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# Minisuperspace limit of the $AdS_3$ WZNW model

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ABSTRACT: We derive the three-point function of the  $AdS_3$  WZNW model in the minisuperspace limit by Wick rotation from the  $H_3^+$  model. The result is expressed in terms of Clebsch-Gordan coefficients of the Lie algebra  $sl(2, \mathbb{R})$ . We also introduce a covariant basis of functions on  $AdS_3$ , which can be interpreted as bulk-boundary propagators.

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## 1. Introduction

The  $AdS_3$  Wess–Zumino–Novikov–Witten model is interesting in particular due to its string theory applications. A conjecture for the spectrum of this model was proposed by Maldacena and Ooguri [1], but the full solution of the model is still missing. In the sense of the conformal bootstrap, a full solution means the computation of the three-point functions of primary fields on the sphere, and the proof of crossing symmetry of the four-point functions. (Equivalently, the computation of operator product expansions of primary fields, and the proof of their associativity.)

The conjectured spectrum of the  $AdS_3$  WZNW model is fairly complicated, as it contains both discrete and continuous series of representations of the symmetry algebra, and their images under the so-called spectral flow automorphism. On the other hand, as a geometrical space,  $AdS_3$  is related by Wick rotation to the Euclidean space  $H_3^+$ , and the  $AdS_3$  WZNW model is often assumed to be related to the  $H_3^+$  WZNW model. The spectrum of the latter model is much simpler, as it contains only a continuous series of representations, and the  $H_3^+$  model has been fully solved [2, 3]. An additional difficulty of the  $AdS_3$  WZNW model is that the group  $AdS_3$ , which is the universal cover of  $SL(2, \mathbb{R})$ , has no realization as a group of finite-dimensional matrices. It follows that writing a simple basis of functions on  $AdS_3$  is more difficult than in the cases of  $SL(2, \mathbb{R})$  or  $H_3^+$ . Similarly, it is in general more complicated to write functions on the Anti-de Sitter space  $AdS_d$

than on its Euclidean version  $H_d^+$ . Some works like [4] which are purportedly about  $AdS_d$  actually deal with  $H_d^+$ , thereby avoiding this difficulty (and other ones).

The presence of discrete representations and the lack of an obvious basis of functions on  $AdS_3$  are two difficulties of the  $AdS_3$  WZNW model which also affect its minisuperspace limit (also known as the zero-mode approximation), where the model reduces to the study of functions on the  $AdS_3$  space. It is therefore interesting to solve the model in this limit. We will do this by starting from the well-understood minisuperspace  $H_3^+$  model [5] and using Wick rotation. The main object we wish to compute is the minisuperspace analog of the operator product expansion, namely the product of functions on  $AdS_3$ . (Equivalently, the minisuperspace three-point function.)

We will start with a study of certain bases of functions on  $AdS_3$ ,  $SL(2, \mathbb{R})$  and  $H_3^+$  (Section 2). In particular we will construct functions on  $AdS_3$  which transform covariantly under the symmetries, and which can be interpreted as bulk-boundary propagators. Then we will study the Clebsch-Gordan coefficients of the Lie algebra  $sl(2, \mathbb{R})$  (Section 3). Due to the symmetries of the  $AdS_3$  WZNW model in the minisuperspace limit, the products of functions on  $AdS_3$  can be expressed in terms of these coefficients. We will check this after obtaining these products of functions by Wick rotation from  $H_3^+$  (Section 4). In conclusion we will comment on the Wick rotation and on the problem of solving the  $AdS_3$  WZNW model (Section 5).

## 2. $H_3^+$ , $SL(2, \mathbb{R})$ , $AdS_3$ and functions thereon

In this section we will review the geometries of the spaces  $H_3^+$ ,  $SL(2, \mathbb{R})$  and  $AdS_3$ , and consider bases of functions on these spaces. While  $H_3^+$  and  $SL(2, \mathbb{R})$  can be viewed as spaces of two-dimensional matrices,  $AdS_3$  cannot, and this will make the descriptions of functions on  $AdS_3$  more complicated.

### 2.1 Geometry and symmetry groups

Let us start with the group  $SL(2, \mathbb{R})$  of real, size two matrices of determinant one. This group is not simply connected, since the subgroup of the matrices

$$g_\tau \equiv \begin{pmatrix} \cos \tau & \sin \tau \\ -\sin \tau & \cos \tau \end{pmatrix} \quad (2.1)$$

is a non-contractible loop. Therefore, there exists a universal covering group, sometimes called  $\widetilde{SL}(2, \mathbb{R})$ , which we will call  $AdS_3$ . If  $SL(2, \mathbb{R})$  elements are parametrized using three real coordinates  $(\rho, \theta, \tau)$  as

$$g = \begin{pmatrix} \cosh \rho \cos \tau + \sinh \rho \cos \theta & \sinh \rho \sin \theta + \cosh \rho \sin \tau \\ \sinh \rho \sin \theta - \cosh \rho \sin \tau & \cosh \rho \cos \tau - \sinh \rho \cos \theta \end{pmatrix}, \quad (2.2)$$

where  $\theta$  and  $\tau$  are  $2\pi$ -periodic, then  $AdS_3$  is obtained by decompactifying  $\tau$ . Elements of  $AdS_3$  can alternatively be parametrized as doublets  $G = (g, I)$  where  $g \in SL(2, \mathbb{R})$  and  $I$  is the integer part of  $\frac{\tau}{2\pi}$ . Writing  $\frac{\tau}{2\pi} = I + F$ , the group multiplication of  $AdS_3$  can be written as

$$(g, I)(g', I') = (gg', I + I' + F(g) + F(g') - F(gg')). \quad (2.3)$$

The  $U(1)$  subgroup of the matrices  $g_\tau \in SL(2, \mathbb{R})$  decompactifies into an  $\mathbb{R}$  subgroup of elements  $G_\tau \in AdS_3$ , which we parametrize as

$$G_\tau = \exp \tau \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (2.4)$$

The group structures of  $AdS_3$  and  $SL(2, \mathbb{R})$  lead to left and right actions by group multiplication, in the  $AdS_3$  case  $(G_L, G_R) \cdot G = G_L G G_R$ , which will be symmetries of the models under consideration. More precisely, the geometrical symmetry group of  $SL(2, \mathbb{R})$  is  $\frac{SL(2, \mathbb{R}) \times SL(2, \mathbb{R})}{\mathbb{Z}_2}$ , where we must divide by the center  $\mathbb{Z}_2 = \{\text{id}, -\text{id}\}$  of  $SL(2, \mathbb{R})$  as its left and right actions are identical. The geometrical symmetry group of  $AdS_3$  is  $\frac{AdS_3 \times AdS_3}{\mathbb{Z}}$ , where the center of  $AdS_3$  is freely generated by  $(-\text{id}, 0) = (\rho = 0, \theta = 0, \tau = \pi)$  and therefore isomorphic to  $\mathbb{Z}$ . These geometrical symmetry groups are not simply connected; their first fundamental groups are  $\mathbb{Z}^2$  in the case of  $SL(2, \mathbb{R})$  and  $\mathbb{Z}$  in the case of  $AdS_3$ . This is the origin of the spectral flow symmetries of the corresponding WZNW models. (See for instance [1].)

Then  $H_3^+$  is the space of hermitian, size two matrices of determinant one, which can be parametrized using three real coordinates  $(\rho, \theta, \tau)$  as

$$h = \begin{pmatrix} e^\tau \cosh \rho & e^{i\theta} \sinh \rho \\ e^{-i\theta} \sinh \rho & e^{-\tau} \cosh \rho \end{pmatrix}. \quad (2.5)$$

We have chosen identical names  $(\rho, \theta, \tau)$  for the coordinates on  $H_3^+$  and  $AdS_3$ , thereby defining a bijection between these two spaces. This bijection gives rise to a map  $\Phi(\rho, \theta, \tau) \rightarrow \Phi(\rho, \theta, i\tau)$  from the analytic functions on  $H_3^+$  to the analytic functions on  $AdS_3$ , which is called the Wick rotation. Our bijection however does not relate the matrix forms of  $H_3^+$  and  $SL(2, \mathbb{R})$  which we have given. Notice that  $H_3^+$  is not a group, rather a group coset, namely  $SL(2, \mathbb{C})/SU(2)$ . The geometrical symmetry group of  $H_3^+$  is  $\frac{SL(2, \mathbb{C})}{\mathbb{Z}_2}$ , whose elements  $k$  act on  $h \in H_3^+$  by  $k \cdot h = khk^\dagger$ .

## 2.2 Functions: $t$ -bases

In both cases  $SL(2, \mathbb{R})$  and  $H_3^+$ , the existence of a matrix realization allows us to write bases of functions which transform very simply under the symmetries. In the case of  $H_3^+$ , we can indeed introduce the following “ $x$ -basis” of functions  $\Phi_x^j(h)$ , parametrized by their spin  $j$  and isospin  $x$ :

$$\Phi_x^j(h) \equiv \frac{2j+1}{\pi} |(\bar{x}, 1)h \begin{pmatrix} x \\ 1 \end{pmatrix}|^{2j} \Rightarrow \Phi_x^j(khk^\dagger) = |cx + d|^{4j} \Phi_{\frac{ax+b}{cx+d}}^j(h), \quad (2.6)$$

where we denote  $k^\dagger = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ . The set  $\{\Phi_x^j | j \in -\frac{1}{2} + i\mathbb{R}_+, x \in \mathbb{C}\}$  is known to be a basis of functions on  $H_3^+$  [5], in the same sense as  $\{e^{ipq} | p \in \mathbb{R}\}$  is a basis of functions on  $\mathbb{R}$ . Similarly, we can introduce the following “ $t$ -basis” of functions  $\Phi_{t_L, t_R}^{j, \eta}(g)$  on  $SL(2, \mathbb{R})$ :

$$\begin{aligned} \Phi_{t_L, t_R}^{j, \eta}(g) &\equiv \frac{2j+1}{\pi} |(1, -t_L)g \begin{pmatrix} t_R \\ 1 \end{pmatrix}|^{2j} \text{sign}^{2\eta}(1, -t_L)g \begin{pmatrix} t_R \\ 1 \end{pmatrix} \Rightarrow \Phi_{t_L, t_R}^{j, \eta}(g_L^{-1} g g_R) \\ &= |(c_R t_R + d_R)(c_L t_L + d_L)|^{2j} \text{sign}^{2\eta}(c_R t_R + d_R)(c_L t_L + d_L) \Phi_{\frac{a_L t_L + b_L}{c_L t_L + d_L}, \frac{a_R t_R + b_R}{c_R t_R + d_R}}^{j, \eta}(g), \quad (2.7) \end{aligned}$$

where we denote  $g_L = \begin{pmatrix} a_L & b_L \\ c_L & d_L \end{pmatrix}$  and  $g_R = \begin{pmatrix} a_R & b_R \\ c_R & d_R \end{pmatrix}$ . The parity  $\eta \in \{0, \frac{1}{2}\}$  is the same for both actions of  $SL(2, \mathbb{R})$  on itself by multiplications from the left and from the right (see the factor

$\text{sign}^{2\eta}(c_R t_R + d_R)(c_L t_L + d_L)$ ), because the parity characterizes the action of the central subgroup  $\mathbb{Z}_2$ . The set  $\{\Phi_{t_L, t_R}^{j, \eta} | j \in -\frac{1}{2} + i\mathbb{R}_+, (t_L, t_R) \in \mathbb{R}^2, \eta \in \{0, \frac{1}{2}\}\} \cup \{\Phi_{t_L, t_R}^{j, \eta} | j \in -1 - \frac{1}{2}\mathbb{N}, (t_L, t_R) \in \mathbb{R}^2, \eta = j \pmod{1}\}$  is known to be a basis of functions on  $SL(2, \mathbb{R})$ .

In the case of  $AdS_3$ , writing a similar "t-basis" of functions is more complicated. We define  $t$ -basis functions  $\Phi_{t_L, t_R}^{j, \alpha}$  on  $AdS_3$  by the assumption that they transform covariantly under the left and right actions of  $AdS_3$  on itself, in a way which generalizes the transformation property of the  $t$ -basis functions  $\Phi_{t_L, t_R}^{j, \eta}$  on  $SL(2, \mathbb{R})$  eq. (2.7). (The  $AdS_3$  parameter  $\alpha \in [0, 1)$  generalizes the  $SL(2, \mathbb{R})$  parameter  $\eta \in \{0, \frac{1}{2}\}$ .) The appropriate generalization of the transformation property has been written in [6] (Section 4.1); it involves a function  $N(G|t)$  on  $AdS_3 \times \mathbb{R}$  such that  $N(G'G|t) = N(G'|Gt) + N(G|t)$  and  $\forall n \in \mathbb{Z}, N((\text{id}, n)|t) = n$ , where if  $G = (g, I) = (\begin{pmatrix} a & b \\ c & d \end{pmatrix}, I)$  then  $Gt \equiv gt = \frac{at+b}{ct+d}$ . For instance,  $N(G|t)$  can be taken as the number of times  $G't$  crosses infinity as  $G'$  moves from  $(\text{id}, 0)$  to  $G$ , in which case  $N(G|t) \in \mathbb{Z}$ , and  $[G] \equiv N(G|t) - \frac{1}{2}\text{sign}(t + \frac{d}{c}) - \frac{1}{2}$  is a  $t$ -independent integer. Then, the axiom for  $\Phi_{t_L, t_R}^{j, \alpha}$  is

$$\Phi_{t_L, t_R}^{j, \alpha}(G_L^{-1} G G_R) = |(c_R t_R + d_R)(c_L t_L + d_L)|^{2j} e^{2\pi i \alpha (N(G_L|t_L) - N(G_R|t_R))} \Phi_{G_L t_L, G_R t_R}^{j, \alpha}(G) \quad (2.8)$$

This axiom is obeyed by

$$\Phi_{t_L, t_R}^{j, \alpha}(G) = \frac{2j+1}{\pi} e^{2\pi i \alpha n(G|t_L, t_R)} |(1, -t_L)g \begin{pmatrix} t_R \\ 1 \end{pmatrix}|^{2j}, \quad (2.9)$$

provided the function  $n(G|t_L, t_R)$  satisfies

$$n(G_L^{-1} G G_R|t_L, t_R) - n(G|G_L t_L, G_R t_R) = N(G_L|t_L) - N(G_R|t_R). \quad (2.10)$$

This implies that the function  $n(\text{id}|t_L, t_R)$  should satisfy

$$n(\text{id}|t_L, G t_R) - n(\text{id}|G^{-1} t_L, t_R) = N(G^{-1}|t_L) + N(G|t_R), \quad (2.11)$$

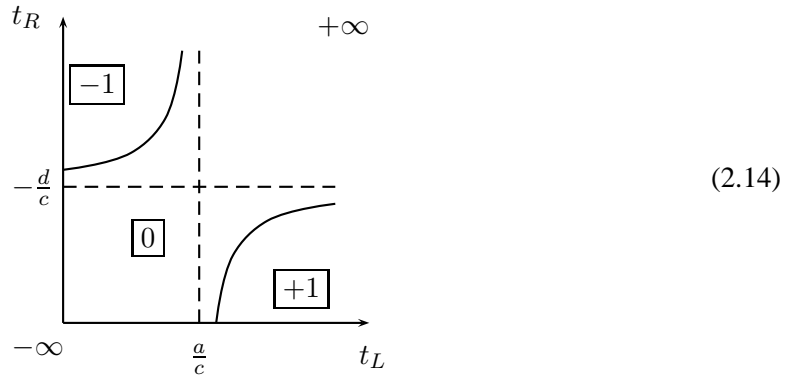
which, using  $Gt = gt = \frac{at+b}{ct+d}$  and the properties of  $N(G|t)$ , amounts to

$$n(\text{id}|t_L, \frac{at_R+b}{ct_R+d}) - n(\text{id}|\frac{dt_L-b}{-ct_L+a}, t_R) = \frac{1}{2}\text{sign}(t_L - \frac{a}{c}) + \frac{1}{2}\text{sign}(t_R + \frac{d}{c}). \quad (2.12)$$

A solution is found to be

$$n(\text{id}|t_L, t_R) = \frac{1}{2}\text{sign}(t_L - t_R), \quad (2.13)$$

then  $n(G|t_L, t_R) - \frac{1}{2}\text{sign}(t_L - t_R)$  is the number of times  $gt_L$  crosses  $t_R$  when  $g$  runs from  $\text{id}$  to  $G$ . Let us now study the behaviour of  $n(G|t_L, t_R)$  as a function of  $t_L, t_R$  for a generic choice of  $G$ . Notice that  $\tilde{n}(t_L, t_R) \equiv n(G|t_L, t_R) + [G] + \frac{1}{2} = \frac{1}{2} [\text{sign}((t_L - \frac{a}{c})(t_R + \frac{d}{c}) + \frac{1}{c^2}) - 1] \text{sign}(t_R + \frac{d}{c})$  takes values  $0, \pm 1$ , and jumps between these values occur on the hyperbola with equation  $(1, -t_L)g \begin{pmatrix} t_R \\ 1 \end{pmatrix} = 0$ . These values and these jumps are shown on the following plot:



We now propose that certain linear combinations of the functions  $\Phi_{t_L, t_R}^{j, \alpha}$  can be interpreted as bulk-boundary propagators. These combinations are

$$\Phi_{(t_L, t_R, N)}^j(G) \equiv \int_0^1 d\alpha e^{-2i\pi\alpha N} \Phi_{t_L, t_R}^{j, \alpha}(G) = \delta_{N, n(G|t_L, t_R)} |(1, -t_L)g\left(\begin{smallmatrix} t_R \\ 1 \end{smallmatrix}\right)|^{2j}. \quad (2.15)$$

We interpret  $(t_L, t_R, N) \in \mathbb{R} \times \mathbb{R} \times \mathbb{Z}$  as coordinates on the boundary of  $AdS_3$ . The action of the symmetry group  $AdS_3 \times AdS_3$  on the boundary is then given by

$$(G_L, G_R) \cdot (t_L, t_R, N) = (g_L t_L, g_R t_R, N - N(G_L|t_L) + N(G_R|t_R)), \quad (2.16)$$

and the behaviour of  $\Phi_{t_L, t_R}^{j, \alpha}$  under the action of  $AdS_3 \times AdS_3$  (2.8) implies the following behaviour of  $\Phi_{(t_L, t_R, N)}^j$ :

$$\Phi_{(t_L, t_R, N)}^j(G_L^{-1} G G_R) = |(c_R t_R + d_R)(c_L t_L + d_L)|^{2j} \Phi_{(G_L, G_R) \cdot (t_L, t_R, N)}^j(G). \quad (2.17)$$

### 2.3 Functions: $m$ -bases

The  $t$ -bases of functions behave simply under symmetry transformations, but  $t$ -bases in  $H_3^+$  and  $AdS_3$  are not related by the Wick rotation. This is because the matrix realizations (2.2) and (2.5) on which the  $t$ -bases are built are themselves not related by the Wick rotation. We will therefore introduce the more complicated “ $m$ -bases” of functions, which are better suited to the Wick rotation. In the case of  $H_3^+$ , the  $m$ -basis functions  $\Phi_{m, \bar{m}}^j(h)$  are defined as

$$\Phi_{m, \bar{m}}^j(h) \equiv \int d^2x x^{-j-1+m} \bar{x}^{-j-1+\bar{m}} \Phi_x^j(h) \quad \text{with } m - \bar{m} \in \mathbb{Z}. \quad (2.18)$$

The numbers  $m, \bar{m}$  can be written in terms of an integer  $n \in \mathbb{Z}$  and a momentum  $p$ , which is imaginary in the  $H_3^+$  model:

$$m = \frac{1}{2}(n + p) \quad , \quad \bar{m} = \frac{1}{2}(-n + p). \quad (2.19)$$

The explicit expression for  $\Phi_{m, \bar{m}}^j(h)$  is found to be

$$\begin{aligned} \Phi_{m, \bar{m}}^j(h) = & -4 \frac{\Gamma(-j + \frac{|n|+p}{2}) \Gamma(-j + \frac{|n|-p}{2})}{\Gamma(|n| + 1) \Gamma(-2j - 1)} e^{-p\tau + in\theta} \sinh^{|n|} \rho \cosh^p \rho \\ & \times F\left(-j + \frac{|n|+p}{2}, j + 1 + \frac{|n|+p}{2}, |n| + 1, -\sinh^2 \rho\right). \end{aligned} \quad (2.20)$$

Notice that this obeys the so-called reflection property

$$\Phi_{m, \bar{m}}^j = R_{m, \bar{m}}^j \Phi_{m, \bar{m}}^{-j-1} \quad , \quad R_{m, \bar{m}}^j = \frac{\Gamma(2j + 1)}{\Gamma(-2j - 1)} \frac{\Gamma(-j + m) \Gamma(-j - \bar{m})}{\Gamma(j + 1 + m) \Gamma(j + 1 - \bar{m})} \quad (2.21)$$

$$= \frac{\Gamma(2j + 1)}{\Gamma(-2j - 1)} \frac{\Gamma(-j + \frac{|n|+p}{2}) \Gamma(-j + \frac{|n|-p}{2})}{\Gamma(j + 1 + \frac{|n|+p}{2}) \Gamma(j + 1 + \frac{|n|-p}{2})}, \quad (2.22)$$

where  $R_{m, \bar{m}}^j = R_{\bar{m}, m}^j$  due to  $n = m - \bar{m} \in \mathbb{Z}$ .

We will use the functions on  $AdS_3$  obtained from the above functions  $\Phi_{m,\bar{m}}^j(h)$  by the Wick rotation  $\tau \rightarrow i\tau$ . In order for the resulting functions to be delta-function normalizable, we now need to assume the momentum  $p$  to be real, instead of imaginary in the  $H_3^+$  case. We do not introduce a new notation for the resulting functions on  $AdS_3$ , but still call them  $\Phi_{m,\bar{m}}^j(G)$  or  $\Phi_{m,\bar{m}}^j$ .

In contrast to  $t$ -basis functions,  $m$ -basis functions on  $AdS_3$  do not transform simply under the action of the  $AdS_3 \times AdS_3$  symmetry group. However, they do transform simply under the action of the  $\mathbb{R} \times \mathbb{R}$  subgroup made of pairs  $(G_{\tau_L}, G_{\tau_R})$ , where  $G_\tau$  was defined by eq. (2.4):

$$\Phi_{m,\bar{m}}^{j,\alpha}(G_{\tau_L} G G_{\tau_R}) = e^{-2i(m\tau_L + \bar{m}\tau_R)} \Phi_{m,\bar{m}}^{j,\alpha}(G). \quad (2.23)$$

Notice that the identity  $G_\pi G G_{-\pi} = G$  implies  $m - \bar{m} \in \mathbb{Z}$ . (In the particular case of  $SL(2, \mathbb{R})$ , we have the additional identity  $g_{2\pi} = \text{id}$ , which implies  $m, \bar{m} \in \frac{1}{2}\mathbb{Z}$ .) Introducing

$$\alpha \in [0, 1) \quad \text{such that} \quad m, \bar{m} \in \alpha + \mathbb{Z}, \quad (2.24)$$

this parameter  $\alpha$  is identical to the parameter  $\alpha$  of the  $t$ -basis functions  $\Phi_{t_L, t_R}^{j,\alpha}$  (2.9). We will look for a relation of the type

$$\Phi_{m,\bar{m}}^j = c^{j,\alpha} \int_{\mathbb{R}} dt_L (1 + t_L^2)^{-j-1} e^{i\pi m} \left( \frac{1-it_L}{1+it_L} \right)^m \int_{\mathbb{R}} dt_R (1 + t_R^2)^{-j-1} \left( \frac{1+it_R}{1-it_R} \right)^{\bar{m}} \Phi_{t_L, t_R}^{j,\alpha} \quad (2.25)$$

where  $c^{j,\alpha}$  is a normalization factor. We can check that the right-hand side of this relation obeys the transformation property (2.23), thanks to the behaviour eq. (2.8) of  $\Phi_{t_L, t_R}^{j,\alpha}$ . To see this it is useful to notice that the integrand in eq. (2.25) is continuous through  $t_L = \infty$  and  $t_R = \infty$ , as can be deduced from the behaviour of the phase factor  $e^{2i\pi\alpha n(G|_{t_L, t_R})}$  of  $\Phi_{t_L, t_R}^{j,\alpha}$ , which is depicted in the diagram (2.14). This makes it possible to perform translations of the variables  $\varphi_L, \varphi_R$  such that  $t_{L,R} = \tan \frac{1}{2}\varphi_{L,R}$ . The normalization factor  $c^{j,\alpha}$  is easily computed in the limit  $\rho \rightarrow \infty$ , where the dependences of the integrand on  $t_L$  and  $t_R$  factorize. We find

$$c^{j,\alpha} = \frac{4^{2j} \sin \pi 2j}{\sin \pi(j - \alpha) \sin \pi(j + \alpha)}. \quad (2.26)$$

### 3. Representations and Clebsch-Gordan coefficients

#### 3.1 Representations of $sl(2, \mathbb{R})$

The minisuperspace limit of the spectrum of the  $AdS_3$  WZNW model is the space of delta-function normalizable functions on  $AdS_3$ . It is subject to the action of the geometrical symmetry group  $\frac{AdS_3 \times AdS_3}{\mathbb{Z}}$ , and therefore of its Lie algebra  $sl(2, \mathbb{R}) \times sl(2, \mathbb{R})$ . Three types of unitary representations of  $sl(2, \mathbb{R})$  appear in the minisuperspace spectrum: continuous representations, and two series of discrete representations. Continuous representations  $C^{j,\alpha}$  are parametrized by a spin  $j$  and a number  $\alpha \in [0, 1)$  such that  $m \in \alpha + \mathbb{Z}$ . Discrete representations  $D^{j,\pm}$  are parametrized by a spin  $j \in (-\frac{1}{2}, \infty)$ , and their states obey  $m \in \pm(j + 1 + \mathbb{N})$ . All these representations of  $sl(2, \mathbb{R})$  extend to representations of the group  $AdS_3$ . However, only representations with  $m \in \frac{1}{2}\mathbb{Z}$ , namely  $C^{j,\alpha}$  with  $\alpha \in \frac{1}{2}\mathbb{Z}$  and  $D^{j,\pm}$  with  $j \in \frac{1}{2}\mathbb{N}$ , extend to representations of the group  $SL(2, \mathbb{R})$ .



More precisely, given the  $sl(2, \mathbb{R})$  algebra with generators  $J^3, J^\pm$  and relations  $[J^3, J^\pm] = \pm J^\pm$ ,  $[J^+, J^-] = -2J^3$ , the spin  $j$  is defined by  $(J^3)^2 - \frac{1}{2}(J^+ J^- + J^- J^+) = j(j+1)$ , and the states  $|m\rangle$  are such that

$$J^3|m\rangle = m|m\rangle \quad , \quad J^+|m\rangle = (m+j+1)|m+1\rangle \quad , \quad J^-|m\rangle = (m-j-1)|m-1\rangle . \quad (3.1)$$

These conventions are incompatible with the unit normalization of the states (which would mean  $\langle m|m'\rangle = \delta_{m,m'}$ ), however they will turn out to agree with the behaviour of our functions  $\Phi_{m,\bar{m}}^j$  eq. (2.20).

The tensor product laws for  $sl(2, \mathbb{R})$  representations are well-known. They are equivalent to knowing the three-point invariants, which we schematically depict here in the cases when they do not vanish:

$$\begin{array}{cccc}
 \begin{array}{c} | \\ \diagdown \quad \diagup \\ \text{2} \\ C \otimes C \otimes C \end{array} & , & \begin{array}{c} | \\ \diagdown \quad \diagup \\ D^- \otimes C \otimes C \end{array} & , & \begin{array}{c} | \\ \diagdown \quad \diagup \\ D^- \otimes D^+ \otimes C \end{array} & , & \begin{array}{c} | \\ \downarrow \\ \diagdown \quad \diagup \\ D^+ \otimes D^+ \otimes D^- \end{array} . & (3.2)
 \end{array}$$

For instance, the first diagram means that any continuous representation  $C^{j,\alpha}$  appears twice in the tensor product  $C^{j_1,\alpha_1} \otimes C^{j_2,\alpha_2}$  of two continuous representations. (The  $m$ -conservation rule  $\alpha = \alpha_1 + \alpha_2 \pmod{1}$  is implicitly assumed.) The fourth diagram means that  $D^{j,+} \subset D^{j_1,+} \otimes D^{j_2,+}$ . (The rule  $j \in j_1 + j_2 + 1 + \mathbb{N}$  is implicitly assumed.) The fourth diagram also means that  $D^{j,-}$  may appear once in  $D^{j_1,-} \otimes D^{j_2,+}$ . (This happens if  $j \in j_2 - j_1 - 1 - \mathbb{N}$ .) We omit the diagrams obtained by reverting the arrows in the second and fourth diagrams, namely  $D^+ \otimes C \otimes C$  and  $D^- \otimes D^- \otimes D^+$ .

### 3.2 Clebsch-Gordan coefficients: $m$ basis

We will rederive the tensor product rules by studying the Clebsch-Gordan coefficients. These coefficients are the three-point invariants, viewed as functions  $C(j_1, j_2, j_3 | m_1, m_2, m_3)$  subject to the equations

$$\sum_{i=1}^3 (m_i + j_i + 1) C(m_i + 1) = \sum_{i=1}^3 (m_i - j_i - 1) C(m_i - 1) = \sum_{i=1}^3 m_i C = 0 . \quad (3.3)$$

It is of course possible to prove a priori that these equations are obeyed by the three-point function  $\left\langle \prod_{i=1}^3 \Phi_{m_i, \bar{m}_i}^{j_i} \right\rangle \equiv \int_{AdS_3} dG \prod_{i=1}^3 \Phi_{m_i, \bar{m}_i}^{j_i}(G)$ . To do this, we would introduce a realization of the Lie algebra  $sl(2, \mathbb{R})$  as first-order differential operators  $D^a$  wrt  $\rho, \theta, \tau$ , such that  $D^+ \Phi_{m,\bar{m}}^j(G) = (m+j+1)\Phi_{m,\bar{m}}^j(G)$ ,  $D^- \Phi_{m,\bar{m}}^j(G) = (m-j-1)\Phi_{m,\bar{m}}^j(G)$  and  $D^3 \Phi_{m,\bar{m}}^j(G) = m\Phi_{m,\bar{m}}^j(G)$ . Then eq. (3.3) would follow from the identity  $\left\langle \prod_{i=1}^3 \Phi_{m_i, \bar{m}_i}^{j_i}(G) \right\rangle = \left\langle \prod_{i=1}^3 \Phi_{m_i, \bar{m}_i}^{j_i}(G_L G) \right\rangle$ . We however abstain from doing this, as we will later explicitly compute the three-point function  $\left\langle \prod_{i=1}^3 \Phi_{m_i, \bar{m}_i}^{j_i} \right\rangle$  and write it in terms of solutions of the equation (3.3).

Given three irreducible representations of  $sl(2, \mathbb{R})$ , there exist zero, one or two linearly independent solutions of the equation (3.3). In the case of three continuous representations, the momenta  $(m_1, m_2)$  belong to a two-dimensional lattice of the type  $\prod_{i=1}^2 (\alpha_i + \mathbb{Z})$  (with of course

$m_3 = -m_1 - m_2$ ), and the coefficients  $m_i \pm (j_i + 1)$  never vanish. In this situation, a solution of eq. (3.3) is determined once the values of  $C$  at two neighbouring points of the lattice are given. In the case when at least one representation is discrete, say  $m_1 \in j_1 + 1 + \mathbb{N}$ , the lattice becomes semi-infinite in one direction, and a solution is determined once the value of  $C$  at one point is given.

Let us introduce the function

$$G \begin{pmatrix} a & b & c \\ e & f & \end{pmatrix} \equiv \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(e)\Gamma(f)} {}_3F_2 \left( \begin{matrix} a & b & c \\ e & f \end{matrix} \middle| 1 \right) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\Gamma(a+n)\Gamma(b+n)\Gamma(c+n)}{\Gamma(e+n)\Gamma(f+n)}, \quad (3.4)$$

where the sum converges provided  $a + b + c - e - f < 0$ , and the poles of  $G$  are the same as those of  $\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(e+f-a-b-c)$ . Knowing the identity

$$(a - e + 1)G \begin{pmatrix} a & b & c \\ e & f & \end{pmatrix} + (b - f)G \begin{pmatrix} a + 1 & b & c \\ e & f + 1 & \end{pmatrix} + (c - 1)G \begin{pmatrix} a & b & c - 1 \\ e - 1 & f & \end{pmatrix} = 0, \quad (3.5)$$

we can use this function for writing a solutions of eq. (3.3):

$$C = \delta(m_1 + m_2 + m_3) G \begin{pmatrix} -j_2 + m_2 & -j_3 - m_3 & -j_{23}^1 \\ 1 + j_1 - j_3 + m_2 & 1 + j_1 - j_2 - m_3 & \end{pmatrix} \equiv \delta(m_1 + m_2 + m_3) g^{23} \quad (3.6)$$

which is well-defined provided  $2 + j_{123} > 0$ , where we use the notations  $j_{123} = j_1 + j_2 + j_3$  and  $j_{23}^1 \equiv j_2 + j_3 - j_1$ . Of course five other solutions of the type  $g^{ab}$  with  $a \neq b \in \{1, 2, 3\}$  can be obtained by permutations of indices. These solutions are not linearly independent, as can be shown with the help of the identity

$$s(b)s(c-a)G \begin{pmatrix} a & b & c \\ e & f & \end{pmatrix} = s(e-a)s(f-a)G \begin{pmatrix} a & a-e+1 & a-f+1 \\ a-b+1 & a-c+1 & \end{pmatrix} \\ - s(c-e)s(c-f)G \begin{pmatrix} c & c-e+1 & c-f+1 \\ c-a+1 & c-b+1 & \end{pmatrix}, \quad (3.7)$$

where  $s(x) \equiv \sin \pi x$ , and we will also use  $c(x) \equiv \cos \pi x$ . Thus we obtain

$$\begin{pmatrix} g^{21} \\ g^{12} \end{pmatrix} = M_{13} \begin{pmatrix} g^{32} \\ g^{23} \end{pmatrix}, \quad M_{13} = \frac{1}{s(j_{12}^3)} \begin{pmatrix} \frac{s(j_2+m_2)s(j_3-m_3)}{s(j_1+m_1)} & \frac{s(j_3+m_3)s(j_3-j_1+m_2)}{s(j_1+m_1)} \\ \frac{s(j_3-m_3)s(j_3-j_1-m_2)}{s(j_1-m_1)} & \frac{s(j_2-m_2)s(j_3+m_3)}{s(j_1-m_1)} \end{pmatrix}. \quad (3.8)$$

Together with the other identities obtained by permuting the indices, this shows that at most two of the solutions  $g^{ab}$  are linearly independent.

Due to our convention  $j \in (-\frac{1}{2}, \infty)$  for discrete representations, we have  $\Re j \geq -\frac{1}{2}$  for all representations of interest. This ensures that the sum in eq. (3.4) converges, so that  $g^{ab}$  is well-defined provided the summand is finite, which occurs unless  $\Gamma(-j_a + m_a)\Gamma(-j_b - m_b)\Gamma(-j_{ab}^c)$  has a pole.

**Case  $C \otimes C \otimes C$ .** In this case, two given solutions say  $g^{23}, g^{32}$  are linearly independent, and they provide a basis of the two-dimensional space of invariants.

**Case  $D^- \otimes C \otimes C$ .** We assume for example  $m_2 = -j_2 - 1 - \ell$  with  $\ell \in \mathbb{N}$ . Some relations of the type of eq. (3.8) simplify, and we find

$$g^{21} = (-1)^\ell \frac{s(j_3 + m_3)}{s(j_1 + m_1)} g^{23} = -\frac{s(j_{13}^2)}{s(2j_2)} g^{31} = -(-1)^\ell \frac{s(j_3 + m_3)}{s(j_1 + m_1)} \frac{s(j_{13}^2)}{s(2j_2)} g^{13}. \quad (3.9)$$

Since the space of invariants is one-dimensional, the two remaining functions  $g^{12}$  and  $g^{32}$  must also be proportional to the other four. However, this proportionality relation is not very simple, as can be seen in the case of the highest-weight state  $\ell = 0$  when  $g^{12}$  and  $g^{32}$  fail to become expressible as products of  $\Gamma$ -functions, in contrast to the other four solutions.

**Case  $D^+ \otimes C \otimes C$ .** The situation is completely analogous to the previous case. We assume for example  $m_2 = j_2 + 1 + \ell$  with  $\ell \in \mathbb{N}$  and find

$$g^{12} = (-1)^\ell \frac{s(j_3 - m_3)}{s(j_1 - m_1)} g^{32} = -\frac{s(j_{13}^2)}{s(2j_2)} g^{13} = -(-1)^\ell \frac{s(j_3 - m_3)}{s(j_1 - m_1)} \frac{s(j_{13}^2)}{s(2j_2)} g^{31}. \quad (3.10)$$

**Case  $D^- \otimes D^- \otimes C$ .** We expect no invariants to exist in this case. Let us check this, assuming for example  $m_2 \in -j_2 - 1 - \ell_2$  and  $m_3 \in -j_3 - 1 - \ell_3$  with  $\ell_2, \ell_3 \in \mathbb{N}$ . Equation (3.9) implies two incompatible relations between  $g^{21}$  and  $g^{31}$ , which must therefore both vanish. An apparent paradox comes from the non-vanishing of  $g^{12}, g^{32}, g^{23}, g^{13}$ . However, these functions do not provide solutions to eq. (3.3), because they become infinite at  $\ell_2 = -1$  or  $\ell_3 = -1$ . For instance, if  $C(m_1, -j_2, m_3) = \infty$ , then  $(m_1 + j_1 + 1)C(m_1 + 1, -j_2 - 1, m_3) + (-j_2 + j_2)C(m_1, -j_2, m_3) + (m_3 + j_3 + 1)C(m_1, -j_2 - 1, m_3 + 1) = 0$  may have an unwanted nonvanishing second term. Therefore, the analysis of  $g^{ab}$  agrees with the representation-theoretic expectations that no invariant exists.

**Case  $D^+ \otimes D^- \otimes C$ .** We assume for example  $m_2 = -j_2 - 1 - \ell_2$  and  $m_3 = j_3 + 1 + \ell_3$  with  $\ell_2, \ell_3 \in \mathbb{N}$ . We find the relations

$$g^{31} = g^{12} = -\frac{s(2j_3)}{s(j_{12}^3)} g^{13} = -\frac{s(2j_2)}{s(j_{13}^2)} g^{21} = \frac{s(2j_2)s(2j_3)}{s(j_{13}^2)s(j_{12}^3)} g^{23}. \quad (3.11)$$

The functions  $g^{13}, g^{21}, g^{23}$  stay finite for any values  $\ell_2, \ell_3 \in \mathbb{Z}$ , and therefore provide three proportional invariants. The functions  $g^{31}$  and  $g^{12}$  become infinite if  $\ell_3 < 0$  and  $\ell_2 < 0$  respectively, so that it is not a priori clear that they provide invariants. That they actually do is guaranteed by the above relations.

**Case  $D^- \otimes D^- \otimes D^+$ .** We assume for example  $m_1 = j_1 + 1 + \ell_1$ ,  $m_2 \in -j_2 - 1 - \ell_2$  and  $m_3 \in -j_3 - 1 - \ell_3$ , with  $\ell_1, \ell_2, \ell_3 \in \mathbb{N}$ . Noticing  $s(2j_2)s(2j_3) = s(j_{13}^2)s(j_{12}^3)$ , we find the relations

$$g^{12} = g^{23} = -\frac{s(2j_3)}{s(j_{12}^3)} g^{13} = -\frac{s(2j_3)}{s(j_{12}^3)} g^{32}. \quad (3.12)$$

These four proportional functions provide the invariant in this case. In particular,  $g^{12}$  and  $g^{13}$  no longer become infinite at  $\ell_2 = -1$  or  $\ell_3 = -1$  respectively, as happened in the case  $D^- \otimes D^- \otimes C$ . The remaining two functions  $g^{21}, g^{31}$  vanish, as they already did in the case  $D^- \otimes D^- \otimes C$ . Notice that the selection rule  $j_1 \in j_2 + j_3 + 1 + \mathbb{N}$  manifests itself as  $g^{23}$  becoming infinite if  $j_1 \in j_2 + j_3 - \mathbb{N}$ , due to a series of poles which correspond to those of  $\Gamma(-j_{23}^1)$ .

### 3.3 Clebsch-Gordan coefficients: $t$ -basis

Our  $m$ -basis invariants  $g^{ab}$  are not symmetric under permutations of the indices, but the equation (3.3) which they solve is. In the  $t$ -basis, we will now show that there exist natural permutation-symmetric invariants, and we will relate them to combinations of the  $g^{ab}$  invariants. A similar analysis was already performed for the Clebsch-Gordan coefficients of  $SO(2, 1) = \frac{SL(2, \mathbb{R})}{\mathbb{Z}_2}$ , in the articles [7, 8]. The representations of  $SO(2, 1)$  correspond to representations of  $sl(2, \mathbb{R})$  such that  $m \in \mathbb{Z}$ , or in other words  $\alpha = 0$ . We will perform the generalization to arbitrary values of  $\alpha$ .

In the  $t$ -basis,  $AdS_3$  invariants should be solutions of

$$C(\{t_i\}) = C\left(\left\{\frac{at_i+b}{ct_i+d}\right\}\right) \prod_{i=1}^3 |ct_i + d|^{2j_i} e^{-2i\pi\alpha_i N(G|t_i)}, \quad (3.13)$$

for any  $AdS_3$  element  $G$  whose projection onto  $SL(2, \mathbb{R})$  is  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ . Solutions exist provided  $\alpha_1 + \alpha_2 + \alpha_3 \in \mathbb{Z}$ , and we will assume  $\alpha_1 + \alpha_2 + \alpha_3 = 0$ . The solutions are [9]

$$C(\{t_i\}) = |t_{12}|^{j_{12}^3} |t_{23}|^{j_{23}^1} |t_{31}|^{j_{31}^2} e^{i\pi(\alpha_{12} \text{sign} t_{12} + \alpha_{23} \text{sign} t_{23} + \alpha_{31} \text{sign} t_{31})}, \quad (3.14)$$

where  $\alpha_{12}, \alpha_{23}, \alpha_{31}$  should obey the equation  $\alpha_{12} - \alpha_{23} = \alpha_2 \pmod{\mathbb{Z}}$  and the two other equations obtained by even permutations thereof. The solutions to such equations are

$$\alpha_{ab} = \frac{1}{3}(\alpha_b - \alpha_a) + \alpha_0, \quad (3.15)$$

where  $\alpha_0$  is an arbitrary constant. (In  $SL(2, \mathbb{R})$  we have  $\alpha_a = \eta_a \in \frac{1}{2}\mathbb{Z}$  and it is more convenient to adopt the convention  $\alpha_{ab} = \eta_a + \eta_b + \alpha_0$ . Equivalently, we can use the above formula provided we assume  $\alpha_a \in \frac{3}{2}\mathbb{Z}$ .) Let us now apply the change of basis (2.25) to  $C(\{t_i\})$ . After the change of integration variables  $t_i = \tan \frac{\varphi_i}{2}$  the  $m$ -basis version of  $C(\{t_i\})$  is

$$C(\{m_i\}) = \prod_{i=1}^3 \left[ \int_{-\pi}^{\pi} d\varphi_i e^{im_i\varphi_i} \right] |\sin \frac{1}{2}\varphi_{12}|^{j_{12}^3} |\sin \frac{1}{2}\varphi_{23}|^{j_{23}^1} |\sin \frac{1}{2}\varphi_{31}|^{j_{31}^2} \\ \times e^{i\pi(\alpha_{12} \text{sign} \sin \frac{1}{2}\varphi_{12} + \alpha_{23} \text{sign} \sin \frac{1}{2}\varphi_{23} + \alpha_{31} \text{sign} \sin \frac{1}{2}\varphi_{31})}, \quad (3.16)$$

where the integrand has the same values at  $\varphi_i = \pi$  and  $\varphi_i = -\pi$ . This integral can be explicitly evaluated by generalizing the computations of [7], and in particular using the formula, valid for  $\varphi \in (-\pi, \pi)$ :

$$\left| \sin \frac{\varphi}{2} \right|^{-2a} e^{i\pi\alpha \text{sign} \sin \frac{1}{2}\varphi} = s(a + \alpha) e^{-i\pi\alpha} 2^{2a} \Gamma(1 - 2a) \sum_{n=-\infty}^{\infty} e^{i(n+\alpha)\varphi} \frac{\Gamma(a + n + \alpha)}{\Gamma(1 - a + n + \alpha)} \quad (3.17)$$

Thus, we find

$$C(\{m_i\}) = -\delta(\sum m_i) 2^{-j_{123}} \frac{1}{\pi^2} \Gamma(1 + j_{12}^3) \Gamma(1 + j_{23}^1) \Gamma(1 + j_{31}^2) e^{-i\pi(\alpha_{12} + \alpha_{23} + \alpha_{31})} g^{\alpha_0} \quad (3.18)$$

where we introduce the invariants

$$g^{\alpha_0} = \pi^2 s\left(\frac{1}{2}j_{12}^3 - \alpha_{12}\right) s\left(\frac{1}{2}j_{23}^1 - \alpha_{23}\right) s\left(\frac{1}{2}j_{31}^2 - \alpha_{31}\right) \\ \times \sum_{n \in \mathbb{Z}} \frac{\Gamma(-\frac{1}{2}j_{12}^3 + \alpha_{12} + n)}{\Gamma(1 + \frac{1}{2}j_{12}^3 + \alpha_{12} + n)} \frac{\Gamma(-\frac{1}{2}j_{23}^1 + m_1 + \alpha_{12} + n)}{\Gamma(1 + \frac{1}{2}j_{23}^1 + m_1 + \alpha_{12} + n)} \frac{\Gamma(-\frac{1}{2}j_{31}^2 - m_2 + \alpha_{12} + n)}{\Gamma(1 + \frac{1}{2}j_{31}^2 - m_2 + \alpha_{12} + n)}, \quad (3.19)$$

which can be expressed in terms of the invariants  $g^{ab}$  (3.6) as

$$g^{\alpha_0} = s(\frac{1}{2}j_{13}^2 - \alpha_{31})s(j_3 - \alpha_3)s(j_1 + \alpha_1)g^{31} + s(\frac{1}{2}j_{13}^2 + \alpha_{31})s(j_1 - \alpha_1)s(j_3 + \alpha_3)g^{13} \quad (3.20)$$

In this formula,  $g^{\alpha_0}$  depends on the parameter  $\alpha_0$  only through  $\alpha_{31}$  eq. (3.15). There are two particularly interesting special values of  $\alpha_0$ , namely 0 and  $\frac{1}{2}$ . In the case  $\alpha_0 = 0$  then  $C(\{t_i\})$  and  $g^0$  are invariant under permutations. In the case  $\alpha_0 = \frac{1}{2}$  then  $C(\{t_i\})$  and  $g^{\frac{1}{2}}$  are odd under permutations i.e. invariant up to a sign. When the three involved representations are continuous,  $g^0$  and  $g^{\frac{1}{2}}$  can serve as a basis of the two-dimensional space of invariants. Notice that in the case of  $SL(2, \mathbb{R})$  only these two values of  $\alpha_0$  are possible.

#### 4. Products of functions

In the conformal bootstrap approach to the  $AdS_3$  WZNW model, all correlation functions can in principle be constructed from the knowledge of three objects: the spectrum, the two-point correlation functions on a sphere, and the operator product expansions – or equivalently the three-point correlation functions on a sphere. We will now determine these objects in the minisuperspace limit. We first recall their definitions. Given  $n$  functions  $\Phi^i(G)$  on  $AdS_3$ , the corresponding correlation function is  $\langle \prod_{i=1}^n \Phi^i \rangle \equiv \int dG \prod_{i=1}^n \Phi^i(G)$  where  $dG$  is the invariant measure. If  $\{\Phi^i\}_{i \in S}$  form an orthogonal basis of the spectrum (that is  $\langle \Phi^i \Phi^j \rangle = 0$  if  $i \neq j$ ), the product of functions is schematically  $\Phi^1 \Phi^2 = \sum_{i \in S} \frac{\langle \Phi^1 \Phi^2 \Phi^i \rangle}{\langle \Phi^i \Phi^i \rangle} \Phi^i$ . This product is obviously associative and commutative.

The functions  $\Phi_{m, \bar{m}}^j(G)$  on  $AdS_3$  are related to corresponding functions  $\Phi_{m, \bar{m}}^j(h)$  on  $H_3^+$  by a Wick rotation, and therefore their correlation functions can be deduced from  $H_3^+$  correlation functions by that Wick rotation. This will involve some subtleties, because the discrete representations which appear in the minisuperspace spectrum on  $AdS_3$  are absent in  $H_3^+$ . But let us first review the products of functions on  $H_3^+$ .

##### 4.1 Products of functions on $H_3^+$

The minisuperspace spectrum of  $H_3^+$  is generated by the functions

$$\{\Phi_{n,p}^j(h) | j \in -\frac{1}{2} + i\mathbb{R}, n \in \mathbb{Z}, p \in i\mathbb{R}\}. \quad (4.1)$$

The correlation functions  $\langle \prod_{i=1}^n \Phi_{n_i, p_i}^{j_i} \rangle \equiv \int dh \prod_{i=1}^n \Phi_{n_i, p_i}^{j_i}(h)$  are obtained by integrating products of such functions with respect to the invariant measure  $dh = \sinh 2\rho d\rho d\theta d\tau$ . The two-point functions can be computed from the expression (2.20) of  $\Phi_{n,p}^j(h)$ :

$$\langle \Phi_{n,p}^j \Phi_{n',p'}^{j'} \rangle = 128\pi^2 \delta_{n+n',0} \delta(p+p') [\delta(j+j'+1) + R_{n,p}^j \delta(j-j')] , \quad (4.2)$$

where the reflection coefficient  $R_{n,p}^j$  was defined in eq. (2.21). A similar direct computation of the three-point function seems complicated. Instead, we will make use of the known  $x$ -basis three-point function [5]

$$\left\langle \prod_{i=1}^3 \Phi_{x_i}^{j_i} \right\rangle = C(j_1, j_2, j_3) |x_{12}|^{2j_{12}^3} |x_{23}|^{2j_{23}^1} |x_{31}|^{2j_{31}^2} , \quad (4.3)$$

$$C(j_1, j_2, j_3) \equiv \pi^{-3} \Gamma(-j_{123} - 1) \frac{\Gamma(-j_{12}^3) \Gamma(-j_{23}^1) \Gamma(-j_{31}^2)}{\Gamma(-2j_1 - 1) \Gamma(-2j_2 - 1) \Gamma(-2j_3 - 1)} . \quad (4.4)$$

The transformation to the  $m$ -basis (2.18) can be performed thanks to an integral formula of Fukuda and Hosomichi [10]. The result can be written in terms of the Clebsch-Gordan coefficients  $g^{ab}$  (3.6):

$$\left\langle \prod_{i=1}^3 \Phi_{m_i, \bar{m}_i}^{j_i} \right\rangle = C(j_1, j_2, j_3) \delta^{(2)}(\sum m_i) K^2 s(j_{13}^2)^2 s(j_{23}^1)^2 \times \left[ g^{23} \bar{g}^{31} + \frac{s(2j_2)}{s(j_{13}^2)} g^{23} \bar{g}^{32} + \frac{s(2j_1)}{s(j_{23}^1)} g^{13} \bar{g}^{31} + g^{13} \bar{g}^{32} \right], \quad (4.5)$$

where  $\bar{g}^{ab}$  denotes  $g^{ab}$  with  $m_i$  replaced by  $\bar{m}_i$ , and we introduced the factor

$$K \equiv \frac{1}{\pi^2} \Gamma(1 + j_{12}^3) \Gamma(1 + j_{13}^2) \Gamma(1 + j_{23}^1). \quad (4.6)$$

It can be checked that the two- and three-point functions have the behaviour under reflection which is expected from the behaviour of  $\Phi_{m, \bar{m}}^j$  (2.21). Now, using the three-point function, products of functions on  $H_3^+$  can be written as

$$\Phi_{m_1, \bar{m}_1}^{j_1} \Phi_{m_2, \bar{m}_2}^{j_2} = \frac{1}{256\pi^2} \int_{-\frac{1}{2} + i\mathbb{R}} dj_3 \left\langle \prod_{i=1}^3 \Phi_{m_i, \bar{m}_i}^{j_i} \right\rangle' \frac{1}{R_{m_3, \bar{m}_3}^{j_3}} \Phi_{-m_3, -\bar{m}_3}^{j_3}, \quad (4.7)$$

where we use the notation  $\left\langle \prod_{i=1}^3 \Phi_{m_i, \bar{m}_i}^{j_i} \right\rangle = \delta^{(2)}(m_1 + m_2 + m_3) \left\langle \prod_{i=1}^3 \Phi_{m_i, \bar{m}_i}^{j_i} \right\rangle'$ .

The invariance of the three-point function (4.5) under permutations of the indices is not manifest, but can be checked using linear relations between the  $g^{ab}$  such as eq. (3.8). It seems that a reasonably simple, manifestly permutation-symmetric expression exists only in the case  $m_i, \bar{m}_i \in \eta_i + \mathbb{Z}$  with  $\eta_i \in \{0, \frac{1}{2}\}$ , which corresponds to functions on  $SL(2, \mathbb{R})$ . In this case, we can use the invariants  $g^0, g^{\frac{1}{2}}$  (3.20), and we find

$$\left\langle \prod_{i=1}^3 \Phi_{m_i, \bar{m}_i}^{j_i} \right\rangle = -\frac{1}{2\pi^5} C(j_1, j_2, j_3) \delta_{\sum m_i, 0} \delta_{\sum \bar{m}_i, 0} \frac{\gamma(1 + j_{12}^3) \gamma(1 + j_{13}^2) \gamma(1 + j_{31}^1)}{\prod_{i=1}^3 s(j_i + \eta_i)} \times \sum_{\epsilon \in \{0, \frac{1}{2}\}} \frac{s(\frac{1}{2} j_{123} + \epsilon)}{s(\frac{1}{2} j_{12}^3 + \eta_3 + \epsilon) s(\frac{1}{2} j_{23}^1 + \eta_1 + \epsilon) s(\frac{1}{2} j_{31}^2 + \eta_2 + \epsilon)} g^\epsilon \bar{g}^\epsilon, \quad (4.8)$$

where we use  $\gamma(x) \equiv \frac{\Gamma(x)}{\Gamma(1-x)}$ . From this, we can reconstruct the  $t$ -basis three-point function

$$\left\langle \prod_{i=1}^3 \Phi_{t_i^L, t_i^R}^{j_i, \eta_i} \right\rangle = \frac{2^{2j_{123}}}{2\pi^4} C(j_1, j_2, j_3) \prod_{i=1}^3 c(j_i - \eta_i) \times |t_{12}^L t_{12}^R|^{j_{12}^3} |t_{23}^L t_{23}^R|^{j_{23}^1} |t_{31}^L t_{31}^R|^{j_{31}^2} e^{i\pi[\eta_3(\text{signt}_{12}^L + \text{signt}_{12}^R) + \eta_1(\text{signt}_{23}^L + \text{signt}_{23}^R) + \eta_2(\text{signt}_{31}^L + \text{signt}_{31}^R)]} \times \sum_{\epsilon \in \{0, \frac{1}{2}\}} s(\frac{1}{2} j_{123} - \epsilon) c(\frac{1}{2} j_{12}^3 + \eta_3 + \epsilon) c(\frac{1}{2} j_{23}^1 + \eta_1 + \epsilon) c(\frac{1}{2} j_{31}^2 + \eta_2 + \epsilon) (\text{signt}_{12}^L t_{12}^R t_{23}^L t_{23}^R t_{31}^L t_{31}^R)^{2\epsilon}. \quad (4.9)$$

Comparing this formula to the  $H_3^+$  three-point function in the  $x$ -basis eq. (4.3), we obtain a confirmation of the lack of a simple relation between the  $x$ -basis in  $H_3^+$  and the  $t$ -basis in  $SL(2, \mathbb{R})$  or  $AdS_3$ .

In the more general case of functions on  $AdS_3$ , the three-point function can still be expressed in terms of the invariants  $g^0$  and  $g^{\frac{1}{2}}$ , but the formula is more complicated than eq. (4.8) and in particular the ‘‘mixed’’ terms  $g^0 \bar{g}^{\frac{1}{2}}$  and  $g^{\frac{1}{2}} \bar{g}^0$  are present. Their absence in the case of  $SL(2, \mathbb{R})$  can be attributed to the exterior automorphism  $\omega$  of  $SL(2, \mathbb{R})$ , namely  $\omega(g) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ , which is such that  $\Phi_{t_L, t_R}^{j, \eta}(\omega(g)) = (-1)^{2\eta} \Phi_{-t_L, -t_R}^{j, \eta}(g)$ . In the case of  $AdS_3$  this action still exists and can be expressed as  $\omega(\rho, \theta, \tau) = (\rho, -\theta, -\tau)$ . But it does not act simply on the function  $\Phi_{t_L, t_R}^{j, \alpha}(G)$ .<sup>1</sup>

## 4.2 Products of functions on $AdS_3$

Functions  $\Phi_{m, \bar{m}}^j$  on  $AdS_3$  are obtained from the corresponding functions on  $H_3^+$  by performing the Wick rotation  $\tau \rightarrow i\tau$  and continuing  $p = m + \bar{m}$  from  $i\mathbb{R}$  to  $\mathbb{R}$ . If we do not modify the value of the spin  $j \in -\frac{1}{2} + i\mathbb{R}$ , this yields functions transforming in the continuous representations of  $AdS_3 \times AdS_3$ , namely  $\Phi_{m, \bar{m}}^j \in C^{j, \alpha} \otimes C^{j, \alpha}$  where  $m, \bar{m} \in \alpha + \mathbb{Z}$ . We may in addition obtain functions transforming in the discrete representation by continuing  $j$  to real values such that  $m, \bar{m} \in \pm j \pm \mathbb{Z}$ . More precisely, functions  $\Phi_{m, \bar{m}}^j \in D^{j, \pm} \otimes D^{j, \pm}$  correspond to

$$\begin{aligned} j \in (-\tfrac{1}{2}, \infty) & \quad \text{and} \quad m, \bar{m} \in \pm(j + 1 + \mathbb{N}) \Leftrightarrow -j - 1 - \tfrac{1}{2}|n| \pm \tfrac{1}{2}p \in \mathbb{N}, 10 \\ \text{or} \quad j \in (-\infty, -\tfrac{1}{2}) & \quad \text{and} \quad m, \bar{m} \in \pm(-j + \mathbb{N}) \Leftrightarrow j - \tfrac{1}{2}|n| \pm \tfrac{1}{2}p \in \mathbb{N}. \end{aligned} \quad (4.11)$$

These two possibilities are related by the reflection  $j \rightarrow -j - 1$  and they are equivalent. We will only consider the first possibility, because our invariants  $g^{ab}$  (3.6) are well-defined for  $\Re j \geq -\frac{1}{2}$ . The set of these discrete and continuous functions is known to generate the space of functions on  $AdS_3$ ; in particular this set is closed under products, as we will check below.

We will now derive the products of functions on  $AdS_3$  by continuing the products of functions on  $H_3^+$  (4.7) to the relevant values of spins  $j$  and momenta  $p$ . We will examine various cases, according to the nature – discrete or continuous – of the fields  $\Phi_{m_1, \bar{m}_1}^{j_1}$  and  $\Phi_{m_2, \bar{m}_2}^{j_2}$ . For example, the case when  $j_1 \in -\frac{1}{2} + i\mathbb{R}$  and  $m_2, \bar{m}_2 \in j_2 + 1 + \mathbb{Z}$  will be denoted  $C \times D^+$ . We will check that the terms which appear in a given product are those which are allowed by the well-known tensor product laws for  $sl(2, \mathbb{R})$  representations (3.2).

**Case  $C \times C$ .** We should continue  $p_1, p_2, p_3$  from imaginary to real values in eq. (4.7). This is problematic only when the integrand, viewed as a function of  $j_3$ , has poles which cross the integration line. Such  $p_i$ -dependent poles of the integrand may come from either of its three factors. The poles coming from the second factor  $\frac{1}{R_{m_3, \bar{m}_3}^{j_3}}$  are easily seen from eq. (2.21), and the poles from the other factors are obtained from these by the reflection<sup>2</sup>  $j_3 \rightarrow -j_3 - 1$ . All these poles fall

<sup>1</sup>Let us give the behaviour of certain objects of section 2.2:  $N(\omega(G)|t) = -N(G| -t)$ ,  $[\omega(G)] = -[G] - 1$  and  $n(\omega(G)|t_L, t_R) = -n(G| -t_L, -t_R)$ .

<sup>2</sup>It is also possible to study the  $m$ -dependent poles of  $\left\langle \prod_{i=1}^3 \Phi_{m_i, \bar{m}_i}^{j_i} \right\rangle$  directly from the formula (4.5). For example,  $\bar{g}^{32}$  has poles at  $j_2 + \bar{m}_2 \in \mathbb{N}$ . But the coefficient of  $\bar{g}^{32}$  is a combination of  $g^{13}$  and  $g^{23}$  of the type  $s(2j_2)g^{23} + s(j_{13}^2)g^{13} = \frac{s(j_3 - m_3)}{s(j_1 - m_1)} s(j_2 + m_2)g^{32}$ , where we used  $j_2 + m_2 \in \mathbb{Z}$  (which follows from  $j_2 + \bar{m}_2 \in \mathbb{N}$ ) and eq. (3.8). This vanishes, unless  $g^{32}$  itself has a pole. This shows that  $\left\langle \prod_{i=1}^3 \Phi_{m_i, \bar{m}_i}^{j_i} \right\rangle$  has simple poles when both  $m_2, \bar{m}_2$  belong to  $-j_2 + \mathbb{N}$ , but not when only  $\bar{m}_2$  does.

on the four dashed half-lines in the following diagram, which depicts the  $j_3$  complex plane. The two half-lines on the left correspond to the poles of  $\frac{1}{R_{m_3, \bar{m}_3}^{j_3}}$ .

$$\begin{array}{ccc}
j_3 \in -\frac{1}{2} + i\mathbb{R} & & \\
\text{-----} \circ & & \circ \text{-----} \\
-1 - \frac{1}{2}(|n_3| - p_3) & & \frac{1}{2}(|n_3| + p_3) \\
\text{-----} & & \text{-----} \\
-1 - \frac{1}{2}(|n_3| + p_3) & & \frac{1}{2}(|n_3| - p_3) \\
\text{-----} \circ & & \circ \text{-----} \\
& & \uparrow
\end{array} \tag{4.12}$$

This diagram assumes  $p_3 \in i\mathbb{R}$ . When  $p_3$  moves to real values, let us consider the poles from the left which may cross the integration line and end up on the right with  $j_3 \in (-\frac{1}{2}, \infty)$ . Such poles belong to the (possibly empty) sets  $j_3 \in -1 - \frac{1}{2}|n_3| \pm \frac{1}{2}p_3 - \mathbb{N} \cap (-\frac{1}{2}, \infty)$ . Therefore, according to eq. (4.10), they correspond to functions  $\Phi_{m_3, \bar{m}_3}^{j_3}$  in the  $D^{j_3, \pm}$  representations. We deduce the formula for the products of two ‘‘continuous’’ functions on  $AdS_3$ :

$$\begin{aligned}
\Phi_{m_1, \bar{m}_1}^{j_1} \Phi_{m_2, \bar{m}_2}^{j_2} &= \frac{1}{256\pi^2} \int_{-\frac{1}{2} + i\mathbb{R}} dj_3 \left\langle \prod_{i=1}^3 \Phi_{m_i, \bar{m}_i}^{j_i} \right\rangle' \frac{1}{R_{m_3, \bar{m}_3}^{j_3}} \Phi_{-m_3, -\bar{m}_3}^{j_3} \\
&+ \frac{2}{256\pi^2} \sum_{j_3 \in -1 - \frac{1}{2}|n_3| \pm \frac{1}{2}p_3 - \mathbb{N} \cap (-\frac{1}{2}, \infty)} \left\langle \prod_{i=1}^3 \Phi_{m_i, \bar{m}_i}^{j_i} \right\rangle' 2\pi i \text{Res} \frac{1}{R_{m_3, \bar{m}_3}^{j_3}} \Phi_{-m_3, -\bar{m}_3}^{j_3}, \tag{4.13}
\end{aligned}$$

where the factor 2 in the discrete term is due to the contribution of the poles with  $j_3 \in (-\infty, -\frac{1}{2})$ . Notice that the general expression (4.5) for the three-point function  $\left\langle \prod_{i=1}^3 \Phi_{m_i, \bar{m}_i}^{j_i} \right\rangle'$  simplifies in the case  $j_3 \in -1 - \frac{1}{2}|n_3| \pm \frac{1}{2}p_3 - \mathbb{N} \cap (-\frac{1}{2}, \infty)$  due to formulas of the type of eqs. (3.9) and (3.10). Examples of simplified expressions are:

$$\left\langle \prod_{i=1}^3 \Phi_{m_i, \bar{m}_i}^{j_i} \right\rangle'_{D^{j_3, +}} = -C(j_1, j_2, j_3) K^2 s(j_{12}^3) s(j_{23}^1) s(j_{31}^2) s(j_{123}) g^{21} \bar{g}^{13}, \tag{4.14}$$

$$g^{21} \bar{g}^{13} = \bar{g}^{21} g^{13} = g^{12} \bar{g}^{23} = \bar{g}^{12} g^{23},$$

$$\left\langle \prod_{i=1}^3 \Phi_{m_i, \bar{m}_i}^{j_i} \right\rangle'_{D^{j_3, -}} = -C(j_1, j_2, j_3) K^2 s(j_{12}^3) s(j_{23}^1) s(j_{31}^2) s(j_{123}) g^{12} \bar{g}^{31}, \tag{4.15}$$

$$g^{12} \bar{g}^{31} = \bar{g}^{12} g^{31} = g^{21} \bar{g}^{32} = \bar{g}^{21} g^{32}.$$

**Case  $D^+ \times C$ .** After moving  $p_i$  to real values as in the previous case  $C \times C$ , we should move  $j_1$  to  $-1 - \frac{1}{2}|n_1| + \frac{1}{2}p_1 - \mathbb{N} \cap (-\frac{1}{2}, \infty)$ . Let us show that no further poles cross the integration line in eq. (4.13) during this operation. We are looking for possible  $j_1$ -dependent poles in  $\left\langle \prod_{i=1}^3 \Phi_{m_i, \bar{m}_i}^{j_i} \right\rangle'$ , viewed as a function of  $j_3$ . We use formulas of the type of eq. (3.10) to obtain

$$\left\langle \prod_{i=1}^3 \Phi_{m_i, \bar{m}_i}^{j_i} \right\rangle'_{D^{j_1, +}} = C(j_1, j_2, j_3) K^2 s(j_{12}^3) s(j_{23}^1) s(j_{31}^2) s(j_{123}) \frac{s(j_{23}^1)}{s(2j_1)} g^{23} \bar{g}^{32}. \tag{4.16}$$



Potential poles come from factors  $\Gamma(1 + j_{23}^1)$  in  $K$  (4.6) and  $\Gamma(-1 - j_{123})\Gamma(-j_{12}^3)\Gamma(-j_{31}^2)$  in  $C(j_1, j_2, j_3)$  (4.4), but they are all cancelled by appropriate sin factors. On the other hand, the poles from the factors  $\Gamma(1 + j_{12}^3)\Gamma(1 + j_{31}^2)$  in  $K$ , from the factor  $\Gamma(-j_{23}^1)$  in  $C(j_1, j_2, j_3)$ , and the poles of  $\Gamma(-j_{23}^1)$  which come from  $g^{23}$  and  $\bar{g}^{32}$ , cannot be reached because  $\Re j_1 \geq -\frac{1}{2}$ .

This shows that the formula (4.13) still holds for products of functions in  $D^+ \times C$ . Of course, simplified expressions for  $\left\langle \prod_{i=1}^3 \Phi_{m_i, \bar{m}_i}^{j_i} \right\rangle'$  can be used for both continuous and discrete values of  $j_3$ . We can moreover check that terms corresponding to  $D^{j_3, +}$  actually vanish. This is due to

$$\left\langle \prod_{i=1}^3 \Phi_{m_i, \bar{m}_i}^{j_i} \right\rangle'_{D^{j_1, +}, D^{j_3, +}} = 0, \quad (4.17)$$

which follows from eq. (4.14) if we notice that  $g^{21} = g^{23} = 0$  in this case due to eq. (3.10). This equation holds for generic values of  $j_2$ , in particular the values  $j_2 \in -\frac{1}{2} + i\mathbb{R}$  which correspond to  $C^{j_2}$ .

**Case  $D^+ \times D^+$ .** The formula (4.13) for the product of functions still holds, but the continuous term  $\int_{-\frac{1}{2} + i\mathbb{R}} dj_3 \cdots$  vanishes due to eq. (4.17). Terms corresponding to  $D^{j_3, +}$  representations also vanish by the same argument, but the equation  $\left\langle \prod_{i=1}^3 \Phi_{m_i, \bar{m}_i}^{j_i} \right\rangle'_{D^{j_1, +}, D^{j_2, +}} = 0$  may fail if a  $D^{j_3, -}$  representation is present, due to poles from the factor  $K^2$  in eq. (4.5). To analyze this matter it is convenient to start with the identity

$$\left\langle \prod_{i=1}^3 \Phi_{m_i, \bar{m}_i}^{j_i} \right\rangle'_{D^{j_2, +}, D^{j_3, -}} = -C(j_1, j_2, j_3) K^2 \frac{s(j_{12}^3)^2 s(j_{31}^2)^2 s(j_{23}^1) s(j_{123})}{s(2j_2) s(2j_3)} g^{13} \bar{g}^{13}. \quad (4.18)$$

We then send  $j_1$  to values corresponding to discrete representations  $D^{j_1, +}$ . Due to momentum conservation we must have  $j_1 \in j_2 - j_3 + \mathbb{Z}$ . If  $j_1 \in j_2 - j_3 - 1 - \mathbb{N}$  then a double pole from  $K^2$  cancels the double zero from  $s(j_{13}^2)^2$  and the result is finite. If  $j_1 \in j_2 - j_3 + \mathbb{N}$  then the simple pole from  $C(j_1, j_2, j_3)$  does not cancel the double zero, and the result vanishes. The formula (4.13) therefore reduces to

$$\Phi_{m_1, \bar{m}_1}^{j_1} \Phi_{m_2, \bar{m}_2}^{j_2} = \frac{2}{256\pi^2} \sum_{j_3 \in j_1 + j_2 + 1 + \mathbb{N}} \left\langle \prod_{i=1}^3 \Phi_{m_i, \bar{m}_i}^{j_i} \right\rangle' 2\pi i \text{Res} \frac{1}{R_{m_3, \bar{m}_3}^{j_3}} \Phi_{-m_3, -\bar{m}_3}^{j_3}. \quad (4.19)$$

**Case  $D^+ \times D^-$ .** The formula (4.13) for the product of functions still holds, and the analysis of eq. (4.18) in the previous case determines which terms may vanish. Nonvanishing  $D^{j_3, +}$  terms occur for  $j_3 \in j_2 - j_1 - 1 - \mathbb{N} \cap (-\frac{1}{2}, \infty)$  and nonvanishing  $D^{j_3, -}$  terms occur for  $j_3 \in j_1 - j_2 - 1 - \mathbb{N} \cap (-\frac{1}{2}, \infty)$ . Depending on the values of  $j_1, j_2$  we can have either  $D^{j_3, +}$  terms, or  $D^{j_3, -}$  terms, or no discrete terms at all in the case  $|j_1 - j_2| \leq \frac{1}{2}$ .

## 5. Conclusion

At the level of symmetry algebras, the Wick rotation from  $H_3^+$  to  $AdS_3$  amounts to a map from  $sl(2, \mathbb{C})$  to  $sl(2, \mathbb{R}) \times sl(2, \mathbb{R})$ , which can be viewed as two different real forms of the same algebra

$sl(2, \mathbb{C})^{\mathbb{C}} = sl(2, \mathbb{C}) \times sl(2, \mathbb{C})$ . In particular, the Wick rotation maps the continuous representation  $C^j$  of  $sl(2, \mathbb{C})$  to the representation  $\int_0^1 d\alpha C^{j,\alpha} \otimes C^{j,\alpha}$  of  $sl(2, \mathbb{R}) \times sl(2, \mathbb{R})$ . The fact that such an irreducible representation is mapped to a reducible one implies that the symmetry constraints are weaker in  $AdS_3$  than in  $H_3^+$ . Namely, the  $H_3^+$  three-point function should be  $\langle \prod_{i=1}^3 \Phi_{m_i, \bar{m}_i}^{j_i} \rangle = C(j_1, j_2, j_3) H(j_i, m_i, \bar{m}_i)$  where  $H$  is determined by  $sl(2, \mathbb{C})$  symmetry; while the  $AdS_3$  three-point function can in principle be  $\langle \prod_{i=1}^3 \Phi_{m_i, \bar{m}_i}^{j_i} \rangle = C'(j_1, j_2, j_3 | \alpha_1, \alpha_2, \alpha_3) H'(j_i, m_i, \bar{m}_i)$  with  $m_i, \bar{m}_i \in \alpha_i + \mathbb{Z}$ , where the  $sl(2, \mathbb{R}) \times sl(2, \mathbb{R})$  symmetry determines  $H'$  but not the  $\alpha_i$ -dependence.

For the full  $H_3^+$  and  $AdS_3$  WZNW models (and not just their minisuperspace limits), the assumption that these models are related by Wick rotation therefore determines part of the  $AdS_3$  structure constants (analogs of  $C'$ ) in terms of the  $H_3^+$  conformal blocks (analogs of  $H$ ). This assumption is therefore rather nontrivial and it should be carefully justified. The best justification may come a posteriori, if an ansatz for the  $AdS_3$  three-point function derived by Wick rotation can be shown to obey crossing symmetry. Such questions did not arise in the minisuperspace limit, as the bases of functions  $\Phi_{m, \bar{m}}^j$  on  $H_3^+$  and  $AdS_3$  are related by Wick rotation by definition, and crossing symmetry amounts to the associativity of the product of functions on these spaces. But proving crossing symmetry – or equivalently the associativity of the operator product expansion – certainly is the most important and difficult task in solving the  $AdS_3$  WZNW model.

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