taking the inner product of (3.1) with  $u$ , and using (2.4) we have

$$\frac{d}{dt} \|u(t)\|^2 + 2\nu \|\nabla u\|^2 = 2\langle f, u \rangle. \quad (3.2)$$

by application of Young's inequality and the Poincaré's Lemma, yields

$$\frac{d}{dt} \|u(t)\|^2 + \nu \|\nabla u\|^2 \leq \frac{\|f\|^2}{\nu\lambda_1}, \quad (3.3)$$

using the Poincaré Lemma and Gronwall's inequality, to get

$$\|u(t)\|^2 \leq e^{-\nu\lambda_1(t-t_0)} \|u(t_0)\|^2 + \frac{1}{\nu^2\lambda_1^2} \|f\|^2 (1 - e^{-\nu\lambda_1(t-t_0)}), \text{ with } 0 < t_0 < t,$$

3) which implies that

$$\|u(t)\|^2 \leq e^{-\nu\lambda_1(t-t_0)} \|u(t_0)\|^2 + \frac{1}{\nu^2\lambda_1^2} \|f\|^2. \quad (3.4)$$

Integrating (3.2) over  $[t_0, t]$  we find that

$$\|u(t)\|^2 + 2\nu \int_{t_0}^t \|A^{\frac{1}{2}} u(s)\|^2 ds \leq \|u(t_0)\|^2 + 2 \int_{t_0}^t \langle f(s), u(s) \rangle ds. \quad (3.5)$$

4) The function  $u$  satisfies the following equality

$$\langle u(t) - u(t_0), v \rangle + \nu \int_{t_0}^t \langle A^{\frac{1}{2}} u(s), A^{\frac{1}{2}} v \rangle ds + \int_{t_0}^t \langle \hat{B}(u(s), u(s)), v \rangle ds = \int_{t_0}^t \langle f, v \rangle ds, \quad (3.6)$$

for all  $v \in V_1$  and for all  $t \geq t_0 \geq 0$ .

The proof of the following theorem is given in [12, 13, 21].

**Theorem 3.1.** *Let  $f \in V'_1$  and  $u_0 \in V_0$  be given. Then for every  $T > 0$ , there exists a weak solution  $u(t)$  of (3.1) from the space  $L^2(0, T; V_1) \cap L^\infty(0, T; V_0)$ , such that  $u(x, 0) = u_0$  and  $u(t)$  satisfies the energy equality (3.6).*

Moreover (see [21]),  $u(\cdot)$  is weakly continuous from  $[0, T]$  into  $V_0$ , the function  $u \in C_w([0, T]; V_0)$  and consequently  $u(x, 0) = u_0(x) \in V_0$ . Let  $W$  is the set of all Leray–Hopf weak solutions  $u(\cdot)$  of equation (3.1) in the space  $L^\infty(0, \infty; V_0) \cap L^2_{loc}[0, \infty; V_1)$  that satisfy the following properties

- $\frac{du}{dt} \in L^{\frac{4}{3}}_{loc}(0, \infty; V'_1)$ ;
- for almost all  $t$  and  $t_0$ , with  $t > t_0 > 0$ , inequalities (3.5, 3.6) are valid.

Let  $X^0$  denote the Fréchet space used to define the Leray–Hopf weak solutions. Thus

$$\varphi \in X^0 = L^\infty(0, \infty; V_0) \cap L^2_{loc}[0, \infty; V_1),$$

where  $\varphi \in C_w[0, \infty; V_0)$  and we let  $\mathfrak{F}^0$  denote a compact, translation invariant set of forcing functions  $f$  in

$$L^\infty C = L^\infty(\mathbb{R}, L^2(\Omega)) \cap C(\mathbb{R}, L^2(\Omega))$$

where the topology on the Fréchet space  $L^\infty C$  is the topology of uniform convergence on bounded sets in  $\mathbb{R}$ .

Then, we use the Leray–Hopf solutions of the 3D Navier–Stokes equations with  $\varepsilon = 0$  to generate a semiflow  $\pi^0$  on  $\mathfrak{F}^0 \times X^0$ , where

$$\pi^0(\tau)(f, \varphi) = (f_\tau, S^0(f, \tau)\varphi) \text{ for } \tau \geq 0,$$

$f_\tau(t) = f(\tau + t)$  and  $u(t) = S^0(f, t)\varphi$  is the Leray–Hopf solution of the 3D Navier–Stokes equations that satisfies  $u(0) = S^0(f, 0)\varphi = \varphi(0)$ . By using the theory of generalized weak solutions, as in Sell [17] or [18], we note that  $\pi^0$  has a global attractor  $\mathfrak{A}_0 \subset \mathfrak{F}^0 \times X^0$  see Theorem 65.12 in [18].

#### 4. THE REGULARIZED NAVIER-STOKES SYSTEM

Using the operators defined in the previous section, we can write the modified system (1.1) in the evolution form

$$\begin{aligned} \partial_t u^\varepsilon + \varepsilon A^2 u^\varepsilon + \nu A u^\varepsilon + B(u^\varepsilon, u^\varepsilon) &= f(x), & \text{in } \Omega \times (0, \infty) \\ u^\varepsilon(x) &= u^\varepsilon, & \text{in } \Omega. \end{aligned} \quad (4.1)$$

For  $\varepsilon > 0$ , we let  $\pi^\varepsilon$  denote the semiflow on  $\mathfrak{F}^0 \times X^0$  generated by the weak solutions of regularized 3D Navier–Stokes equations of (4.1). Thus

$$\pi^\varepsilon(\tau) = (f_\tau, S^\varepsilon(f, \tau)\varphi), \quad (4.2)$$

where  $u_0^\varepsilon = \varphi$  and

$$u^\varepsilon(t) = S^\varepsilon(f, t)\varphi = S^\varepsilon(f, t)u_0^\varepsilon \quad (4.3)$$

is the weak solution of (4.1) that satisfies  $u^\varepsilon(0) = \varphi(0) = u_0^\varepsilon(0)$ .

**Theorem 4.1.** *Let  $\Omega \subset \mathbb{R}^3$  and let  $f \in L^2(0, T; V_0)$  and  $u_0^\varepsilon \in V_1$  be given. Then there exists a unique weak solution of (4.1) which satisfies  $u^\varepsilon \in L^\infty(0, T; V_0) \cap L^2[0, T; V_1)$  and  $u^\varepsilon \in C_w([0, T]; V_0)$ ,  $\forall T > 0$ .*

For the proof see, J. Lions [12, Remark 6.11].

**Proposition 4.2.**  *$f \in L^2(0, T; V_0) \cap L^\infty(0, T; H^{-4})$  and  $u^\varepsilon \in V_0$  then the weak solution  $u^\varepsilon(t)$  of the modified Navier–Stokes equations satisfy*

- i)  $u^\varepsilon$  is uniformly bounded in  $L^2(0, T; V_2)$ ,
- ii)  $u^\varepsilon$  is uniformly bounded in  $L^\infty(0, T; V_0)$ ,
- iii)  $\frac{d}{dt}u^\varepsilon(t)$  is uniformly bounded in  $L^\infty(0, T; H^{-4})$ .

*Proof.* Taking the inner product of (4.1) by  $u^\varepsilon \in V_2$ , we obtain

$$\frac{d}{dt} \|u^\varepsilon\|^2 + 2\varepsilon \|Au^\varepsilon\|^2 + 2\nu \|\nabla u^\varepsilon\|^2 = 2(f, u^\varepsilon). \quad (4.4)$$

Applying Young's inequality and using the Poincaré Lemma, we obtain

$$\frac{d}{dt} \|u^\varepsilon\|^2 + 2\varepsilon \|Au^\varepsilon\|^2 + \nu \|\nabla u^\varepsilon\|^2 \leq \frac{\|f\|^2}{\nu\lambda_1}. \quad (4.5)$$

If we drop the positive term associated with  $\varepsilon$ , we obtain

$$\frac{d}{dt} \|u^\varepsilon\|^2 + \nu\lambda_1 \|u^\varepsilon\|^2 \leq \frac{\|f\|^2}{\nu\lambda_1}. \quad (4.6)$$

Hence, integrating the above inequality over  $[0, t]$ , we obtain *i)*

$$\|u^\varepsilon(t)\|^2 \leq \|u^\varepsilon\|^2 e^{-\nu\lambda_1 t} + \frac{1}{\nu\lambda_1} \int_0^t \|f(s)\|^2 e^{-\nu\lambda_1(t-s)} ds.$$

By discarding the term  $2\varepsilon \|Au^\varepsilon\|^2$  in (4.3) and integrating from 0 to  $T$ , we have

$$\int_0^T \|\nabla u^\varepsilon\|^2 dt \leq \frac{1}{\nu} [\|u^\varepsilon\|^2 + \frac{1}{\nu\lambda_1} \int_0^T \|f\|^2 dt], \quad (4.7)$$

we obtain *ii)*. Taking the duality pairing with (4.1) by  $v$  to get

$$|\langle \partial_t u^\varepsilon, v \rangle| \leq |\langle f(x), v \rangle| + \varepsilon |\langle A^2 u^\varepsilon, v \rangle| + \nu |\langle Au^\varepsilon, v \rangle| + |\langle B(u^\varepsilon, u^\varepsilon), v \rangle|. \quad (4.8)$$

For  $v \in H^4$ , we have

$$|\langle A^2 u^\varepsilon, v \rangle| = |\langle u^\varepsilon, A^2 v \rangle| \leq c_2 \|u^\varepsilon\| \|v\|_{H^4} \quad \text{and} \quad |\langle Au^\varepsilon, v \rangle| = |\langle u^\varepsilon, Av \rangle| \leq c_3 \|u^\varepsilon\| \|v\|_{H^4}.$$

We estimate the trilinear term by choosing  $m_1 = m_3 = 0$  and  $m_2 = 3$  in (2.2),

$$|\langle B(u^\varepsilon, u^\varepsilon), v \rangle| = |b(u^\varepsilon, v, u^\varepsilon)| \leq c_0 \|u^\varepsilon\|^2 \|v\|_{H^4}.$$

Thus (4.8) becomes

$$|\langle \partial_t u^\varepsilon(t), v \rangle| \leq \left( \|f(t)\|_{H^{-4}} + c_2 \varepsilon \|u^\varepsilon(t)\| + c_3 \nu \|u^\varepsilon(t)\| + c_0 \|u^\varepsilon(t)\|^2 \right) \|v\|_{H^4}.$$

This gives

$$\text{ess sup}_{t \in [0, T]} \|\partial_t u^\varepsilon(t)\|_{H^{-4}} \leq \text{ess sup}_{t \in [0, T]} \|f(t)\|_{H^{-4}} + c_2 \sup_{t \in [0, T]} \|u^\varepsilon(t)\| + c_3 \nu \sup_{t \in [0, T]} \|u^\varepsilon(t)\| + c_0 \sup_{t \in [0, T]} \|u^\varepsilon(t)\|^2.$$

Using the result *i)*, we can conclude *iii)*.  $\square$

We need three forms of convergence for appropriate subsequences.

**Lemma 4.3.** *There exist a sequence  $\varepsilon_n$  Such that For*

- (i)  $u^{\varepsilon_n} \rightharpoonup u$  in  $L^2(0, T; V_1)$  weakly;
  - (ii)  $u^{\varepsilon_n} \rightharpoonup u$  in  $L^\infty(0, T; V_0)$  weak-star;
  - (iii)  $u^{\varepsilon_n} \rightarrow u$  in  $L^2(0, T; V_0)$  strongly,
- as  $\varepsilon_n \rightarrow 0$ .

*Proof.* It follows from Proposition 4.2 (i) and (ii), that easily  $u^{\varepsilon_n} \in L^2(0, T; V_1) \cap L^\infty(0, T; V_0)$  with bounds independent of  $\varepsilon_n$ . The convergence results (i) and (ii) are satisfied by the weak solutions. Hence the sequence  $u^{\varepsilon_n}$  converges strongly in  $L^2(0, T; V_0)$  and so the item (iii) is true for  $u^{\varepsilon_n}$ .  $\square$

As a consequence of Proposition 4.2, we have



- Proposition 4.4.**  $\forall w \in L^2(0, T; V_1), \forall w' \in L^2(0, T; V_1')$
- a)  $\lim_{n \rightarrow \infty} \int_0^T (\partial_t u^{\varepsilon_n}(t), w) dt = \int_0^T (\partial_t u(t), w(t)) dt,$
  - b)  $\lim_{n \rightarrow \infty} \int_0^T (\nabla u^{\varepsilon_n}(t), \nabla w(t)) dt = \int_0^T (\nabla u(t), \nabla w(t)) dt,$
  - c)  $\lim_{n \rightarrow \infty} \int_0^T b(u^{\varepsilon_n}(t), u^{\varepsilon_n}(t), w(t)) dt = \int_0^T b(u(t), u(t), w(t)) dt.$
- the strong convergence of  $u^{\varepsilon_n} \rightarrow u$  in  $L^2(0, T; V_0)$  implies c).

**Theorem 4.5.** [22, Lemma 4.2]

The injection of  $\mathcal{X} = \{u \in L^2(0, T; V_1) \cap L^\infty(0, T; V_0); \frac{du}{dt} \in L^\infty(0, T; H^{-4})\}$  into  $\mathcal{Y} = \{u \in L^2(0, T; V_0) \cap C(0, T; H^{-4})\}$  is compact.

**Theorem 4.6.** Let  $\Omega \subset \mathbb{R}^3$  and  $u^\varepsilon(t)$ , be the weak solution of problem (4.1) given by Theorem 4.1 Then as  $\varepsilon \rightarrow 0$ , the solution  $u^\varepsilon$  converges to the weak solution of the Navier-Stokes equations.

*Proof.* From i) and ii) of Proposition 4.2 we can deduce that the weak solutions  $u^{\varepsilon_n} \in L^2(0, T; V_1) \cap L^\infty(0, T; V_0)$  are bounded in independent of  $\varepsilon_n$ . This establishes items (i) and (ii) of Lemma 4.3. Additionally,  $\frac{du^{\varepsilon_n}}{dt} \in L^\infty(0, T; H^{-4})$  with bounds independent of  $\varepsilon_n$  indicates that  $u^{\varepsilon_n} \in \mathcal{X}$ . The compactness Theorem 4.5 implies the strong convergence in (iii) of Lemma 4.3.  $\square$

Taking the inner product of (4.1) with a test function  $\varphi \in \mathcal{D}(0, T; \mathcal{D}(A^2))$  [21] and then integrate by parts to get,

$$\begin{aligned} & (u^{\varepsilon_n}(T), \varphi(T)) - (u^{\varepsilon_n}(0), \varphi(0)) - \int_0^T (u^{\varepsilon_n}(t), \varphi'(t)) dt \\ & + \varepsilon_n \int_0^T (\Delta u^{\varepsilon_n}(t), \Delta \varphi(t)) dt + \nu \int_0^T (\nabla u^{\varepsilon_n}(t), \nabla \varphi(t)) dt \\ & + \int_0^T b(u^{\varepsilon_n}(t), u^{\varepsilon_n}(t), \varphi(t)) dt = \int_0^T \langle f(t), \varphi(t) \rangle dt. \end{aligned} \quad (4.9)$$

Using Proposition 4.4 for take the equation (4.9) to the limit as  $\varepsilon_n \rightarrow 0$ ,

$$- \int_0^T (u, \varphi') dt + \nu \int_0^T (\nabla u, \nabla \varphi) dt + \int_0^T b(u, u, \varphi) dt = \int_0^T \langle f, \varphi \rangle dt.$$

The term  $\varepsilon_n \int_0^T (\Delta u^{\varepsilon_n}(t), \Delta \varphi(t)) dt \rightarrow 0$  as  $\varepsilon_n \rightarrow 0$ , since the weak solution  $u^{\varepsilon_n} \in L^2(0, T; V_1)$  with bound uniform in  $\varepsilon_n$  and

$$\varepsilon_n \int_0^T |(\Delta u^{\varepsilon_n}, \Delta \varphi)| dt \leq \varepsilon_n \int_0^T |(u^{\varepsilon_n}, A\varphi)| dt \leq c\varepsilon_n.$$

Since  $u \in L^2(0, T; V_1) \cap L^\infty(0, T; V_0)$ , we can conclude that  $u$  is indeed the weak solution for the conventional Navier-Stokes equations. We need the following fact

**Lemma 4.7.** For fixed  $\varepsilon > 0$ , if  $u_0^\varepsilon \in V_1$  satisfies

$$\|A^{\frac{1}{2}} u_0^\varepsilon\|^2 \leq R_1 \text{ and } \|f\|_\infty^2 \leq \frac{\nu^2 \lambda_1}{4} R_1, \quad (4.10)$$

where  $R_1 = \frac{8^{-\frac{1}{2}} \nu^{\frac{3}{2}} \lambda_1^{\frac{1}{2}}}{3\sqrt{3}c_1^{\frac{1}{2}}}$ , then there is a constant  $C_0$  positive such that

$$\|A^{\frac{1}{2}} u^\varepsilon(t)\|^2 \leq C_0 \text{ for all } t \geq 0.$$

$C_0$  independent of  $\varepsilon$  is given in the proof.

*Proof.* Taking the inner product of (4.1) with  $Au^\varepsilon$ , we find

$$\frac{1}{2} \frac{d}{dt} \|A^{\frac{1}{2}} u^\varepsilon\|^2 + \varepsilon (A^2 u^\varepsilon, Au^\varepsilon) + \nu \|Au^\varepsilon\|^2 + b(u^\varepsilon, u^\varepsilon, Au^\varepsilon) = (f, Au^\varepsilon).$$

by using (2.3), to obtain

$$|b(u^\varepsilon, u^\varepsilon, Au^\varepsilon)| \leq c_1 \|A^{\frac{1}{2}} u^\varepsilon\|^{\frac{3}{2}} \|Au^\varepsilon\|^{\frac{1}{2}} \|Au^\varepsilon\|.$$

Due to (2.4) and (2.5), we get

$$(A^2 u^\varepsilon, Au^\varepsilon) = (A^{\frac{1}{2}} Au^\varepsilon, A^{\frac{1}{2}} Au^\varepsilon) = \|A^{\frac{3}{2}} u^\varepsilon\|^2.$$

We have the following inequality

$$\frac{1}{2} \frac{d}{dt} \|A^{\frac{1}{2}} u^\varepsilon\|^2 + \varepsilon \|A^{\frac{3}{2}} u^\varepsilon\|^2 + \frac{\nu}{2} \|Au^\varepsilon\|^2 \leq \frac{1}{\nu} \|f\|_\infty^2 + \frac{27c_1^4}{2\nu^3} \|A^{\frac{1}{2}} u^\varepsilon\|^6. \quad (4.11)$$

Since  $\lambda_1 \|A^{\frac{1}{2}} v\|^2 \leq \|Av\|^2$ , we obtain

$$\frac{d}{dt} \|A^{\frac{1}{2}} u^\varepsilon\|^2 + 2\varepsilon \|A^{\frac{3}{2}} u^\varepsilon\|^2 + (\nu\lambda_1 - \frac{27c_1^4}{\nu^3} \|A^{\frac{1}{2}} u^\varepsilon\|^4) \|A^{\frac{1}{2}} u^\varepsilon\|^2 \leq \frac{2}{\nu} \|f\|_\infty^2.$$

We drop the positive term  $2\varepsilon \|A^{\frac{3}{2}} u^\varepsilon\|^2$  to get

$$\frac{d}{dt} \|A^{\frac{1}{2}} u^\varepsilon\|^2 + (\nu\lambda_1 - \frac{27c_1^4}{\nu^3} \|A^{\frac{1}{2}} u^\varepsilon\|^4) \|A^{\frac{1}{2}} u^\varepsilon\|^2 \leq \frac{2}{\nu} \|f\|_\infty^2.$$

From the definition of  $R_1$ , we have

$$\frac{27c_1^4}{\nu^2} \|A^{\frac{1}{2}} u^\varepsilon\|^4 \leq \frac{\nu\lambda_1}{2} \text{ for } 0 \leq t \leq T. \quad (4.12)$$

By using (4.10), (4.11) and (4.12), together with the Gronwall inequality we obtain

$$\|A^{\frac{1}{2}} u^\varepsilon\|^2 \leq \|A^{\frac{1}{2}} u_0^\varepsilon\|^2 \exp(-\frac{\nu\lambda_1}{2} T) + \frac{4}{\nu^2 \lambda_1} \|f\|_\infty^2,$$

we denote

$$C_0 = R_1 \exp(-\frac{\nu\lambda_1}{2} T) + \frac{4}{\nu^2 \lambda_1} \|f\|_\infty^2$$

this completes the proof of Lemma 4.7.  $\square$

From this proof we can easily deduce the following results.

**Lemma 4.8.** *For fixed  $\varepsilon > 0$  and for all  $0 \leq t \leq T$ ,  $u_0^\varepsilon \in B_{R_1} = \{u_0^\varepsilon \in V_1; \|u_0^\varepsilon\|_1 \leq R_1\}$  there is a constant  $C_1$  positive independent of  $\varepsilon$  such that*

$$\int_0^T \|Au^\varepsilon(t)\|^2 dt \leq C_1. \quad (4.13)$$

*Proof.* We drop the positive term  $\varepsilon \|A^{\frac{3}{2}} u^\varepsilon\|^2$  in inequality (4.11) and integrate from 0 to  $T$  we get

$$\int_0^T \|Au^\varepsilon(s)\|^2 ds \leq \frac{2C_0}{\nu} + \frac{2\|f\|_\infty^2}{\nu^2} + \frac{27c_1^4 R_1^6}{\nu^2} = C_1. \quad \square$$

**Theorem 4.9.** *For fixed  $\varepsilon \geq 0$ , and  $f \in L^\infty C$  a time independent functions,  $\pi^\varepsilon$  is a continuous family of semiflows on  $X^0$ .*

*Proof.* Let convergent sequences  $\varepsilon_n$ ,  $\varphi^n$  and  $f^n$ , with limits  $\varepsilon_n \rightarrow \varepsilon_0$ , (especially with  $\varepsilon_0 = 0$ ),  $\varphi^n \rightarrow \varphi_0$  in the  $X^0$ -topology and  $f^n \rightarrow f^0$  in the  $L^\infty C$ -topology as  $n \rightarrow \infty$ , then

$$S^{\varepsilon_n}(f^n, t)\varphi^n \rightarrow S^{\varepsilon_0}(f^0, t)\varphi^0. \quad (4.14)$$

Let

$$S^{\varepsilon_n}(f^n, t)\varphi^n - S^{\varepsilon_0}(f^0, t)\varphi^0 = u^{\varepsilon_n}(t) - u^{\varepsilon_0}(t), \quad (4.15)$$

we obtain for  $w_n = u^{\varepsilon_n}(t) - u^{\varepsilon_0}(t)$  and  $g_n = f^n - f^0$

$$\partial_t w_n + \varepsilon_n A^2 w_n + A w_n + B(u^{\varepsilon_n}, u^{\varepsilon_n}) - B(u^{\varepsilon_0}, u^{\varepsilon_0}) = g_n. \quad (4.16)$$

By taking inner product with  $w_n$  for above equation we get

$$\frac{1}{2} \frac{d}{dt} \|w_n\|^2 + \varepsilon_n \|A w_n\|^2 + \nu \|A^{\frac{1}{2}} w_n\|^2 = b(w_n, w_n, u^{\varepsilon_n}) + (g_n, w_n). \quad (4.17)$$

Using Young's inequality, we obtain

$$(g_n, w_n) \leq \frac{1}{2\nu} \|g_n\|^2 + \frac{\nu}{2} \|A^{\frac{1}{2}} w_n\|^2,$$

thanks to (2.2) with  $(m_3, m_2, m_1) = (0, 1, 1)$ , we find

$$|b(w_n, u^{\varepsilon_n}, w_n)| \leq c_0 \|A^{\frac{1}{2}} w_n\| \|A u^{\varepsilon_n}\| \|w_n\| \quad (4.18)$$

this give

$$|b(w_n, u^{\varepsilon_n}, w_n)| \leq \frac{\nu}{2} \|A^{\frac{1}{2}} w_n\|^2 + \frac{c_4}{2} \|A u^{\varepsilon_n}\|^2 \|w_n\|^2. \quad (4.19)$$

Combining all these inequalities in (4.17), we have

$$\frac{1}{2} \frac{d}{dt} \|w_n\|^2 + \varepsilon_n \|A w_n\|^2 \leq \frac{c_4}{2} \|A u^{\varepsilon_n}\|^2 \|w_n\|^2 + \frac{1}{2\nu} \|g_n\|^2.$$

We drop the positive term  $\varepsilon_n \|A w_n\|^2$  to obtain the following differential inequality

$$\frac{d}{dt} \|w_n(t)\|^2 \leq c_4 \|A u^{\varepsilon_n}\|^2 \|w_n\|^2 + \frac{1}{\nu} \|g_n\|^2. \quad (4.20)$$

Applying now Gronwall's inequality to (4.20), for  $t \geq 0$  we have

$$\begin{aligned} \|w_n(t)\|^2 &\leq \|w_n(0)\|^2 \exp c_4 \int_0^t \|A u^{\varepsilon_n}\|^2 ds \\ &\quad + \frac{1}{\nu} \int_0^t \|g_n(h)\|^2 \exp c_4 \int_0^t \|A u^{\varepsilon_n}\|^2 ds dh \end{aligned} \quad (4.21)$$

using estimate (4.13) we find

$$\|w_n(t)\|^2 \leq C_2 \|w_n(0)\|^2 + \frac{TC_2}{\nu} \|g_n\|^2 \quad (4.22)$$

for all  $t$  in compact sets in  $[0, \infty)$ ,  $C_2 = \exp c_4 C_1$ . Since  $f^n \rightarrow f^0$  in the  $L^\infty C$ -topology and  $\varphi^n \rightarrow \varphi_0$  in the  $X^0$ -topology this means that  $\|g_n\| \rightarrow 0$  and  $\|w_n(0)\| \rightarrow 0$  as  $n \rightarrow \infty$ , it follows from (4.22) that

$$\|S^{\varepsilon_n}(f^n, t)\varphi^n - S^{\varepsilon_0}(f^0, t)\varphi^0\| \leq C_2 \|u_0^{\varepsilon_n} - u_0^{\varepsilon_0}\|^2 + \frac{TC_2}{\nu} \|f^n - f\|^2 \rightarrow 0, \text{ as } n \rightarrow \infty.$$

It follows that  $\pi^\varepsilon$  is a continuous semiflows on  $X^0$ . Hence  $\pi^\varepsilon$  approximates  $\pi^0$  on  $B_{R_1}$  uniformly on  $[0, T]$ .  $\square$

Regarding the existence of the attractor  $\mathfrak{A}_\varepsilon$  when  $\varepsilon > 0$ , we use especially the related papers of Chepyzhov and Vishik, such as [14] to show that the system (4.1) possesses a global attractor. For  $\varepsilon > 0$ , we consider the trajectory space  $\mathcal{K}_\varepsilon$  of the modified Navier-Stokes equations (4.1).  $\mathcal{K}_\varepsilon$  is the union of all weak solutions  $u^\varepsilon \in X^0$  that satisfy (4.1), see (6.163) in [12]. Using the described scheme in [14], we construct the spaces  $\mathcal{S}_b$

$$\mathcal{S}_b = \{v(\cdot) \in L^\infty(0, T; V_0) \cap L_b^2(0, T; V_1), \partial_t v(\cdot) \in L_b^2(0, T; D(A)')\}$$

with norm

$$\|v\|_{\mathcal{S}_b} = \|v\|_{L_b^2(0, T; V_1)} + \|v\|_{L^\infty(0, T; V_0)} + \|\partial_t v\|_{L_b^2(0, T; D(A)' )}$$

where

$$\|v\|_{L_b^2(0, T; V_1)} = \sup_{t \geq 0} \left( \int_t^{t+1} \|v(s)\|_1^2 ds \right)^{\frac{1}{2}}, \quad \|v\|_{L^\infty(0, T; V_0)} = \operatorname{ess\,sup}_{t \geq 0} \|v\|$$

and

$$\|\partial_t v\|_{L_b^2(0, T; D(A)' )} = \sup_{t \geq 0} \left( \int_t^{t+1} \|\partial_t v(s)\|_{D(A)'}^2 ds \right)^{\frac{1}{2}}.$$

We need a topology in the space  $\mathcal{K}_\varepsilon$ . We define on  $X^0$  the following sequential topology which we denote  $\Gamma$ .

By definition, a sequence of functions  $\{v_n\} \subseteq X^0$  converges to a function  $v \in X^0$  in the topology  $\Gamma$  as  $n \rightarrow \infty$  if, for any  $T > 0$ ,  $v_n \rightarrow v$  weakly in  $L^2(0, T; V_1)$ ;  $v_n \rightarrow v$  weak-\* in  $L^\infty(0, T; V_0)$  and  $v_n \rightarrow v$  strongly in  $L^2(0, T; V_0)$ , as  $n \rightarrow \infty$ .

We consider the topology  $\Gamma$  on  $\mathcal{K}_\varepsilon$ . It is easy to prove that the space  $\mathcal{K}_\varepsilon$  is closed in  $\Gamma$ . From the definition of  $\mathcal{K}_\varepsilon$ , it follows that  $\pi^\varepsilon \mathcal{K}_\varepsilon \subset \mathcal{K}_\varepsilon$  for all  $t \geq 0$ .

**Proposition 4.10.** *If  $u^\varepsilon(t)$  is a solution of (4.1), then the following inequalities hold for all  $t > 0$*

$$\|u^\varepsilon(t)\|^2 \leq e^{-\nu\lambda_1 t} \|u_0^\varepsilon\|^2 + \frac{\|f\|^2}{\nu^2 \lambda_1^2}, \quad (4.23)$$

$$\int_t^{t+1} \|u^\varepsilon(s)\|^2 ds \leq \frac{e^{-\nu\lambda_1 t}}{\nu\lambda_1} \|u_0^\varepsilon\|^2 + \frac{\|f\|^2}{\nu^2 \lambda_1^2}, \quad (4.24)$$

$$\nu \int_t^{t+1} \|u^\varepsilon(s)\|_1^2 ds \leq \frac{e^{-\nu\lambda_1 t}}{\nu\lambda_1} \|u_0^\varepsilon\|^2 + \frac{\|f\|^2}{\nu^2 \lambda_1^2} + \frac{\|f\|^2}{\nu\lambda_1}. \quad (4.25)$$

*Proof.* Taking the inner product of (4.1) by  $u^\varepsilon \in V_2$ , we obtain

$$\frac{d}{dt} \|u^\varepsilon\|^2 + 2\varepsilon \|Au^\varepsilon\|^2 + 2\nu \|\nabla u^\varepsilon\|^2 = 2(f, u^\varepsilon). \quad (4.26)$$

Applying Young's inequality and using the Poincaré Lemma, we obtain

$$\frac{d}{dt} \|u^\varepsilon\|^2 + 2\varepsilon \|u^\varepsilon\|_2^2 + \nu \|\nabla u^\varepsilon\|^2 \leq \frac{\|f\|^2}{\nu\lambda_1}. \quad (4.27)$$

Using the Poincaré inequality, dropping the positive term  $2\varepsilon \|u^\varepsilon\|_2^2$  and using Gronwall's inequality over  $[0, t]$ , we obtain (4.23). Integrating (4.23) over  $[t, t+1]$  we find (4.24). Integrating (4.27) over  $[t, t+1]$  we find

$$\nu \int_t^{t+1} \|\nabla u^\varepsilon(s)\|^2 ds \leq \frac{\|f\|^2}{\nu\lambda_1} + \|u^\varepsilon(t)\|^2.$$

Applying inequality (4.23), we have (4.25).  $\square$

**Proposition 4.11.** *Let  $f \in V_0$ . Then any solution  $u_\varepsilon(t)$  of (4.1) satisfies*

$$\int_t^{t+1} \|\partial_t u_\varepsilon(s)\|_{D(A^2)}^2 ds \leq \frac{e^{-\nu\lambda_1 t}}{\nu\lambda_1} \|u_\varepsilon(0)\|^2 + \frac{\|f\|^2}{\nu^2\lambda_1^2} + C_3, \quad (4.28)$$

$C_3$  is a positive constant independent of  $\varepsilon$ .

*Proof.* Taking the duality pairing with (4.1) by  $v$  to get

$$|\langle \partial_t u_\varepsilon, v \rangle| \leq |\langle f(x), v \rangle| + \varepsilon |\langle A^2 u_\varepsilon, v \rangle| + \nu |\langle Au_\varepsilon, v \rangle| + |\langle B(u_\varepsilon, u_\varepsilon), v \rangle|. \quad (4.29)$$

For  $v \in V_4$ , we have

$$|\langle A^2 u_\varepsilon, v \rangle| = |\langle Au_\varepsilon, Av \rangle| \leq c_5 \|u_\varepsilon\| \|v\|_4.$$

and

$$|\langle Au_\varepsilon, v \rangle| = |\langle u_\varepsilon, Av \rangle| \leq c_6 \|u_\varepsilon\| \|v\|_4.$$

We estimate the trilinear term by choosing  $m_1 = m_3 = 0$  and  $m_2 = 3$  in (2.2),

$$|\langle B(u_\varepsilon, u_\varepsilon), v \rangle| = |b(u_\varepsilon, v, u_\varepsilon)| \leq c_0 \|u_\varepsilon\|^2 \|v\|_4.$$

Thus (4.29) becomes

$$|\langle \partial_t u_\varepsilon(t), v \rangle| \leq (\|f(t)\|_{V_4'} + c_5 \varepsilon \|u_\varepsilon(t)\| + c_6 \nu \|u_\varepsilon(t)\| + c_0 \|u_\varepsilon(t)\|^2) \|v\|_4.$$

This gives

$$\operatorname{ess\,sup}_{t \in [0, T]} \|\partial_t u_\varepsilon(t)\|_{V_4'} \leq \operatorname{ess\,sup}_{t \in [0, T]} \|f(t)\|_{V_4'} + c_5 \sup_{t \in [0, T]} \|u_\varepsilon(t)\| + c_6 \nu \sup_{t \in [0, T]} \|u_\varepsilon(t)\| + c_0 \sup_{t \in [0, T]} \|u_\varepsilon(t)\|^2.$$

Integrating this inequality over  $[t, t+1]$ , we find

$$\int_t^{t+1} \|\partial_t u_\varepsilon(t)\|_{V_4'}^2 ds \leq \operatorname{ess\,sup}_{t \in [0, T]} \|f(t)\|_{V_4'} + (c_5 + c_6 \nu) \sup_{t \in [0, T]} \int_t^{t+1} \|u_\varepsilon(s)\| ds + c_0 \sup_{t \in [0, T]} \int_t^{t+1} \|u_\varepsilon(s)\|^2 ds.$$

Applying inequality (4.24), we have that

$$\begin{aligned} \int_t^{t+1} \|\partial_t u_\varepsilon(s)\|_{D(A^2)}^2 ds &\leq \frac{c_7 e^{-\nu\lambda_1 t}}{\nu\lambda_1} \|u_\varepsilon(0)\|^2 + \frac{c_7 \|f\|^2}{\nu^2\lambda_1^2} + \|f(t)\|_{D(A^2)'} \\ &\quad + \frac{c_0 \|f\|^4}{\nu^4\lambda_1^4} + c_0 e^{-\nu 2\lambda_1 t} \|u_\varepsilon(0)\|^4 + 2c_0 \frac{\|f\|^2}{\nu^2\lambda_1^2} e^{-\nu\lambda_1 t} \|u_\varepsilon(0)\|^2 \\ &\leq \frac{c_7 e^{-\nu\lambda_1 t}}{\nu\lambda_1} \|u_\varepsilon(0)\|^2 + \frac{c_7 \|f\|^2}{\nu^2\lambda_1^2} + C_3, \end{aligned}$$

where  $c_7 = (c_5 + c_6 \nu)$  and

$$C_3 = \|f(t)\|_{D(A^2)'} + \frac{c_0 \|f\|^4}{\nu^4\lambda_1^4} + c_0 e^{-\nu 2\lambda_1 t} \|u_\varepsilon(0)\|^4 + 2c_0 \frac{\|f\|^2}{\nu^2\lambda_1^2} e^{-\nu\lambda_1 t} \|u_\varepsilon(0)\|^2. \quad \square$$

Moreover, due to estimates (4.23) and (4.28), we also have the uniform estimate.

**Proposition 4.12.** *If  $f \in V_0$ , then any solution  $u_\varepsilon(t)$  of problem (4.1) satisfies the inequality*

$$\|\pi^\varepsilon(u_\varepsilon)\|_{\mathcal{S}_b}^2 \leq \frac{c_7 e^{-\nu\lambda_1 t}}{\nu\lambda_1} \|u_\varepsilon(0)\|^2 + \frac{c_7 \|f\|^2}{\nu^2\lambda_1^2} + C_4 \quad (4.30)$$

where the positive constant  $C_4$  is independent of  $\varepsilon$ .

From Proposition 4.10 it follows that  $\mathcal{K}_\varepsilon \subset \mathcal{S}_b$  for all  $\varepsilon > 0$  and for all  $\tau > 0$ . Also Proposition 4.10 implies that the semigroup  $\pi^\varepsilon$  has absorbing set in  $\mathcal{K}_\varepsilon$  for all  $\varepsilon > 0$  and for all  $\tau > 0$  (We note, that this absorbing set does not depend on  $\varepsilon$ , since the constant  $C_4$  in (4.30) is independent of  $\varepsilon$ ), bounded in  $\mathcal{S}_b$  and inequality (4.30) implies that absorbing set is compact in  $\Gamma$ . The continuity of  $\pi^\varepsilon$  is proved. These facts are sufficient to state that  $\pi^\varepsilon$  has a global attractor  $\mathfrak{A}_\varepsilon$ . Such that  $\mathfrak{A}_\varepsilon \subset \mathfrak{F}^0 \times X^0$ , bounded in  $\mathcal{S}_b$  and compact in  $\Gamma$ . For a more detailed, see [14].

## 5. UPPER SEMICONTINUITY OF ATTRACTORS

We now prove the robustness property for the global attractor  $\mathfrak{A}_\varepsilon$ . We have shown in Theorem 4.9 the continuity of the family of semiflows  $\pi^\varepsilon$  on  $X^0$ . Having done this, We can simply invoke Theorem 23.14 in [18] to complete the proof of the robustness for the family of attractors  $\mathfrak{A}_\varepsilon$  at  $\varepsilon = 0$ . Clearly, it is sufficient to show that the small  $\varepsilon_0$ -neighbourhood of attractor  $\mathfrak{A}_0$  is an absorbing set and that  $\pi^\varepsilon$  approximates  $\pi^0$  on  $B_{R_1} = \{u^\varepsilon(0) \in V_1; \|u_0^\varepsilon\|_1 \leq R_1\}$  uniformly on compact sets of  $[0, \infty)$ .

**Theorem 5.1.** *The family of attractors  $\{\mathfrak{A}_\varepsilon, 0 < \varepsilon \leq 1\}$  of the continuous family of semiflows  $\pi^\varepsilon$  generated by the weak solutions of regularized 3D Navier-Stokes equations (1.1) is upper semicontinuous with respect to  $\varepsilon$  at  $\varepsilon = 0$ .*

*Proof.* Let  $N_{\varepsilon_0}(\mathfrak{A}_0)$  be the  $\varepsilon_0$ -neighborhood of  $\mathfrak{A}_0$ . Since  $\mathfrak{A}_0$  is a global attractor, for any bounded set  $B_{R_0} = \{u(0) \in V_0; \|u(0)\| \leq R_0\} \subset V_0$ , we have

$$d_{X^0}(\pi^0 B_{R_0}, \mathfrak{A}_0) \rightarrow 0, \text{ as } t \rightarrow \infty. \quad (5.1)$$

Thus, there exists  $\varepsilon_0 > 0$  and  $t > t_{\varepsilon_0}$  such that

$$d_{X^0}(\pi^0 B_{R_0}, \mathfrak{A}_0) \leq \frac{\varepsilon_0}{2}, \text{ for } t \geq t_{\varepsilon_0}. \quad (5.2)$$

Consequently

$$\pi^0(t) B_{R_0} \subset N_{\varepsilon_0}(\mathfrak{A}_0), \text{ for } t \geq t_{\varepsilon_0}. \quad (5.3)$$

This shows that  $N_{\varepsilon_0}(\mathfrak{A}_0)$  is an absorbing set. To establish the second step, we use Lemma 4.7 which states that any ball  $B_{R_1} = \{u_0^\varepsilon \in V_1; \|A^{\frac{1}{2}} u_0^\varepsilon(0)\| \leq R_1\}$  in  $V_1$ , with radius  $R_1 > \frac{1}{\nu^2 \lambda_1} \|f\|_\infty^2$  will satisfy

$$\pi^\varepsilon(t) B_{R_1} \subset B_{R_1}, \text{ for } t \geq 0. \quad (5.4)$$

This means if  $u_{\varepsilon_0} \in B_{R_1}$ , then  $\pi^\varepsilon(t) u_0^\varepsilon$  is defined and belongs to  $B_{R_1}$  for  $t \geq 0$ . The ball  $B_{R_1}$  is therefore invariant under the map  $\pi^\varepsilon$ . Since  $\pi^\varepsilon$  approximates  $\pi^0$  on  $B_{R_1}$  uniformly on  $[0, T]$ , we have for any  $\varepsilon_0 > 0$ , there are  $\varepsilon_1 > 0$  and  $\tau_0 > 0$  such that

$$\pi^\varepsilon(B_{R_0} \cap B_{R_1}) \subset N_{\varepsilon_0}(\mathfrak{A}_0), \text{ for } 0 < \varepsilon < \varepsilon_1, t \geq \tau_0. \quad (5.5)$$

Since the attractor  $\mathfrak{A}_\varepsilon$  is contained in  $B_{R_0} \cap B_{R_1}$ , an open neighborhood in the  $X^0$  Fréchet space [18, Item (2) Theorem 23.14], we have

$$\pi^\varepsilon(\mathfrak{A}_\varepsilon) \subset N_{\varepsilon_0}(\mathfrak{A}_0), \text{ for } 0 < \varepsilon < \varepsilon_1, t \geq \tau_0. \quad (5.6)$$

Since  $\mathfrak{A}_\varepsilon$  is an invariant set, we deduce that

$$\mathfrak{A}_\varepsilon \subset N_{\varepsilon_0}(\mathfrak{A}_0), \text{ for } 0 < \varepsilon < \varepsilon_1, t \geq \tau_0. \quad (5.7)$$

Moreover, since  $\varepsilon_0$  is arbitrary, we obtain the upper semicontinuity of  $\mathfrak{A}_\varepsilon$ , at  $\varepsilon = 0$

$$d_{X^0}(\mathfrak{A}_\varepsilon, \mathfrak{A}_0) \rightarrow 0, \text{ as } \varepsilon \rightarrow 0. \quad (5.8)$$

□

One can modify the argument described above so that the final result will have broader applicability by allowing the family of forcing functions to vary with  $\varepsilon$ , for  $\varepsilon > 0$ . Thus, we consider the regularized Navier-Stokes system (1.1) with a perturbed external force  $f^\varepsilon$  in place of  $f$ , for  $\varepsilon > 0$ . Then (4.1) becomes

$$\begin{aligned} \partial_t u^\varepsilon + \varepsilon A^2 u^\varepsilon + \nu A u^\varepsilon + B(u^\varepsilon, u^\varepsilon) &= f^\varepsilon(x), & \text{in } \Omega \times (0, \infty) \\ u^\varepsilon(x) &= u_0^\varepsilon, & \text{in } \Omega. \end{aligned} \quad (5.9)$$

We show that the trajectory attractor of the perturbed system (5.9) coincides with the trajectory attractor  $\mathfrak{A}_\varepsilon$  of the unperturbed system (1.1). Our results rely on the work of Hale ([14]) who show that the limit behaviour is valid even through  $\mathfrak{F}^\varepsilon$ , where  $\mathfrak{F}^\varepsilon$  denote a compact, translation invariant set of perturbed forcing functions to vary with  $\varepsilon$ , for  $\varepsilon > 0$  and satisfy the condition

$$\omega(\mathcal{H}^+(f^\varepsilon)) = \omega(\mathcal{H}^+(f)). \quad (5.10)$$

Thus we would use  $\mathfrak{F}^\varepsilon$  in place of  $\mathfrak{F}^0$ , for  $\varepsilon > 0$ . Moreover, by using a metric  $d$  on the  $L^\infty C$ -toplogy, see [18] for some samples, we can note that (5.10) is equivalent to saying that for every  $\delta > 0$  there is an  $\varepsilon_1 > 0$  and  $T_\delta = T(\delta) \geq 0$  such that

$$d_{X^0}(f^\varepsilon, \mathfrak{F}^0) \leq \delta, \text{ for } 0 < \varepsilon \leq \varepsilon_1 \text{ and } f^\varepsilon \in \mathfrak{F}^\varepsilon$$

for any  $t \geq T_\delta$ , that is

$$\mathfrak{F}^\varepsilon \subset N_\delta(\mathfrak{F}^0), \text{ for } 0 < \varepsilon \leq \varepsilon_1, \quad (5.11)$$

where  $N_\delta$  denotes the  $\delta$ -neighborhood of  $\mathfrak{F}^0$  in  $L^\infty C$ . The resulting argument for robustness will then depend on two parameters  $\lambda = (\varepsilon, \delta)$ , where  $\lambda \rightarrow (0, 0)$ .

The following statement generalizes Theorem 5.1

**Theorem 5.2.** *Under the above conditions, the trajectory attractor of the perturbed 3D Navier-Stokes system (5.9) coincides with the trajectory attractor  $\mathfrak{A}_\varepsilon$  of the non-perturbed system (4.1). Moreover, the perturbed attractor of (5.9) is upper semicontinuous with respect to  $\varepsilon$  at  $\varepsilon = 0$ .*

*Proof.* The existence of trajectory attractor  $\mathfrak{A}_\varepsilon$  is treated above. The proof follows from formulas (5.10), (5.11) and Theorem 5.1. □

Our future investigations will be concerned with the corresponding problem but with  $p$ -Laplacian and perturbed external force.

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