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Analysis of a new class of Forward Semi-Lagrangian schemes for the 1D Vlasov-Poisson Equations

Thomas Respaud* Eric Sonnendrücker †

December 25, 2009

Abstract

The Vlasov equation is a kinetic model describing the evolution of charged particles, and is coupled with Poisson's equation, which rules the evolution of the self-consistent electric field. In this paper, we introduce a new class of forward Semi-Lagrangian schemes for the Vlasov-Poisson system based on a Cauchy Kovalevsky (CK) procedure for the numerical solution of the characteristic curves. Exact conservation properties of the first moments of the distribution function for the schemes are derived and a convergence study is performed that applies as well for the CK scheme as for a more classical Verlet scheme. A L^1 convergence of the schemes will be proved. Error estimates (in $O(\Delta t^2 + \Delta x^2 + \frac{\Delta x^2}{\Delta t})$ for Verlet) are obtained, where Δt and Δx are the discretisation parameters.

Keywords: Semi-Lagrangian method, Convergence, L^1 Stability, Conservation of moments

Contents

1 Introduction **2**

2 The continuous problem **3**
2.1 The Vlasov-Poisson model . . . 3
2.2 Existence, uniqueness and regularity of the solution of the continuous problem . . . 4

3 The discrete problem **5**
3.1 Definitions and notations . . . 5
3.2 The numerical scheme . . . 6
3.2.1 Verlet algorithm . . . 7
3.2.2 Cauchy Kovalevsky procedure . . . 7
3.3 Exact conservation number of particles and momentum . . . 9
3.3.1 B-spline interpolation . . . 9
3.3.2 Particle conservation . . . 10
3.3.3 Momentum conservation . . . 10

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4 Convergence analysis **14**
4.1 Decomposition of the error 14
4.2 A priori estimates 15
4.2.1 Stability for linear splines 15
4.2.2 Towards Gronwall 15
4.3 End of the proof 23

5 Numerical results **24**

6 Conclusion **25**

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1 Introduction

To describe the dynamics of charged particles in a plasma or in a propagating beam, the Vlasov equation can be used to calculate the response of the plasma to the electromagnetic fields. The unknown $f(t, x, v)$ which depends on the time t, the space x and the velocity v represents the distribution function of the studied particles. The coupling with the self-consistent electric fields is taken into account through the Poisson equation.

The numerical solution of such systems is most of the time performed using Particle In Cell (PIC) methods, in which the plasma is approximated by macro-particles (see [4]). They are advanced in time with the electromagnetic fields which are computed on a grid. However, despite their capability to treat complex problems, PIC methods are inherently noisy, which becomes problematic when low density or highly turbulent regions are studied. Hence, numerical methods which discretize the Vlasov equation on a grid of phase space can offer a good alternative to PIC methods (see [6, 11, 12, 17, 5]). The so-called Eulerian methods can deal with strongly nonlinear processes without additional complexity, and are well suited for parallel computation (see [14]). Moreover, semi-Lagrangian methods which have first been introduced in meteorology (see [16, 18, 19]), try to take advantage of both Lagrangian and Eulerian approaches. Indeed, they allow a relatively accurate description of phase space using a fixed mesh and avoid traditional step size restriction using the invariance of the distribution function along the trajectories.

Traditional semi-Lagrangian schemes follow the characteristics backward in time. In [7], following the idea of Reich [15], we introduced a forward Semi-Lagrangian scheme for the Vlasov-Poisson system based on a forward numerical solution of the characteristics using a classical Verlet or Runge-Kutta (order 2 and 4) scheme. The Verlet scheme can only be applied for specific differential equations, as for example the characteristics of the Vlasov-Poisson system, but not for more general cases, as the characteristics of the guiding centre or the gyrokinetic approximation of the Vlasov equation. Therefore an alternative to Verlet is necessary. On the other hand, Runge-Kutta schemes, which can be used in the general case, are very costly in our context, especially when going to higher order, as they require a deposition of the charge and the solution of the Poisson equation at intermediate time steps. We propose here, a new scheme for the characteristics based on a Cauchy-Kowalevsky (CK) procedure, that can be written up to an arbitrary order. Second and third order are developed in the present paper. We shall also dicuss the conservation of the first moment in for both Verlet and CK algorithms.

A proof of the convergence of PIC method for the Vlasov-Poisson system was performed by Cottet and Raviart [9]. Proofs of convergence and stability of the classical Semi-Lagrangian method applied to the same model were obtained by Besse and Mehrenberger [1]. These estimates are made in L^2 norm, since L^∞ seems out of reach as they explain. They manage to do it because

they deal with split methods, and thus only consider constant coefficient transport. In order to prove convergence in more general cases, the L^1 norm seems appropriate, as it enables to use the partition of unity property of the splines. Moreover, Després [10] explains possible advantages of studying L^1 convergence instead of more common L^2.

We propose here a proof of L^1 convergence of the forward semi-Lagrangian scheme with both Verlet and CK algorithm for the characteristics in the particular case of linear spline interpolation. We also obtain error estimates in time and space.

This paper is organized as follows. In the first part, the continuous problem is presented. In the second part, the discrete problem and the numerical schemes to solve it are introduced. We also prove the exact conservation of the first moment with respect to v at the discrete level for both CK and Verlet schemes. Then, in the main section of the paper, the convergence of our numerical schemes are proved. Finally some numerical results are presented.

2 The continuous problem

2.1 The Vlasov-Poisson model

Let us consider $f(t,x,v) \geq 0$ being the distribution function of electrons in phase-space, and $E(t,x)$ the self consistent electric field. The dimensionless Vlasov Poisson system reads

$$\frac{\partial f}{\partial t} + v\partial_x f + E(t,x)\partial_v f = 0, \tag{2.1}$$

$$\partial_x E(t,x) = \rho(t,x) = \int_{\mathbb{R}} f(t,x,v)dv - 1, \tag{2.2}$$

where x and v are the phase space independent variables. A periodic plasma is considered, of period L. So $x \in [0,L]$, $v \in \mathbb{R}$, $t \geq 0$. The functions f and E are submitted to the following conditions

$$f(t,0,v) = f(t,L,v), \forall v \in \mathbb{R}, t \geq 0, \tag{2.3}$$

$$E(t,0) = E(t,L) \Leftrightarrow \frac{1}{L}\int_0^L \int_{\mathbb{R}} f(t,x,v)\,dx\,dv = 1, \forall t \geq 0, \tag{2.4}$$

which translates the global neutrality of the plasma. In order to get a well-posed problem, a zero-mean electrostatic condition has to be added,

$$\int_0^L E(t,x)\,dx = 0, \quad \forall t \geq 0, \tag{2.5}$$

and an initial condition

$$f(0,x,v) = f_0(x,v), \quad \forall x \in [0,L], v \in \mathbb{R}. \tag{2.6}$$

Assuming that the electric field is smooth enough, equations (2.1), (2.3) and (2.6) can be solved in the classical sense as follows.

The first order differential system

$$\begin{aligned} \frac{dX}{dt}(t;(x,v),s) &= V(t;(x,v),s), \\ \frac{dV}{dt}(t;(x,v),s) &= E(t,X(t;(x,v),s)), \end{aligned} \tag{2.7}$$

where (X(t;(x,v),s),V(t;(x,v),s)) are the characteristic curves, solutions of (2.7) at time t with the initial condition

$$X(s;(x,v),s) = x, V(t;(x,v),s) = v. \tag{2.8}$$

For the existence, the uniqueness and the regularity of the solutions of this differential system, the reader is referred to [3].

The solution of problem (2.1), (2.6) is then given by

$$f(t,x,v) = f_0(X(0;(x,v),t), V(0;(x,v),t)), \quad \forall x \in [0,L], v \in \mathbb{R}, t \geq 0. \tag{2.9}$$

Since

$$\frac{\partial(X,V)}{\partial(x,v)} = 1,$$

it follows that

$$\frac{1}{L}\int_0^L \int_{\mathbb{R}} f(t,x,v)\,dv\,dx = \frac{1}{L}\int_0^L \int_{\mathbb{R}} f_0(x,v)\,dv\,dx = 1.$$

According to previous considerations, an equivalent form of the periodic Vlasov-Poisson problem is to find (f,E), smooth enough, periodic with respect to x, with period L, and solving the equations (2.2), (2.7), (2.8) and (2.9). Introducing the electrostatic potential $\varphi \equiv \varphi(t,x)$ such that $E(t,x) = -\partial_x\varphi(t,x)$, and setting $G = G(x,y)$ the fundamental solution of the Laplacian operator in one dimension. That is $-\partial_x^2 G(x,y) = \delta_0(x-y)$ with periodic boundary conditions. It comes

$$E(t,x) = \int_0^L K(x,y)(\int_{\mathbb{R}} f(t,y,v)dv - 1)\,dy,$$

where

$$\begin{aligned} K(x,y) = -\partial_x G(x,y) &= (\frac{y}{L} - 1), \quad 0 \leq x < y, \\ &= \frac{y}{L}, \quad y < x \leq L. \end{aligned}$$

2.2 Existence, uniqueness and regularity of the solution of the continuous problem

Theorem 1 *Assuming that $f_0 \in W^{1,\infty}_{c,per_x}(\mathbb{R}_x \times \mathbb{R}_v)$ ($W^{1,\infty}_{c,per_x}(\mathbb{R}_x \times \mathbb{R}_v$ being the Sobolev space of functions with first derivatives in L^∞, compactly supported in v and periodical in x), positive, periodic with respect to the variable x with period L, and $Q(0) \leq R$, with $R > 0$ defined as follows*

$$Q(t) := 1 + sup|v| : x \in [0,L], \tau \in [0,t] | f(\tau,x,v) \neq 0,$$

and

$$\frac{1}{L}\int_0^L \int_{\mathbb{R}} f_0(x,v)dvdx = 1,$$

then the periodic Vlasov-Poisson system has a unique classical solution (f,E), periodic in x, with period L, for all t in $[0,T]$, such that

$$f \in W^{1,\infty}(0,T; W^{1,\infty}_{c,per_x}(\mathbb{R}_x \times \mathbb{R}_v)),$$

$$E \in W^{1,\infty}(0,T; W^{1,\infty}_{per_x}(\mathbb{R})),$$

and there exists a constant $C = C(R, f_0)$ dependent of R and f_0 such that

$$Q(T) \leq CT.$$

Moreover if we assume that $f_0 \in W^{m,\infty}_{c,per_x}(\mathbb{R}_x \times \mathbb{R}_v)$, then $(f,E) \in W^{m,\infty}(0,T; W^{m,\infty}_{c,per_x}(\mathbb{R}_x \times \mathbb{R}_v)) \times W^{1,\infty}(0,T; W^{1,\infty}_{per_x}(\mathbb{R}))$, for all finite time T.

For the proof, the reader is referred to the references [3, 13]

3 The discrete problem

3.1 Definitions and notations

Let $\Omega = [0, L] \times [-R, R]$, with $R > Q(T)$, and M_h a cartesian mesh of the phase-space Ω. M_h is given by a first increasing sequence $(x_i)_{i\in[0..N_x]}$ of the interval $[0, L]$ and a second one $(v_j)_{j\in[0..N_v]}$ of the interval [-R,R]. Let $\Delta x_i = x_{i+1} - x_i$ the physical space step and $\Delta v_j = v_{j+1} - v_j$ the velocity space step. In order to simplify the study, a regular mesh will be used, i.e $\Delta x_i = \Delta x = \frac{L}{N_x+1}$, and $\Delta v_j = \Delta v = \frac{2R}{N_v+1}$, where N_x, N_v belong to $\mathbb{N}$. Then h is defined being $\max(\Delta x, \Delta v)$.

For each function g defined on all the points $(x_i, v_j) \in M_h$ we will set $g_{i,j} := g(x_i, v_j)$, and the sequence is completed on $\mathbb{Z} \times \mathbb{Z}$ by periodicity, having $g_{i,j} := g_{i[N_x+1],j[N_v+1]}$. The sequence (x_i, v_j) will also be defined on the whole set $\mathbb{Z} \times \mathbb{Z}$, by $x_i := i\Delta x$, and $v_j := -R + j\Delta v$. The set of all Ω-periodic functions (L-periodic in x and $2R$-periodic in v) will be denoted $P(\Omega)$.

Now let $f = (f_{i,j})_{(i,j)\in\mathbb{Z}\times\mathbb{Z}}$ be a periodic continuous grid-function with respectively the period N_x+1 and N_v+1. If f is a function defined on the points M_h, a discrete grid-function $\tilde{f}$ defined by $\tilde{f}_{i,j} := f(x_i, v_j)$ for all $(i, j) \in [0, N_x] \times [0, N_v]$. In order to lighten notations, f will be kept instead of $\tilde{f}$. Let $L^2_h(\Omega)$ (resp $L^1_h(\Omega)$), the set of grid-functions whose $||.||_{L^2_h(\Omega)}$ (resp $||.||_{L^1_h(\Omega)}$) is bounded

$$||f||_{L^2_h(\Omega)} = (\Delta x \Delta v \sum_{i=0}^{N_x} \sum_{j=0}^{N_v} |f_{i,j}|^2)^{\frac{1}{2}},$$

$$||f||_{L^1_h(\Omega)} = \Delta x \Delta v \sum_{i=0}^{N_x} \sum_{j=0}^{N_v} |f_{i,j}|.$$

As it was precised in the introduction, a L^2 convergence analysis for backward Semi-Lagrangian schemes was performed in [1], helped with the fact that a Strang splitting is used, so that the transport velocity in each direction becomes constant for each split step. In paper [7], it is explained that for split methods, which treat only constant transport, forward and backward methods are in fact exactly the same. So, all the L^2 results exposed in [1] are also valid for our method when a Strang splitting is used. In this paper, we are interested in the convergence of the non split method. Due to the partition of unity property of B-splines, the L^1 norm seems the most appropriate for this kind of study. For this reason we shall study the convergence of our sheme in the L^1 norm.

Remark 3.1 *If $f \in L^2_h(\Omega)$, then $f \in L^1_h(\Omega)$ thanks to Cauchy Schwarz inequality, as Ω is bounded.*

In the sequel, an ending time T is fixed, and a uniform time discretization $(t^n)_{n\leq N_T}$ of the interval $[0, T]$, with time step $\Delta t = t^{n+1} - t^n$. At each point $(x_i, v_j) \in M_h$, an approximation $f_h(t^n, x_i, v_j)$ of the exact distribution function $f(t^n, x_i, v_j)$ at time $t^n = n\Delta t$ is defined. The approximation function $f_h(t^n)$ is then given at each point of $\mathbb{R}_x \times \mathbb{R}_v$ thanks to an interpolation operator R_h defined on a uniform grid:

$$R_h : \quad L^1(\Omega) \cap P(\Omega) \quad \longrightarrow \quad L^1(\Omega) \cap P(\Omega), \qquad f \qquad \mapsto \quad R_h f = \sum_{(i,j)\in\mathbb{Z}\times\mathbb{Z}} f_{i,j}\Psi_{i,j},$$

where $\Psi_{i,j}$ will be linear spline functions for our convergence study. In numerical results, since linear interpolation is quite diffusive, cubic splines will be used. In order to get a convergent scheme, the operator R_h must satisfy some approximation properties which will be detailed later.

3.2 The numerical scheme

The electric field operator for the real-valued function $g \in L^1([0,L] \times \mathbb{R})$ is defined this way:

$$E[g](x) = \int_0^L K(x,y)(\int_{\mathbb{R}} g(y,v)dv - 1). \tag{3.10}$$

The approximate function f_h is solution on the grid of the following Vlasov equation:

$$\frac{\partial f_h}{\partial t}(t,x,v) + v\frac{\partial f_h}{\partial x}(t,x,v) + E_h(t,x)\frac{\partial f_h}{\partial v}(t,x,v) = 0.$$

This function follows approximate characteristics, solutions of

$$\begin{aligned} \frac{dX_h}{dt}(t;(x,v),s) &= V_h(t;(x,v),s), \\ \frac{dV_h}{dt}(t;(x,v),s) &= E_h(t,X(t;(x,v),s)), \end{aligned} \tag{3.11}$$

where E_h is defined exactly through f_h with (3.10). So we get:

$$\forall t \in [t^n, t^{n+1}[: f_h(t,x,v) = \sum_{k,l} \omega_{k,l}^n S_h(x - X_h(t;(x_k,v_l),t^n))S_h(v - V_h(t,(x_k,v_l),t^n)),$$

where f_h is given on the mesh at time t^n by:

$$f_h(t^n, x_i, v_j) = \sum_{k,l} \omega_{k,l}^n S_h(x_i - x_k)S_h(v_j - v_l) \quad \forall n.$$

These are the interpolation conditions enabling to define the spline coefficients $\omega_{k,l}^n$ and thus f_h everywhere.

The distribution function is updated this way. The ending point of the characteristic starting from (x_i, v_j) is computed: $(X_h(t^{n+1};(x_i,v_j),t^n), V_h(t^{n+1};(x_i,v_j),t^n)), \forall (i,j)$. Then, since f_h is constant along the approximate characteristics, the value is deposited on the nearest grid points, the number of which depending on the degree of the splines used for the interpolation. The linear B-spline S is defined as follows

$$S(x) = \begin{cases} (1-|x|) & \text{if } 0 \le |x| \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

The spline S_h actually used, which shall be defined with the size of the mesh will be:

$$S_h(x) = S(\frac{x}{h}).$$

From now on, S_h will be denoted by S for the sake of simplicity. This step, which is in fact the resolution of a linear system based on the construction of the approximate solution, but trivial in the linear case. It is imposed that this function is continuous at the grid points, so we have to solve:

$$f_h(t^{n+1}, x_i, v_j) = \sum_{k,l} \omega_{k,l}^n S(x_i - X_h(t^{n+1};(x_k,v_l),t^n))S(v_j - V_h(t^{n+1};(x_k,v_l),t^n)) \quad \forall (i,j),$$

with $\omega_{k,l}^n = f_h(t^n, x_k, v_l)$. In order to do that we have to compute

$$(X_h(t^{n+1}; (x_i, v_j), t^n), V_h(t^{n+1}; (x_i, v_j), t^n)), \forall (i, j),$$

solution of the differential system (3.11). Since this requires an explicit solution of that system, any standard ODE solver such as Verlet, Runge-Kutta or others can be used. Our analysis will be based on the Verlet algorithm, which is of second order, and on a Cauchy Kovalevsky procedure, which can be of any order, as an alternative to more costly Runge-Kutta solvers. But we will consider only the second of third order, because higher ones would be useless here, as we shall be explained.

3.2.1 Verlet algorithm

Starting at time t^n from the gid point (x_k, v_l)

- Step 1: $\forall k, l,\ x_{k,l}^{n+\frac{1}{2}} - x_k \ = \ \frac{\Delta t}{2}\, v_l$,
- Step 2: compute the electric field at time $t^{n+\frac{1}{2}}$
  - deposition of the particles $x_{k,l}^{n+\frac{1}{2}}$ on the spatial grid x_i for the density ρ_h: $\rho_h(x_i, t^{n+\frac{1}{2}}) = \sum_{k,l} \omega_{k,l}^n S(x_i - x_{k,l}^{n+\frac{1}{2}})$, like in a PIC method.
  - solve the Poisson equation on the grid x_i: $E(x_i, t^{n+\frac{1}{2}})$.
- Step 3: $\forall k, l,\ v_{k,l}^{n+1} - v_l \ = \ \Delta t\, E(x_{k,l}^{n+\frac{1}{2}}, t^{n+\frac{1}{2}})$,
- Step 4: $\forall k, l,\ x_{k,l}^{n+1} - x_{k,l}^{n+\frac{1}{2}} \ = \ \frac{\Delta t}{2}\, v_{k,l}^{n+1}$.

This is the way the algorithm is implemented. In our convergence study, the slight difference is that an exact solution of Poisson's equation, based on the Green formula (3.10) is used.

3.2.2 Cauchy Kovalevsky procedure

The idea is to get high order approximations of the characteristics using Taylor expansions in time. And then, using the charge conservation equation, and higher velocity moments of the Vlasov equation, to replace time derivatives with terms containing only spatial derivatives and moments at time t_n which can be easily computed. Up to third order these Taylor expansions in time lead to

$$\begin{aligned} X^{n+1} &= X^n + \Delta t V^n + \frac{\Delta t^2}{2} E^n(X^n) + \frac{\Delta t^3}{6} \frac{d}{dt} E(X(t), t)_{|t=t^n}. \\ V^{n+1} &= V^n + \Delta t E^n(X^n) + \frac{\Delta t^2}{2} \frac{d}{dt} E(X(t), t)_{|t=t^n} + \frac{\Delta t^3}{6} \frac{d^2}{dt^2} E(X(t), t)_{|t=t^n}. \end{aligned}$$

In order to be able to compute all terms of these expansion we need the three first total time derivatives of $E(X(t), t)$.

$$\begin{aligned} \frac{d}{dt} E(X(t), t) &= \frac{\partial E}{\partial t}(X(t), t) + \frac{dX}{dt}(t) \frac{\partial E}{\partial x}(X(t), t) \\ &= -J(X(t), t) + \bar{J}(t) + V(t)\rho(X(t), t), \end{aligned}$$

where $\rho(x,t) = \int f(x,v,t)\,dv - 1$, $J(x,t) = \int f(x,v,t)v\,dv$ and $\bar{J}(t) = \frac{1}{L}\int_0^L J(x,t)\,dx$. Indeed, the Poisson equation yields $\frac{\partial E}{\partial x} = \rho$ and integrating the Vlasov equation with respect to velocity, yields the charge conservation equation $\frac{\partial \rho}{\partial t} + \frac{\partial J}{\partial x} = 0$. Hence taking the derivative of the Poisson equation with respect to time and using this equation we get

$$\frac{\partial}{\partial x}(\frac{\partial E}{\partial t} + J) = 0.$$

From which we obtain, as $\int_0^L E(x,t)\,dx = 0$ that

$$\frac{\partial E}{\partial t} = -J + \bar{J}.$$

The second order total derivative in time of E reads

$$\begin{aligned}\frac{d^2}{dt^2}E(X(t),t) &= -\frac{\partial J}{\partial t}(X(t),t) - V(t)\frac{\partial J}{\partial x}(X(t),t) + \frac{d\bar{J}}{dt}(t)\\ &+ E(X(t),t)\rho(X(t),t) + V(t)(\frac{\partial \rho}{\partial t}(X(t),t) + V(t)\frac{\partial \rho}{\partial x}(X(t),t)).\end{aligned}$$

In order to use this expression, we need $\frac{\partial J}{\partial t}, \frac{\partial J}{\partial x}, \frac{\partial \rho}{\partial t}, \frac{\partial \rho}{\partial x}, \frac{d\bar{J}}{dt}(t)$. The Cauchy-Kovalevsky procedure consists in getting rid of time derivatives, replacing them with space derivatives obtained from the equation, in our case, we use the velocity moments of the Vlasov equation. First for ρ, we use the charge conservation equation:

$$\frac{\partial \rho}{\partial t}(X(t),t) = -\frac{\partial J}{\partial x}(X(t),t). \tag{3.12}$$

In order to get the time derivative of the current J, we need to use the Vlasov equation, multiply it with v, and integrate it with respect to v, so that we get:

$$\frac{\partial J}{\partial t} = \frac{\partial}{\partial x}I_2 + E\int_{\mathbb{R}} \frac{\partial f}{\partial v} v\,dv = 0,$$

where $I_n(x,t) = \int_{\mathbb{R}} f(x,v,t)v^n dv$ so that, using that f is compactly supported and integrating by parts:

$$\frac{\partial J}{\partial t}(X(t),t) = -\frac{\partial I_2}{\partial x}(X(t),t) + E(X(t),t)(1+\rho(X(t),t)). \tag{3.13}$$

Let us prove now that $\frac{d\bar{J}}{dt}(t) = 0$:

$$\begin{aligned}\frac{d\bar{J}}{dt}(t) &= \frac{1}{L}\int_0^L \frac{\partial \bar{J}}{\partial t}(t,x)\,dx,\\ &= \frac{1}{L}\int_0^L (-\frac{\partial I_2}{\partial x}(t,x) + E(t,x)(1+\rho(t,x))\,dx,\\ &= \frac{1}{L}([I_2(t,0) - I_2(t,L)] + \int_0^L E(t,x)dx + \int_0^L E(t,x)\rho(t,x)\,dx\\ &= \frac{1}{L}[E^2(t,L) - E^2(t,0)],\\ &= 0.\end{aligned}$$

thanks to periodicity, in fact (2.3), (2.5).

We finally get the following third order Cauchy Kovalevsky (CK3) time algorithm, using (3.12), (3.13):

$$\begin{aligned} X^{n+1} &= X^n + \Delta t V^n + \frac{\Delta t^2}{2} E^n(X^n) + \frac{\Delta t^3}{6}(V^n \rho^n(X^n) - J^n(X^n) + \bar{J}), \\ V^{n+1} &= V^n + \Delta t E^n(X^n) + \frac{\Delta t^2}{2}(V^n \rho^n(X^n) - J^n(X^n) + \bar{J}) \\ &+ \frac{\Delta t^3}{6}(\frac{\partial I_2}{\partial x}(X^n, t^n) - E^n(X^n) - 2V^n \frac{\partial J}{\partial x}(X^n, t^n) + (V^n)^2 \frac{\partial \rho}{\partial x}(X^n, t^n)). \end{aligned}$$

Let us now introduce a notation, which will be useful later:

$$X^{n+1} = X^n + \Delta t V^n + \frac{\Delta t^2}{2} E^n(X^n) + \frac{\Delta t^3}{6} \phi^n(X^n, V^n),$$

and

$$V^{n+1} = V^n + \Delta t E^n(X^n) + \frac{\Delta t^2}{2} \phi^n(X^n, V^n) + \frac{\Delta t^3}{6} \varphi^n(X^n, V^n),$$

where ϕ, φ are naturally defined.

Remark 3.2 *Obviously, in order to get a second order algorithm (CK2), we just keep the terms until Δt^2 included.*

3.3 Exact conservation number of particles and momentum

3.3.1 B-spline interpolation

First, let us recall some useful properties of B-splines interpolation. The linear space of B-splines of order $m+1$ writes, denoting by $s^{(m)}$ the mth derivative of s,

$$S_{m+1,\Delta_x} = \{s(x) \in C^{m-1}(\mathbb{R}), \quad s^{(m+1)}(x) = 0, \forall x \in (x_i, x_{i+1}), \forall i \in \mathbb{R}\},$$

if $m+1$ is even, and

$$S_{m+1,\Delta_x} = \{s(x) \in C^{m-1}(\mathbb{R}), \quad s^{(m+1)}(x) = 0, \forall x \in (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}), \forall i \in \mathbb{R}\},$$

if $m+1$ is odd.

The space of B-spline functions in two dimensions is defined as the tensor product of $1D$ spaces. Let us precise the interpolation operator:

$$R_{h_{i,j}}(f)(x,v) = \omega_{i,j}(f) S(x - x_i) S(v - v_j),$$

$$R_h f(x,v) = \sum_{i,j} R_{h_{i,j}}(f)(x,v).$$

Now come the properties:

- $\mathbb{S}_{m+1,h} = Span(S_{m+1}(. - x_i) S_{m+1}(. - v_j); \forall (i,j) \in \mathbb{Z})$,
- $\mathbb{S}_{m+1,h} \subset W^{k,p} \quad 1 \leq p \leq \infty \quad 0 \leq k \leq m$,
- Stability: $||R_h f||_{L^p(\Omega)} \leq C ||f||_{L^p(\Omega)} \quad \forall f \in L^p(\Omega) \cap P(\Omega), \quad 1 \leq p \leq \infty \quad$ (i),

- Consistency and accuracy: $\exists C > 0 \quad | \quad ||f - R_h f||_{W^{k,p}(\Omega)} \leq C h^{m+1-k} |f|_{W^{m+1,p}(\Omega)} \quad \forall f \in W^{m+1,p}(\Omega) \cap P(\Omega) \quad 1 \leq p \leq \infty \quad 0 \leq k \leq m$ (ii),
- $\sum_i S_m(. - x_i) = 1$ (iii), $\int S_m(u) du = \Delta x$ (iv).

Let us also precise particle and momentum conservation. The proof for the mass is independent of the spline degree, and the one for the first moment will only be shown for linear splines, even though it has been checked for the first three splines.

3.3.2 Particle conservation

The discrete algorithm preserves the total number of particles, as the following computation shows:

$$\begin{aligned}
m^{n+1} &= \int f_h(t^{n+1}, x, v)\, dx\, dv, \\
&= \sum_{i,j} \omega_{i,j}^{n+1} \int S(x - x_i) S(v - v_j)\, dx\, dv, \\
&= \Delta x \Delta v \sum_{i,j} f_h^{n+1}(x_i, v_j), \\
&= \Delta x \Delta v \sum_{i,j} \sum_{k,l} \omega_{k,l}^n S(x_i - X(t^{n+1}; (x_k, v_l), t^n) S(v_j - V(t^{n+1}; (x_k, v_l), t^n), \\
&= \Delta x \Delta v \sum_{k,l} \omega_{k,l}^n = \Delta x \Delta v \sum_{i,j} f^n(x_i, v_j) = m^n,
\end{aligned}$$

thanks to partition of unity property (iii).

Let us precise the way the interpolation operator acts:

$$\begin{aligned}
f_h(t^{n+1}, x, v) &= R_h(\sum_{i,j} \omega_{i,j}^n S(x - X_h(t^{n+1}; (x_i, v_j), t^n) S(v - V_h(t^{n+1}; (x_i, v_j), t^n), \\
&= \sum_{i,j} \omega_{i,j}^{n+1} S(x - x_i) S(v - v_j), \qquad (3.14)
\end{aligned}$$

by definition of $\omega_{i,j}^n$. This implies a kind of continuity of f_h at time t^n on the grid points.

3.3.3 Momentum conservation

Let us precise that in this paragraph $(X_h(t; (x_i, v_j), t^n), V_h(t; (x_i, v_j), t^n))$ will be denoted $(X_{i,j}(t), V_{i,j}(t))$, and that Poisson will not be solved exactly. The aim here is to prove that

$$\sum_{i,j} v_j f_h(t^n, x_i, v_j) = \sum_{i,j} v_j f_h(t^{n+1}, x_i, v_j) \quad \forall n. \qquad (3.15)$$

Let us distinguish two phases: the transport one and the deposition one. Let us start with the deposition phase, where we have to get:

$$\sum_{i,j,k,l} \omega_{i,j}^{n+1} v_l S_1(x_k - x_i) S_1(v_l - v_j) = \sum_{i,j,k,l} \omega_{i,j}^n v_l S_1(x_k - X_{i,j}(t^{n+1})) S_1(v_l - V_{i,j}(t^{n+1})).$$

First, the left hand side, using the partition of unity property (iii),

$$\begin{aligned}\sum_{i,j,k,l} \omega_{i,j}^{n+1} v_l S_1(x_k - x_i) S_1(v_l - v_j) &= \sum_{i,j} \omega_{i,j}^{n+1} \sum_l v_l S(v_l - v_j), \\ &= \sum_{i,j} \omega_{i,j}^{n+1} v_j.\end{aligned}$$

For the right hand side we need to evaluate $\sum_l v_l S_1(v_l - V_{i,j}(t^{n+1}))$. So we need to know where $V_{i,j}(t^{n+1})$ is. Let us suppose $V_{i,j}(t^{n+1}) = v_p + \alpha\Delta v, \quad \alpha \in [0, 1[$

$$\begin{aligned}\sum_l v_l S_1(v_l - V_{i,j}(t^{n+1})) &= v_p S_1(\alpha\Delta v) + v_{p+1} S_1(\Delta v - \alpha\Delta v), \\ &= (p + \alpha\Delta v), \\ &= V_{i,j}(t^{n+1}).\end{aligned}$$

So we finally get for the deposition phase:

$$\sum_{i,j} \omega_{i,j}^{n+1} v_j = \sum_{i,j} \omega_{i,j}^n V(t^{n+1}; (x_i, v_j), t^n). \tag{3.16}$$

Remark 3.3 *This proof is given for linear splines, but was also checked for quadratic and cubic ones. The transport phase is independent from the spline degree.*

There remains to prove that

$$\sum_{i,j} \omega_{i,j}^n v_j = \sum_{i,j} \omega_{i,j}^n V(t^{n+1}; (x_i, v_j), t^n), \tag{3.17}$$

which corresponds to the transport phase. Note that this phase exists also in PIC methods, and the following proof of conservation of moments is adapted from [4].

Verlet We have with our Verlet algorithm:

$$V(t^{n+1}; (x_i, v_j), t^n) = v_j + \Delta t E^{n+\frac{1}{2}}(X(t^{n+\frac{1}{2}}, (x_i, v_j), t^n)). \tag{3.18}$$

Using a regularisation of the electric field based on the same splines used for the distribution function:

$$E^{n+\frac{1}{2}}(X(t^{n+\frac{1}{2}}, (x_i, v_j), t^n)) = \sum_k E^{n+\frac{1}{2}}(x_k) S(x_k - X(t^{n+\frac{1}{2}}, (x_i, v_j), t^n)).$$

To get (3.17) using (3.18) we just have to prove that

$$\sum_{i,j,k} \omega_{i,j}^n E^{n+\frac{1}{2}}(x_k) S(x_k - X(t^{n+\frac{1}{2}}, (x_i, v_j), t^n)) = 0,$$

and

$$\begin{aligned}\sum_{i,j,k} \omega_{i,j}^n E^{n+\frac{1}{2}}(x_k) S(x_k - X(t^{n+\frac{1}{2}}, (x_i, v_j), t^n)) &= \sum_k E^{n+\frac{1}{2}}(x_k) \sum_{i,j} \omega_{i,j}^n S(x_k - X(t^{n+\frac{1}{2}}, (x_i, v_j), t^n)), \\ &= \sum_k E^{n+\frac{1}{2}}(x_k) \rho^{n+\frac{1}{2}}(x_k), \\ &= 0,\end{aligned}$$

for most of the centered algorithms used to solve Poisson numerically, like the following centered finite difference one on staggered mesh, with linear regularisation:

$$E^{n+\frac{1}{2}}(x_{k+\frac{1}{2}}) - E^{n+\frac{1}{2}}(x_{k-\frac{1}{2}}) = \Delta x \rho^{n+\frac{1}{2}}(x_k) \quad \forall k,$$

and

$$E^{n+\frac{1}{2}}(x) = \sum_i E^{n+\frac{1}{2}}(x_i) S_1(x - x_i),$$

we get:

$$\begin{aligned}
\sum_k E^{n+\frac{1}{2}}(x_k)\rho^{n+\frac{1}{2}}(x_k) &= \sum_k \sum_i E^{n+\frac{1}{2}}(x_k) E^{n+\frac{1}{2}}(x_i)((S_1(x_{k+\frac{1}{2}} - x_i) - S_1(x_{k+\frac{1}{2}} - x_i)), \\
&= \sum_i E^{n+\frac{1}{2}}(x_i)(\frac{E^{n+\frac{1}{2}}(x_{i-1}) - E^{n+\frac{1}{2}}(x_{i+1})}{2}), \\
&= \frac{1}{2}(\sum_i E^{n+\frac{1}{2}}(x_i) E^{n+\frac{1}{2}}(x_{i+1}) - \sum_i E^{n+\frac{1}{2}}(x_i) E^{n+\frac{1}{2}}(x_{i-1})), \\
&= 0,
\end{aligned}$$

thanks to periodicity.

To conclude, using (3.16) and (3.17), we get (3.15), which is what was wanted.

CK algorithm. We still have to prove that:

$$\sum_{i,j} \omega_{i,j}^n v_j = \sum_{i,j} \omega_{i,j}^n V(t^{n+1}; (x_i, v_j), t^n).$$

That means for the third order scheme:

$$\begin{aligned}
\sum_{i,j} \omega_{i,j}^n E^n(x_i) &= 0, \qquad (i) \\
\sum_{i,j} \omega_{i,j}^n \varphi^n(x_i, v_j) &= 0, \qquad (ii) \\
\sum_{i,j} \omega_{i,j}^n \phi^n(x_i, v_j) &= 0, \qquad (iii)
\end{aligned}$$

each number being linked with the order of the algorithm.

First order. Using the same strategy (regularisation of the electric field and centered algorithm):

$$\begin{aligned}
\sum_{i,j} \omega_{i,j}^n E^n(x_i) &= \sum_{i,j,k,l} \omega_{i,j}^n E^n(x_k) S(x_i - x_k), \\
&= \sum_{k,l} E^n(x_k)\rho^n(x_k) = 0. \qquad (3.19)
\end{aligned}$$

Second order. For the second order, we need $J_i^n, \rho_i^n, \bar{J}$

$$\rho_i^n = \Delta v \sum_{k,l} \omega_{k,l}^n S(x_i - x_k) - 1,$$

$$J_i^n - \bar{J} = \Delta v \sum_{k,l} \omega_{k,l}^n v_l S(x_i - x_k) - \frac{\Delta x \Delta v}{L} \sum_{k,l} \omega_{k,l}^n v_l,$$

$$\sum_{i,j} \omega_{i,j}^n v_j \rho^n(x_i) = \Delta v \sum_{i,j,k,l} \omega_{i,j}^n \omega_{k,l}^n v_j S(x_i - x_k) - \sum_{i,j} \omega_{i,j}^n v_j, \tag{3.20}$$

$$\begin{aligned} \sum_{i,j} \omega_{i,j}^n J^n(x_i) &= \Delta v \sum_{i,j,k,l} \omega_{i,j}^n \omega_{k,l}^n v_l S(x_i - x_k) - \frac{\Delta x \Delta v}{L} \sum_{i,jk,l} \omega_{k,l}^n \omega_{i,j}^n v_l, \\ &= \Delta v \sum_{i,j,k,l} \omega_{i,j}^n \omega_{k,l}^n v_j S(x_i - x_k) - \sum_{k,l} \omega_{k,l}^n v_l, \end{aligned} \tag{3.21}$$

using mass conservation and $\sum_{i,j} \omega_{i,j}^0 = L$ (3.20) and (3.21) are the same, just changing (i,j) and (k,l). So

$$\sum_{i,j} \omega_{i,j}^n \varphi^n(x_i, v_j) = 0.$$

Third order. Here we need $I_2^n(x_i)$

$$\begin{aligned} I_2(t^n, x_i) &= \int_{\mathbb{R}} v^2 \sum_{k,l} \omega_{k,l}^n S(x_k - x_i) S(v - v_l)\, dv, \\ &= \sum_{k,l} \omega_{k,l}^n S(x_k - x_i) \int \mathbb{R}(v^2 S(v) + 2 v v_l S(v) + v_l^2 S(v))\, dv, \\ &= \sum_{k,l} \omega_{k,l}^n S(x_k - x_i)(\alpha + \Delta v\, v_l^2). \end{aligned}$$

In Φ we still have $\sum_{i,j} \omega_{i,j}^n E^n(x_i) = 0$. We also have three terms in $\frac{\partial}{\partial_x}$ which will be approached with a centered finite difference formula:

$$\begin{aligned} \sum_{i,j} \omega_{i,j}^n \frac{\partial}{\partial_x} I_2(t^n, x_i) &= \frac{1}{2\Delta x} \sum_{i,j} \omega_{i,j}^n (I_2(t^n, x_{i+1} - I_2(t^n, x_{i-1})), \\ &= \frac{1}{2\Delta x} \sum_{i,j,k,l} \omega_{i,j}^n \omega_{k,l}^n (\alpha + v_l^2)(S(x_{i+1} - x_k) - S(x_{i-1} - x_k)), \end{aligned} \tag{3.22}$$

$$2 \sum_{i,j} \omega_{i,j}^n v_j \frac{\partial J}{\partial_x}(t^n, (x_i) = \frac{1}{\Delta x} \sum_{i,j,k,l} \omega_{i,j}^n \omega_{k,l}^n v_j v_l (S(x_{i+1} - x_k) - S(x_{i-1} - x_k)), \tag{3.23}$$

$$\sum_{i,j} \omega_{i,j}^n v_j^2 \frac{\partial \rho}{\partial_x}(t^n, (x_i) = \frac{1}{2\Delta x} \sum_{i,j,k,l} \omega_{i,j}^n \omega_{k,l}^n v_j^2 (S(x_{i+1} - x_k) - S(x_{i-1} - x_k)). \tag{3.24}$$

Adding (3.22), (3.23) and (3.24) and using (3.19) we have:

$$\sum_{i,j} \omega_{i,j}^n \phi^n(x_i) = \frac{1}{\Delta x} \sum_{i,j,k,l} \omega_{i,j}^n \omega_{k,l}^n (\alpha + v_l^2 + 2v_j v_l + v_j^2)(S(x_{i+1} - x_k) - S(x_{i-1} - x_k)),$$

$$\begin{aligned} \sum_{i,j,k,l} \omega_{i,j}^n \omega_{k,l}^n (\alpha + v_l^2 + 2v_j v_l + v_j^2) S(x_{i+1} - x_k) &= \sum_{i,j,k,l} \omega_{k,l}^n \omega_{i,j}^n (\alpha + v_l^2 + 2v_j v_l + v_j^2) S(x_{k+1} - x_i), \\ &= \sum_{i,j,k,l} \omega_{i,j}^n \omega_{k,l}^n (\alpha + v_l^2 + 2v_j v_l + v_j^2) S(x_k - x_{i-1}), \end{aligned}$$

just changing (i, j) and (k, l) and $S(x_{k+1} - x_i) = S(x_k - x_{i-1})$. So we get:

$$\sum_{i,j} \omega_{i,j}^n \phi^n(x_i) = 0.$$

Remark 3.4 *We can see that the conservation of the first moment in v implies that numerically $\frac{d\bar{J}}{dt} = 0$, which is $\frac{\Delta x \Delta v}{L} \sum_{i,j}^n \omega_{i,j}^n v_j$, that is constant.*

4 Convergence analysis

Theorem 2 *Assume that $f_0 \in W^{2,\infty}_{c,per_x}(\mathbb{R}_x \times \mathbb{R}_v)$, positive, periodic with respect to the variable x, with period L, and compactly supported in velocity.*

Then the numerical solution of the Vlasov Poisson system (f_h, E_h), computed by the numerical scheme introduced in section 3.2 converges towards the solution (f, E) of the periodic Vlasov-Poisson system, and there exists a constant $C = C(||f||_{W^{1,\infty}(0,T;W^{2,\infty}(\Omega))})$ independent of Δt and h such that for Verlet and CK2 algorithms:

$$||f - f_h||_{l^\infty(0,T;L^1(\Omega))} + ||E - E_h||_{l^\infty(0,T;L^\infty([0,L]))} \leq C(\Delta t^2 + h^2 + \frac{h^2}{\Delta t}).$$

For CK3, we have:

$$||f - f_h||_{l^\infty(0,T;L^1(\Omega))} + ||E - E_h||_{l^\infty(0,T;L^\infty([0,L]))} \leq C(\Delta t^3 + \Delta t\, h^2 + \Delta t^2\, h + h^2 + \frac{h^2}{\Delta t}).$$

4.1 Decomposition of the error

Let f be the exact solution of the Vlasov Poisson equation and f_h the approximate solution equation fixed few paragraphs ago. In order to apply a discrete Gronwall inequality we express the l^1 error at time t^{n+1}

$$e^{n+1}(i,j) = |f(t^{n+1}, x_i, v_j) - f_h(t^{n+1}, x_i, v_j)| \quad \forall (i,j),$$
$$e^{n+1} = \Delta x \Delta v \sum_{i,j} e^{n+1}(i,j).$$

Then $f(t^{n+1}, x_k, v_l) - f_h(t^{n+1}, x_k, v_l)$ can be decomposed as

$$\begin{aligned} f(t^{n+1}, x_k, v_l) - f_h(t^{n+1}, x_k, v_l) &= f(t^{n+1}, x_k, v_l) - R_h f(t^{n+1}, x_k, v_l) + \\ &\quad R_h f(t^{n+1}, x_k, v_l) - R_h \tilde{f}_h(t^{n+1}, x_k, v_l) + \\ &\quad R_h \tilde{f}_h(t^{n+1}, x_k, v_l) - R_h f_h(t^{n+1}, x_k, v_l), \end{aligned} \tag{4.25}$$

where $\tilde{f}_h$ is the computed function f_h, but which follow the exact characteristics. Since f_h^{n+1} already belongs to the image of R_h, we have $R_h f_h(t^{n+1}, x_k, v_l) = f_h(t^{n+1}, x_k, v_l)$.

In order to estimate e^{n+1}, the three terms of the right hand side of the previous equation have to be dealt with. These estimations are developped in the following subsection.

4.2 A priori estimates

4.2.1 Stability for linear splines

Let us translate the useful properties in that case, and give a few more results about the operator R_h.

Lemma 1 *The R_h operator is consistent, that is, using property (i), for $1 \leq p \leq \infty$, and $0 \leq k \leq 1$*

$$\exists C > 0 \quad | \quad ||f - R_h f||_{W^{k,p}(\Omega)} \leq Ch^{2-k}|f|_{W^{2,p}(\Omega)} \quad \forall f \in W^{2,p}(\Omega) \cap P(\Omega).$$

This result is a classical property of B-splines.

Lemma 2 *With linear splines, if $\omega_{i,j}(f_0) \geq 0 \quad \forall (i,j) \quad$ then $\forall n \quad \omega_{i,j}(f^n) \geq 0 \quad \forall (i,j)$.*

Proof: With linear interpolation, we get in fact $\omega_{i,j}(f^n) = f^n(x_i, v_j)$, so if f_0 is positive, $\omega_{i,j}(f_0)$ is also, and since $f^{n+1}(x_i, v_j)$ is a sum of positive contributions coming from the $f^n(x_k, v_l)$ which are positive by a recurrence hypothesis, it will also be positive, and so $\omega_{i,j}(f^{n+1})$ is positive for all (i,j), and reccurently for all n.

Lemma 3 *Stability: Let f belong to $C(\Omega) \cap P(\Omega)$, then we have:*

$$||R_h f||_{L^1(\Omega)} = ||f||_{L^1_h(\Omega)}.$$

Proof:

$$\begin{aligned}
||R_h f||_{L^1(\Omega)} &= \int_0^L \int_{\mathbb{R}} |R_h f(x,v)| dv dx, \\
&= \int_0^L \int_{\mathbb{R}} \sum_{i,j} \omega_{i,j}(f) S(x - x_i) S(v - v_j), \\
&= \Delta x \Delta v \sum_{i,j} \omega_{i,j}(f), \\
&= ||f||_{L^1_h(\Omega)},
\end{aligned}$$

using $\int S(x)dx = \Delta x$ and the positivity of f thanks to lemma (3) and the positivity of f_0.

4.2.2 Towards Gronwall

Let us precise that in this subsection, some lemmas are valid for all the time algorithms we use, and when they are not, the lemmas will be proved in each case successively. For the Cauchy Kovalevsky procedure, the proofs will be done for (CK 3), since their adaptation to lower orders is trivial. We will now give estimates about the three right-hand side terms of the error e^{n+1} (4.25):

Lemma 4

$$||f - R_h f||_{L^1_h(\Omega)} \leq Ch^2. \tag{4.26}$$

Proof: Thanks to Lemma 3

$$\begin{aligned}
||f - R_h f||_{L^1_h(\Omega)} &= ||R_h(f - R_h f)||_{L^1(\Omega)}, \\
&\leq C||f - R_h f||_{L^1(\Omega)}, \\
&\leq C'(||f||_{L^\infty(0,T;W^{2,\infty}(\Omega)})h^2,
\end{aligned}$$

thanks to the property (ii) of spline interpolation.

Lemma 5

$$||R_h f^{n+1} - R_h \tilde{f_h}^{n+1}||_{L^1_h(\Omega)} \leq e^n. \tag{4.27}$$

Proof: We compute

$$\begin{aligned}
||R_h f^{n+1} &- R_h \tilde{f_h}^{n+1}||_{L^1_h(\Omega)} = \Delta x \Delta v \sum_{k,l} |(R_h f^{n+1} - R_h \tilde{f_h}^{n+1})(x_k, v_l)|, \\
&= \Delta x \Delta v \sum_{k,l} |\sum_{i,j} (f^{n+1}(x_i, v_j) - \tilde{f_h}^{n+1}(x_i, v_j)) S(x_k - x_i) S(v_l - v_j)|, \\
&\leq \Delta x \Delta v \sum_{i,j} |\omega_{i,j}(f^n) - \omega_{i,j}(f_h^n)| \sum_{k,l} S(x_k - X(t^{n+1}; (x_i, v_j), t^n)) S(v_l - V(t^{n+1}; (x_i, v_j), t^n)), \\
&\leq \Delta x \Delta v \sum_{i,j} |\omega_{i,j}(f^n) - \omega_{i,j}(f_h^n)|, \\
&\leq e^n,
\end{aligned}$$

thanks once more to the partition of unity (iii) and $f(x_i, v_j, t^n) = \omega_{i,j}(f^n)$.

Lemma 6

$$\begin{aligned}
||R_h \tilde{f_h}^{n+1} - R_h {f_h}^{n+1}||_{L^1_h(\Omega)} &\leq C \max_{i,j} (|X(t^{n+1}; (x_i, v_j), t^n) - X_h(t^{n+1}; (x_i, v_j), t^n)|, \\
&\quad + |V(t^{n+1}; (x_i, v_j), t^n) - V_h(t^{n+1}; (x_i, v_j), t^n)|).
\end{aligned} \tag{4.28}$$

Proof:

$$\begin{aligned}
&||R_h \tilde{f_h}^{n+1} - R_h f_h^{n+1}||_{L^1_h(\Omega)} = \Delta x \Delta v \sum_{k,l} |(R_h \tilde{f_h}^{n+1} - R_h {f_h}^{n+1})(x_k, v_l)|, \\
= \ & \Delta x \Delta v \sum_{k,l} |\sum_{i,j} \omega_{i,j}(f_h)^n (S(x_k - X(t^{n+1}; (x_i, v_j), t^n)) S(v_l - V(t^{n+1}; (x_i, v_j), t^n)), \\
- \ & S(x_k - X_h(t^{n+1}; (x_i, v_j), t^n)) S(v_l - V_h(t^{n+1}; (x_i, v_j), t^n)))|.
\end{aligned}$$

Then, we use the fact that S_1 is 1-Lipschitzian, and compactly supported, so that:

$$\begin{aligned}
&(S(x_k - X(t^{n+1}; (x_i, v_j), t^n)) S(v_l - V(t^{n+1}; (x_i, v_j), t^n)) \\
&\qquad - S(x_k - X_h(t^{n+1}; (x_i, v_j), t^n)) S(v_l - V_h(t^{n+1}; (x_i, v_j), t^n)))| \\
&= (S(x_k - X(t^{n+1}; (x_i, v_j), t^n)) - S(x_k - X_h(t^{n+1}; (x_i, v_j), t^n))) S(v_l - V(t^{n+1}; (x_i, v_j), t^n)) \\
&\quad - (S(v_l - V(t^{n+1}; (x_i, v_j), t^n)) - S(v_l - V_h(t^{n+1}; (x_i, v_j), t^n))) S(x_k - X_h(t^{n+1}; (x_i, v_j), t^n)).
\end{aligned}$$

Then, using the Lipschitz property and the support of S_1, and the property (i):

$$\begin{aligned}
&\sum_{k,l} |(S(x_k - X(t^{n+1}; (x_i, v_j), t^n)) - S(x_k - X_h(t^{n+1}; (x_i, v_j), t^n))) S(v_l - V(t^{n+1}; (x_i, v_j), t^n))| \\
&\qquad \leq |X(t^{n+1}; (x_i, v_j), t^n) - X_h(t^{n+1}; (x_i, v_j), t^n)|,
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{k,l} |(S(v_l - V(t^{n+1}; (x_i, v_j), t^n)) - S(v_l - V_h(t^{n+1}; (x_i, v_j), t^n))) S(x_k - X_h(t^{n+1}; (x_i, v_j), t^n))| \\
&\qquad \leq |V(t^{n+1}; (x_i, v_j), t^n) - V_h(t^{n+1}; (x_i, v_j), t^n)|.
\end{aligned}$$

So that we get:

$$||R_h\tilde{f_h}^{n+1} - R_h f_h^{n+1}||_{L^1_h(\Omega)} \leq \Delta x \Delta v \sum_{i,j} |\omega_{i,j}(f_h)^n|(|X(t^{n+1};(x_i,v_j),t^n) - X_h(t^{n+1};(x_i,v_j),t^n)|$$
$$+ |V(t^{n+1};(x_i,v_j),t^n) - V_h(t^{n+1};(x_i,v_j),t^n))|),$$
$$\leq C \max_{i,j}(|X(t^{n+1};(x_i,v_j),t^n) - X_h(t^{n+1};(x_i,v_j),t^n)| + |V(t^{n+1};(x_i,v_j),t^n) - V_h(t^{n+1};(x_i,v_j),t^n)|).$$

thanks to particle conservation ($\sum_{i,j}\omega_{i,j}(f_h^n) = \sum_{i,j}\omega_{i,j}(f_0)$) and positivity of $\omega_{i,j}(f_h^n)$ for all n, where (X,V) are the exact characteristics, solutions of the differential system (2.7), and (X_h,V_h) the approximate characteristics defined in (3.11).

To move on, we need another lemma which enables to control the difference between exact and computed characteristics. It clearly depends on the algorithm we use. Let us first give the lemma concerning Verlet algorithm.

Lemma 7 *: Verlet*

If $E \in W^{2,\infty}([0,t]\times\mathbb{R})$, and with (X,V) calculated exactly with the differential system (2.7)*, and (X_h,V_h) computed with E_h and a Verlet algorithm:*

$$\begin{aligned} |X(t^{n+1};(x_i,v_j),t^n) &- X_h(t^{n+1};(x_i,v_j),t^n)| + |V(t^{n+1};(x_i,v_j),t^n) - V_h(t^{n+1};(x_i,v_j),t^n)| \\ &\leq C\Delta t^3 + C\Delta t^2||(E-E_h)(t^{n+\frac{1}{2}})||_{L^\infty} + \Delta t||(E-E_h)(t^{n+\frac{1}{2}})||_{L^\infty}. \end{aligned}$$

Proof: The strategy follows the work of M. Bostan and N. Crouseilles ([2]).

Let us recall how $(x,v)^{n+1}$ is computed from $(x,v)^n$ with the Verlet algorithm

$$x^{n+\frac{1}{2}} = x^n + \frac{\Delta t}{2}v^n,$$

$$v^{n+1} = v^n + \Delta t E_h(t^{n+\frac{1}{2}}, x^n + \frac{\Delta t}{2}v^n),$$

$$x^{n+1} = x^{n+\frac{1}{2}} + \frac{\Delta t}{2}v^{n+1}.$$

Let us begin with the characteristics in v:

$$\begin{aligned} (V_h - V)(t^{n+1};(x^n,v^n),t^n) &= v^n + \Delta t E_h(t^{n+\frac{1}{2}}, x^n + \frac{\Delta t}{2}v^n) - v^n - \int_{t^n}^{t^{n+1}} E(s, X(s;(x^n,v^n),t^n)\,ds, \\ &= -\int_{t^n}^{t^{n+1}} (E(s, X(s;(x^n,v^n),t^n) - E(t^{n+\frac{1}{2}}, x^n + \frac{\Delta t}{2}v^n))\,ds, \\ &- \Delta t(E(t^{n+\frac{1}{2}}, x^n + \frac{\Delta t}{2}v^n) - E_h(t^{n+\frac{1}{2}}, x^n + \frac{\Delta t}{2}v^n)). \end{aligned} \tag{4.29}$$

Let us take care of the integral term in (4.29)

$$\begin{aligned} E(s, X(s;(x^n,v^n),t^n) &= E(t^{n+\frac{1}{2}}, x^n + \frac{\Delta t}{2}v^n) + (s - t^{n+\frac{1}{2}})E'(t^{n+\frac{1}{2}}, x^n + \frac{\Delta t}{2}v^n) \\ &+ \int_{t^{n+\frac{1}{2}}}^{s} (s-u)E''(u, X(u;(x^n,v^n),t^n))\,du. \end{aligned} \tag{4.30}$$

Thanks to Taylor's formula. Let us precise that $E'(s,X(s)) = \frac{d}{ds}E(s,X(s))$, the same for E''. Then using (4.30) in (4.29), we get

$$\int_{t^n}^{t^{n+1}} (E(s,X(s;(x^n,v^n),t^n)) - E(t^{n+\frac{1}{2}}, x^n + \frac{\Delta t}{2}v^n))\,ds = E'(t^{n+\frac{1}{2}}, x^n + \frac{\Delta t}{2}v^n)[(\frac{s-t^{n+\frac{1}{2}}}{2})^2]_{t^n}^{t^{n+1}}$$
$$+ \int_{t^n}^{t^{n+1}} \int_{t^{n+\frac{1}{2}}}^{s} (s-u)E''(u,X(u;(x,v),t^n)\,du\,ds.$$

The term between brackets being zero. Then, as $E \in W^{2,\infty}$,

$$\begin{aligned} |\int_{t^n}^{t^{n+1}} \int_{t^{n+\frac{1}{2}}}^{s} (s-u)E''(u,X(u;(x^n,v^n),t^n)\,du\,ds| &\leq C(||E''||_{L^\infty}) \int_{t^n}^{t^{n+1}} \int_{t^{n+\frac{1}{2}}}^{s} (s-u)\,du\,ds, \\ &\leq C \int_{t^n}^{t^{n+1}} [(\frac{s-u}{2})^2]_{t^{n+\frac{1}{2}}}^{s}, \\ &\leq \frac{C}{2} \int_{t^n}^{t^{n+1}} (s-t^{n+\frac{1}{2}})^2\,ds, \\ &\leq \frac{C\Delta t^3}{24} \leq C'(||E''||_{L^\infty})\Delta t^3. \end{aligned} \tag{4.31}$$

Now let us deal with the second term of (4.29), since E is bounded:

$$\begin{aligned} |x^n + \frac{\Delta t}{2}v^n - X(t^{n+\frac{1}{2}};(x^n,v^n),t^n)| &= |\int_{t^n}^{t^{n+\frac{1}{2}}} v^n - V(s;(x^n,v^n),t^n)\,ds|, \\ &\leq \int_{t^n}^{t^{n+\frac{1}{2}}} (s-t^n)V'(u;(x^n,v^n),t^n)\,ds \quad u \in [t^n,s], \\ &\leq C(||E||_{L^\infty}) \int_{t^n}^{t^{n+\frac{1}{2}}} (s-t^n)\,ds \leq C'\Delta t^2. \end{aligned} \tag{4.32}$$

$$\begin{aligned} |E(t^{n+\frac{1}{2}}, X(t^{n+\frac{1}{2}};(x^n,v^n),t^n)) &- E_h(t^{n+\frac{1}{2}}, x^{n+\frac{1}{2}})| \leq ||(E-E_h)(t^{n+\frac{1}{2}})||_\infty \\ &+ |E(t^{n+\frac{1}{2}}, X(t^{n+\frac{1}{2}};(x^n,v^n),t^n)) - E(t^{n+\frac{1}{2}}, x^{n+\frac{1}{2}})|, \end{aligned} \tag{4.33}$$

$$\begin{aligned} |E(t^{n+\frac{1}{2}}, X(t^{n+\frac{1}{2}};(x^n,v^n),t^n)) - E(t^{n+\frac{1}{2}}, x^{n+\frac{1}{2}})| &\leq C(||E'||_{L^\infty})|X(t^{n+\frac{1}{2}};(x^n,v^n),t^n) - x^{n+\frac{1}{2}}|, \\ &\leq C(||E'||_{L^\infty})\Delta t^2, \end{aligned} \tag{4.34}$$

using a Taylor expansion, $E \in W^{1,\infty}$ and (3.23).

So, using (4.31), (4.33) and (4.34), we get

$$|(V_h - V)(t^{n+1};(x,v),t^n)| \leq C\Delta t^3 + \Delta t||(E-E_h)(t^{n+\frac{1}{2}})||_{L^\infty(\Omega)}. \tag{4.35}$$

Let us now deal with the characteristics in X:

$$\begin{aligned} (X_h - X)(t^{n+1};(x^n,v^n),t^n) &= \frac{\Delta t}{2}v^n + \frac{\Delta t}{2}v^{n+1} - \int_{t^n}^{t^{n+1}} V(s;(x^n,v^n),t^n)\,ds, \\ &= -\Delta t(V(t^{n+\frac{1}{2}};(x^n,v^n),t^n) - \frac{1}{2}v^n - \frac{1}{2}v^{n+1}), \\ &- \int_{t^n}^{t^{n+1}} (V(s;(x^n,v^n),t^n) - V(t^{n+\frac{1}{2}};(x^n,v^n),t^n)))\,ds, \end{aligned} \tag{4.36}$$

so once again we have to control two terms.

For the first one, thanks to Taylor's inequality, like for X, it comes:

$$|\int_{t^n}^{t^{n+1}} (V(s;(x^n,v^n),t^n) - V(t^{n+\frac{1}{2}};(x^n,v^n),t^n))\,ds| \leq C(||E||_{L^\infty})\Delta t^3. \tag{4.37}$$

Let us precise that this is nothing else than the error in the mid-point formula for numerical integration. Now, the second term in (4.36):

$$\begin{aligned} V(t^{n+\frac{1}{2}};(x^n,v^n),t^n)) &= v^n + \int_{t^n}^{t^{n+\frac{1}{2}}} E(s,X(s;(x^n,v^n),t^n))\,ds, \\ &= v^n + \frac{\Delta t}{2} E(t^{n+\frac{1}{2}}, X(t^{n+\frac{1}{2}};(x^n,v^n),t^n)), \\ &+ \int_{t^n}^{t^{n+\frac{1}{2}}} (E(s,X(s;(x^n,v^n),t^n)) - E(t^{n+\frac{1}{2}}, X(t^{n+\frac{1}{2}};(x^n,v^n),t^n)))\,ds, \end{aligned}$$

and

$$|\int_{t^n}^{t^{n+\frac{1}{2}}} (E(s,X(s;(x^n,v^n),t^n)) - E(t^{n+\frac{1}{2}}, X(t^{n+\frac{1}{2}};(x^n,v^n),t^n)))\,ds| \leq C(||E'||_{L^\infty})\Delta t^2,$$

with the mid-point error formula. On the other hand

$$\begin{aligned} V(t^{n+\frac{1}{2}};(x^n,v^n),t^n) - \frac{1}{2}v^n - \frac{1}{2}v^{n+1} &= \frac{1}{2}v^n - \frac{1}{2}v^{n+1} \\ &+ \frac{\Delta t}{2} E(t^{n+\frac{1}{2}}, X(t^{n+\frac{1}{2}};(x^n,v^n),t^n)) + O(\Delta t^2). \end{aligned}$$

Since

$$\frac{1}{2}(v^n - v^{n+1}) = -\frac{\Delta t}{2} E_h(t^{n+\frac{1}{2}}, x^{n+\frac{1}{2}}),$$

we have

$$\begin{aligned} |V(t^{n+\frac{1}{2}};(x^n,v^n),t^n) - \frac{1}{2}v^n - \frac{1}{2}v^{n+1}| &= |\frac{\Delta t}{2}(E(t^{n+\frac{1}{2}}, X(t^{n+\frac{1}{2}};(x^n,v^n),t^n)) - E_h(t^{n+\frac{1}{2}}, x^{n+\frac{1}{2}})|, \\ &\leq \frac{\Delta t}{2}(||(E-E_h)(t^{n+\frac{1}{2}})||_{L^\infty(\Omega)} + C\Delta t^2). \end{aligned} \tag{4.38}$$

To conclude, using (4.37), (4.38), we have:

$$|(X_h - X)(t^{n+1};(x,v),t^n)| \leq C\Delta t^3 + C\Delta t^2 ||(E-E_h)(t^{n+\frac{1}{2}})||_{L^\infty(\Omega)}. \tag{4.39}$$

Finally, using (4.35) and (4.39), we get:

$$||R_h \tilde{f_h}^{n+1} - R_h f_h^{n+1}||_{L^1_h(\Omega)} \leq C\Delta t^3 + C\Delta t^2 ||(E-E_h)(t^{n+\frac{1}{2}})||_{L^\infty} + \Delta t ||(E-E_h)(t^{n+\frac{1}{2}})||_{L^\infty}.$$

Lemma 8 *: CK3*

If $E \in W^{4,\infty}([0,t]\times\mathbb{R})$, and with (X,V) calculated exactly with the differential system (2.7), and (X_h,V_h) computed with E_h, ρ_h, J_h and a CK3 algorithm:

$$\begin{aligned} &|X(t^{n+1};(x_i,v_j),t^n) - X_h(t^{n+1};(x_i,v_j),t^n)| + |V(t^{n+1};(x_i,v_j),t^n) - V_h(t^{n+1};(x_i,v_j),t^n)| \\ &\quad \leq C\Delta t^4 + C(\Delta t||(E^n - E_h^n)||_{l^\infty(\Omega} + \Delta t^2||(\phi^n - \phi_h^n)||_{l^\infty(\Omega)} + \Delta t^3||(\varphi^n - \varphi_h^n)||_{l^\infty(\Omega)}). \end{aligned}$$

Proof: This proof just relies on Taylor expansions and computations already made:

$$\begin{aligned} X(t^{n+1};(x_i,v_j),t^n) &= x_i + \Delta t v_j + \frac{\Delta t^2}{2}E^n(x_i) + \frac{\Delta t^3}{6}\phi^n(x_i,v_j) + O(\Delta t^4), \\ X_h(t^{n+1};(x_i,v_j),t^n) &= x_i + \Delta t v_j + \frac{\Delta t^2}{2}E_h^n(x_i) + \frac{\Delta t^3}{6}\phi_h^n(x_i,v_j) + O(\Delta t^4), \end{aligned}$$

and

$$\begin{aligned} V(t^{n+1};(x_i,v_j),t^n) &= v_j + \Delta t E^n(x_i) + \frac{\Delta t^2}{2}\phi^n(x_i,v_j) + \frac{\Delta t^3}{6}\varphi^n(x_i,v_j) + O(\Delta t^4), \\ V(t^{n+1};(x_i,v_j),t^n) &= v_j + \Delta t E_h^n(x_i) + \frac{\Delta t^2}{2}\phi_h^n(x_i,v_j) + \frac{\Delta t^3}{6}\varphi_h^n(x_i,v_j) + O(\Delta t^4), \end{aligned}$$

and the lemma follows by simple subtraction.

In both cases we need to control the difference between the exact and approximate fields. Let us begin with the Verlet algorithm.

Lemma 9 *: Verlet.*
If $E \in W^{2,\infty}([0,t]\times\mathbb{R})$, it comes

$$||(E-E_h)(t^{n+\frac{1}{2}})||_{L^\infty(\Omega)} \leq C(h^2 + \Delta t^2 + e^n).$$

Proof: First

$$\begin{aligned} E(t^{n+\frac{1}{2}},x) &= \int_0^L K(x,y)(\int_{\mathbb{R}} f(t^{n+\frac{1}{2}},y,v)dv - 1)dy, \\ E_h(t^{n+\frac{1}{2}},x) &= \int_0^L K(x,y)(\int_{\mathbb{R}} f_h(t^{n+\frac{1}{2}},y,v)dv - 1)dy. \end{aligned}$$

On the other hand

$$\begin{aligned} E(t^{n+\frac{1}{2}},x) - E_h(t^{n+\frac{1}{2}},x) &= \int_0^L K(x,y)(\int_{\mathbb{R}} (f(t,y,v) - f_h(t,y,v)dv)dy, \\ &= \int_0^L K(x,y)(\int_{\mathbb{R}} (f(t^{n+\frac{1}{2}},y,v) - R_h f(t^{n+\frac{1}{2}},y,v)dv)dy \\ &+ \int_0^L K(x,y)(\int_{\mathbb{R}} (R_h f(t^{n+\frac{1}{2}},y,v) - R_h \tilde{f}(t^{n+\frac{1}{2}},y,v)dv)dy \\ &+ \int_0^L K(x,y)(\int_{\mathbb{R}} (R_h \tilde{f}(t^{n+\frac{1}{2}},y,v) - f_h(t^{n+\frac{1}{2}},y,v)dv)dy, \end{aligned}$$

where once again $\tilde{f}$ is the function which has the same values as f at time t^n, but which follow the computed (approximate) characteristics. We have three terms to control. The first is controlled using property (ii) of consistency and accuracy:

$$\begin{aligned} |E(t^{n+\frac{1}{2}},x) - E_h(t^{n+\frac{1}{2}},x)| &\leq C||K||_\infty h^2 \\ &+ |\int_0^L K(x,y)(\int_{\mathbb{R}} (R_h f(t^{n+\frac{1}{2}},y,v) - R_h \tilde{f}(t^{n+\frac{1}{2}},y,v)dv)dy| \\ &+ \Delta x|\sum_i K(x,y_i^{n+\frac{1}{2}})\int_{\mathbb{R}} R_h \tilde{f}(t^{n+\frac{1}{2}},y_i^{n+\frac{1}{2}},v) - f_h(t^{n+\frac{1}{2}},y_i^{n+\frac{1}{2}},v)dv| + O(h^2), \quad (4.40) \end{aligned}$$

with a mid-point quadrature of order 2, made on the third term, since, the domain in x is periodic and setting for a fixed v

$$y_i^{n+\frac{1}{2}} = y_i + \frac{\Delta t}{2} v,$$

so that the $(y_i^{n+\frac{1}{2}})_i$ are equidistant.

Then, since the two function f_h and $R_h\tilde{f}$ follow the same approximate characteristics, still using a mid-point formula with the domain in v which is also periodic:

$$\begin{aligned}\int_{\mathbb{R}} R_h\tilde{f}(t^{n+\frac{1}{2}}, y_i^{n+\frac{1}{2}}, v) - f_h(t^{n+\frac{1}{2}}, y_i^{n+\frac{1}{2}}, v)dv &= \int_{\mathbb{R}} R_h\tilde{f}(t^n, y_i, \tilde{v}) - f_h(t^n, y_i, \tilde{v})d\tilde{v}, \\ &= \Delta v \sum_j R_h\tilde{f}(t^n, y_i, v_j) - f_h(t^n, y_i, v_j) + O(h^2).\end{aligned}$$

So that, since at time t^n $R_h f = f$ on the mesh:

$$\begin{aligned}|\int_0^L K(x,y)(\int_{\mathbb{R}}(R_h\tilde{f}(t^{n+\frac{1}{2}}, y, v) - f_h(t^{n+\frac{1}{2}}, y, v)dvdy| &\leq \Delta x\Delta v||K||_\infty \sum_{i,j} e^n(i,j) + O(h^2), \\ &\leq Ce^n + O(h^2). \qquad (4.41)\end{aligned}$$

There remains a last term to control:

$$\begin{aligned}&\int_0^L K(x,y)(\int_{\mathbb{R}}(R_h f(t^{n+\frac{1}{2}}, y, v) - R_h\tilde{f}(t^{n+\frac{1}{2}}, y, v)dv)dy \\ &\quad= \int_0^L K(x,y)\int_{\mathbb{R}}\sum_{k,l}\omega_{k,l}^n(S(y - X(t^{n+\frac{1}{2}}, (x_k, v_l), t^n))S(v - V(t^{n+\frac{1}{2}}, (x_k, v_l), t^n)) \\ &\qquad - S(y - X_h(t^{n+\frac{1}{2}}, (x_k, v_l), t^n))S(v - V_h(t^{n+\frac{1}{2}}, (x_k, v_l), t^n)))dvdy, \\ &\quad= \Delta x\Delta v(\sum_{k,l}\omega_{k,l}^n\sum_{i,j}K(x, y_i)(S(y_i - X(t^{n+\frac{1}{2}}, (x_k, v_l), t^n))S(v_j - V(t^{n+\frac{1}{2}}, (x_k, v_l), t^n)) \\ &\qquad - S(y_i - X_h(t^{n+\frac{1}{2}}, (x_k, v_l), t^n))S(v_j - V_h(t^{n+\frac{1}{2}}, (x_k, v_l), t^n))) + O(h^2).\end{aligned}$$

Then, using the form of $K(x,y)$, partition of unity and a result about conservation of the first moment:

$$\sum_l v_l S_1(v_l - V_{i,j}(t^{n+\frac{1}{2}})) = V(t^{n+\frac{1}{2}}, (x_i, v_j), t^n),$$

and

$$\Delta x\Delta v|\sum_{i,j} K(x,y_i)(S(y_i - X(t^{n+\frac{1}{2}},(x_k,v_l),t^n))S(v_j - V(t^{n+\frac{1}{2}},(x_k,v_l),t^n))$$
$$- S(y_i - X_h(t^{n+\frac{1}{2}},(x_k,v_l),t^n))S(v_j - V_h(t^{n+\frac{1}{2}},(x_k,v_l),t^n)))|$$
$$\leq \Delta x\Delta v|\sum_i \frac{y_i}{L}(S(y_i - X(t^{n+\frac{1}{2}},(x_k,v_l),t^n)) - S(y_i - X_h(t^{n+\frac{1}{2}},(x_k,v_l),t^n)))|$$
$$+ \Delta x\Delta v|\sum_{i|x<y_i}(S(y_i - X(t^{n+\frac{1}{2}},(x_k,v_l),t^n)) - S(y_i - X_h(t^{n+\frac{1}{2}},(x_k,v_l),t^n)))|,$$
$$\leq \Delta x\Delta v\sum_i |X(t^{n+\frac{1}{2}},(x_k,v_l),t^n) - X_h(t^{n+\frac{1}{2}},(x_k,v_l),t^n)|$$
$$+ \Delta x\Delta v|\sum_{i|x<y_i}(X(t^{n+\frac{1}{2}},(x_k,v_l),t^n) - X_h(t^{n+\frac{1}{2}},(x_k,v_l),t^n))S'(y_i - X(t^{n+\frac{1}{2}},(x_k,v_l),t^n)|,$$
$$\leq C\Delta x\Delta v|X(t^{n+\frac{1}{2}},(x_k,v_l),t^n) - X_h(t^{n+\frac{1}{2}},(x_k,v_l),t^n))|,$$

using once more the Lipschitz property and the fact that S_1 is compactly supported.

We already saw in the proof of Lemma 7 equation (4.32) that

$$|X(t^{n+\frac{1}{2}},(x_k,v_l),t^n) - X_h(t^{n+\frac{1}{2}},(x_k,v_l),t^n)| \leq C\Delta t^2.$$

Finally,

$$\begin{aligned} |\int_0^L K(x,y)(\int_{\mathbb{R}}(R_h f(t^{n+\frac{1}{2}},y,v) &- R_h\tilde{f}(t^{n+\frac{1}{2}},y,v)dv)dy|, \\ &\leq C\Delta t^2(\Delta x\Delta v\sum_{k,l}\omega_{k,l}^n) + O(h^2), \\ &\leq C'(\Delta t^2 + h^2). \end{aligned} \tag{4.42}$$

To conclude, using (4.40), (4.41) and (4.42), we get:

$$||(E - E_h)(t^{n+\frac{1}{2}})||_{L^\infty(\Omega)} \leq C(h^2 + \Delta t^2 + e^n).$$

Lemma 10 *CK3 With the same hypothesis as in lemma 8, we have :*

$$||R_h\tilde{f_h}^{n+1} - R_h f_h{}^{n+1}||_{L_h^1(\Omega)} \leq C(e^n(\Delta t + \frac{\Delta t^3}{\Delta x}) + \Delta t^2 h^2 + \Delta t^3 h + \Delta t^4.$$

Proof: Here, we need to evaluate the difference between the $l^\infty(\Omega)$ norms of the exact and approximate values of ϕ, φ and ψ, so to say the one between:

- E^n and E_h^n, ρ^n and ρ_h^n, J^n and J_h^n,
- their first spatial derivative and the one of I_2^n and I_{2h}^n.

Let us start with

$$\begin{aligned} (E^n - E_h^n)(x_i) &= \int_0^L K(x,y)(\int_{\mathbb{R}}(f(t^n,y,v) - f_h(t^n,y,v)\,dv)\,dy, \\ &\leq Ce^n + O(h^2). \end{aligned}$$

simply using the same quadrature as with Verlet in both directions, and the fact that K is bounded. So that:

$$||E^n - E_h^n||_{l^\infty([0,L])} \leq Ce^n + O(h^2), \tag{4.43}$$

$$\begin{aligned} ||\rho^n - \rho_h^n||_{l^1([0,L])} &= \Delta x \sum_i |\int_{\mathbb{R}} (f^n - f_h^n(x_i, v)\, dv|, \\ &\leq Ce^n + O(h^2). \end{aligned}$$

So that using the equivalence of discrete norms, we get:

$$\begin{aligned} ||\rho^n - \rho_h^n||_{l^\infty([0,L])} &\leq C||\rho^n - \rho_h^n||_{l^1([0,L])}, \\ &\leq Ce^n + O(h^2), \end{aligned}$$

and

$$\begin{aligned} ||J^n - J_h^n||_{l^1([0,L])} &= \Delta x \sum_i |\int_{\mathbb{R}} v(f^n - f_h^n(x_i, v)\, dv|, \\ &\leq Ce^n + O(h^2), \end{aligned}$$

using the same arguments and the fact that f is compactly supported, so that:

$$\begin{aligned} ||J^n - J_h^n||_{l^\infty([0,L])} &\leq C||J^n - J_h^n||_{l^1([0,L])}, \\ &\leq Ce^n + O(h^2). \end{aligned}$$

Let us precise that the same bound is obviously also valid for $\bar{J}$. So that we get, still using that Ω is bounded, we have:

$$||\phi^n - \phi_h^n||_{l^\infty([0,L])} \leq Ce^n + O(h^2).$$

For the first spatial derivative of these three terms, we can use the same strategy of finite difference. Let us do it with E:

$$\frac{\partial(E - E_h)}{\partial x}(t^n, x_i) = \frac{(E_{i+1}^n - E_{i+1,h}^n) - (E_{i-1}^n - E_{i-1,h}^n)}{2\Delta x} + O(\Delta x^2),$$

so that using (4.43) we get:

$$||\frac{\partial(E - E_h)}{\partial x}(t^n)||_{l^\infty([0,L])} \leq C\frac{e^n}{\Delta x} + O(h).$$

The same bound is also valid for ρ^n, J^n and for I_2^n just bounding v^2 in its integral definition. So that we get:

$$||\varphi^n - \varphi_h^n||_{l^\infty([0,L])} \leq C\frac{e^n}{\Delta x} + O(h).$$

4.3 End of the proof

We can now apply Gronwall inequality since

Verlet For Verlet, using lemmas (4),(5),(7),(9), we get

$$\begin{aligned} e^{n+1} &\leq C(h^2 + \Delta t^3 + (\Delta t^2 + \Delta t)(h^2 + \Delta t^2 + e^n)) + e^n, \\ &\leq (1 + C'\Delta t)e^n + C(h^2 + \Delta t^3 + (\Delta t^2 + \Delta t)(h^2 + \Delta t^2)), \end{aligned}$$

So that

$$e^n \leq \exp(C'T)e^0 + C(h^2 + \Delta t^2 + \frac{h^2}{\Delta t}),$$

which is what was expected. Taking $\Delta t = Ch^\alpha$, we find the best global order with $\alpha = \frac{2}{3}$ being $\frac{4}{3}$.

CK2. For CK2 , using lemmas (4),(5),(8),(10), we get

$$\begin{aligned} e^{n+1} &\leq C(h^2 + \Delta t^3 + (\Delta t^2 + \Delta t)(h^2 + e^n)) + e^n, \\ &\leq (1 + C'\Delta t)e^n + C(h^2 + \Delta t^3 + \Delta t h^2), \end{aligned}$$

and the same conclusion as with Verlet, which is what was expected.

CK3. In the same way, we get

$$\begin{aligned} e^{n+1} &\leq C(h^2 + \Delta t^4 + e^n(\Delta t + \frac{\Delta t^3}{\Delta x}) + \Delta t h^2 + \Delta t^3 h) + e^n, \\ &\leq e^n(1 + C(\Delta t + \frac{\Delta t^3}{\Delta x})) + C(h^2 + \Delta t^3 h + \Delta t^4). \end{aligned}$$

Here we need to bound $\frac{\Delta t^2}{\Delta x}$ in order to get:

$$e^n \leq \exp(C'T)e^0 + C(\Delta t^3 + \Delta t^2 h + \frac{h^2}{\Delta t}).$$

The best order would be reached taking $\alpha = \frac{1}{3}$ and would be $\frac{5}{3}$, but $\frac{\Delta t^2}{\Delta x}$ would not be bounded. We see that in order to reach real high orders, we would need to use a deposition with splines of higher degrees. But this analysis cannot be generalized easily because of problems of stability. The fourth order CK algorithm has been written, but the result was the same as CK3 because of the same limitation.

5 Numerical results

In order to validate our new schemes, we have tested them on two standard test cases of plasma physics, the two stream instability and the bump on tail instability. We also compared them to the classical and knowledgeably robust Verlet scheme. Notice, that because of the diffusitivity of linear splines, we have used cubic splines for the distribution function.

For the two stream instability, the initial condition is given by

$$f_0(x, v) = \frac{1}{\sqrt{2\pi}} e^{-v^2/2} v^2 [1 - \alpha \cos(kx)],$$

with $k = 0.2$ and $\alpha = 0.05$. The computational domain is $[0, 2\pi/k] \times [-9, 9]$ which is sampled by $N_x = N_v = 128$ points. We used a time step $\Delta t = 0.1$ in the results of the left hand side of Figure 1 and of $\Delta t = 0.3$ on the right-hand side of the figure. We display the L^2 norm which reveals the dissipation of the scheme, the total momentum and the total energy. All of those are

conserved in the continuous Vlasov-Poisson system. We do not display the number of particules which is conserved with and even better accuracy than the momentum. The momentum is exactly conserved by the scheme and up to about 10^{-13} in the simulation. This is due to roundoff errors and the truncation of the velocity space. The L^2 norm cannot be exactly conserved by any scheme using a phase space grid as soon as the grid does not resolve anymore the filaments. The Verlet scheme is our reference scheme here, and we observe that the results obtained with the CK schemes are very close, especially for the smallest time step. Moreover conservation properties are better for the third order CK3 than for the second order CK2.

For the bump-on-tail instability test case the initial condition writes

$$f_0(x,v) = \tilde{f}(v)[1 + \alpha \cos(kx)],$$

with

$$\tilde{f}(v) = n_p \exp(-v^2/2) + n_b \exp\left(-\frac{|v-u|^2}{2v_t^2}\right)$$

on the interval $[0, 20\pi]$, with periodic conditions in space. The initial condition f_0 is a Maxwellian distribution function which has a bump on the Maxwell distribution tail; the parameters of this bump are the following

$$n_p = \frac{9}{10(2\pi)^{1/2}}, n_b = \frac{2}{10(2\pi)^{1/2}}, u = 4.5, v_t = 0.5,$$

whereas the numerical parameters are $N_x = 128, N_v = 128, v_{max} = 9, \Delta t = 0.2$. The results are displayed in Figure 2. Here the momentum does not vanish and is exactly conserved. The conclusion for the L^2 norm and the total energy is the same as in the Two stream instability test case. The potential or electric energy is a classical diagnostic for the bump on tail instability. The oscillations go on for a long time with all three time schemes, even though there is a slight energy increase for the CK2 scheme.

6 Conclusion

In this paper, the proof of L^1 convergence has been reached for linear spline interpolation. The originality, except from the choice of the L^1 norm is that the convergence has been reached for a non split method. The computation of the characteristics has been made with the Verlet algorithm, or with a CK procedure, but the proof can be adapted to other algorithms such as Runge Kutta of any order. There remains for the moment some problems using splines of higher orders, especially concerning stability. This prevents us from reaching real high order algorithms. Numerical experiments that can be seen in [7], and confirmed here seem to prove that the method is also stable and convergent for cubic splines. Nevertheless, there remains a problem to preserve the l^1 norm of the coefficients $\omega_{i,j}$, since some of them can become non positive in the solution of the linear system with splines of degree higher than 2. Another way of tackling the problem will probably be needed.

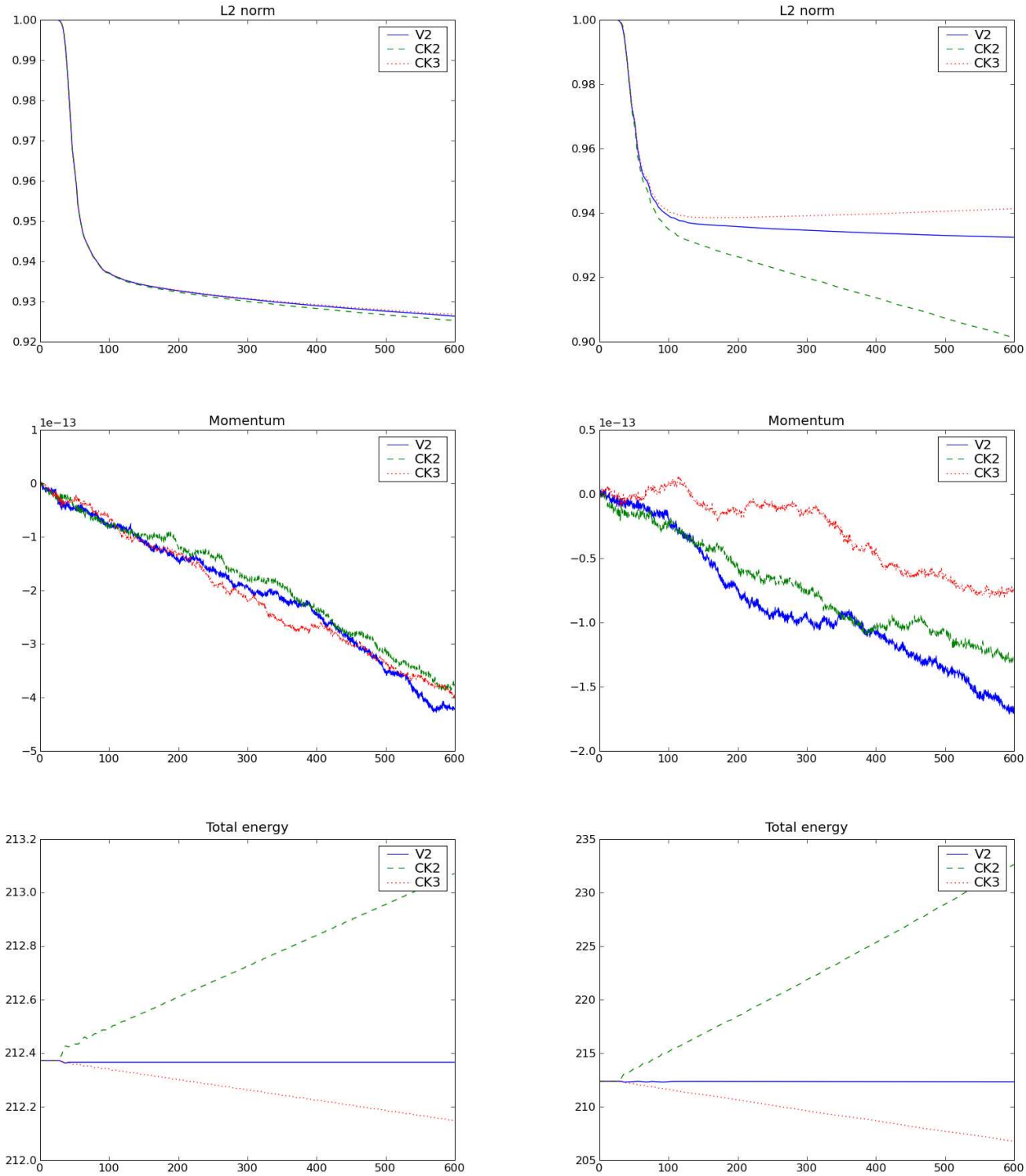

Figure 1: Two stream instability.

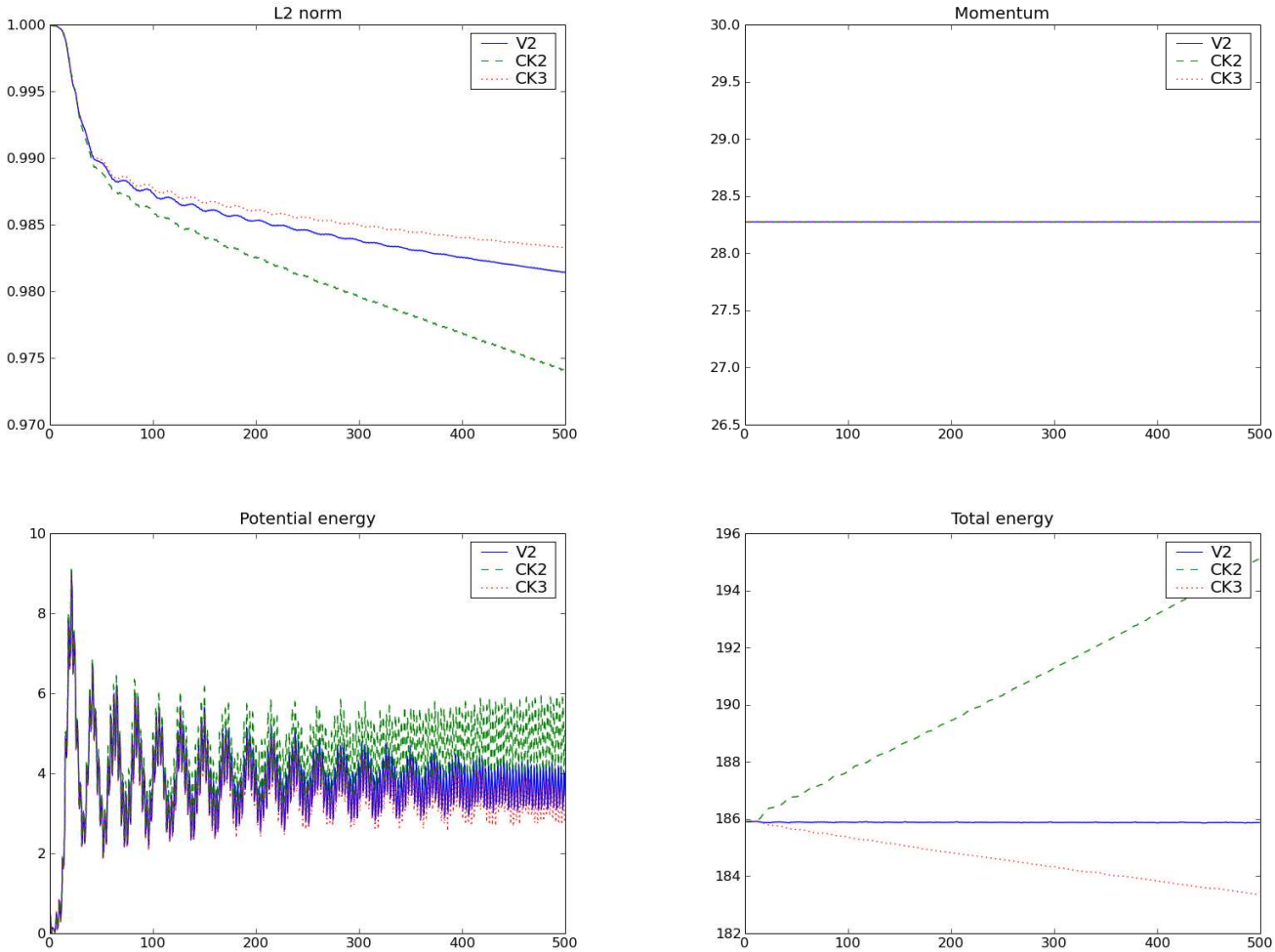

Figure 2: Bump on tail instability.

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