Minisuperspace limit of the AdS3 WZNW model

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Preprint typeset in JHEP style - HYPER VERSION LPTA/09-100

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ABSTRACT: We derive the three-point function of the AdS_3 WZNW model in the minisuperspace limit by Wick rotation from the H_3^+ model. The result is expressed in terms of Clebsch-Gordan coefficients of the Lie algebra $s\ell(2,\mathbb{R})$. We also introduce a covariant basis of functions on AdS_3, which can be interpreted as bulk-boundary propagators.

Contents

1. Introduction

The AdS_3 Wess–Zumino–Novikov–Witten model is interesting in particular due to its string theory applications. A conjecture for the spectrum of this model was proposed by Maldacena and Ooguri [1], but the full solution of the model is still missing. In the sense of the conformal bootstrap, a full solution means the computation of the three-point functions of primary fields on the sphere, and the proof of crossing symmetry of the four-point functions. (Equivalently, the computation of operator product expansions of primary fields, and the proof of their associativity.)

The conjectured spectrum of the AdS_3 WZNW model is fairly complicated, as it contains both discrete and continuous series of representations of the symmetry algebra, and their images under the so-called spectral flow automorphism. On the other hand, as a geometrical space, AdS_3 is related by Wick rotation to the Euclidean space H_3^+, and the AdS_3 WZNW model is often assumed to be related to the H_3^+ WZNW model. The spectrum of the latter model is much simpler, as it contains only a continuous series of representations, and the H_3^+ model has been fully solved [2, 3]. An additional difficulty of the AdS_3 WZNW model is that the group AdS_3, which is the universal cover of $SL(2,\mathbb{R})$, has no realization as a group of finite-dimensional matrices. It follows that writing a simple basis of functions on AdS_3 is more difficult than in the cases of $SL(2,\mathbb{R})$ or H_3^+. Similarly, it is in general more complicated to write functions on the Anti-de Sitter space AdS_d

than on its Euclidean version H_d^+. Some works like [4] which are purportedly about AdS_d actually deal with H_d^+, thereby avoiding this difficulty (and other ones).

The presence of discrete representations and the lack of an obvious basis of functions on AdS_3 are two difficulties of the AdS_3 WZNW model which also affect its minisuperspace limit (also known as the zero-mode approximation), where the model reduces to the study of functions on the AdS_3 space. It is therefore interesting to solve the model in this limit. We will do this by starting from the well-understood minisuperspace H_3^+ model [5] and using Wick rotation. The main object we wish to compute is the minisuperspace analog of the operator product expansion, namely the product of functions on AdS_3. (Equivalently, the minisuperspace three-point function.)

We will start with a study of certain bases of functions on AdS_3, $SL(2,\mathbb{R})$ and H_3^+ (Section 2). In particular we will construct functions on AdS_3 which transform covariantly under the symmetries, and which can be interpreted as bulk-boundary propagators. Then we will study the Clebsch-Gordan coefficients of the Lie algebra $s\ell(2,\mathbb{R})$ (Section 3). Due to the symmetries of the AdS_3 WZNW model in the minisuperspace limit, the products of functions on AdS_3 can be expressed in terms of these coefficients. We will check this after obtaining these products of functions by Wick rotation from H_3^+ (Section 4). In conclusion we will comment on the Wick rotation and on the problem of solving the AdS_3 WZNW model (Section 5).

2. H_3^+, $SL(2,\mathbb{R})$, AdS_3 and functions thereon

In this section we will review the geometries of the spaces H_3^+, $SL(2,\mathbb{R})$ and AdS_3, and consider bases of functions on these spaces. While H_3^+ and $SL(2,\mathbb{R})$ can be viewed as spaces of two-dimensional matrices, AdS_3 cannot, and this will make the descriptions of functions on AdS_3 more complicated.

2.1 Geometry and symmetry groups

Let us start with the group $SL(2,\mathbb{R})$ of real, size two matrices of determinant one. This group is not simply connected, since the subgroup of the matrices

$$g_\tau \equiv \left(\begin{smallmatrix} \cos\tau & \sin\tau \\ -\sin\tau & \cos\tau \end{smallmatrix} \right) \tag{2.1}$$

is a non-contractible loop. Therefore, there exists a universal covering group, sometimes called $\widetilde{SL}(2,\mathbb{R})$, which we will call AdS_3. If $SL(2,\mathbb{R})$ elements are parametrized using three real coordinates (ρ,θ,τ) as

$$g = \begin{pmatrix} \cosh\rho\cos\tau + \sinh\rho\cos\theta & \sinh\rho\sin\theta + \cosh\rho\sin\tau \\ \sinh\rho\sin\theta - \cosh\rho\sin\tau & \cosh\rho\cos\tau - \sinh\rho\cos\theta \end{pmatrix} , \tag{2.2}$$

where θ and τ are 2π-periodic, then AdS_3 is obtained by decompactifying τ. Elements of AdS_3 can alternatively be parametrized as doublets $G = (g, I)$ where $g \in SL(2,\mathbb{R})$ and I is the integer part of $\frac{\tau}{2\pi}$. Writing $\frac{\tau}{2\pi} = I + F$, the group multiplication of AdS_3 can be written as

$$(g, I)(g', I') = (gg', I + I' + F(g) + F(g') - F(gg')) . \tag{2.3}$$

The $U(1)$ subgroup of the matrices $g_\tau \in SL(2,\mathbb{R})$ decompactifies into an $\mathbb{R}$ subgroup of elements $G_\tau \in AdS_3$, which we parametrize as

$$G_\tau = \exp \tau \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right) \ . \tag{2.4}$$

The group structures of AdS_3 and $SL(2,\mathbb{R})$ lead to left and right actions by group multiplication, in the AdS_3 case $(G_L, G_R) \cdot G = G_L G G_R$, which will be symmetries of the models under consideration. More precisely, the geometrical symmetry group of $SL(2,\mathbb{R})$ is $\frac{SL(2,\mathbb{R})\times SL(2,\mathbb{R})}{\mathbb{Z}_2}$, where we must divide by the center $\mathbb{Z}_2 = \{\mathrm{id}, -\mathrm{id}\}$ of $SL(2,\mathbb{R})$ as its left and right actions are identical. The geometrical symmetry group of AdS_3 is $\frac{AdS_3\times AdS_3}{\mathbb{Z}}$, where the center of AdS_3 is freely generated by $(-\mathrm{id}, 0) = (\rho = 0, \theta = 0, \tau = \pi)$ and therefore isomorphic to $\mathbb{Z}$. These geometrical symmetry groups are not simply connected; their first fundamental groups are $\mathbb{Z}^2$ in the case of $SL(2,\mathbb{R})$ and $\mathbb{Z}$ in the case of AdS_3. This is the origin of the spectral flow symmetries of the corresponding WZNW models. (See for instance [1].)

Then H_3^+ is the space of hermitian, size two matrices of determinant one, which can be parametrized using three real coordinates (ρ, θ, τ) as

$$h = \begin{pmatrix} e^\tau \cosh\rho & e^{i\theta} \sinh\rho \\ e^{-i\theta} \sinh\rho & e^{-\tau} \cosh\rho \end{pmatrix} \ . \tag{2.5}$$

We have chosen identical names (ρ, θ, τ) for the coordinates on H_3^+ and AdS_3, thereby defining a bijection between these two spaces. This bijection gives rise to a map $\Phi(\rho, \theta, \tau) \to \Phi(\rho, \theta, i\tau)$ from the analytic functions on H_3^+ to the analytic functions on AdS_3, which is called the Wick rotation. Our bijection however does not relate the matrix forms of H_3^+ and $SL(2,\mathbb{R})$ which we have given. Notice that H_3^+ is not a group, rather a group coset, namely $SL(2,\mathbb{C})/SU(2)$. The geometrical symmetry group of H_3^+ is $\frac{SL(2,\mathbb{C})}{\mathbb{Z}_2}$, whose elements k act on $h \in H_3^+$ by $k \cdot h = khk^\dagger$.

2.2 Functions: t-bases

In both cases $SL(2,\mathbb{R})$ and H_3^+, the existence of a matrix realization allows us to write bases of functions which transform very simply under the symmetries. In the case of H_3^+, we can indeed introduce the following "x-basis" of functions $\Phi_x^j(h)$, parametrized by their spin j and isospin x:

$$\Phi_x^j(h) \equiv \frac{2j+1}{\pi} \left|(\bar{x}, 1) h \left(\begin{smallmatrix} x \\ 1 \end{smallmatrix} \right)\right|^{2j} \ \Rightarrow \ \Phi_x^j(khk^\dagger) = |cx+d|^{4j} \Phi^j_{\frac{ax+b}{cx+d}}(h) \ , \tag{2.6}$$

where we denote $k^\dagger = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)$. The set $\{\Phi_x^j | j \in -\frac{1}{2} + i\mathbb{R}_+, x \in \mathbb{C}\}$ is known to be a basis of functions on H_3^+ [5], in the same sense as $\{e^{ipq} | p \in \mathbb{R}\}$ is a basis of functions on $\mathbb{R}$. Similarly, we can introduce the following "t-basis" of functions $\Phi^{j,\eta}_{t_L,t_R}(g)$ on $SL(2,\mathbb{R})$:

$$\begin{aligned} \Phi^{j,\eta}_{t_L,t_R}(g) &\equiv \frac{2j+1}{\pi} \left|(1, -t_L) g \left(\begin{smallmatrix} t_R \\ 1 \end{smallmatrix} \right)\right|^{2j} \mathrm{sign}^{2\eta}(1, -t_L) g \left(\begin{smallmatrix} t_R \\ 1 \end{smallmatrix} \right) \ \Rightarrow \ \Phi^{j,\eta}_{t_L,t_R}(g_L^{-1} g g_R) \\ &= |(c_R t_R + d_R)(c_L t_L + d_L)|^{2j} \, \mathrm{sign}^{2\eta}(c_R t_R + d_R)(c_L t_L + d_L) \, \Phi^{j,\eta}_{\frac{a_L t_L + b_L}{c_L t_L + d_L}, \frac{a_R t_R + b_R}{c_R t_R + d_R}}(g) \ , \end{aligned} \tag{2.7}$$

where we denote $g_L = \left(\begin{smallmatrix} a_L & b_L \\ c_L & d_L \end{smallmatrix} \right)$ and $g_R = \left(\begin{smallmatrix} a_R & b_R \\ c_R & d_R \end{smallmatrix} \right)$. The parity $\eta \in \{0, \frac{1}{2}\}$ is the same for both actions of $SL(2,\mathbb{R})$ on itself by multiplications from the left and from the right (see the factor

$\mathrm{sign}^{2\eta}(c_R t_R + d_R)(c_L t_L + d_L)$), because the parity characterizes the action of the central subgroup $\mathbb{Z}_2$. The set $\{\Phi^{j,\eta}_{t_L,t_R} | j \in -\frac{1}{2} + i\mathbb{R}_+, (t_L,t_R) \in \mathbb{R}^2, \eta \in \{0, \frac{1}{2}\}\} \cup \{\{\Phi^{j,\eta}_{t_L,t_R} | j \in -1 - \frac{1}{2}\mathbb{N}, (t_L,t_R) \in \mathbb{R}^2, \eta = j \mod 1\}$ is known to be a basis of functions on $SL(2,\mathbb{R})$.

In the case of AdS_3, writing a similar "t-basis" of functions is more complicated. We define t-basis functions $\Phi^{j,\alpha}_{t_L,t_R}$ on AdS_3 by the assumption that they transform covariantly under the left and right actions of AdS_3 on itself, in a way which generalizes the transformation property of the t-basis functions $\Phi^{j,\eta}_{t_L,t_R}$ on $SL(2,\mathbb{R})$ eq. (2.7). (The AdS_3 parameter $\alpha \in [0,1)$ generalizes the $SL(2,\mathbb{R})$ parameter $\eta \in \{0, \frac{1}{2}\}$.) The appropriate generalization of the transformation property has been written in [6] (Section 4.1); it involves a function $N(G|t)$ on $AdS_3 \times \mathbb{R}$ such that $N(G'G|t) = N(G'|Gt) + N(G|t)$ and $\forall n \in \mathbb{Z},\ N((\mathrm{id}, n)|t) = n$, where if $G = (g, I) = (\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), I)$ then $Gt \equiv gt = \frac{at+b}{ct+d}$. For instance, $N(G|t)$ can be taken as the number of times $G't$ crosses infinity as G' moves from $(\mathrm{id}, 0)$ to G, in which case $N(G|t) \in \mathbb{Z}$, and $[G] \equiv N(G|t) - \frac{1}{2}\mathrm{sign}(t + \frac{d}{c}) - \frac{1}{2}$ is a t-independent integer. Then, the axiom for $\Phi^{j,\alpha}_{t_L,t_R}$ is

$$\Phi^{j,\alpha}_{t_L,t_R}(G_L^{-1} G G_R) = |(c_R t_R + d_R)(c_L t_L + d_L)|^{2j}\, e^{2\pi i \alpha (N(G_L|t_L) - N(G_R|t_R))} \Phi^{j,\alpha}_{G_L t_L, G_R t_R}(G) \tag{2.8}$$

This axiom is obeyed by

$$\Phi^{j,\alpha}_{t_L,t_R}(G) = \frac{2j+1}{\pi} e^{2\pi i \alpha n(G|t_L,t_R)} \left|(1, -t_L) g \left(\begin{smallmatrix} t_R \\ 1 \end{smallmatrix}\right)\right|^{2j} , \tag{2.9}$$

provided the function $n(G|t_L, t_R)$ satisfies

$$n(G_L^{-1} G G_R | t_L, t_R) - n(G | G_L t_L, G_R t_R) = N(G_L|t_L) - N(G_R|t_R) . \tag{2.10}$$

This implies that the function $n(\mathrm{id}|t_L, t_R)$ should satisfy

$$n(\mathrm{id}|t_L, Gt_R) - n(\mathrm{id}|G^{-1}t_L, t_R) = N(G^{-1}|t_L) + N(G|t_R) , \tag{2.11}$$

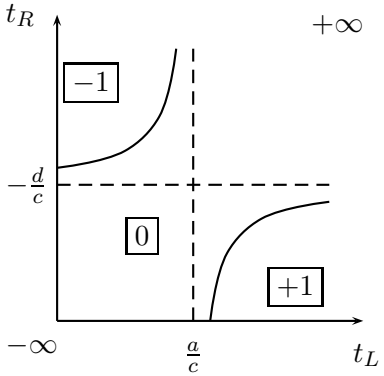
which, using $Gt = gt = \frac{at+b}{ct+d}$ and the properties of $N(G|t)$, amounts to

$$n(\mathrm{id}|t_L, \tfrac{at_R+b}{ct_R+d}) - n(\mathrm{id}|\tfrac{dt_L-b}{-ct_L+a}, t_R) = \tfrac{1}{2}\mathrm{sign}(t_L - \tfrac{a}{c}) + \tfrac{1}{2}\mathrm{sign}(t_R + \tfrac{d}{c}) . \tag{2.12}$$

A solution is found to be

$$n(\mathrm{id}|t_L, t_R) = \tfrac{1}{2}\mathrm{sign}(t_L - t_R) , \tag{2.13}$$

then $n(G|t_L, t_R) - \frac{1}{2}\mathrm{sign}(t_L - t_R)$ is the number of times gt_L crosses t_R when g runs from id to G. Let us now study the behaviour of $n(G|t_L, t_R)$ as a function of t_L, t_R for a generic choice of G. Notice that $\tilde{n}(t_L, t_R) \equiv n(G|t_L, t_R) + [G] + \frac{1}{2} = \frac{1}{2}\left[\mathrm{sign}\left((t_L - \frac{a}{c})(t_R + \frac{d}{c}) + \frac{1}{c^2}\right) - 1\right] \mathrm{sign}(t_R + \frac{d}{c})$ takes values $0, \pm 1$, and jumps between these values occur on the hyperbola with equation $(1, -t_L) g \left(\begin{smallmatrix} t_R \\ 1 \end{smallmatrix}\right) = 0$. These values and these jumps are shown on the following plot:



(2.14)

We now propose that certain linear combinations of the functions $\Phi^{j,\alpha}_{t_L,t_R}$ can be interpreted as bulk-boundary propagators. These combinations are

$$\Phi^j_{(t_L,t_R,N)}(G) \equiv \int_0^1 d\alpha\, e^{-2i\pi\alpha N}\Phi^{j,\alpha}_{t_L,t_R}(G) = \delta_{N,n(G|t_L,t_R)}\left|(1,-t_L)g\begin{pmatrix}t_R\\1\end{pmatrix}\right|^{2j} . \quad (2.15)$$

We interpret $(t_L, t_R, N) \in \mathbb{R}\times\mathbb{R}\times\mathbb{Z}$ as coordinates on the boundary of AdS_3. The action of the symmetry group $AdS_3 \times AdS_3$ on the boundary is then given by

$$(G_L, G_R)\cdot(t_L, t_R, N) = (g_L t_L, g_R t_R, N - N(G_L|t_L) + N(G_R|t_R)) , \quad (2.16)$$

and the behaviour of $\Phi^{j,\alpha}_{t_L,t_R}$ under the action of $AdS_3\times AdS_3$ (2.8) implies the following behaviour of $\Phi^j_{(t_L,t_R,N)}$:

$$\Phi^j_{(t_L,t_R,N)}(G_L^{-1}GG_R) = |(c_R t_R + d_R)(c_L t_L + d_L)|^{2j}\,\Phi^j_{(G_L,G_R)\cdot(t_L,t_R,N)}(G) . \quad (2.17)$$

2.3 Functions: m-bases

The t-bases of functions behave simply under symmetry transformations, but t-bases in H_3^+ and AdS_3 are not related by the Wick rotation. This is because the matrix realizations (2.2) and (2.5) on which the t-bases are built are themselves not related by the Wick rotation. We will therefore introduce the more complicated "m-bases" of functions, which are better suited to the Wick rotation. In the case of H_3^+, the m-basis functions $\Phi^j_{m,\bar m}(h)$ are defined as

$$\Phi^j_{m,\bar m}(h) \equiv \int d^2x\, x^{-j-1+m}\bar x^{-j-1+\bar m}\Phi^j_x(h) \quad \text{with} \quad m-\bar m\in\mathbb{Z} . \quad (2.18)$$

The numbers $m, \bar m$ can be written in terms of an integer $n \in \mathbb{Z}$ and a momentum p, which is imaginary in the H_3^+ model:

$$m = \frac{1}{2}(n+p) \quad , \quad \bar m = \frac{1}{2}(-n+p) . \quad (2.19)$$

The explicit expression for $\Phi^j_{m,\bar m}(h)$ is found to be

$$\Phi^j_{m,\bar m}(h) = -4\frac{\Gamma(-j+\frac{|n|+p}{2})\Gamma(-j+\frac{|n|-p}{2})}{\Gamma(|n|+1)\Gamma(-2j-1)}e^{-p\tau+in\theta}\sinh^{|n|}\rho\,\cosh^p\rho \times F(-j+\tfrac{|n|+p}{2}, j+1+\tfrac{|n|+p}{2}, |n|+1, -\sinh^2\rho) . \quad (2.20)$$

Notice that this obeys the so-called reflection property

$$\Phi^j_{m,\bar m} = R^j_{m,\bar m}\Phi^{-j-1}_{m,\bar m} \quad , \quad R^j_{m,\bar m} = \frac{\Gamma(2j+1)}{\Gamma(-2j-1)}\frac{\Gamma(-j+m)\Gamma(-j-\bar m)}{\Gamma(j+1+m)\Gamma(j+1-\bar m)} \quad (2.21)$$

$$= \frac{\Gamma(2j+1)}{\Gamma(-2j-1)}\frac{\Gamma(-j+\frac{|n|+p}{2})\Gamma(-j+\frac{|n|-p}{2})}{\Gamma(j+1+\frac{|n|+p}{2})\Gamma(j+1+\frac{|n|-p}{2})} , \quad (2.22)$$

where $R^j_{m,\bar m} = R^j_{\bar m,m}$ due to $n = m-\bar m\in\mathbb{Z}$.

We will use the functions on AdS_3 obtained from the above functions $\Phi^j_{m,\bar{m}}(h)$ by the Wick rotation $\tau \to i\tau$. In order for the resulting functions to be delta-function normalizable, we now need to assume the momentum p to be real, instead of imaginary in the H_3^+ case. We do not introduce a new notation for the resulting functions on AdS_3, but still call them $\Phi^j_{m,\bar{m}}(G)$ or $\Phi^j_{m,\bar{m}}$.

In contrast to t-basis functions, m-basis functions on AdS_3 do not transform simply under the action of the $AdS_3 \times AdS_3$ symmetry group. However, they do transform simply under the action of the $\mathbb{R} \times \mathbb{R}$ subgroup made of pairs (G_{τ_L}, G_{τ_R}), where G_τ was defined by eq. (2.4):

$$\Phi^{j,\alpha}_{m,\bar{m}}(G_{\tau_L} G G_{\tau_R}) = e^{-2i(m\tau_L + \bar{m}\tau_R)} \Phi^{j,\alpha}_{m,\bar{m}}(G) \ . \tag{2.23}$$

Notice that the identity $G_\pi G G_{-\pi} = G$ implies $m - \bar{m} \in \mathbb{Z}$. (In the particular case of $SL(2,\mathbb{R})$, we have the additional identity $g_{2\pi} = \mathrm{id}$, which implies $m, \bar{m} \in \frac{1}{2}\mathbb{Z}$.) Introducing

$$\alpha \in [0,1) \qquad \text{such that} \qquad m, \bar{m} \in \alpha + \mathbb{Z} \ , \tag{2.24}$$

this parameter α is identical to the parameter α of the t-basis functions $\Phi^{j,\alpha}_{t_L,t_R}$ (2.9). We will look for a relation of the type

$$\Phi^j_{m,\bar{m}} = c^{j,\alpha} \int_{\mathbb{R}} dt_L \ (1+t_L^2)^{-j-1} e^{i\pi m} \left(\tfrac{1-it_L}{1+it_L}\right)^m \int_{\mathbb{R}} dt_R \ (1+t_R^2)^{-j-1} \left(\tfrac{1+it_R}{1-it_R}\right)^{\bar{m}} \Phi^{j,\alpha}_{t_L,t_R} \tag{2.25}$$

where $c^{j,\alpha}$ is a normalization factor. We can check that the right-hand side of this relation obeys the transformation property (2.23), thanks to the behaviour eq. (2.8) of $\Phi^{j,\alpha}_{t_L,t_R}$. To see this it is useful to notice that the integrand in eq. (2.25) is continuous through $t_L = \infty$ and $t_R = \infty$, as can be deduced from the behaviour of the phase factor $e^{2i\pi\alpha n(G|t_L,t_R)}$ of $\Phi^{j,\alpha}_{t_L,t_R}$, which is depicted in the diagram (2.14). This makes it possible to perform translations of the variables φ_L, φ_R such that $t_{L,R} = \tan\frac{1}{2}\varphi_{L,R}$. The normalization factor $c^{j,\alpha}$ is easily computed in the limit $\rho \to \infty$, where the dependences of the integrand on t_L and t_R factorize. We find

$$c^{j,\alpha} = \frac{4^{2j} \sin \pi 2j}{\sin \pi(j-\alpha) \ \sin \pi(j+\alpha)} \ . \tag{2.26}$$

3. Representations and Clebsch-Gordan coefficients

3.1 Representations of $s\ell(2,\mathbb{R})$

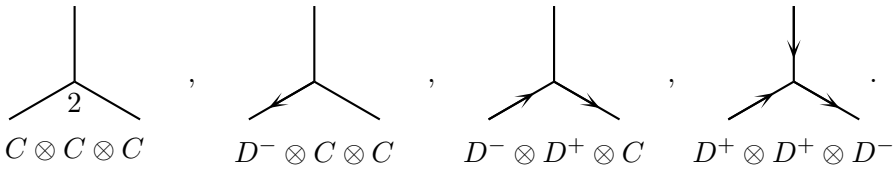
The minisuperspace limit of the spectrum of the AdS_3 WZNW model is the space of delta-function normalizable functions on AdS_3. It is subject to the action of the geometrical symmetry group $\frac{AdS_3 \times AdS_3}{\mathbb{Z}}$, and therefore of its Lie algebra $s\ell(2,\mathbb{R}) \times s\ell(2,\mathbb{R})$. Three types of unitary representations of $s\ell(2,\mathbb{R})$ appear in the minisuperspace spectrum: continuous representations, and two series of discrete representations. Continuous representations $C^{j,\alpha}$ are parametrized by a spin j and a number $\alpha \in [0,1)$ such that $m \in \alpha + \mathbb{Z}$. Discrete representations $D^{j,\pm}$ are parametrized by a spin $j \in (-\frac{1}{2}, \infty)$, and their states obey $m \in \pm(j+1+\mathbb{N})$. All these representations of $s\ell(2,\mathbb{R})$ extend to representations of the group AdS_3. However, only representations with $m \in \frac{1}{2}\mathbb{Z}$, namely $C^{j,\alpha}$ with $\alpha \in \frac{1}{2}\mathbb{Z}$ and $D^{j,\pm}$ with $j \in \frac{1}{2}\mathbb{N}$, extend to representations of the group $SL(2,\mathbb{R})$.

More precisely, given the $s\ell(2,\mathbb{R})$ algebra with generators $J^3, J^\pm$ and relations $[J^3, J^\pm] = \pm J^\pm$, $[J^+, J^-] = -2J^3$, the spin j is defined by $(J^3)^2 - \frac{1}{2}(J^+J^- + J^-J^+) = j(j+1)$, and the states $|m\rangle$ are such that

$$J^3|m\rangle = m|m\rangle \quad , \quad J^+|m\rangle = (m+j+1)|m+1\rangle \quad , \quad J^-|m\rangle = (m-j-1)|m-1\rangle \ . \tag{3.1}$$

These conventions are incompatible with the unit normalization of the states (which would mean $\langle m|m'\rangle = \delta_{m,m'}$), however they will turn out to agree with the behaviour of our functions $\Phi^j_{m,\bar{m}}$ eq. (2.20).

The tensor product laws for $s\ell(2,\mathbb{R})$ representations are well-known. They are equivalent to knowing the three-point invariants, which we schematically depict here in the cases when they do not vanish:



(3.2)

For instance, the first diagram means that any continuous representation $C^{j,\alpha}$ appears twice in the tensor product $C^{j_1,\alpha_1} \otimes C^{j_2,\alpha_2}$ of two continuous representations. (The m-conservation rule $\alpha = \alpha_1 + \alpha_2 \mod 1$ is implicitly assumed.) The fourth diagram means that $D^{j,+} \subset D^{j_1,+} \otimes D^{j_2,+}$. (The rule $j \in j_1 + j_2 + 1 + \mathbb{N}$ is implicitly assumed.) The fourth diagram also means that $D^{j,-}$ may appear once in $D^{j_1,-} \otimes D^{j_2,+}$. (This happens if $j \in j_2 - j_1 - 1 - \mathbb{N}$.) We omit the diagrams obtained by reverting the arrows in the second and fourth diagrams, namely $D^+ \otimes C \otimes C$ and $D^- \otimes D^- \otimes D^+$.

3.2 Clebsch-Gordan coefficients: m basis

We will rederive the tensor product rules by studying the Clebsch-Gordan coefficients. These coefficients are the three-point invariants, viewed as functions $C(j_1, j_2, j_3|m_1, m_2, m_3)$ subject to the equations

$$\sum_{i=1}^3 (m_i + j_i + 1) C(m_i + 1) = \sum_{i=1}^3 (m_i - j_i - 1) C(m_i - 1) = \sum_{i=1}^3 m_i C = 0 \ . \tag{3.3}$$

It is of course possible to prove a priori that these equations are obeyed by the three-point function $\left\langle \prod_{i=1}^3 \Phi^{j_i}_{m_i,\bar{m}_i} \right\rangle \equiv \int_{AdS_3} dG \prod_{i=1}^3 \Phi^{j_i}_{m_i,\bar{m}_i}(G)$. To do this, we would introduce a realization of the Lie algebra $s\ell(2,\mathbb{R})$ as first-order differential operators D^a wrt ρ, θ, τ, such that $D^+\Phi^j_{m,\bar{m}}(G) = (m+j+1)\Phi^j_{m,\bar{m}}(G)$, $D^-\Phi^j_{m,\bar{m}}(G) = (m-j-1)\Phi^j_{m,\bar{m}}(G)$ and $D^3\Phi^j_{m,\bar{m}}(G) = m\Phi^j_{m,\bar{m}}(G)$. Then eq. (3.3) would follow from the identity $\left\langle \prod_{i=1}^3 \Phi^{j_i}_{m_i,\bar{m}_i}(G) \right\rangle = \left\langle \prod_{i=1}^3 \Phi^{j_i}_{m_i,\bar{m}_i}(G_L G) \right\rangle$. We however abstain from doing this, as we will later explicitly compute the three-point function $\left\langle \prod_{i=1}^3 \Phi^{j_i}_{m_i,\bar{m}_i} \right\rangle$ and write it in terms of solutions of the equation (3.3).

Given three irreducible representations of $s\ell(2,\mathbb{R})$, there exist zero, one or two linearly independent solutions of the equation (3.3). In the case of three continuous representations, the momenta (m_1, m_2) belong to a two-dimensional lattice of the type $\prod_{i=1}^2 (\alpha_i + \mathbb{Z})$ (with of course

$m_3 = -m_1 - m_2$), and the coefficients $m_i \pm (j_i + 1)$ never vanish. In this situation, a solution of eq. (3.3) is determined once the values of C at two neighbouring points of the lattice are given. In the case when at least one representation is discrete, say $m_1 \in j_1 + 1 + \mathbb{N}$, the lattice becomes semi-infinite in one direction, and a solution is determined once the value of C at one point is given.

Let us introduce the function

$$G\begin{pmatrix} a & b & c \\ e & f \end{pmatrix} \equiv \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(e)\Gamma(f)}\, {}_3F_2\left(\begin{matrix} a & b & c \\ e & f \end{matrix}\middle| 1\right) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\Gamma(a+n)\Gamma(b+n)\Gamma(c+n)}{\Gamma(e+n)\Gamma(f+n)}\,, \quad (3.4)$$

where the sum converges provided $a+b+c-e-f < 0$, and the poles of G are the same as those of $\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(e+f-a-b-c)$. Knowing the identity

$$(a-e+1)G\begin{pmatrix} a & b & c \\ e & f \end{pmatrix} + (b-f)G\begin{pmatrix} a+1 & b & c \\ e & f+1 \end{pmatrix} + (c-1)G\begin{pmatrix} a & b & c-1 \\ e-1 & f \end{pmatrix} = 0\,, \quad (3.5)$$

we can use this function for writing a solutions of eq. (3.3):

$$C = \delta(m_1+m_2+m_3)\, G\begin{pmatrix} -j_2+m_2 & -j_3-m_3 & -j_{23}^1 \\ 1+j_1-j_3+m_2 & 1+j_1-j_2-m_3 \end{pmatrix} \equiv \delta(m_1+m_2+m_3) g^{23}, \quad (3.6)$$

which is well-defined provided $2 + j_{123} > 0$, where we use the notations $j_{123} = j_1 + j_2 + j_3$ and $j_{23}^1 \equiv j_2 + j_3 - j_1$. Of course five other solutions of the type g^{ab} with $a \neq b \in \{1,2,3\}$ can be obtained by permutations of indices. These solutions are not linearly independent, as can be shown with the help of the identity

$$\begin{aligned} s(b)s(c-a)G\begin{pmatrix} a & b & c \\ e & f \end{pmatrix} &= s(e-a)s(f-a)G\begin{pmatrix} a & a-e+1 & a-f+1 \\ a-b+1 & a-c+1 \end{pmatrix} \\ &\quad - s(c-e)s(c-f)G\begin{pmatrix} c & c-e+1 & c-f+1 \\ c-a+1 & c-b+1 \end{pmatrix}\,, \quad (3.7) \end{aligned}$$

where $s(x) \equiv \sin \pi x$, and we will also use $c(x) \equiv \cos \pi x$. Thus we obtain

$$\begin{pmatrix} g^{21} \\ g^{12} \end{pmatrix} = M_{13} \begin{pmatrix} g^{32} \\ g^{23} \end{pmatrix} \quad , \quad M_{13} = \frac{1}{s(j_{12}^3)} \begin{pmatrix} \frac{s(j_2+m_2)s(j_3-m_3)}{s(j_1+m_1)} & \frac{s(j_3+m_3)s(j_3-j_1+m_2)}{s(j_1+m_1)} \\ \frac{s(j_3-m_3)s(j_3-j_1-m_2)}{s(j_1-m_1)} & \frac{s(j_2-m_2)s(j_3+m_3)}{s(j_1-m_1)} \end{pmatrix}. \quad (3.8)$$

Together with the other identities obtained by permuting the indices, this shows that at most two of the solutions g^{ab} are linearly independent.

Due to our convention $j \in (-\frac{1}{2}, \infty)$ for discrete representations, we have $\Re j \geq -\frac{1}{2}$ for all representations of interest. This ensures that the sum in eq. (3.4) converges, so that g^{ab} is well-defined provided the summand is finite, which occurs unless $\Gamma(-j_a + m_a)\Gamma(-j_b - m_b)\Gamma(-j_{ab}^c)$ has a pole.

Case $C \otimes C \otimes C$. In this case, two given solutions say g^{23}, g^{32} are linearly independent, and they provide a basis of the two-dimensional space of invariants.

Case $D^- \otimes C \otimes C$**.** We assume for example $m_2 = -j_2 - 1 - \ell$ with $\ell \in \mathbb{N}$. Some relations of the type of eq. (3.8) simplify, and we find

$$g^{21} = (-1)^\ell \frac{s(j_3+m_3)}{s(j_1+m_1)} g^{23} = -\frac{s(j_{13}^2)}{s(2j_2)} g^{31} = -(-1)^\ell \frac{s(j_3+m_3)}{s(j_1+m_1)} \frac{s(j_{13}^2)}{s(2j_2)} g^{13} \,. \tag{3.9}$$

Since the space of invariants is one-dimensional, the two remaining functions g^{12} and g^{32} must also be proportional to the other four. However, this proportionality relation is not very simple, as can be seen in the case of the highest-weight state $\ell = 0$ when g^{12} and g^{32} fail to become expressible as products of Γ-functions, in contrast to the other four solutions.

Case $D^+ \otimes C \otimes C$**.** The situation is completely analogous to the previous case. We assume for example $m_2 = j_2 + 1 + \ell$ with $\ell \in \mathbb{N}$ and find

$$g^{12} = (-1)^\ell \frac{s(j_3-m_3)}{s(j_1-m_1)} g^{32} = -\frac{s(j_{13}^2)}{s(2j_2)} g^{13} = -(-1)^\ell \frac{s(j_3-m_3)}{s(j_1-m_1)} \frac{s(j_{13}^2)}{s(2j_2)} g^{31} \,. \tag{3.10}$$

Case $D^- \otimes D^- \otimes C$**.** We expect no invariants to exist in this case. Let us check this, assuming for example $m_2 \in -j_2 - 1 - \ell_2$ and $m_3 \in -j_3 - 1 - \ell_3$ with $\ell_2, \ell_3 \in \mathbb{N}$. Equation (3.9) implies two incompatible relations between g^{21} and g^{31}, which must therefore both vanish. An apparent paradox comes from the non-vanishing of $g^{12}, g^{32}, g^{23}, g^{13}$. However, these functions do not provide solutions to eq. (3.3), because they become infinite at $\ell_2 = -1$ or $\ell_3 = -1$. For instance, if $C(m_1, -j_2, m_3) = \infty$, then $(m_1+j_1+1)C(m_1+1, -j_2-1, m_3)+(-j_2+j_2)C(m_1, -j_2, m_3)+(m_3+j_3+1)C(m_1, -j_2-1, m_3+1) = 0$ may have an unwanted nonvanishing second term. Therefore, the analysis of g^{ab} agrees with the representation-theoretic expectations that no invariant exists.

Case $D^+ \otimes D^- \otimes C$**.** We assume for example $m_2 = -j_2 - 1 - \ell_2$ and $m_3 = j_3 + 1 + \ell_3$ with $\ell_2, \ell_3 \in \mathbb{N}$. We find the relations

$$g^{31} = g^{12} = -\frac{s(2j_3)}{s(j_{12}^3)} g^{13} = -\frac{s(2j_2)}{s(j_{13}^2)} g^{21} = \frac{s(2j_2)s(2j_3)}{s(j_{13}^2)s(j_{12}^3)} g^{23} \,. \tag{3.11}$$

The functions g^{13}, g^{21}, g^{23} stay finite for any values $\ell_2, \ell_3 \in \mathbb{Z}$, and therefore provide three proportional invariants. The functions g^{31} and g^{12} become infinite if $\ell_3 < 0$ and $\ell_2 < 0$ respectively, so that it is not a priori clear that they provide invariants. That they actually do is guaranteed by the above relations.

Case $D^- \otimes D^- \otimes D^+$**.** We assume for example $m_1 = j_1 + 1 + \ell_1$, $m_2 \in -j_2 - 1 - \ell_2$ and $m_3 \in -j_3 - 1 - \ell_3$, with $\ell_1, \ell_2, \ell_3 \in \mathbb{N}$. Noticing $s(2j_2)s(2j_3) = s(j_{13}^2)s(j_{12}^3)$, we find the relations

$$g^{12} = g^{23} = -\frac{s(2j_3)}{s(j_{12}^3)} g^{13} = -\frac{s(2j_3)}{s(j_{12}^3)} g^{32} \,. \tag{3.12}$$

These four proportional functions provide the invariant in this case. In particular, g^{12} and g^{13} no longer become infinite at $\ell_2 = -1$ or $\ell_3 = -1$ respectively, as happened in the case $D^- \otimes D^- \otimes C$. The remaining two functions g^{21}, g^{31} vanish, as they already did in the case $D^- \otimes D^- \otimes C$. Notice that the selection rule $j_1 \in j_2+j_3+1+\mathbb{N}$ manifests itself as g^{23} becoming infinite if $j_1 \in j_2+j_3-\mathbb{N}$, due to a series of poles which correspond to those of $\Gamma(-j_{23}^1)$.

3.3 Clebsch-Gordan coefficients: t-basis

Our m-basis invariants g^{ab} are not symmetric under permutations of the indices, but the equation (3.3) which they solve is. In the t-basis, we will now show that there exist natural permutation-symmetric invariants, and we will relate them to combinations of the g^{ab} invariants. A similar analysis was already performed for the Clebsch-Gordan coefficients of $SO(2,1) = \frac{SL(2,\mathbb{R})}{\mathbb{Z}_2}$, in the articles [7, 8]. The representations of $SO(2,1)$ correspond to representations of $s\ell(2,\mathbb{R})$ such that $m \in \mathbb{Z}$, or in other words $\alpha = 0$. We will perform the generalization to arbitrary values of α.

In the t-basis, AdS_3 invariants should be solutions of

$$C(\{t_i\}) = C(\{\tfrac{at_i+b}{ct_i+d}\}) \prod_{i=1}^{3} |ct_i+d|^{2j_i} e^{-2i\pi\alpha_i N(G|t_i)} \,, \tag{3.13}$$

for any AdS_3 element G whose projection onto $SL(2,\mathbb{R})$ is $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Solutions exist provided $\alpha_1+\alpha_2+\alpha_3 \in \mathbb{Z}$, and we will assume $\alpha_1+\alpha_2+\alpha_3 = 0$. The solutions are [9]

$$C(\{t_i\}) = |t_{12}|^{j_{12}^3} |t_{23}|^{j_{23}^1} |t_{31}|^{j_{31}^2} \, e^{i\pi(\alpha_{12}\mathrm{sign}t_{12}+\alpha_{23}\mathrm{sign}t_{23}+\alpha_{31}\mathrm{sign}t_{31})} \,, \tag{3.14}$$

where $\alpha_{12}, \alpha_{23}, \alpha_{31}$ should obey the equation $\alpha_{12}-\alpha_{23} = \alpha_2 \mod \mathbb{Z}$ and the two other equations obtained by even permutations thereof. The solutions to such equations are

$$\alpha_{ab} = \tfrac{1}{3}(\alpha_b - \alpha_a) + \alpha_0 \,, \tag{3.15}$$

where α_0 is an arbitrary constant. (In $SL(2,\mathbb{R})$ we have $\alpha_a = \eta_a \in \frac{1}{2}\mathbb{Z}$ and it is more convenient to adopt the convention $\alpha_{ab} = \eta_a + \eta_b + \alpha_0$. Equivalently, we can use the above formula provided we assume $\alpha_a \in \frac{3}{2}\mathbb{Z}$.) Let us now apply the change of basis (2.25) to $C(\{t_i\})$. After the change of integration variables $t_i = \tan\frac{\varphi_i}{2}$ the m-basis version of $C(\{t_i\})$ is

$$C(\{m_i\}) = \prod_{i=1}^{3} \left[\int_{-\pi}^{\pi} d\varphi_i \, e^{im_i\varphi_i} \right] |\sin\tfrac{1}{2}\varphi_{12}|^{j_{12}^3} |\sin\tfrac{1}{2}\varphi_{23}|^{j_{23}^1} |\sin\tfrac{1}{2}\varphi_{31}|^{j_{13}^2} \times e^{i\pi\left(\alpha_{12}\mathrm{sign}\sin\frac{1}{2}\varphi_{12}+\alpha_{23}\mathrm{sign}\sin\frac{1}{2}\varphi_{23}+\alpha_{31}\mathrm{sign}\sin\frac{1}{2}\varphi_{31}\right)} \,, \tag{3.16}$$

where the integrand has the same values at $\varphi_i = \pi$ and $\varphi_i = -\pi$. This integral can be explicitly evaluated by generalizing the computations of [7], and in particular using the formula, valid for $\varphi \in (-\pi, \pi)$:

$$\left|\sin\tfrac{\varphi}{2}\right|^{-2a} e^{i\pi\alpha\mathrm{sign}\sin\frac{1}{2}\varphi} = s(a+\alpha) e^{-i\pi\alpha} 2^{2a} \Gamma(1-2a) \sum_{n=-\infty}^{\infty} e^{i(n+\alpha)\varphi} \frac{\Gamma(a+n+\alpha)}{\Gamma(1-a+n+\alpha)} \tag{3.17}$$

Thus, we find

$$C(\{m_i\}) = -\delta(\textstyle\sum m_i) 2^{-j_{123}} \frac{1}{\pi^2} \Gamma(1+j_{12}^3)\Gamma(1+j_{23}^1)\Gamma(1+j_{31}^2) \, e^{-i\pi(\alpha_{12}+\alpha_{23}+\alpha_{31})} \, g^{\alpha_0} \tag{3.18}$$

where we introduce the invariants

$$g^{\alpha_0} = \pi^2 s(\tfrac{1}{2}j_{12}^3 - \alpha_{12}) s(\tfrac{1}{2}j_{23}^1 - \alpha_{23}) s(\tfrac{1}{2}j_{31}^2 - \alpha_{31}) \times \sum_{n\in\mathbb{Z}} \frac{\Gamma(-\frac{1}{2}j_{12}^3 + \alpha_{12} + n)}{\Gamma(1+\frac{1}{2}j_{12}^3 + \alpha_{12} + n)} \frac{\Gamma(-\frac{1}{2}j_{13}^2 + m_1 + \alpha_{12} + n)}{\Gamma(1+\frac{1}{2}j_{13}^2 + m_1 + \alpha_{12} + n)} \frac{\Gamma(-\frac{1}{2}j_{23}^1 - m_2 + \alpha_{12} + n)}{\Gamma(1+\frac{1}{2}j_{23}^1 - m_2 + \alpha_{12} + n)} \,, \tag{3.19}$$

which can be expressed in terms of the invariants g^{ab} (3.6) as

$$g^{\alpha_0} = s(\tfrac{1}{2}j_{13}^2 - \alpha_{31})s(j_3 - \alpha_3)s(j_1 + \alpha_1)g^{31} + s(\tfrac{1}{2}j_{13}^2 + \alpha_{31})s(j_1 - \alpha_1)s(j_3 + \alpha_3)g^{13} \quad (3.20)$$

In this formula, g^{α_0} depends on the paramter α_0 only through α_{31} eq. (3.15). There are two particularly interesting special values of α_0, namely 0 and $\frac{1}{2}$. In the case $\alpha_0 = 0$ then $C(\{t_i\})$ and g^0 are invariant under permutations. In the case $\alpha_0 = \frac{1}{2}$ then $C(\{t_i\})$ and $g^{\frac{1}{2}}$ are odd under permutations i.e. invariant up to a sign. When the three involved representations are continuous, g^0 and $g^{\frac{1}{2}}$ can serve as a basis of the two-dimensional space of invariants. Notice that in the case of $SL(2,\mathbb{R})$ only these two values of α_0 are possible.

4. Products of functions

In the conformal bootstrap approach to the AdS_3 WZNW model, all correlation functions can in principle be constructed from the knowledge of three objects: the spectrum, the two-point correlation functions on a sphere, and the operator product expansions – or equivalently the three-point correlation functions on a sphere. We will now determine these objects in the minisuperspace limit. We first recall their definitions. Given n functions $\Phi^i(G)$ on AdS_3, the corresponding correlation function is $\left\langle \prod_{i=1}^n \Phi^i \right\rangle \equiv \int dG \ \prod_{i=1}^n \Phi^i(G)$ where dG is the invariant measure. If $\{\Phi^i\}_{i\in S}$ form an orthogonal basis of the spectrum (that is $\left\langle \Phi^i \Phi^j \right\rangle = 0$ if $i \neq j$), the product of functions is schematically $\Phi^1\Phi^2 = \sum_{i\in S} \frac{\left\langle \Phi^1\Phi^2\Phi^i \right\rangle}{\left\langle \Phi^i\Phi^i \right\rangle}\Phi^i$. This product is obviously associative and commutative.

The functions $\Phi^j_{m,\bar{m}}(G)$ on AdS_3 are related to corresponding functions $\Phi^j_{m,\bar{m}}(h)$ on H_3^+ by a Wick rotation, and therefore their correlation functions can be deduced from H_3^+ correlation functions by that Wick rotation. This will involve some subtleties, because the discrete representations which appear in the minisuperspace spectrum on AdS_3 are absent in H_3^+. But let us first review the products of functions on H_3^+.

4.1 Products of functions on H_3^+

The minisuperspace spectrum of H_3^+ is generated by the functions

$$\{\Phi^j_{n,p}(h) | j \in -\tfrac{1}{2} + i\mathbb{R},\ n \in \mathbb{Z},\ p \in i\mathbb{R}\} \ . \quad (4.1)$$

The correlation functions $\left\langle \prod_{i=1}^n \Phi^{j_i}_{n_i,p_i} \right\rangle \equiv \int dh \ \prod_{i=1}^n \Phi^{j_i}_{n_i,p_i}(h)$ are obtained by integrating products of such functions with respect to the invariant measure $dh = \sinh 2\rho \, d\rho \, d\theta \, d\tau$. The two-point functions can be computed from the expression (2.20) of $\Phi^j_{n,p}(h)$:

$$\left\langle \Phi^j_{n,p} \Phi^{j'}_{n',p'} \right\rangle = 128\pi^2 \delta_{n+n',0} \ \delta(p+p') \left[\delta(j+j'+1) + R^j_{n,p}\delta(j-j') \right] \ , \quad (4.2)$$

where the reflection coefficient $R^j_{n,p}$ was defined in eq. (2.21). A similar direct computation of the three-point function seems complicated. Instead, we will make use of the known x-basis three-point function [5]

$$\left\langle \prod_{i=1}^3 \Phi^{j_i}_{x_i} \right\rangle = C(j_1,j_2,j_3) |x_{12}|^{2j^3_{12}} |x_{23}|^{2j^1_{23}} |x_{31}|^{2j^2_{31}} \ , \quad (4.3)$$

$$C(j_1,j_2,j_3) \equiv \pi^{-3}\Gamma(-j_{123}-1) \frac{\Gamma(-j^3_{12})\Gamma(-j^1_{23})\Gamma(-j^2_{31})}{\Gamma(-2j_1-1)\Gamma(-2j_2-1)\Gamma(-2j_3-1)} \ . \quad (4.4)$$

The transformation to the m-basis (2.18) can be performed thanks to an integral formula of Fukuda and Hosomichi [10]. The result can be written in terms of the Clebsch-Gordan coefficients g^{ab} (3.6):

$$\left\langle \prod_{i=1}^{3} \Phi^{j_i}_{m_i,\bar{m}_i} \right\rangle = C(j_1,j_2,j_3)\ \delta^{(2)}(\textstyle\sum m_i)\ K^2 s(j^2_{13})^2 s(j^1_{23})^2 \\ \times \left[g^{23}\bar{g}^{31} + \frac{s(2j_2)}{s(j^2_{13})} g^{23}\bar{g}^{32} + \frac{s(2j_1)}{s(j^1_{23})} g^{13}\bar{g}^{31} + g^{13}\bar{g}^{32} \right] , \quad (4.5)$$

where $\bar{g}^{ab}$ denotes g^{ab} with m_i replaced by $\bar{m}_i$, and we introduced the factor

$$K \equiv \frac{1}{\pi^2}\Gamma(1+j^3_{12})\Gamma(1+j^2_{13})\Gamma(1+j^1_{23}) . \quad (4.6)$$

It can be checked that the two- and three-point functions have the behaviour under reflection which is expected from the behaviour of $\Phi^j_{m,\bar{m}}$ (2.21). Now, using the three-point function, products of functions on H_3^+ can be written as

$$\Phi^{j_1}_{m_1,\bar{m}_1}\Phi^{j_2}_{m_2,\bar{m}_2} = \frac{1}{256\pi^2}\int_{-\frac{1}{2}+i\mathbb{R}} dj_3 \left\langle \prod_{i=1}^{3} \Phi^{j_i}_{m_i,\bar{m}_i} \right\rangle' \frac{1}{R^{j_3}_{m_3,\bar{m}_3}} \Phi^{j_3}_{-m_3,-\bar{m}_3} , \quad (4.7)$$

where we use the notation $\left\langle \prod_{i=1}^{3} \Phi^{j_i}_{m_i,\bar{m}_i} \right\rangle = \delta^{(2)}(m_1+m_2+m_3) \left\langle \prod_{i=1}^{3} \Phi^{j_i}_{m_i,\bar{m}_i} \right\rangle'$.

The invariance of the three-point function (4.5) under permutations of the indices is not manifest, but can be checked using linear relations between the g^{ab} such as eq. (3.8). It seems that a reasonably simple, manifestly permutation-symmetric expression exists only in the case $m_i, \bar{m}_i \in \eta_i + \mathbb{Z}$ with $\eta_i \in \{0, \frac{1}{2}\}$, which corresponds to functions on $SL(2,\mathbb{R})$. In this case, we can use the invariants $g^0, g^{\frac{1}{2}}$ (3.20), and we find

$$\left\langle \prod_{i=1}^{3} \Phi^{j_i}_{m_i,\bar{m}_i} \right\rangle = -\tfrac{1}{2\pi^5} C(j_1,j_2,j_3)\ \delta_{\sum m_i,0}\delta_{\sum \bar{m}_i,0}\ \frac{\gamma(1+j^3_{12})\gamma(1+j^1_{23})\gamma(1+j^2_{31})}{\prod_{i=1}^{3} s(j_i+\eta_i)} \\ \times \sum_{\epsilon\in\{0,\frac{1}{2}\}} \frac{s(\frac{1}{2}j_{123}+\epsilon)}{s(\frac{1}{2}j^3_{12}+\eta_3+\epsilon)s(\frac{1}{2}j^1_{23}+\eta_1+\epsilon)s(\frac{1}{2}j^2_{31}+\eta_2+\epsilon)}\ g^\epsilon \bar{g}^\epsilon , \quad (4.8)$$

where we use $\gamma(x) \equiv \frac{\Gamma(x)}{\Gamma(1-x)}$. From this, we can reconstruct the t-basis three-point function

$$\left\langle \prod_{i=1}^{3} \Phi^{j_i,\eta_i}_{t^L_i,t^R_i} \right\rangle = \frac{2^{2j_{123}}}{2\pi^4} C(j_1,j_2,j_3) \prod_{i=1}^{3} c(j_i-\eta_i) \times \\ \left|t^L_{12}t^R_{12}\right|^{j^3_{12}} \left|t^L_{23}t^R_{23}\right|^{j^1_{23}} \left|t^L_{31}t^R_{31}\right|^{j^2_{31}} e^{i\pi\left[\eta_3(\mathrm{sign}t^L_{12}+\mathrm{sign}t^R_{12})+\eta_1(\mathrm{sign}t^L_{23}+\mathrm{sign}t^R_{23})+\eta_2(\mathrm{sign}t^L_{31}+\mathrm{sign}t^R_{31})\right]} \times \\ \sum_{\epsilon\in\{0,\frac{1}{2}\}} s(\tfrac{1}{2}j_{123}-\epsilon)c(\tfrac{1}{2}j^3_{12}+\eta_3+\epsilon)c(\tfrac{1}{2}j^1_{23}+\eta_1+\epsilon)c(\tfrac{1}{2}j^2_{31}+\eta_2+\epsilon)\ \left(\mathrm{sign}t^L_{12}t^R_{12}t^L_{23}t^R_{23}t^L_{31}t^R_{31}\right)^{2\epsilon} . \quad (4.9)$$

Comparing this formula to the H_3^+ three-point function in the x-basis eq. (4.3), we obtain a confirmation of the lack of a simple relation between the x-basis in H_3^+ and the t-basis in $SL(2,\mathbb{R})$ or AdS_3.

In the more general case of functions on AdS_3, the three-point function can still be expressed in terms of the invariants g^0 and $g^{\frac{1}{2}}$, but the formula is more complicated than eq. (4.8) and in particular the "mixed" terms $g^0\bar{g}^{\frac{1}{2}}$ and $g^{\frac{1}{2}}\bar{g}^0$ are present. Their absence in the case of $SL(2,\mathbb{R})$ can be attributed to the exterior automorphism ω of $SL(2,\mathbb{R})$, namely $\omega(g) = \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right) g \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$, which is such that $\Phi^{j,\eta}_{t_L,t_R}(\omega(g)) = (-1)^{2\eta}\Phi^{j,\eta}_{-t_L,-t_R}(g)$. In the case of AdS_3 this action still exists and can be expressed as $\omega(\rho,\theta,\tau) = (\rho,-\theta,-\tau)$. But it does not act simply on the function $\Phi^{j,\alpha}_{t_L,t_R}(G)$.[1]

4.2 Products of functions on AdS_3

Functions $\Phi^j_{m,\bar{m}}$ on AdS_3 are obtained from the corresponding functions on H_3^+ by performing the Wick rotation $\tau \to i\tau$ and continuing $p = m + \bar{m}$ from $i\mathbb{R}$ to $\mathbb{R}$. If we do not modify the value of the spin $j \in -\frac{1}{2} + i\mathbb{R}$, this yields functions transforming in the continuous representations of $AdS_3 \times AdS_3$, namely $\Phi^j_{m,\bar{m}} \in C^{j,\alpha} \otimes C^{j,\alpha}$ where $m, \bar{m} \in \alpha + \mathbb{Z}$. We may in addition obtain functions transforming in the discrete representation by continuing j to real values such that $m, \bar{m} \in \pm j \pm \mathbb{Z}$. More precisely, functions $\Phi^j_{m,\bar{m}} \in D^{j,\pm} \otimes D^{j,\pm}$ correspond to

$$j \in (-\tfrac{1}{2}, \infty) \qquad \text{and} \qquad m, \bar{m} \in \pm(j+1+\mathbb{N}) \Leftrightarrow -j-1-\tfrac{1}{2}|n| \pm \tfrac{1}{2}p \in \mathbb{N} \qquad (4.10)$$

$$\text{or} \quad j \in (-\infty, -\tfrac{1}{2}) \qquad \text{and} \qquad m, \bar{m} \in \pm(-j+\mathbb{N}) \Leftrightarrow j - \tfrac{1}{2}|n| \pm \tfrac{1}{2}p \in \mathbb{N}\,. \qquad (4.11)$$

These two possibilities are related by the reflection $j \to -j-1$ and they are equivalent. We will only consider the first possibility, because our invariants g^{ab} (3.6) are well-defined for $\Re j \geq -\frac{1}{2}$. The set of these discrete and continuous functions is known to generate the space of functions on AdS_3; in particular this set is closed under products, as we will check below.

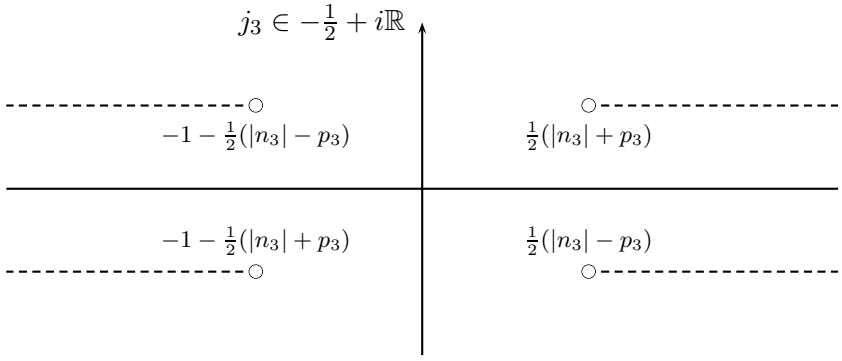
We will now derive the products of functions on AdS_3 by continuing the products of functions on H_3^+ (4.7) to the relevant values of spins j and momenta p. We will examine various cases, according to the nature – discrete or continuous – of the fields $\Phi^{j_1}_{m_1,\bar{m}_1}$ and $\Phi^{j_2}_{m_2,\bar{m}_2}$. For example, the case when $j_1 \in -\frac{1}{2} + i\mathbb{R}$ and $m_2, \bar{m}_2 \in j_2 + 1 + \mathbb{Z}$ will be denoted $C \times D^+$. We will check that the terms which appear in a given product are those which are allowed by the well-known tensor product laws for $s\ell(2,\mathbb{R})$ representations (3.2).

Case $C \times C$. We should continue p_1, p_2, p_3 from imaginary to real values in eq. (4.7). This is problematic only when the integrand, viewed as a function of j_3, has poles which cross the integration line. Such p_i-dependent poles of the integrand may come from either of its three factors. The poles coming from the second factor $\frac{1}{R^{j_3}_{m_3,\bar{m}_3}}$ are easily seen from eq. (2.21), and the poles from the other factors are obtained from these by the reflection[2] $j_3 \to -j_3 - 1$. All these poles fall

[1] Let us give the behaviour of certain objects of section 2.2: $N(\omega(G)|t) = -N(G|-t)$, $[\omega(G)] = -[G]-1$ and $n(\omega(G)|t_L,t_R) = -n(G|-t_L,-t_R)$.

[2] It is also possible to study the m-dependent poles of $\left\langle \prod_{i=1}^3 \Phi^{j_i}_{m_i,\bar{m}_i} \right\rangle$ directly from the formula (4.5). For example, $\bar{g}^{32}$ has poles at $j_2 + \bar{m}_2 \in \mathbb{N}$. But the coefficient of $\bar{g}^{32}$ is a combination of g^{13} and g^{23} of the type $s(2j_2)g^{23} + s(j_{13}^2)g^{13} = \frac{s(j_3-m_3)}{s(j_1-m_1)}s(j_2+m_2)g^{32}$, where we used $j_2 + m_2 \in \mathbb{Z}$ (which follows from $j_2 + \bar{m}_2 \in \mathbb{N}$) and eq. (3.8). This vanishes, unless g^{32} itself has a pole. This shows that $\left\langle \prod_{i=1}^3 \Phi^{j_i}_{m_i,\bar{m}_i} \right\rangle$ has simple poles when both $m_2, \bar{m}_2$ belong to $-j_2 + \mathbb{N}$, but not when only $\bar{m}_2$ does.

on the four dashed half-lines in the following diagram, which depicts the j_3 complex plane. The two half-lines on the left correspond to the poles of $\frac{1}{R^{j_3}_{m_3,\bar{m}_3}}$.



(4.12)

This diagram assumes $p_3 \in i\mathbb{R}$. When p_3 moves to real values, let us consider the poles from the left which may cross the integration line and end up on the right with $j_3 \in (-\frac{1}{2}, \infty)$. Such poles belong to the (possibly empty) sets $j_3 \in -1 - \frac{1}{2}|n_3| \pm \frac{1}{2}p_3 - \mathbb{N} \cap (-\frac{1}{2}, \infty)$. Therefore, according to eq. (4.10), they correspond to functions $\Phi^{j_3}_{m_3,\bar{m}_3}$ in the $D^{j_3,\pm}$ representations. We deduce the formula for the products of two "continuous" functions on AdS_3:

$$\Phi^{j_1}_{m_1,\bar{m}_1}\Phi^{j_2}_{m_2,\bar{m}_2} = \frac{1}{256\pi^2}\int_{-\frac{1}{2}+i\mathbb{R}} dj_3 \left\langle \prod_{i=1}^{3} \Phi^{j_i}_{m_i,\bar{m}_i} \right\rangle' \frac{1}{R^{j_3}_{m_3,\bar{m}_3}} \Phi^{j_3}_{-m_3,-\bar{m}_3}$$
$$+ \frac{2}{256\pi^2} \sum_{j_3 \in -1-\frac{1}{2}|n_3| \pm \frac{1}{2}p_3 - \mathbb{N}\cap(-\frac{1}{2},\infty)} \left\langle \prod_{i=1}^{3} \Phi^{j_i}_{m_i,\bar{m}_i} \right\rangle' 2\pi i \mathrm{Res} \frac{1}{R^{j_3}_{m_3,\bar{m}_3}} \Phi^{j_3}_{-m_3,-\bar{m}_3} \ , \quad (4.13)$$

where the factor 2 in the discrete term is due to the contribution of the poles with $j_3 \in (-\infty, -\frac{1}{2})$. Notice that the general expression (4.5) for the three-point function $\left\langle \prod_{i=1}^{3} \Phi^{j_i}_{m_i,\bar{m}_i} \right\rangle'$ simplifies in the case $j_3 \in -1 - \frac{1}{2}|n_3| \pm \frac{1}{2}p_3 - \mathbb{N} \cap (-\frac{1}{2}, \infty)$ due to formulas of the type of eqs. (3.9) and (3.10). Examples of simplified expressions are:

$$\left\langle \prod_{i=1}^{3} \Phi^{j_i}_{m_i,\bar{m}_i} \right\rangle' \underset{D^{j_3,+}}{=} -C(j_1,j_2,j_3)K^2 s(j^3_{12})s(j^1_{23})s(j^2_{31})s(j_{123})\, g^{21}\bar{g}^{13} \ , \quad (4.14)$$
$$g^{21}\bar{g}^{13} = \bar{g}^{21}g^{13} = g^{12}\bar{g}^{23} = \bar{g}^{12}g^{23} \ ,$$
$$\left\langle \prod_{i=1}^{3} \Phi^{j_i}_{m_i,\bar{m}_i} \right\rangle' \underset{D^{j_3,-}}{=} -C(j_1,j_2,j_3)K^2 s(j^3_{12})s(j^1_{23})s(j^2_{31})s(j_{123})\, g^{12}\bar{g}^{31} \ , \quad (4.15)$$
$$g^{12}\bar{g}^{31} = \bar{g}^{12}g^{31} = g^{21}\bar{g}^{32} = \bar{g}^{21}g^{32} \ .$$

Case $D^+ \times C$. After moving p_i to real values as in the previous case $C \times C$, we should move j_1 to $-1 - \frac{1}{2}|n_1| + \frac{1}{2}p_1 - \mathbb{N} \cap (-\frac{1}{2}, \infty)$. Let us show that no further poles cross the integration line in eq. (4.13) during this operation. We are looking for possible j_1-dependent poles in $\left\langle \prod_{i=1}^{3} \Phi^{j_i}_{m_i,\bar{m}_i} \right\rangle'$, viewed as a function of j_3. We use formulas of the type of eq. (3.10) to obtain

$$\left\langle \prod_{i=1}^{3} \Phi^{j_i}_{m_i,\bar{m}_i} \right\rangle' \underset{D^{j_1,+}}{=} C(j_1,j_2,j_3)K^2 s(j^3_{12})s(j^1_{23})s(j^2_{31})s(j_{123}) \frac{s(j^1_{23})}{s(2j_1)} g^{23}\bar{g}^{32} \ . \quad (4.16)$$

Potential poles come from factors $\Gamma(1+j_{23}^1)$ in K (4.6) and $\Gamma(-1-j_{123})\Gamma(-j_{12}^3)\Gamma(-j_{31}^2)$ in $C(j_1,j_2,j_3)$ (4.4), but they are all cancelled by appropriate $\sin$ factors. On the other hand, the poles from the factors $\Gamma(1+j_{12}^3)\Gamma(1+j_{31}^2)$ in K, from the factor $\Gamma(-j_{23}^1)$ in $C(j_1,j_2,j_3)$, and the poles of $\Gamma(-j_{23}^1)$ which come from g^{23} and $\bar{g}^{32}$, cannot be reached because $\Re j_1 \geq -\frac{1}{2}$.

This shows that the formula (4.13) still holds for products of functions in $D^+ \times C$. Of course, simplified expressions for $\left\langle \prod_{i=1}^3 \Phi^{j_i}_{m_i,\bar{m}_i} \right\rangle'$ can be used for both continuous and discrete values of j_3. We can moreover check that terms corresponding to $D^{j_3,+}$ actually vanish. This is due to

$$\left\langle \prod_{i=1}^3 \Phi^{j_i}_{m_i,\bar{m}_i} \right\rangle' \underset{D^{j_1,+},D^{j_3,+}}{=} 0 \,, \tag{4.17}$$

which follows from eq. (4.14) if we notice that $g^{21} = g^{23} = 0$ in this case due to eq. (3.10). This equation holds for generic values of j_2, in particular the values $j_2 \in -\frac{1}{2} + i\mathbb{R}$ which correspond to C^{j_2}.

Case $D^+ \times D^+$. The formula (4.13) for the product of functions still holds, but the continuous term $\int_{-\frac{1}{2}+i\mathbb{R}} dj_3 \cdots$ vanishes due to eq. (4.17). Terms corresponding to $D^{j_3,+}$ representations also vanish by the same argument, but the equation $\left\langle \prod_{i=1}^3 \Phi^{j_i}_{m_i,\bar{m}_i} \right\rangle' \underset{D^{j_1,+},D^{j_2,+}}{=} 0$ may fail if a $D^{j_3,-}$ representation is present, due to poles from the factor K^2 in eq. (4.5). To analyze this matter it is convenient to start with the identity

$$\left\langle \prod_{i=1}^3 \Phi^{j_i}_{m_i,\bar{m}_i} \right\rangle' \underset{D^{j_2,+},D^{j_3,-}}{=} -C(j_1,j_2,j_3)K^2 \frac{s(j_{12}^3)^2 s(j_{31}^2)^2 s(j_{23}^1) s(j_{123})}{s(2j_2)s(2j_3)} g^{13}\bar{g}^{13} \,. \tag{4.18}$$

We then send j_1 to values corresponding to discrete representations $D^{j_1,+}$. Due to momentum conservation we must have $j_1 \in j_2 - j_3 + \mathbb{Z}$. If $j_1 \in j_2 - j_3 - 1 - \mathbb{N}$ then a double pole from K^2 cancels the double zero from $s(j_{13}^2)^2$ and the result is finite. If $j_1 \in j_2 - j_3 + \mathbb{N}$ then the simple pole from $C(j_1,j_2,j_3)$ does not cancel the double zero, and the result vanishes. The formula (4.13) therefore reduces to

$$\Phi^{j_1}_{m_1,\bar{m}_1}\Phi^{j_2}_{m_2,\bar{m}_2} = \frac{2}{256\pi^2} \sum_{j_3 \in j_1+j_2+1+\mathbb{N}} \left\langle \prod_{i=1}^3 \Phi^{j_i}_{m_i,\bar{m}_i} \right\rangle' 2\pi i \mathrm{Res} \frac{1}{R^{j_3}_{m_3,\bar{m}_3}} \Phi^{j_3}_{-m_3,-\bar{m}_3} \,. \tag{4.19}$$

Case $D^+ \times D^-$. The formula (4.13) for the product of functions still holds, and the analysis of eq. (4.18) in the previous case determines which terms may vanish. Nonvanishing $D^{j_3,+}$ terms occur for $j_3 \in j_2 - j_1 - 1 - \mathbb{N} \cap (-\frac{1}{2}, \infty)$ and nonvanishing $D^{j_3,-}$ terms occur for $j_3 \in j_1 - j_2 - 1 - \mathbb{N} \cap (-\frac{1}{2}, \infty)$. Depending on the values of j_1, j_2 we can have either $D^{j_3,+}$ terms, or $D^{j_3,-}$ terms, or no discrete terms at all in the case $|j_1 - j_2| \leq \frac{1}{2}$.

5. Conclusion

At the level of symmetry algebras, the Wick rotation from H_3^+ to AdS_3 amounts to a map from $s\ell(2,\mathbb{C})$ to $s\ell(2,\mathbb{R}) \times s\ell(2,\mathbb{R})$, which can be viewed as two different real forms of the same algebra

$s\ell(2,\mathbb{C})^{\mathbb{C}} = s\ell(2,\mathbb{C}) \times s\ell(2,\mathbb{C})$. In particular, the Wick rotation maps the continuous representation C^j of $s\ell(2,\mathbb{C})$ to the representation $\int_0^1 d\alpha\, C^{j,\alpha} \otimes C^{j,\alpha}$ of $s\ell(2,\mathbb{R}) \times s\ell(2,\mathbb{R})$. The fact that such an irreducible representation is mapped to a reducible one implies that the symmetry constraints are weaker in AdS_3 than in H_3^+. Namely, the H_3^+ three-point function should be $\left\langle \prod_{i=1}^3 \Phi^{j_i}_{m_i,\bar{m}_i} \right\rangle = C(j_1,j_2,j_3) H(j_i,m_i,\bar{m}_i)$ where H is determined by $s\ell(2,\mathbb{C})$ symmetry; while the AdS_3 three-point function can in principle be $\left\langle \prod_{i=1}^3 \Phi^{j_i}_{m_i,\bar{m}_i} \right\rangle = C'(j_1,j_2,j_3|\alpha_1,\alpha_2,\alpha_3) H'(j_i,m_i,\bar{m}_i)$ with $m_i, \bar{m}_i \in \alpha_i + \mathbb{Z}$, where the $s\ell(2,\mathbb{R}) \times s\ell(2,\mathbb{R})$ symmetry determines H' but not the α_i-dependence.

For the full H_3^+ and AdS_3 WZNW models (and not just their minisuperspace limits), the assumption that these models are related by Wick rotation therefore determines part of the AdS_3 structure constants (analogs of C') in terms of the H_3^+ conformal blocks (analogs of H). This assumption is therefore rather nontrivial and it should be carefully justified. The best justification may come a posteriori, if an ansatz for the AdS_3 three-point function derived by Wick rotation can be shown to obey crossing symmetry. Such questions did not arise in the minisuperspace limit, as the bases of functions $\Phi^j_{m,\bar{m}}$ on H_3^+ and AdS_3 are related by Wick rotation by definition, and crossing symmetry amounts to the associativity of the product of functions on these spaces. But proving crossing symmetry – or equivalently the associativity of the operator product expansion – certainly is the most important and difficult task in solving the AdS_3 WZNW model.

Acknowledgments

I am very grateful to Kazuo Hosomichi for decisive help in constructing the t-basis of functions on AdS_3. Moreover I wish to thank Nicolas Crampé and Sabine Morisset for comments on the draft of this article.

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