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Almost reducibility of analytic quasi-periodic cocycles

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Astract: Let $G \subset GL(n, \mathbb{C})$ a classical Lie group, \mathcal{G} the Lie algebra associated to G , $\omega \in \mathbb{R}^d$ a diophantine vector, $A \in \mathcal{G}$ and a map $F \in C_r^\omega(\mathbb{T}^d, \mathcal{G})$ which is analytic on a neighbourhood of the torus of radius $r \leq \frac{1}{2}$, and $r' \in]0, r[$. There exists ϵ depending only on $n, d, A, r - r'$ and on the diophantine class of ω such that if $\|F\|_r \leq \epsilon$, then the quasi-periodic cocycle generated by $A + F$ is almost reducible in $C_{r'}^\omega(2\mathbb{T}^d, G)$. If G is a complex Lie group or $n = 2$, almost reducibility holds in $C_{r'}^\omega(\mathbb{T}^d, G)$ and reducible cocycles are dense near constant cocycles in a real analytic topology.

Introduction

0.1 Statement of the main result

Let $n \geq 1, d \geq 1, \omega \in \mathbb{R}^d$ and $0 < \kappa < 1, \tau > \max(1, d - 1)$. Suppose ω is diophantine with constant κ and exponent τ , that is

$$\forall m \in \mathbb{Z}^d \setminus \{0\}, |\langle m, \omega \rangle| \geq \frac{\kappa}{|m|^\tau} \quad (1)$$

One can assume without loss of generality that $\sup |\omega_i| \leq 1$. Denote by $\mathbb{T}^d := \mathbb{R}^d / \mathbb{Z}^d$ the d -torus and by $2\mathbb{T}^d := \mathbb{R}^d / (2\mathbb{Z}^d)$ the double torus.

Definition: Let \mathcal{G} be a Lie algebra, G the Lie group associated to \mathcal{G} and $A : 2\mathbb{T}^d \rightarrow \mathcal{G}$. The quasi-periodic cocycle associated to A is the map $X : 2\mathbb{T}^d \times \mathbb{R} \rightarrow G$ defined for all $(\theta, t) \in 2\mathbb{T}^d \times \mathbb{R}$ by

$$\frac{d}{dt} X^t(\theta) = A(\theta + t\omega) X^t(\theta); \quad X^0(\theta) = Id \quad (2)$$

We say it is a constant cocycle if A is constant.

Remark: Terminology is explained by the fact that A is the envelope of a quasi-periodic function. For all $\theta \in 2\mathbb{T}^d, t \mapsto A(\theta + t\omega)$ is indeed a quasi-periodic function.

A constant cocycle is always of the form $t \mapsto e^{tA}$.

We shall introduce an equivalence relation on cocycles. In order to do this, we will have to mention the regularity of the applications. Let us give the following definitions and notations:

Definition: A function f is *analytic on an r -neighbourhood of the torus* (resp. *double torus*) if f is holomorphic on $\{x = (x_1, \dots, x_d) \in \mathbb{C}^d, \sup_j |\operatorname{Im} x_j| < r\}$ and 1-periodic (resp. 2-periodic) in $\operatorname{Re} x_j$ for all $1 \leq j \leq d$.

For all subset E of $gl(n, \mathbb{C})$, denote by $C_r^\omega(\mathbb{T}^d, E)$ the set of functions which are analytic on an r -neighbourhood of the torus and whose restriction to \mathbb{T}^d takes its values in E ; let $C_r^\omega(2\mathbb{T}^d, E)$ be the set of functions which are analytic on an r -neighbourhood of the double torus and whose restriction to $2\mathbb{T}^d$ takes its values in E . For all $f \in C_r^\omega(2\mathbb{T}^d, E)$, denote

$$|f|_r = \sup_{|\operatorname{Im} x| < r} \|f(x)\| \quad (3)$$

where $\|\cdot\|$ stands for the operator norm.

Notation: For a function $f \in C^1(2\mathbb{T}^d, \mathbb{C})$, for all $\theta \in 2\mathbb{T}^d$ we will denote by

$$\partial_\omega f(\theta) = \frac{d}{dt} f(\theta + t\omega)|_{t=0} \quad (4)$$

the derivative of f in the direction ω .

Definition: Let \mathcal{G} be a Lie algebra and G the Lie group associated to \mathcal{G} . Let $r, r' > 0$ and $A, B \in C_{r'}^\omega(2\mathbb{T}^d, \mathcal{G})$. We say that A and B are *conjugate in $C_{r'}^\omega(2\mathbb{T}^d, G)$* if there exists $Z \in C_{r'}^\omega(2\mathbb{T}^d, G)$ such that for all $\theta \in 2\mathbb{T}^d$,

$$\partial_\omega Z(\theta) = A(\theta)Z(\theta) - Z(\theta)B(\theta)$$

If B is constant in θ , we say that A is *reducible in $C_{r'}^\omega(2\mathbb{T}^d, G)$* , or *reducible by Z to B* .

Remark: Let X be the quasi-periodic cocycle associated to A . The map A is reducible by Z to B if and only if

$$\forall(t, \theta), X^t(\theta) = Z(\theta + t\omega)^{-1} e^{tB} Z(\theta) \quad (5)$$

Reducibility is also equivalent to the fact that the map from $2\mathbb{T}^d \times \mathbb{R}^n$ to itself:

$$\begin{pmatrix} \theta \\ v \end{pmatrix} \mapsto \begin{pmatrix} \theta + \omega \\ X^1(\theta)v \end{pmatrix} \quad (6)$$

is conjugate to a map χ such that

$$\frac{d\chi}{d\theta} \begin{pmatrix} \theta \\ v \end{pmatrix} \equiv \begin{pmatrix} \bar{1} \\ 0 \end{pmatrix} \quad (7)$$

The aim of this paper is to show that for

$$G = GL(n, \mathbb{C}), GL(n, \mathbb{R}), SL(n, \mathbb{R}), Sp(n, \mathbb{R}), O(n), U(n)$$

in the neighbourhood of a constant cocycle, every cocycle which is analytic on an r -neighbourhood of the torus and G -valued is almost reducible in $C_{r'}^\omega(2\mathbb{T}^d, G)$ for all $0 < r' < r \leq \frac{1}{2}$. The radius of the neighbourhood only depends on the dimensions n, d , on the diophantine parameters κ, τ , on the constant cocycle and on the loss of analyticity $r - r'$.

We shall prove the following theorem, for G among the groups cited above and \mathcal{G} the Lie algebra associated to G :

Theorem 0.1 *Let $0 < r' < r \leq \frac{1}{2}$, $A \in \mathcal{G}$, $F \in C_r^\omega(\mathbb{T}^d, \mathcal{G})$. There is $\epsilon_0 < 1$ depending only on $n, d, \kappa, \tau, A, r - r'$ such that if*

$$|F|_r \leq \epsilon_0$$

then for all $\epsilon > 0$, there exists $\bar{A}_\epsilon, \bar{F}_\epsilon \in C_{r'}^\omega(2\mathbb{T}^d, \mathcal{G})$, $Z_\epsilon \in C_{r'}^\omega(2\mathbb{T}^d, G)$ such that for all $\theta \in 2\mathbb{T}^d$,

$$\partial_\omega Z_\epsilon(\theta) = (A + F(\theta))Z_\epsilon(\theta) - Z_\epsilon(\theta)(\bar{A}_\epsilon(\theta) + \bar{F}_\epsilon(\theta))$$

with

- \bar{A}_ϵ reducible in $C_{r'}^\omega(2\mathbb{T}^d, G)$,
- $|\bar{F}_\epsilon|_{r'} \leq \epsilon$,
- and $|Z_\epsilon - Id|_{r'} \leq 2\epsilon_0^{\frac{1}{2}}$.

Moreover, in dimension 2 or if $G = GL(n, \mathbb{C})$ or $U(n)$, $Z_\epsilon, \bar{A}_\epsilon, \bar{F}_\epsilon$ are continuous on \mathbb{T}^d .

There is a loss of analyticity in this result, but it is arbitrarily small.

0.2 Generalisation and consequences

Theorem 0.1 means that in the neighbourhood of a constant cocycle, all cocycles are almost reducible i.e they are arbitrarily close to a reducible cocycle, which suggests that reductibility is a predominant phenomenon. Reductibility implies almost reducibility, however the reverse is not true: there are non reducible cocycles even close to a constant cocycle (see [5]).

Almost reducibility is an interesting notion since the dynamics of an almost reducible cocycle are quite well known on a very long time.

Theorem 0.1 holds also if one chooses F in a class which is bigger than $C_r^\omega(\mathbb{T}^d, \mathcal{G})$, i.e the class of functions in $C_r^\omega(2\mathbb{T}^d, \mathcal{G})$ satisfying some "nice periodicity properties" with respect to the matrix A .

In dimension 2 or if \mathcal{G} is complex, this result can be rephrased as density of reducible cocycles in the neighbourhood of constant cocycles:

Theorem 0.2 *Let $\mathcal{G} = gl(n, \mathbb{C}), u(n), gl(2, \mathbb{R}), sl(2, \mathbb{R})$ or $o(2)$. Let $0 < r' < r \leq \frac{1}{2}$ and $A \in \mathcal{G}, F \in C_r^\omega(\mathbb{T}^d, \mathcal{G})$. There is ϵ_0 depending only on $r - r', n, d, \kappa, \tau, A$ such that if*

$$|F|_r \leq \epsilon_0$$

then for all $\epsilon > 0$ there exists $H \in C_{r'}^\omega(\mathbb{T}^d, \mathcal{G})$ which is reducible in $C_{r'}^\omega(\mathbb{T}^d, \mathcal{G})$ and such that

$$|A + F - H|_{r'} \leq \epsilon$$

0.3 Well-known results

A similar result, for smooth cocycles with values in compact Lie groups, was obtained by R. Krikorian in [8] (th.5.1.1). For cocycles over a rotation on the circle, analyticity is far better controlled (see for instance [1]) since it is then possible to use global methods. In this article, we are considering the case of a torus of arbitrary dimension. The KAM-type method that is being used here had already given birth to full-measure reducibility results for cocycles with values in $SL(2, \mathbb{R})$ ([5], [6]).

A result close to Theorem 0.1 in the case when $G = GL(n, \mathbb{R})$ had already been proven in [3] by L.H.Eliasson:

Let $A \in gl(n, \mathbb{R})$ and $F \in C_r^\omega(\mathbb{T}^d, gl(n, \mathbb{R}))$. There is $\epsilon_0 < 1$ depending only on $n, d, \kappa, \tau, \|A\|$ such that if $|F|_r \leq \epsilon_0$, then for all $\epsilon > 0$, there exists $0 < r_\epsilon < r$, $Z_\epsilon \in C_{r_\epsilon}^\omega(2\mathbb{T}^d, GL(n, \mathbb{R}))$ such that for all $\theta \in 2\mathbb{T}^d$,

$$\partial_\omega Z_\epsilon(\theta) = (A + F(\theta))Z_\epsilon(\theta) - Z_\epsilon(\theta)(A_\epsilon + F_\epsilon(\theta))$$

with $A_\epsilon \in gl(n, \mathbb{R}), F_\epsilon \in C_{r_\epsilon}^\omega(2\mathbb{T}^d, gl(n, \mathbb{R}))$ and $|F_\epsilon|_{r_\epsilon} \leq \epsilon$.

Remark: This result merely states almost reducibility in $C_0^\omega(2\mathbb{T}^d, GL(n, \mathbb{R}))$, since the sequence (r_ϵ) might well tend to 0.

The achievement of 0.1 is to state almost reducibility in a more general algebraic framework, but also, and mostly, to show that almost reducibility holds in a fixed neighbourhood of a torus even when this torus has dimension greater than 1.

Note that, as was the case in [3], one cannot avoid to lose periodicity in 0.1 if G is a real group with dimension greater than 2. The notion of "nice periodicity properties" that will be given aims at limiting this loss to a period doubling. In comparison with the real framework, the symplectic framework introduces new constraints in the elimination of resonances, but these constraints have no consequences on the construction of a renormalization; therefore there is no more loss of periodicity here than in the case when $G = GL(n, \mathbb{R})$. As before in [2], a single period doubling is sufficient in the case when G is a real symplectic group.

0.4 Sketch of the paper

The proof of Theorems 0.1 and 0.2 is a refinement of the method in [3]; it is based on a KAM scheme. Here are the main steps:

- Construction of a renormalization Φ of order R, \bar{N} (Proposition 2.3) for $R, N \in \mathbb{N} \setminus \{0\}$.

In dimension 2, Φ will be such that for all H continuous on \mathbb{T}^d , $\Phi H \Phi^{-1}$ is continuous on \mathbb{T}^d .

- Resolution of the homological equation (Proposition 3.2): if \tilde{A} has a spectrum fulfilling some diophantine conditions and \tilde{F} is a function with nice periodicity properties with respect to \tilde{A} , then there exists a solution \tilde{X} of equation

$$\partial_\omega \tilde{X} = [\tilde{A}, \tilde{X}] + \tilde{F}^{R\bar{N}}; \quad \hat{X}(0) = 0$$

having the same periodicity properties as \tilde{F} ; it takes its values in the same Lie algebra as does \tilde{F} . Moreover, it can be well controlled by losing some analyticity.

- Inductive lemma (Proposition 5.2): If $\tilde{F} \in C_r^\omega(2\mathbb{T}^d, \mathcal{G})$ has some periodicity properties (with respect to \tilde{A}), if

$$\partial_\omega \Psi = \bar{A}\Psi - \Psi\tilde{A}$$

and $\bar{F} = \Psi\tilde{F}\Psi^{-1}$, then there exists $Z \in C_r^\omega(2\mathbb{T}^d, G)$ such that

$$\partial_\omega Z = (\bar{A} + \bar{F})Z - Z(\bar{A}' + \bar{F}') \tag{8}$$

with \bar{A}' reducible, \bar{F}' is much smaller than \bar{F} , Z is close to the identity and $\Psi'^{-1}\bar{F}'\Psi'$ has periodicity properties with respect to A' which are similar to the properties of \tilde{F} .

The estimate of \bar{F}' depends on $\tilde{F} - \tilde{F}^{R\bar{N}}$, on the renormalization Φ , and on the solution \tilde{X} of the homological equation.

- Iteration of the inductive lemma (Theorem 5.4): We shall iterate Lemma 5.2 so as to obtain estimates of analytic functions on a sequence of neighbourhoods of the torus not tending to 0, by means of a numerical lemma (Lemma 5.3), to reduce the perturbation arbitrarily.

Remark: The renormalization is defined in a way similar to [3], however here it will remove resonances up to an order $R\bar{N}$ which is much greater than the value

of the parameter \bar{N} appearing in the estimates. To iterate the inductive lemma without letting the degree of analyticity tend to 0, parameter R will be used to define a renormalization map of order R, \bar{N} where \bar{N} does not depend on the loss of analyticity. This way, the renormalization map will stay under control on a neighbourhood of the torus which will not have to fade totally.

0.5 Notations

Denote by $\langle \cdot, \cdot \rangle$ the complex euclidian scalar product, taking it antilinear in the second variable. For a linear operator M , we shall call M^* its adjoint, which is identical to the transpose of M if M is real. Also denote by $M_{\mathcal{N}}$ the nilpotent part of M , as follows: let $M = PAP^{-1}$ with A in Jordan normal form, let A_D be the diagonal part of A , then $M_{\mathcal{N}} = P(A - A_D)P^{-1}$. To simplify the writing, if $A : 2\mathbb{T}^d \rightarrow GL(n, \mathbb{C})$, we will denote by A^{-1} the map $\theta \mapsto A(\theta)^{-1}$. For all $m = (m_1, \dots, m_d) \in \frac{1}{2}\mathbb{Z}^d$, we shall denote $|m| = |m_1| + \dots + |m_d|$. The letter J will stand for matrix $J = \begin{pmatrix} 0 & -Id \\ Id & 0 \end{pmatrix}$.

1 Nice periodicity properties

A few definitions will first be given. The notion of "triviality with respect to a decomposition" will make the construction of the renormalization easier; the "nice periodicity properties" have been introduced in [3] and are used in the real case to make sure that only one period doubling will be needed in iterating the inductive lemma.

1.1 Invariant decompositions

Definition: $\mathcal{L} = \{L_1, \dots, L_R\}$ is a *decomposition* of \mathbb{C}^n if $\mathbb{C}^n = \bigoplus_j L_j$.

Definition: Let $\mathcal{L}, \mathcal{L}'$ be decompositions of \mathbb{C}^n . Then \mathcal{L} is said to be *finer than* \mathcal{L}' if for all $L \in \mathcal{L}$, there is $L' \in \mathcal{L}'$ such that $L \subset L'$; \mathcal{L} is said *strictly finer than* \mathcal{L}' if \mathcal{L} is finer than \mathcal{L}' and $\mathcal{L} \neq \mathcal{L}'$.

Definition: Let $A \in gl(n, \mathbb{C})$; then $\mathcal{L} = \{L_1, \dots, L_s\}$ is an *A-decomposition*, or else *A-invariant decomposition*, if it is a decomposition of \mathbb{C}^n and for all i , $AL_i \subset L_i$. Subsets L_i are called *subspaces* of \mathcal{L} .

Remark: An *A-decomposition* is always less fine than the Jordan decomposition into generalized eigenspaces. Therefore, if operators A and A' have the same decomposition into generalized eigenspaces, then an *A-decomposition* is an *A'-decomposition*.

Notation: Let \mathcal{L} be an *A-decomposition*. For all $L \in \mathcal{L}$, denote by $\sigma(A|_L)$ the spectrum of the restriction of A to subspace L .

Definition: Let $\kappa' \geq 0$. Let $\mathcal{L}_{A, \kappa'}$ be the unique *A-decomposition* \mathcal{L} such that for all $L \neq L' \in \mathcal{L}$, $\alpha \in \sigma(A|_L)$ and $\beta \in \sigma(A|_{L'}) \Rightarrow |\alpha - \beta| > \kappa'$ and such that no *A-decomposition* strictly finer than \mathcal{L} has this property.

Definition: Let $A \in gl(n, \mathbb{C})$. Denote by \mathcal{L}_A the decomposition of \mathbb{C}^n which is the set of all generalized eigenspaces of A .

Remark: Note that \mathcal{L}_A does not have to coincide with $\mathcal{L}_{A,0}$. In general \mathcal{L}_A is finer than $\mathcal{L}_{A,0}$.

Definition: Let \mathcal{L} be a decomposition of \mathbb{C}^n . For all $u \in \mathbb{C}^n$, there is a unique decomposition $u = \sum_{L \in \mathcal{L}} u_L$ such that $u_L \in L$ for all $L \in \mathcal{L}$. For all $L \in \mathcal{L}$, the *projection on L with respect to \mathcal{L}* , denoted by $P_L^\mathcal{L}$, is the map defined by $P_L^\mathcal{L}u = u_L$.

Remark: Let $A \in gl(n, \mathbb{C})$ and $\kappa' > 0$. If \mathcal{L} is an A -decomposition which is less fine than $\mathcal{L}_{A,\kappa'}$, then one has the following lemma, which can be found in [3], appendix, Lemma A¹:

Lemma 1.1 *There is a constant $C_0 \geq 1$ depending only on n such that for all subspace $L \in \mathcal{L}$,*

$$\|P_L^\mathcal{L}\| \leq C_0 \left(\frac{1 + \|A_{\mathcal{N}}\|}{\kappa'} \right)^{n(n+1)} \quad (9)$$

In what follows, C_0 will always stand for this constant fixed in Lemma 1.1.

Definition: An (A, κ', γ) -*decomposition* is an A -decomposition \mathcal{L} such that for all $L \in \mathcal{L}$, the projection on L with respect to \mathcal{L} satisfies

$$\|P_L^\mathcal{L}\| \leq C_0 \left(\frac{1 + \|A_{\mathcal{N}}\|}{\kappa'} \right)^\gamma \quad (10)$$

Remark: For $A \in gl(n, \mathbb{C})$, one always has $A = \sum_{L, L' \in \mathcal{L}} P_L^\mathcal{L} A P_{L'}^\mathcal{L}$. In particular, if \mathcal{L} is an A -decomposition, then $A = \sum_{L \in \mathcal{L}} P_L^\mathcal{L} A P_L^\mathcal{L}$.

Definitions: Let \mathcal{L} be a decomposition. We say that

- \mathcal{L} is a *real decomposition* if for all $L \in \mathcal{L}$, $\bar{L} \in \mathcal{L}$;
- \mathcal{L} is a *symplectic decomposition* if it is a decomposition of \mathbb{C}^n with even n and for all $L \in \mathcal{L}$, there is a unique $L' \in \mathcal{L}$ such that $\langle L, JL' \rangle \neq 0$;
- \mathcal{L} is a *unitary decomposition* if for all $L \neq L' \in \mathcal{L}$, $\langle L, L' \rangle = 0$.

Remark:

- If A is a real matrix, then for all $\kappa' \geq 0$, $\mathcal{L}_{A,\kappa'}$ is a real decomposition.

¹Lemma A from [3] gives in fact an estimate which depends on $\|A\|$, but the proof shows clearly that the estimate in fact only depends on $A_{\mathcal{N}}$.

- For all L , there is at least one L' such that $\langle L, JL' \rangle \neq 0$. This comes from the fact that the symplectic form $\langle \cdot, J \cdot \rangle$ is non-degenerate.
- If $A \in sp(n, \mathbb{R})$, then any A -decomposition \mathcal{L} which is less fine than $\mathcal{L}_{A,0}$ is a real and symplectic decomposition. To see this, let $L, L' \in \mathcal{L}$ such that $\langle L, JL' \rangle \neq 0$; let $v \in L, v' \in L'$ be eigenvectors of A such that $\langle v, Jv' \rangle \neq 0$ and λ, λ' their associated eigenvalues. Then

$$\lambda \langle v, Jv' \rangle = \langle Av, Jv' \rangle = \langle v, A^* Jv' \rangle = -\langle v, JAv' \rangle = -\bar{\lambda}' \langle v, Jv' \rangle \quad (11)$$

and since $\langle v, Jv' \rangle \neq 0$, then $\lambda = -\bar{\lambda}'$.

- If $A \in U(n)$, then any decomposition which is less fine than $\mathcal{L}_{A,0}$ is unitary.
- If \mathcal{L} is unitary, then for every $L \in \mathcal{L}$, $P_L^\mathcal{L}$ is an orthogonal projection so

$$\| P_L^\mathcal{L} \| \leq 1 \quad (12)$$

1.2 Triviality and nice periodicity properties with respect to a decomposition

Definition: Let \mathcal{L} be a decomposition of \mathbb{C}^n . We say a map Ψ is *trivial with respect to* \mathcal{L} if there exist $\{m_L, L \in \mathcal{L}\} \subset \frac{1}{2}\mathbb{Z}^d$ such that for all $\theta \in 2\mathbb{T}^d$,

$$\Psi(\theta) = \sum_{L \in \mathcal{L}} e^{2i\pi \langle m_L, \theta \rangle} P_L^\mathcal{L} \quad (13)$$

Definition: We say that the function Ψ is *trivial* if there exists a decomposition \mathcal{L} such that Ψ is trivial with respect to \mathcal{L} .

Remark:

- If Ψ is trivial with respect to \mathcal{L} and \mathcal{L}' is finer than \mathcal{L} , then Ψ is trivial with respect to \mathcal{L}' .
- If $\Phi, \Psi : 2\mathbb{T}^d \rightarrow GL(n, \mathbb{C})$ are trivial with respect to \mathcal{L} , then the product $\Phi\Psi$ is trivial with respect to \mathcal{L} .
- If Φ is trivial with respect to an A -decomposition \mathcal{L} , then for all $\theta \in 2\mathbb{T}^d$, $[A, \Phi(\theta)] = 0$.

Lemma 1.2 *Let \mathcal{L} be a real decomposition of \mathbb{C}^n , $\{m_L, L \in \mathcal{L}\} \subset \frac{1}{2}\mathbb{Z}^d$ and Ψ defined by*

$$\Psi(\theta) = \sum_{L \in \mathcal{L}} e^{2i\pi \langle m_L, \theta \rangle} P_L^\mathcal{L} \quad (14)$$

Then Ψ is real if and only if for all L , $m_L = -m_{\bar{L}}$. Moreover, if Ψ is real, then Ψ takes its values in $SL(n, \mathbb{R})$.

Proof: Assume that for all $L \in \mathcal{L}$, $m_L = -m_{\bar{L}}$. Let $u \in \mathbb{R}^n$. Then

$$\overline{\Psi(\theta)u} = \sum_{L \in \mathcal{L}} e^{2i\pi\langle -m_L, \theta \rangle} \overline{P_L^\mathcal{L}u} = \sum_{L \in \mathcal{L}} e^{2i\pi\langle m_{\bar{L}}, \theta \rangle} P_{\bar{L}}^\mathcal{L}u = \Psi(\theta)u \quad (15)$$

so $\Psi(\theta)$ is real.

Now suppose that Ψ is real. Then for all θ ,

$$\sum_{L \in \mathcal{L}} e^{2i\pi\langle m_L, \theta \rangle} P_L^\mathcal{L} = \sum_{L \in \mathcal{L}} e^{2i\pi\langle -m_L, \theta \rangle} \overline{P_L^\mathcal{L}} = \sum_{L \in \mathcal{L}} e^{2i\pi\langle -m_L, \theta \rangle} P_{\bar{L}}^\mathcal{L} \quad (16)$$

so $m_L = -m_{\bar{L}}$.

Suppose Ψ is real; then for all L , $m_L = -m_{\bar{L}}$ so $\Psi(\theta)$ is the exponential of a trace-zero matrix, so it has determinant 1. \square

Remark: Any map which is trivial with respect to a unitary decomposition is unitary: let \mathcal{L} be a unitary decomposition, let Φ be trivial with respect to \mathcal{L} and let $L, L' \in \mathcal{L}$. Then for all $u \in \mathcal{L}, v \in \mathcal{L}'$,

$$\langle \Phi(\theta)u, \Phi(\theta)v \rangle = \langle e^{2i\pi\langle m_L, \theta \rangle} u, e^{2i\pi\langle m_{L'}, \theta \rangle} v \rangle = \langle u, v \rangle \quad (17)$$

Lemma 1.3 *Let \mathcal{L} be a real and symplectic decomposition and $\{m_L, L \in \mathcal{L}\}$ be a family of elements of $\frac{1}{2}\mathbb{Z}^d$. Let $\Psi = \sum_{L \in \mathcal{L}} e^{2i\pi\langle m_L, \cdot \rangle} P_L^\mathcal{L}$. Then Ψ takes its values in $Sp(n, \mathbb{R})$ if and only if*

- for all L , $m_L = -m_{\bar{L}}$
- and if $\langle L, JL' \rangle \neq 0$, then $m_L = m_{L'}$.

Proof: By Lemma 1.2, Ψ is real if and only if for all L , $m_L = -m_{\bar{L}}$. Assume now Ψ is real.

We show first that if for all $L, L' \in \mathcal{L}$, $\langle L, JL' \rangle \neq 0 \Rightarrow m_L = m_{L'}$, then Ψ takes its values in $Sp(n, \mathbb{R})$. Let $u, v \in \mathbb{R}^n$. Then

$$\langle u, \Psi(\theta)^* J \Psi(\theta) v \rangle = \langle \Psi(\theta)u, J \Psi(\theta)v \rangle = \sum_L e^{2i\pi\langle m_L - m_{M(L)}, \theta \rangle} \langle P_L^\mathcal{L}u, J P_{M(L)}^\mathcal{L}v \rangle \quad (18)$$

where $M(L)$ stands for the unique subspace such that $\langle L, JM(L) \rangle \neq 0$. Assume that if $\langle L, JL' \rangle \neq 0$, then $m_L = m_{L'}$. This implies that

$$\langle u, \Psi(\theta)^* J \Psi(\theta) v \rangle = \sum_L \langle P_L^\mathcal{L}u, J P_{M(L)}^\mathcal{L}v \rangle = \langle u, Jv \rangle \quad (19)$$

so $\Psi(\theta) \in Sp(n, \mathbb{R})$.

Now we will show that if $\Psi(\theta) \in Sp(n, \mathbb{R})$ and if $\langle L, JL' \rangle \neq 0$, then $m_L = m_{L'}$. Suppose $\Psi(\theta) \in Sp(n, \mathbb{R})$. For any two vectors u, v ,

$$\langle u, Jv \rangle = \langle u, \Psi(\theta)^* J \Psi(\theta) v \rangle = \langle \Psi(\theta) u, J \Psi(\theta) v \rangle \quad (20)$$

If $u \in L$ and $v \in m(L)$ satisfy $\langle u, Jv \rangle \neq 0$, then

$$\langle u, Jv \rangle = \langle \Psi(\theta) u, J \Psi(\theta) v \rangle = e^{2i\pi \langle m_L - m_{M(L)}, \theta \rangle} \langle u, Jv \rangle \quad (21)$$

so $m_L = m_{M(L)}$. \square

We will now define the periodicity properties.

Definition: Let \mathcal{L} be a decomposition of \mathbb{C}^n . We say that $F \in C^0(2\mathbb{T}^d, gl(n, \mathbb{R}))$ has *nice periodicity properties with respect to \mathcal{L}* if there exists a map Φ which is trivial with respect to \mathcal{L} and such that $\Phi^{-1} F \Phi$ is continuous on \mathbb{T}^d .

To make the family (m_L) explicit, we say that F has *nice periodicity properties with respect to \mathcal{L} and (m_L)* .

Remark:

- If $F \in C^0(2\mathbb{T}^d, gl(n, \mathbb{R}))$ has nice periodicity properties with respect to a decomposition \mathcal{L} and Φ is trivial with respect to \mathcal{L} , then $\Phi F \Phi^{-1}$ has nice periodicity properties with respect to \mathcal{L} .
- If \mathcal{L}' is a decomposition of \mathbb{C}^n which is finer than \mathcal{L} and F has nice periodicity properties with respect to \mathcal{L} , then F has nice periodicity properties with respect to \mathcal{L}' .
- Let \mathcal{L} be a decomposition of \mathbb{C}^n and $(m_L)_{L \in \mathcal{L}}$ be a family of elements of $\frac{1}{2}\mathbb{Z}^d$. If $F_1, F_2 \in C^0(2\mathbb{T}^d, gl(n, \mathbb{R}))$ have nice periodicity properties with respect to \mathcal{L} and (m_L) , then the product $F_1 F_2$ has nice periodicity properties with respect to \mathcal{L} and (m_L) .

2 Removing the resonances

In the following we will have to solve a homological equation and estimate the solution on a neighbourhood of the torus; in order to have a sufficient estimate, one will assume that the coefficients of the equation satisfy some diophantine conditions:

Let $A \in gl(n, \mathbb{R})$ and $0 < \kappa' < 1$. Let $N \in \mathbb{N}$.

Definition: Let $z \in \mathbb{C}, \nu \in \{1, 2\}$. We say that z is *diophantine modulo ν with respect to ω , with constant κ' , exponent τ and order N* if for every $m \in \frac{1}{\nu}\mathbb{Z}^d$ such that $0 < |m| \leq N$,

$$|z - 2i\pi\langle m, \omega \rangle| \geq \frac{\kappa'}{|m|^\tau} \quad (22)$$

This property will be denoted by

$$z \in DC_{\omega, \nu}^N(\kappa', \tau) \quad (23)$$

Note that

$$DC_{\omega, 2}^N(\kappa', \tau) \subset DC_{\omega, 1}^N(\kappa', \tau) \quad (24)$$

and that every real number z is in $DC_{\omega, 2}^N(\frac{\kappa}{2^\tau}, \tau)$ since for all $m \in \frac{1}{2}\mathbb{Z}^d$,

$$|z - 2i\pi\langle m, \omega \rangle| = (|z|^2 + (2\pi|\langle m, \omega \rangle|)^2)^{\frac{1}{2}} \geq \frac{\pi\kappa}{|2m|^\tau} \geq \frac{\kappa}{|2m|^\tau} \quad (25)$$

Remark: In the definition above, the condition is required only for non vanishing m , so (22) has a meaning.

Definition: A is said to have $DC_{\omega}^N(\kappa', \tau)$ spectrum if

$$\begin{cases} \forall \alpha, \beta \in \sigma(A), \alpha - \beta \in DC_{\omega, 1}^N(\kappa', \tau) \\ \forall \alpha, \beta \in \sigma(A), \alpha \neq \beta \Rightarrow \alpha - \beta \in DC_{\omega, 2}^N(\kappa', \tau) \end{cases} \quad (26)$$

Let $N \in \mathbb{N}$. Let A in a Lie algebra \mathcal{G} . The aim is to show that there exists $\kappa' > 0$, $\tilde{A} \in \mathcal{G}$ such that \tilde{A} has $DC_{\omega}^N(\kappa', \tau)$ spectrum and A and \tilde{A} are conjugate (in the acceptance of cocycles, following the definition given in the introduction). To achieve this, one has to find a family (m_1, \dots, m_n) satisfying

$$\begin{cases} \forall \alpha_j, \alpha_k \in \sigma(A), \alpha_j - \alpha_k + 2i\pi\langle m_j - m_k, \omega \rangle \in DC_{\omega, 1}^N(\kappa', \tau) \\ \forall \alpha_j, \alpha_k \in \sigma(A), \alpha_j \neq \alpha_k \Rightarrow \alpha_j - \alpha_k + 2i\pi\langle m_j - m_k, \omega \rangle \in DC_{\omega, 2}^N(\kappa', \tau) \end{cases} \quad (27)$$

We shall construct the so-called renormalization map Φ conjugating (in the sense of cocycles) A to the matrix obtained from A by substituting an eigenvalue α_j by $\alpha_j + 2i\pi\langle m_j, \omega \rangle$, then we will prove that Φ is G -valued.

2.1 Diophantine conditions

Lemma 2.1 *Let $\{\alpha_1, \dots, \alpha_n\} \subset \mathbb{C}$. Let $\tilde{N} \in \mathbb{N}$ and $\kappa' \leq \frac{\kappa}{n(8N)^\tau}$. There exists $m_1, \dots, m_n \in \frac{1}{2}\mathbb{Z}^d$ such that $\sup_j |m_j| \leq \tilde{N}$, and such that letting for all j , $\tilde{\alpha}_j = \alpha_j - 2i\pi\langle m_j, \omega \rangle$, then*

$$\{\alpha_1, \dots, \alpha_n\} = \overline{\{\alpha_1, \dots, \alpha_n\}} \Rightarrow \forall j, k, \alpha_j = \bar{\alpha}_k \Rightarrow m_j = -m_k \quad (28)$$

$$\forall j, k, \alpha_j = -\bar{\alpha}_k \Rightarrow m_j = m_k \quad (29)$$

$$\forall j, k, |\alpha_j - \alpha_k| \leq \kappa' \Rightarrow m_j = m_k \quad (30)$$

$$\forall j, |Im\tilde{\alpha}_j| \leq |Im\alpha_j| \quad (31)$$

$$\forall j, k, \alpha_j = \bar{\alpha}_k \Rightarrow \tilde{\alpha}_j - \tilde{\alpha}_k \in DC_{\omega,1}^{\tilde{N}}(\kappa', \tau) \quad (32)$$

and

$$\forall j, k, \alpha_j \neq \bar{\alpha}_k \Rightarrow \tilde{\alpha}_j - \tilde{\alpha}_k \in DC_{\omega,2}^{\tilde{N}}(\kappa', \tau) \quad (33)$$

and such that if not all m_j vanish, then there exist j, k such that

$$|\alpha_j - \alpha_k| \geq \kappa', \quad |\tilde{\alpha}_j - \tilde{\alpha}_k| < \kappa' \quad (34)$$

Moreover, there exist $m_1, \dots, m_n \in \mathbb{Z}^d$, with $|m_j| \leq \tilde{N}$ for all j , fulfilling conditions (29), (30), (31), such that

$$\forall j, k, \tilde{\alpha}_j - \tilde{\alpha}_k \in DC_{\omega,1}^{\tilde{N}}(\kappa', \tau) \quad (35)$$

and such that if not all m_j vanish, then there exist j, k such that (34) holds.

Proof: We shall proceed in two steps. The first step consists in removing resonances which might occur between two eigenvalues whose imaginary parts are nearly opposite to each other. Once this first lot of resonances is removed, the second step consists in removing the resonances which might occur between two eigenvalues whose imaginary parts are far from opposite.

- Let $1 \leq j \leq n$. Suppose that there is an $m \in \mathbb{Z}^d, 0 < |m| \leq \tilde{N}$ such that

$$|2Im\alpha_j - 2\pi\langle m, \omega \rangle| < \frac{\kappa'}{|m|^\tau} \quad (36)$$

then let $\alpha'_j = \alpha_j - 2i\pi\langle \frac{m}{2}, \omega \rangle$. Otherwise, let $\alpha'_j = \alpha_j$. Note that if $|\alpha_j - \alpha_k| \leq \kappa'$ and if there exist $m_j \neq m_k$ such that

$$|2Im\alpha_j - 2\pi\langle m_j, \omega \rangle| < \frac{\kappa'}{|m_j|^\tau}; \quad |2Im\alpha_k - 2\pi\langle m_k, \omega \rangle| < \frac{\kappa'}{|m_k|^\tau} \quad (37)$$

then

$$|2i\pi\langle m_j - m_k, \omega \rangle| \leq \frac{\kappa}{|m_j - m_k|^\tau} \quad (38)$$

which is impossible since ω is diophantine. Therefore conditions (28) to (32) hold with $\alpha'_j = \tilde{\alpha}_j$ and m_j such that $\alpha_j - \alpha'_j = 2i\pi\langle m_j, \omega \rangle$.

• Let I_{-r}, \dots, I_r be the finest partition of $\{1, \dots, n\}$ such that

$$|Im(\alpha'_j - \alpha'_k)| \leq \kappa' \Rightarrow \exists -r \leq r' \leq r \mid j, k \in I_{r'} \quad (39)$$

and choose the indices in such a way that

$$r' < r'' \Rightarrow \forall j \in I_{r'}, \forall k \in I_{r''}, Im\alpha'_j \leq Im\alpha'_k \quad (40)$$

Note that I_0 might be empty. We will proceed by induction on r' to prove the following property $\mathcal{P}(r')$:

There are $m'_1, m'_{-1}, \dots, m'_{r'}, m'_{-r'} \in \mathbb{Z}^d$ with $\sup_{|j| \leq r'} |m'_j| \leq \tilde{N}$ such that properties (28) to (33) hold for all $-r' \leq r_1, r_2 \leq r', j \in I_{r_1}, k \in I_{r_2}$ with m'_j instead of m_j and α'_j instead of α_j .

• Case $r' = 0$: if I_0 is empty, then $\mathcal{P}(0)$ trivially holds. Assume I_0 is non empty. Then for all $j, k \in I_0$ and all $m \in \frac{1}{2}\mathbb{Z}^d$ such that $0 < |m| \leq \tilde{N}$,

$$|\alpha'_j - \alpha'_k - 2i\pi\langle m, \omega \rangle| \geq |Im(\alpha'_j - \alpha'_k) - 2\pi\langle m, \omega \rangle| \geq \frac{\kappa}{|m|^\tau} - n\kappa' \geq \kappa' \quad (41)$$

so $\alpha'_j - \alpha'_k \in DC_{\omega, 2}^{\tilde{N}}(\kappa', \tau)$ and $\mathcal{P}(0)$ holds true.

• Let $r' \leq r - 1$. Assume $\mathcal{P}(r')$ holds. Consider $I_{r'+1}$ and $I_{-r'-1}$. There are two possible cases.

• There exist $-r' \leq r'' \leq r', j \in I_{r''}, k \in I_{r'+1}$ and $m \in \mathbb{Z}^d$ such that $|m| \leq \tilde{N}$ and

$$|\alpha'_j - \alpha'_k - 2i\pi\langle m_{r''} + m, \omega \rangle| < \frac{\kappa'}{|m|^\tau} \quad (42)$$

- The case above does not hold.

In the first case, let $m'_{r'+1} = m = -m'_{-r'-1}$. In the second case, let $m'_{r'+1} = m'_{-r'-1} = 0$.

Now $m'_{r'+1}$ and $m'_{-r'-1}$ are independent from j, k . To see this, suppose there are $j_1, j_2 \in I_{r_1}, k_1, k_2 \in I_{r_2}, m_1 \neq m_2 \in \mathbb{Z}^d$ such that for $l = 1, 2$,

$$|\alpha'_{j_l} - \alpha'_{k_l} - 2i\pi\langle m_l, \omega \rangle| < \frac{\kappa'}{|m_l|^\tau} \quad (43)$$

Then

$$|2\pi\langle m_1 - m_2, \omega \rangle| \leq \frac{\kappa}{|m_1 - m_2|^\tau} \quad (44)$$

which is impossible. Therefore $\mathcal{P}(r' + 1)$ holds true.

- Once $m'_1, \dots, m'_r, m'_{-1}, \dots, m'_{-r} \in \mathbb{Z}^d$ are defined, conditions (28) to (33) hold with, for all $j \in I_{r'}$, $\tilde{\alpha}_j = \alpha'_j - 2i\pi\langle m'_{r'}, \omega \rangle$ and m_j such that $\alpha_j - \tilde{\alpha}_j = 2i\pi\langle m_j, \omega \rangle$. Condition (34) is obvious by construction.

- By proceeding only with the second step, one gets $m_1, \dots, m_n \in \mathbb{Z}^d$, with $|m_j| \leq \tilde{N}$ for all j , satisfying conditions (29), (30), (31), such that

$$\forall j, k, \tilde{\alpha}_j - \tilde{\alpha}_k \in DC_{\omega,1}^{\tilde{N}}(\kappa', \tau) \quad (45)$$

and such that if not all m_j vanish, then there are j, k such that (34) holds true. \square

Lemma 2.2 *Let $\{\alpha_1, \dots, \alpha_n\} \subset \mathbb{C}$. For every $R, N \in \mathbb{N}, N \geq 2, R \geq 1$, there exists $\bar{N} \in [N, R^{\frac{1}{2}n(n-1)}N]$ and $m_1, \dots, m_n \in \frac{1}{2}\mathbb{Z}^d$ with*

$$\sup_j |m_j| \leq 2\bar{N} \quad (46)$$

such that letting $\tilde{\alpha}_j = \alpha_j - 2i\pi\langle m_j, \omega \rangle$ and

$$\kappa'' = \frac{\kappa}{n(8R^{\frac{1}{2}n(n-1)+1}N)^\tau} \quad (47)$$

conditions (28) to (31) of Lemma 2.1 hold for $\kappa' = \kappa''$, and such that

$$\forall j, k, \tilde{\alpha}_j - \tilde{\alpha}_k \in DC_{\omega,1}^{R\bar{N}}(\kappa'', \tau) \quad (48)$$

and

$$\forall j, k, \alpha_j \neq \bar{\alpha}_k \Rightarrow \tilde{\alpha}_j - \tilde{\alpha}_k \in DC_{\omega,2}^{R\bar{N}}(\kappa'', \tau) \quad (49)$$

Moreover, there exist $m_1, \dots, m_n \in \mathbb{Z}^d$ with $|m_j| \leq \bar{N}$ for all j such that conditions (29), (30), (31) and (48) hold true.

Proof: If α_j satisfy for all j, k

$$\begin{cases} \alpha_j = \bar{\alpha}_k \Rightarrow \alpha_j - \alpha_k \in DC_{\omega,1}^{RN}(\kappa'', \tau) \\ \alpha_j \neq \bar{\alpha}_k \Rightarrow \alpha_j - \alpha_k \in DC_{\omega,2}^{RN}(\kappa'', \tau) \end{cases} \quad (50)$$

then we are done with $\bar{N} = N$ and $m_1 = \dots = m_n = 0$.

Suppose (50) does not hold. Then apply Lemma 2.1 with $\tilde{N} = RN, \kappa' = \kappa''$ to get m_1^1, \dots, m_n^1 such that

$$\begin{cases} \forall j, k, \alpha_j = \bar{\alpha}_k \Rightarrow m_j^1 = -m_k^1 \\ \forall j, k, \alpha_j = -\bar{\alpha}_k \Rightarrow m_j^1 = m_k^1 \\ \forall j, k, |\alpha_j - \alpha_k| \leq \kappa'' \Rightarrow m_j^1 = m_k^1 \\ \forall j, |Im\alpha_j - 2i\pi\langle m_j^1, \omega \rangle| \leq |Im\alpha_j| \end{cases} \quad (51)$$

and

$$\begin{cases} \alpha_j = \bar{\alpha}_k \Rightarrow \alpha_j - \alpha_k - 2i\pi\langle m_j^1 - m_k^1, \omega \rangle \in DC_{\omega,1}^{RN}(\kappa'', \tau) \\ \alpha_j \neq \bar{\alpha}_k \Rightarrow \alpha_j - \alpha_k - 2i\pi\langle m_j^1 - m_k^1, \omega \rangle \in DC_{\omega,2}^{RN}(\kappa'', \tau) \end{cases} \quad (52)$$

and such that there exist j_1, k_1 satisfying $|Im(\alpha_{j_1} - \alpha_{k_1}) - 2i\pi\langle m_{j_1}^1 - m_{k_1}^1, \omega \rangle| < \kappa''$.

Assume there are m_1^r, \dots, m_n^r such that $\sup |m_j^r| \leq (R + R^2 + \dots + R^r)N$ and that for all j, k ,

$$\begin{cases} \forall j, k, \alpha_j = \bar{\alpha}_k \Rightarrow m_j^r = -m_k^r \\ \forall j, k, \alpha_j = -\bar{\alpha}_k \Rightarrow m_j^r = m_k^r \\ \forall j, k, |\alpha_j - \alpha_k| \leq \kappa'' \Rightarrow m_j^r = m_k^r \\ \forall j, |Im\alpha_j - 2i\pi\langle m_j^r, \omega \rangle| \leq |Im\alpha_j| \end{cases} \quad (53)$$

and

$$\begin{cases} \alpha_j = \bar{\alpha}_k \Rightarrow \alpha_j - \alpha_k - 2i\pi\langle m_j^r - m_k^r, \omega \rangle \in DC_{\omega,1}^{R^r N}(\kappa'', \tau) \\ \alpha_j \neq \bar{\alpha}_k \Rightarrow \alpha_j - \alpha_k - 2i\pi\langle m_j^r - m_k^r, \omega \rangle \in DC_{\omega,2}^{R^r N}(\kappa'', \tau) \end{cases} \quad (54)$$

and suppose there exist distinct $(j_1, k_1), \dots, (j_r, k_r)$ such that for all $l \leq r$,

$$|Im\alpha_{j_l} - Im\alpha_{k_l} - 2i\pi\langle m_{j_l}^r - m_{k_l}^r, \omega \rangle| < \kappa'' \quad (55)$$

If moreover one has for all j, k

$$\begin{cases} \alpha_j = \bar{\alpha}_k \Rightarrow \alpha_j - \alpha_k - 2i\pi\langle m_j^r - m_k^r, \omega \rangle \in DC_{\omega,1}^{R^{r+1}N}(\kappa'', \tau) \\ \alpha_j \neq \bar{\alpha}_k \Rightarrow \alpha_j - \alpha_k - 2i\pi\langle m_j^r - m_k^r, \omega \rangle \in DC_{\omega,2}^{R^{r+1}N}(\kappa'', \tau) \end{cases} \quad (56)$$

then the process ends with $\bar{N} = R^r N$ and $m_j = m_j^r$ since it is true that

$$|m_j^r| \leq (R + R^2 + \cdots + R^r)N \leq R^r N \frac{1 - \frac{1}{R^r}}{1 - \frac{1}{R}} \leq 2R^r N \quad (57)$$

Otherwise, iterate once more Lemma 2.1 with $\tilde{N} = R^{r+1}N$ and $\alpha_j - 2i\pi\langle m_j^r, \omega \rangle$ in place of α_j to get $m_1^{r+1}, \dots, m_n^{r+1}$ such that $\sup |m_j^{r+1}| \leq (R + R^2 + \cdots + R^{r+1})N$ and for all j, k ,

$$\left\{ \begin{array}{l} \forall j, k, \alpha_j = \bar{\alpha}_k \Rightarrow m_j^{r+1} = -m_k^{r+1} \\ \forall j, k, \alpha_j = -\bar{\alpha}_k \Rightarrow m_j^{r+1} = m_k^{r+1} \\ \forall j, k, |\alpha_j - \alpha_k| \leq \kappa'' \Rightarrow m_j^{r+1} = m_k^{r+1} \\ \forall j, |Im\alpha_j - 2i\pi\langle m_j^{r+1}, \omega \rangle| \leq |Im\alpha_j| \end{array} \right. \quad (58)$$

and

$$\left\{ \begin{array}{l} \alpha_j = \bar{\alpha}_k \Rightarrow \alpha_j - \alpha_k - 2i\pi\langle m_j^{r+1} - m_k^{r+1}, \omega \rangle \in DC_{\omega,1}^{R^{r+1}N}(\kappa'', \tau) \\ \alpha_j \neq \bar{\alpha}_k \Rightarrow \alpha_j - \alpha_k - 2i\pi\langle m_j^{r+1} - m_k^{r+1}, \omega \rangle \in DC_{\omega,2}^{R^{r+1}N}(\kappa'', \tau) \end{array} \right. \quad (59)$$

and that there exist distinct $(j_1, k_1), \dots, (j_{r+1}, k_{r+1})$ such that for all $l \leq r+1$,

$$|Im\alpha_{j_l} - Im\alpha_{k_l} - 2i\pi\langle m_{j_l}^{r+1} - m_{k_l}^{r+1}, \omega \rangle| < \kappa'' \quad (60)$$

Therefore, for all $1 \leq l \leq r+1$,

$$|\alpha_{j_l} - \alpha_{k_l} - 2i\pi\langle m_{j_l}^{r+1} - m_{k_l}^{r+1}, \omega \rangle| < \kappa'' \quad (61)$$

This implies that for all $m \in \frac{1}{2}\mathbb{Z}^d$ such that $0 < |m| \leq R\bar{N}$ and for all $l, 1 \leq l \leq r+1$,

$$|\alpha_{j_l} - \alpha_{k_l} - 2i\pi\langle m_{j_l}^{r+1} - m_{k_l}^{r+1}, \omega \rangle - 2i\pi\langle m, \omega \rangle| \geq \frac{\kappa}{2^{\tau+1}(R\bar{N})^\tau} - \kappa'' \geq \kappa'' \quad (62)$$

so for all $l \leq r+1$,

$$\alpha_{j_l} - \alpha_{k_l} - 2i\pi\langle m_{j_l}^{r+1} - m_{k_l}^{r+1}, \omega \rangle \in DC_{\omega,2}^{R\bar{N}}(\kappa'', \tau) \quad (63)$$

Therefore, after $\bar{r} \leq \frac{n(n-1)}{2}$ steps, one gets conditions (48) and (49) with $m_j = m_j^{\bar{r}}$ and $\tilde{\alpha}_j = \alpha_j - 2i\pi\langle m_j, \omega \rangle$ and $|Im\alpha_j - 2i\pi\langle m_j, \omega \rangle| \leq |Im\alpha_j|$. It is true that $|m_j^{\bar{r}}| \leq 2\bar{N}$ and conditions (28) to (31) of Lemma 2.1 are also satisfied.

Lemma 2.1 implies that if conditions (28) and (49) are not required, then one can get $m_1, \dots, m_n \in \mathbb{Z}^d$. \square

2.2 Renormalization

Now the preceding lemmas will be used to define the renormalization map Φ which will conjugate A to a matrix with $DC_\omega^{RN}(\kappa'', \tau)$ spectrum for some κ'' , with R, N arbitrarily great and Φ bounded independently of R .

In all that follows, \mathcal{G} will be a Lie algebra among

$$gl(n, \mathbb{C}), gl(n, \mathbb{R}), sp(n, \mathbb{R}), sl(n, \mathbb{R}), o(n), u(n)$$

and G will be the Lie group associated to \mathcal{G} .

Proposition 2.3 *Let $A \in \mathcal{G}$, $R \geq 1$ and $N \in \mathbb{N}$. There exists $\bar{N} \in [N, R^{\frac{1}{2}n(n-1)}N]$ such that if*

$$\kappa'' = \frac{\kappa}{n(8R^{\frac{1}{2}n(n-1)+1}N)\tau} \quad (64)$$

then there exists a map Φ which is trivial with respect to $\mathcal{L}_{A, \kappa''}$ and G -valued and such that

1. for all $r' \geq 0$,

$$|\Phi|_{r'} \leq nC_0 \left(\frac{1 + \|A_{\mathcal{N}}\|}{\kappa''} \right)^{n(n+1)} e^{4\pi\bar{N}r'}, \quad |\Phi^{-1}|_{r'} \leq nC_0 \left(\frac{1 + \|A_{\mathcal{N}}\|}{\kappa''} \right)^{n(n+1)} e^{4\pi\bar{N}r'} \quad (65)$$

2. Let \tilde{A} be such that

$$\forall \theta \in 2\mathbb{T}^d, \quad \partial_\omega \Phi(\theta) = A\Phi(\theta) - \Phi(\theta)\tilde{A} \quad (66)$$

then

$$\|\tilde{A} - A\| \leq 4\pi\bar{N} \quad (67)$$

and \tilde{A} has $DC_\omega^{R\bar{N}}(\kappa'', \tau)$ spectrum.

3. If $\mathcal{G} = gl(n, \mathbb{C})$ or $u(n)$, Φ is defined on \mathbb{T}^d .

4. If $\mathcal{G} = o(n)$ or $u(n)$, then

$$|\Phi|_{r'} \leq ne^{4\pi\bar{N}r'}, \quad |\Phi^{-1}|_{r'} \leq ne^{4\pi\bar{N}r'} \quad (68)$$

Proof: Let $\{\alpha_1, \dots, \alpha_n\} = \sigma(A)$. Two cases must be considered:

- If $\mathcal{G} = gl(n, \mathbb{C})$ or $u(n)$, Lemma 2.2 gives \bar{N} and $m_j \in \mathbb{Z}^d$ for $j = 1, \dots, n$ such that

$$N \leq \bar{N} \leq R^{\frac{1}{2}n(n-1)}N; \sup_j |m_j| \leq 2\bar{N} \quad (69)$$

and such that conditions (29) to (31) of Lemma 2.1 hold with $\kappa' = \kappa''$, as well as conditions (48).

- If $\mathcal{G} = gl(n, \mathbb{R}), sp(n, \mathbb{R}), sl(n, \mathbb{R})$ or $o(n)$, Lemma 2.2 gives \bar{N} and $m_j \in \frac{1}{2}\mathbb{Z}^d$ for $j = 1, \dots, n$ such that

$$N \leq \bar{N} \leq R^{\frac{1}{2}n(n-1)}N; \sup_j |m_j| \leq 2\bar{N} \quad (70)$$

and such that conditions (28) to (31) of Lemma 2.1 hold with $\kappa' = \kappa''$, as well as conditions (48) and (49).

For all j there is a unique $L \in \mathcal{L}_{A, \kappa''}$ such that $\alpha_j \in \sigma(A|_L)$. Let $m_L = m_j$. Then m_L is independent of j thanks to property (30).

For all $\theta \in 2\mathbb{T}^d$, let

$$\Phi(\theta) = \sum_{L \in \mathcal{L}_{A, \kappa''}} e^{2i\pi \langle m_L, \theta \rangle} P_L^{\mathcal{L}_{A, \kappa''}} \quad (71)$$

By construction of the (m_L) , Φ is defined on \mathbb{T}^d if $\mathcal{G} = gl(n, \mathbb{C})$ or $u(n)$. Let us prove that Φ is G -valued.

- if $\mathcal{G} = gl(n, \mathbb{C})$, this is trivial;
- if $\mathcal{G} = u(n)$, Φ has unitary values.
- if $\mathcal{G} = gl(n, \mathbb{R})$, this comes from Lemma 1.2, since $\mathcal{L}_{A, \kappa''}$ is a real decomposition and according to Lemma 2.2, for all $L \in \mathcal{L}_{A, \kappa''}$, $m_L = -m_{\bar{L}}$.
- if $\mathcal{G} = o(n)$, the map Φ has values in real unitary matrices, i.e orthogonal matrices.
- if $\mathcal{G} = sp(n, \mathbb{R})$, $\mathcal{L}_{A, \kappa''}$ is a symplectic decomposition. Lemma 2.2 ensures that for all $L \in \mathcal{L}_{A, \kappa''}$, $m_L = -m_{\bar{L}}$ and

$$\forall L, L' \in \mathcal{L}_{A, \kappa''}, \langle L, JL' \rangle \neq 0 \Rightarrow m_L = m_{L'} \quad (72)$$

Therefore Lemma 1.3 implies that for all θ the matrix $\Phi(\theta)$ is in $Sp(n, \mathbb{R})$.

Properties (48) and (49) ensure that \tilde{A} has $DC_\omega^{R\bar{N}}(\kappa'', \tau)$ spectrum. Moreover, for all $L \in \tilde{\mathcal{L}}$, $|m_L| \leq 2\bar{N}$. The estimate of each $P_L^{\tilde{\mathcal{L}}}$ recalled in Lemma 1.1 implies that Φ satisfies the estimate

$$|\Phi|_{r'} \leq nC_0 \left(\frac{1 + \|A_{\mathcal{N}}\|}{\kappa''} \right)^{n(n+1)} e^{4\pi\bar{N}r'} \quad (73)$$

and so does Φ^{-1} since

$$\Phi^{-1} = \sum_{L \in \mathcal{L}_{A, \kappa''}} e^{-2i\pi\langle m_L, \cdot \rangle} P_L^{\mathcal{L}_{A, \kappa''}} \quad (74)$$

Now if \mathcal{G} is $o(n)$ or $u(n)$, then every projection $P_L^{\mathcal{L}_{A, \kappa''}}$ has norm 1 and therefore Φ and Φ^{-1} satisfy (68). By definition of \tilde{A} ,

$$\forall L \in \mathcal{L}', \quad \sigma(\tilde{A}|_L) = \sigma(A|_L) - 2i\pi\langle m_L, \omega \rangle \quad (75)$$

and by property (31),

$$\forall \alpha \in \sigma(A|_L), \quad |\alpha - 2i\pi\langle m_L, \omega \rangle| \leq |\alpha| \quad (76)$$

Let P be such that PAP^{-1} is in Jordan normal form, let α_j be the eigenvalues of A and p_j the columns of P , then for all j ,

$$\|(\tilde{A} - A)p_j\| = \|2i\pi\langle m_j, \omega \rangle p_j\| \leq 4\pi\bar{N}\|p_j\| \quad (77)$$

So $\|\tilde{A} - A\| \leq 4\pi\bar{N}$, whence property (67). \square

Definition: A map Φ satisfying the conclusion of Proposition 2.3 will be called a *renormalization of A of order R, \bar{N}* .

In dimension 2, the renormalization map Φ satisfies the following property: for every function H continuous on \mathbb{T}^d and with values in $gl(2, \mathbb{C})$, $\Phi H \Phi^{-1}$ and $\Phi^{-1} H \Phi$ are continuous on \mathbb{T}^d .

Dimension 2 has, indeed, the particularity that every decomposition \mathcal{L} of \mathbb{R}^2 at most two subspaces L_1, L_2 , in which case $m_{L_1} + m_{L_2} \in \mathbb{Z}^d$ (if the decomposition is trivial, $m_L = 0$). In any case, $\sum_{L \in \mathcal{L}} m_L \in \mathbb{Z}^d$.

3 Homological equation

Solving the homological equation is a first step towards reducing the perturbation.

Notation: For every function $F \in L^2(2\mathbb{T}^d)$ and every $N \in \mathbb{N}$, we will denote by F^N and call truncation of F at order N the function that one obtains by truncating the Fourier series of F :

$$F^N(\theta) = \sum_{|m| \leq N} \hat{F}(m) e^{2i\pi\langle m, \theta \rangle}$$

The following lemma will be useful in the solving of the homological equation.

Lemma 3.1 *Let f, g be trigonometric polynomial with g real on \mathbb{R}^d . Let $r > 0, r' \in]0, r[$ and suppose that there exists C such that $|f|_{r'} \leq C|g|_r$. Then for all $m \in \frac{1}{2}\mathbb{Z}^d$,*

$$|f e^{2i\pi\langle m, \cdot \rangle}|_{r'} \leq C |g e^{2i\pi\langle m, \cdot \rangle}|_r \quad (78)$$

Proof: Since g is real,

$$\forall m \in \mathbb{Z}^d, \overline{\hat{g}(-m)} = \hat{g}(m) \quad (79)$$

so for all x and all $y \in [-r, r]^d$,

$$g(x-iy) = \sum_m \hat{g}(m) e^{2i\pi\langle m, x-iy \rangle} = \sum_m \overline{\hat{g}(-m)} e^{2i\pi\langle -m, -x+iy \rangle} = \overline{\sum_m \hat{g}(-m) e^{2i\pi\langle -m, x+iy \rangle}} = \overline{g(x+iy)} \quad (80)$$

which implies that for all x, y ,

$$|g(x-iy)| = |g(x+iy)| \quad (81)$$

Let us show that for every $m \in \mathbb{Z}^d$,

$$|g|_r e^{2\pi|m|r} = |g e^{2i\pi\langle m, \cdot \rangle}|_r \quad (82)$$

By the maximum principle,

$$|g|_r = \sup_{x; |y_j| \leq r, 1 \leq j \leq d} |g(x+iy)| = \sup_{x; |y_j| = r, 1 \leq j \leq d} |g(x+iy)| \quad (83)$$

Let y_0 such that

$$|g|_r = \sup_x |g(x + iy_0)| \quad (84)$$

then, for m having only one non-zero component m_j , either

$$|g|_r e^{2\pi|m|r} = \sup_x |g(x + iy_0)| |e^{2i\pi\langle m, x+iy_0 \rangle}| = |ge^{2i\pi\langle m, \cdot \rangle}|_r \quad (85)$$

if m_j et $(y_0)_j$ have opposite signs, or

$$|g|_r e^{2\pi|m|r} = \sup_x |g(x - iy_0)| |e^{2i\pi\langle m, x-iy_0 \rangle}| = |ge^{2i\pi\langle m, \cdot \rangle}|_r \quad (86)$$

if m_j et $(y_0)_j$ have the same sign, whence (82) if m has only one non-zero component. Assume that

$$|g|_r e^{2\pi|m|r} = |ge^{2i\pi\langle (m_1, \dots, m_{j-1}, 0, \dots, 0), \cdot \rangle}|_r e^{2\pi(|m_j| + \dots + |m_d|)r} \quad (87)$$

and that $|ge^{2i\pi\langle (m_1, \dots, m_{j-1}, 0, \dots, 0), \cdot \rangle}|_r$ is reached at \bar{y} . Let $\delta_j \in \{-1, 1\}$ be such that m_j and $\delta_j \bar{y}_j$ have opposite signs. Then

$$\begin{aligned} |g|_r e^{2\pi|m|r} &= |ge^{2i\pi\langle (m_1, \dots, m_{j-1}, 0, \dots, 0), \cdot \rangle}|_r e^{2\pi(|m_j| + \dots + |m_d|)r} \\ &= \sup_{x, y_k, k \neq j} |g(x + i(y_1, \dots, \bar{y}_j, \dots, y_d))e^{2i\pi\langle (m_1, \dots, m_{j-1}, 0, \dots, 0), x+i(y_1, \dots, \bar{y}_j, \dots, y_d) \rangle}| e^{2\pi(|m_j| + \dots + |m_d|)r} \\ &= \sup_{x, y_k, k \neq j} |g(x + i(y_1, \dots, \delta_j \bar{y}_j, \dots, y_d))e^{2i\pi\langle (m_1, \dots, m_{j-1}, 0, \dots, 0), x+i(y_1, \dots, \delta_j \bar{y}_j, \dots, y_d) \rangle}| e^{2i\pi m_j(x_j + i\delta_j \bar{y}_j)} | e^{2\pi(|m_{j+1}| + \dots + |m_d|)r} \\ &= \sup_{x, y_k, k \neq j} |g(x + i(y_1, \dots, \delta_j \bar{y}_j, \dots, y_d))e^{2i\pi\langle (m_1, \dots, m_j, 0, \dots, 0), x+i(y_1, \dots, \delta_j \bar{y}_j, \dots, y_d) \rangle}| e^{2\pi(|m_{j+1}| + \dots + |m_d|)r} \\ &= |ge^{2i\pi\langle (m_1, \dots, m_j, 0, \dots, 0), \cdot \rangle}|_r e^{2\pi(|m_{j+1}| + \dots + |m_d|)r} \end{aligned} \quad (88)$$

and (82) is obtained through a simple iteration. Thus

$$|fe^{2i\pi\langle m, \cdot \rangle}|_{r'} \leq |f|_{r'} e^{2\pi|m|r'} \leq C|g|_r e^{2\pi|m|r} = C|ge^{2i\pi\langle m, \cdot \rangle}|_r \quad \square \quad (89)$$

Remark: If f, g are matrix-valued trigonometric polynomials, $f = (f_{j,k}), g = (g_{j,k})$, and g has real coefficients on \mathbb{R}^d , a similar statement holds. For if

$$|f|_{r'} = \sup_{x, |y_j| \leq r'} \|f(x + iy)\| \leq C|g|_r = C \sup_{x, |y_j| \leq r} \|g(x + iy)\| \quad (90)$$

as the norm of the greatest coefficient is equivalent to the operator norm, one has

$$\sup_{j,k} |f_{j,k}|_{r'} \leq CC' \sup_{j,k} |g_{j,k}|_r \quad (91)$$

for some C' only depending on the dimension of the matrices. So from Lemma 3.1, since there exists j_0, k_0 such that

$$\forall j, k, |f_{j,k}|_{r'} \leq CC' |g_{j_0, k_0}|_r \quad (92)$$

then

$$\sup_{j,k} |f_{j,k} e^{2i\pi\langle m, \cdot \rangle}|_{r'} \leq CC' \sup_{j,k} |g_{j,k} e^{2i\pi\langle m, \cdot \rangle}|_r \quad (93)$$

and as the norms are equivalent, the statement also holds in operator norm:

$$|f e^{2i\pi\langle m, \cdot \rangle}|_{r'} \leq CC'' |g e^{2i\pi\langle m, \cdot \rangle}|_r \quad (94)$$

for some C'' depending only on the dimension of the matrices.

Proposition 3.2 *Let*

- $N \in \mathbb{N}$,
- $\kappa' \in]0, \kappa]$,
- $\gamma \geq n(n+1)$,
- $0 < r' < r$.

Let $\tilde{A} \in \mathcal{G}$ have $DC_{\omega}^N(\kappa', \tau)$ spectrum. Let $\tilde{F} \in C_r^{\omega}(2\mathbb{T}^d, \mathcal{G})$ with nice periodicity properties with respect to an $(\tilde{A}, \kappa', \gamma)$ -decomposition \mathcal{L} . Then equation

$$\forall \theta \in 2\mathbb{T}^d, \partial_{\omega} \tilde{X}(\theta) = [\tilde{A}, \tilde{X}(\theta)] + \tilde{F}^N(\theta) - \hat{\tilde{F}}(0); \hat{\tilde{X}}(0) = 0 \quad (95)$$

has a solution $\tilde{X} \in C_{r'}^{\omega}(2\mathbb{T}^d, \mathcal{G})$ such that

- *if \tilde{F} has nice periodicity properties with respect to \mathcal{L} et (m_L) , then \tilde{X} has nice periodicity properties with respect to \mathcal{L} and (m_L) ; in particular, if \tilde{F} is defined on \mathbb{T}^d , then so is \tilde{X} ,*
- *if Φ is trivial with respect to \mathcal{L} , then there exist C', D depending only on n, d, τ such that*

$$|\Phi^{-1} \tilde{X} \Phi|_{r'} \leq C' \left(\frac{1 + \|\tilde{A}_{\mathcal{M}}\|}{(r - r')\kappa'} \right)^{2n^2\gamma + D} |\Phi^{-1} \tilde{F} \Phi|_r \quad (96)$$

Moreover, the truncation of \tilde{X} at order N is unique.

Proof: • Let $C \in GL(n, \mathbb{C})$ be such that $C^{-1}\tilde{A}C$ is in Jordan normal form. Solving equation (95) is equivalent to solving

$$\forall \theta \in 2\mathbb{T}^d, \partial_\omega C^{-1}\tilde{X}(\theta)C = [C^{-1}\tilde{A}C, C^{-1}\tilde{X}(\theta)C] + C^{-1}(\tilde{F}^N(\theta) - \hat{\tilde{F}}(0))C; \hat{\tilde{X}}(0) = 0 \quad (97)$$

Equation (97) can be decomposed along coefficients $x_{j,k}(\theta)$ of $C^{-1}\tilde{X}(\theta)C$: for all j, k , and all $\theta \in 2\mathbb{T}^d$,

$$\partial_\omega x_{j,k}(\theta) = (\tilde{\alpha}_j - \tilde{\alpha}_k)x_{j,k}(\theta) + \delta_1 x_{j,k+1}(\theta) + \delta_2 x_{j-1,k}(\theta) + (C^{-1}(\tilde{F}^N(\theta) - \hat{\tilde{F}}(0))C)_{j,k} \quad (98)$$

where δ_1 and δ_2 are 0 or 1 (with $\delta_2 = 0$ if $j = 1$ and $\delta_1 = 0$ if $k = n$). Developing into Fourier series, one gets for all $m \in \frac{1}{\nu}\mathbb{Z}^d$, with $\nu = 1$ or 2 according to the periodicity of $(C^{-1}(\tilde{F}^N - \hat{\tilde{F}}(0))C)_{j,k}$,

$$i\langle m, \omega \rangle \hat{x}_{j,k}(m) = (\tilde{\alpha}_j - \tilde{\alpha}_k)\hat{x}_{j,k}(m) + \delta_1 \hat{x}_{j,k+1}(m) + \delta_2 \hat{x}_{j-1,k}(m) + \hat{f}(m) \quad (99)$$

where $\hat{f}(m)$ stands for the m -th Fourier coefficient of the function $(C^{-1}(\tilde{F}^N - \hat{\tilde{F}}(0))C)_{j,k}$. The diophantine conditions given by Proposition 2.3 allow the existence of a solution to the set of equations (98), therefore (97) has a solution \tilde{X}^2 .

• Now we shall see that \tilde{X}^N is unique. Suppose that \tilde{X} and \tilde{Y} are both solutions of (95). Then

$$\partial_\omega(\tilde{X} - \tilde{Y}) = [\tilde{A}, \tilde{X} - \tilde{Y}]; \hat{\tilde{X}}(0) - \hat{\tilde{Y}}(0) = 0 \quad (100)$$

The diophantine conditions on \tilde{A} imply that the truncation at order N of any solution of (100) is constant, and condition $\hat{\tilde{X}}(0) - \hat{\tilde{Y}}(0) = 0$ implies that it vanishes, so $\tilde{X}^N = \tilde{Y}^N$.

• To check that \tilde{X} is \mathcal{G} -valued, it is enough to show it for \tilde{X}^N , since one can assume that $\tilde{X} = \tilde{X}^N$.

- if $\mathcal{G} = gl(n, \mathbb{C})$, this is trivial.
- if $\mathcal{G} = gl(n, \mathbb{R})$, this comes from the unicity of the solution up to order N , since \tilde{X} and its complex conjugate are solutions of the same equation.

²To solve the set of equations (98), proceed as follows: first solve (98) for $j = 1, k = n$; the solution of (98) for $(1, k)$ gives the solution of (98) for $(1, k - 1)$; the solution of (98) for (j, n) gives the solution of (98) for $(j + 1, n)$; the solutions of (98) for $(j - 1, k)$ and $(j, k + 1)$ give the solution of (98) for any (j, k) .

- if $\mathcal{G} = sp(n, \mathbb{C})$, first note that

$$\begin{aligned} \forall \theta \in 2\mathbb{T}^d, \partial_\omega \tilde{X}(\theta)^* J &= [\tilde{X}(\theta)^*, \tilde{A}^*] J + \tilde{F}^N(\theta)^* J - \hat{F}(0)^* J \\ &= -\tilde{X}(\theta)^* J \tilde{A} - \tilde{A}^* \tilde{X}(\theta)^* J - J \tilde{F}^N(\theta) + J \hat{F}(0) \end{aligned} \quad (101)$$

and

$$\begin{aligned} \forall \theta \in 2\mathbb{T}^d, \partial_\omega (J \tilde{X}(\theta)) &= J[\tilde{A}, \tilde{X}(\theta)] + J \tilde{F}^N(\theta) - J \hat{F}(0) \\ &= -\tilde{A}^* J \tilde{X}(\theta) - J \tilde{X}(\theta) \tilde{A} + J \tilde{F}^N(\theta) - J \hat{F}(0) \end{aligned} \quad (102)$$

so

$$\begin{aligned} \forall \theta \in 2\mathbb{T}^d, \partial_\omega J(\tilde{X}(\theta)^* J + J \tilde{X}(\theta)) &= -J(\tilde{X}(\theta)^* J + J \tilde{X}(\theta)) \tilde{A} - J \tilde{A}^* (\tilde{X}(\theta)^* J + J \tilde{X}(\theta)) \\ &= [\tilde{A}, J(\tilde{X}(\theta)^* J + J \tilde{X}(\theta))] \end{aligned} \quad (103)$$

Diophantine conditions on \tilde{A} imply that $\tilde{X}^* J + J \tilde{X}$ is constant. Condition $\hat{X}(0) = 0$ implies that for every $\theta \in 2\mathbb{T}^d$, $\tilde{X}(\theta)^* J + J \tilde{X}(\theta) = 0$, so \tilde{X} takes its values in $sp(n, \mathbb{C})$.

- if $\mathcal{G} = u(n)$, proceed as in the $sp(n, \mathbb{C})$ case, showing this time that $\tilde{X}^* + \tilde{X}$ is constant and thus is zero.
- if $\mathcal{G} = sp(n, \mathbb{R})$ or $o(n)$, use the previous cases and the fact that $sp(n, \mathbb{R}) = sp(n, \mathbb{C}) \cap gl(n, \mathbb{R})$ and $o(n) = u(n) \cap gl(n, \mathbb{R})$.
- if $\mathcal{G} = sl(n, \mathbb{R})$, note that the trace of \tilde{X} is solution of

$$\forall \theta \in 2\mathbb{T}^d, \partial_\omega (Tr \tilde{X}(\theta)) = Tr[\tilde{A}, \tilde{X}(\theta)] = Tr(\tilde{A} \tilde{X}(\theta)) - Tr(\tilde{X}(\theta) \tilde{A}) = 0 \quad (104)$$

so it is a constant, and as $Tr \hat{X}(0) = 0$, it is identical to zero.

- As for periodicity properties, equation (95) decomposes into blocks:

$$\begin{aligned} \forall L, L' \in \mathcal{L}, \\ \partial_\omega (P_L^\mathcal{L} \tilde{X} P_{L'}^\mathcal{L}) &= P_L^\mathcal{L} \tilde{A} P_{L'}^\mathcal{L} \tilde{X} P_{L'}^\mathcal{L} - P_L^\mathcal{L} \tilde{X} P_{L'}^\mathcal{L} \tilde{A} P_{L'}^\mathcal{L} + P_L^\mathcal{L} (\tilde{F}^N - \hat{F}(0)) P_{L'}^\mathcal{L} \end{aligned} \quad (105)$$

which again decomposes into Fourier coefficients: for $0 < |m| \leq N$,

$$2i\pi\langle m, \omega \rangle (P_L^\mathcal{L} \hat{X}(m) P_{L'}^\mathcal{L}) = P_L^\mathcal{L} \tilde{A} P_L^\mathcal{L} \hat{X}(m) P_{L'}^\mathcal{L} - P_L^\mathcal{L} \hat{X}(m) P_{L'}^\mathcal{L} \tilde{A} P_{L'}^\mathcal{L} + P_L^\mathcal{L} \hat{F}(m) P_{L'}^\mathcal{L} \quad (106)$$

Let (m_L) be a family such that \tilde{F} has nice periodicity properties with respect to \mathcal{L} and (m_L) . If m is not in $\mathbb{Z}^d + m_L - m_{L'}$, then $P_L^\mathcal{L} \hat{F}(m) P_{L'}^\mathcal{L} = 0$ and since \tilde{X}^N is unique, $P_L^\mathcal{L} \hat{X}(m) P_{L'}^\mathcal{L} = 0$. For $|m| > N$ one can assume $\hat{X}(m) = 0$. Therefore \tilde{X} also has nice periodicity properties with respect to \mathcal{L} et (m_L) .

• Finally let us prove the estimate (96). Let $m \in \frac{1}{2}\mathbb{Z}^d$, $|m| \leq N$. First we shall prove that for all $L, L' \in \mathcal{L}$,

$$\|P_L^\mathcal{L} \hat{X}(m) P_{L'}^\mathcal{L}\| \leq C' \frac{(1 + \|\tilde{A}_N\|)^{n^2-1} |m|^{(n^2-1)\tau}}{\kappa^{(n^2-1)}} \|P_L^\mathcal{L} \hat{F}(m) P_{L'}^\mathcal{L}\| (\|P_L^\mathcal{L}\| \|P_{L'}^\mathcal{L}\|)^{n^2-1} \quad (107)$$

where C' only depends on n . The proof will be inspired by [3], Lemma 2. Let $\mathcal{A}_{L,L'}$ be the linear operator from $gl(n, \mathbb{C})$ into itself such that for all $M \in gl(n, \mathbb{C})$,

$$\mathcal{A}_{L,L'} M = \tilde{A} P_L^\mathcal{L} M - M P_{L'}^\mathcal{L} \tilde{A} \quad (108)$$

Decomposing (95) into blocks, then into Fourier series, one obtains for all $L, L' \in \mathcal{L}$ and all $m \in \frac{1}{2}\mathbb{Z}^d$ such that $0 < |m| \leq N$,

$$(P_L^\mathcal{L} \hat{X}(m) P_{L'}^\mathcal{L}) = (2i\pi\langle m, \omega \rangle - \mathcal{A}_{L,L'})^{-1} P_L^\mathcal{L} \hat{F}(m) P_{L'}^\mathcal{L} \quad (109)$$

Write $\mathcal{A}_{L,L'}$ as an n^2 -dimensional matrix. Let $A_D \in gl(n^2, \mathbb{C})$ be a diagonal matrix and $A_N \in gl(n^2, \mathbb{C})$ a nilpotent matrix such that

$$(2i\pi\langle m, \omega \rangle - \mathcal{A}_{L,L'}) = A_D - A_N \quad (110)$$

Then A_N coincides with the operator

$$A_N : B \mapsto (\tilde{A} P_L^\mathcal{L})_N B - B (P_{L'}^\mathcal{L} \tilde{A})_N \quad (111)$$

Moreover,

$$(2i\pi\langle m, \omega \rangle - \mathcal{A}_{L,L'})^{-1} = A_D^{-1} (I + A_N A_D^{-1} + \dots + (A_N A_D^{-1})^{n^2-1}) \quad (112)$$

We will estimate $(2i\pi\langle m, \omega \rangle - \mathcal{A}_{L,L'})^{-1}$, for $m \in \mathbb{Z}^d$ if $L = \bar{L}'$ and $m \in \frac{1}{2}\mathbb{Z}^d$ if $L \neq \bar{L}'$. Each coefficient of $A_D^{-1} (A_N A_D^{-1})^{j-1}$ has the form $\frac{p}{q}$ with $|p| \leq \|A_N\|^{j-1}$ and $q = \beta_1 \dots \beta_j$ where β_i are eigenvalues of $2i\pi\langle m, \omega \rangle - \mathcal{A}_{L,L'}$. Now

$$\sigma(\mathcal{A}_{L,L'}) = \{\alpha - \alpha' \mid \alpha \in \sigma(\tilde{A}|_L), \alpha' \in \sigma(\tilde{A}|_{L'})\} \quad (113)$$

and for all $\alpha \in \sigma(\tilde{A}|_L), \alpha' \in \sigma(\tilde{A}|_{L'})$,

$$|\alpha - \alpha' - 2i\pi\langle m, \omega \rangle| \geq \frac{\kappa'}{|m|^\tau} \quad (114)$$

for all $m \in \mathbb{Z}^d$ if $L = \bar{L}'$ and all $m \in \frac{1}{2}\mathbb{Z}^d$ if $L \neq \bar{L}'$ Thus

$$\|(2i\pi\langle m, \omega \rangle - \mathcal{A}_{L,L'})^{-1}\| \leq n^2 2^{n^2} (1 + \|\tilde{A}_N\| (\|P_L^\mathcal{L}\| + \|P_{L'}^\mathcal{L}\|))^{n^2-1} \left(\frac{|m|^\tau}{\kappa'}\right)^{n^2-1} \quad (115)$$

and (109) implies (107).

• The estimate (107) implies that

$$|P_L^\mathcal{L} \tilde{X} P_{L'}^\mathcal{L}|_{r'} \leq C' \frac{(1 + \|\tilde{A}_N\|)^{n^2-1}}{\kappa'^{(n^2-1)}} \sum_m |m|^{(n^2-1)\tau} |P_L^\mathcal{L} \tilde{F} P_{L'}^\mathcal{L}|_r e^{-2\pi|m|r} e^{2\pi|m|r'} (\|P_L^\mathcal{L}\| \|P_{L'}^\mathcal{L}\|)^{n^2-1} \quad (116)$$

where C' only depends on n . Now

$$\begin{aligned} \sum_m |m|^{(n^2-1)\tau} e^{-2\pi|m|(r-r')} &\leq C_d \sum_{M \geq 1} M^{(n^2-1)\tau+d} e^{-2\pi M(r-r')} \\ &\leq C_d \int_0^\infty t^{(n^2-1)\tau+d} e^{-2\pi t(r-r')} dt \leq \frac{C_d}{(2\pi(r-r'))^{(n^2-1)\tau+d+1}} \end{aligned} \quad (117)$$

where C_d only depends on d , so

$$|P_L^\mathcal{L} \tilde{X} P_{L'}^\mathcal{L}|_{r'} \leq \frac{C''}{(r-r')^{(n^2-1)\tau+d+1}} \frac{(1 + \|\tilde{A}_N\|)^{n^2-1}}{\kappa'^{(n^2-1)}} |P_L^\mathcal{L} \tilde{F} P_{L'}^\mathcal{L}|_r (\|P_L^\mathcal{L}\| \|P_{L'}^\mathcal{L}\|)^{n^2-1} \quad (118)$$

where C'' only depends on n, d, τ .

Let $(m'_L)_{L \in \mathcal{L}}$ a family of elements of $\frac{1}{2}\mathbb{Z}^d$ and Φ defined by

$$\Phi = \sum_{L \in \mathcal{L}} P_L^\mathcal{L} e^{2i\pi\langle m'_L, \cdot \rangle} \quad (119)$$

then

$$|\Phi^{-1} \tilde{X} \Phi|_{r'} = \left| \sum_{L, L' \in \mathcal{L}} P_L^\mathcal{L} \tilde{X} e^{2i\pi\langle m'_L - m'_{L'}, \cdot \rangle} P_{L'}^\mathcal{L} \right|_{r'} \quad (120)$$

thus Lemma 3.1 applied to (118) gives

$$|\Phi^{-1}\tilde{X}\Phi|_{r'} \leq \frac{C_3}{(r-r')^{(n^2-1)\tau+d+1}} \frac{(1+\|\tilde{A}_{\mathcal{N}}\|)^{n^2-1}}{\kappa'^{(n^2-1)}} \sum_{L,L' \in \mathcal{L}} \|P_L^{\mathcal{L}}\|^{n^2} |\tilde{F}e^{2i\pi\langle m'_L - m'_{L'}, \cdot \rangle}|_r \|P_{L'}^{\mathcal{L}}\|^{n^2} \quad (121)$$

where C_3 only depends on n, d, τ ; now, since for every $L \in \mathcal{L}$

$$\|P_L^{\mathcal{L}}\| \leq C_0 \left(\frac{1 + \|\tilde{A}_{\mathcal{N}}\|}{\kappa'} \right)^\gamma \quad (122)$$

then

$$|\Phi^{-1}\tilde{X}\Phi|_{r'} \leq \frac{C_4}{(r-r')^{(n^2-1)\tau+d+1}} \left(\frac{1 + \|\tilde{A}_{\mathcal{N}}\|}{\kappa'} \right)^{n^2(2\gamma+1)} \sum_{L,L'} |P_L^{\mathcal{L}}\Phi^{-1}\tilde{F}\Phi P_{L'}^{\mathcal{L}}|_r \quad (123)$$

where C_4 only depends on n, d, τ , whence (96). \square

Remark: The loss of analyticity $r - r'$ is needed in order to have good estimates of the solution. Note that when $\mathcal{G} = o(n)$ or $u(n)$, then $\tilde{A}_{\mathcal{N}}$ is zero, thus the estimate does not depend on \tilde{A} .

4 Inductive lemma without renormalization

4.1 Auxiliary lemmas

The first lemma will be used to iterate the inductive lemma without having to perform renormalization at each step, which will greatly improve the final estimates.

Lemma 4.1 *Let*

- $\kappa' \in]0, 1[$, $C > 0$,
- $\tilde{F} \in \mathcal{G}$,
- $\tilde{\epsilon} = \|\tilde{F}\|$,
- $\tilde{N} \in \mathbb{N}$,
- $\tilde{A} \in \mathcal{G}$ with $DC_\omega^{\tilde{N}}(\kappa', \tau)$ spectrum.

There exists a constant c only depending on $n\tau$ such that if $\tilde{\epsilon}$ satisfies

$$\tilde{\epsilon} \leq c \left(\frac{C^\tau \kappa'}{1 + \|\tilde{A}\|} \right)^{2n} \quad (124)$$

and

$$\tilde{N} \leq \frac{|\log \tilde{\epsilon}|}{C} \quad (125)$$

then $\tilde{A} + \tilde{F}$ has $DC_{\omega}^{\tilde{N}}(\frac{3\kappa'}{4}, \tau)$ spectrum.

Proof: If $\tilde{\alpha} \in \sigma(\tilde{A} + \tilde{F})$, by Lemma 6.1 given as an appendix, there exists $\alpha \in \sigma(\tilde{A})$ such that $|\alpha - \tilde{\alpha}| \leq 2n(\|\tilde{A}\| + 1)\tilde{\epsilon}^{\frac{1}{n}}$.

By assumption \tilde{A} has $DC_{\omega}^{\tilde{N}}(\kappa', \tau)$ spectrum. Thus for all $\alpha, \beta \in \sigma(\tilde{A} + \tilde{F})$ and all $m \in \mathbb{Z}^d, 0 < |m| \leq \tilde{N}$,

$$|\alpha - \beta - 2i\pi\langle m, \omega \rangle| \geq \frac{\kappa'}{|m|^\tau} - 4n(\|\tilde{A}\| + 1)\tilde{\epsilon}^{\frac{1}{n}} \quad (126)$$

and if $\alpha \neq \bar{\beta}$, (126) holds for every $m \in \frac{1}{2}\mathbb{Z}^d, 0 < |m| \leq \tilde{N}$. Therefore it is enough to show that

$$4n\tilde{N}^\tau(\|\tilde{A}\| + 1)\tilde{\epsilon}^{\frac{1}{n}} \leq \frac{\kappa'}{4} \quad (127)$$

Now there is a constant $c \leq 1$ which only depends on $n\tau$ such that if $\tilde{\epsilon} \leq c$, then

$$\tilde{\epsilon} (|\log \tilde{\epsilon}|)^{n\tau} \leq \tilde{\epsilon}^{\frac{1}{2}} \quad (128)$$

so if

$$\tilde{\epsilon} \leq c \left(\frac{C^\tau \kappa'}{16n(\|\tilde{A}\| + 1)} \right)^{2n} \quad (129)$$

by assumption (125), then

$$4n(\|\tilde{A}\| + 1)\tilde{\epsilon}^{\frac{1}{n}}\tilde{N}^\tau \leq 4n(\|\tilde{A}\| + 1)\tilde{\epsilon}^{\frac{1}{2n}}C^{-\tau} \leq \frac{\kappa'}{4} \quad (130)$$

which proves the Lemma. \square

The following lemma will be used to avoid doubling the period more than once.

Lemma 4.2 *Let $A, A' \in gl(n, \mathbb{R})$ and $H : 2\mathbb{T}^d \rightarrow gl(n, \mathbb{R})$. Assume that H has nice periodicity properties with respect to an A -decomposition \mathcal{L} and assume*

$$\forall L, L' \in \mathcal{L}, P_L^{\mathcal{L}}(A' - A)P_{L'}^{\mathcal{L}} \neq 0 \Rightarrow P_L^{\mathcal{L}}HP_{L'}^{\mathcal{L}} \in C^0(\mathbb{T}^d, gl(n, \mathbb{R})) \quad (131)$$

Then H has nice periodicity properties with respect to an A' -decomposition which is less fine than \mathcal{L} .

Proof: Define a decomposition \mathcal{L}' of \mathbb{C}^n as follows: for all $L, L' \in \mathcal{L}$,

$$(\exists L_0 \in \mathcal{L}' \mid L \subset L_0, L' \subset L_0) \Leftrightarrow P_L^{\mathcal{L}}HP_{L'}^{\mathcal{L}} \in C^0(\mathbb{T}^d, gl(n, \mathbb{R})) \quad (132)$$

Let (m_L) be a family such that H has nice periodicity properties with respect to \mathcal{L} and $(m_{L'})$. For all $L' \in \mathcal{L}'$, let L be a subspace of \mathcal{L} contained in L' and let $\bar{m}_{L'} = m_L$; the class of $\bar{m}_{L'}$ in the equivalence relation

$$m \sim m' \Leftrightarrow m - m' \in \mathbb{Z}^d \quad (133)$$

does not depend on a particular choice of L . Then for all $L' \in \mathcal{L}'$,

$$e^{2i\pi\langle \bar{m}_{L'}, \cdot \rangle} P_{L'}^{\mathcal{L}'} = \sum_{L \in \mathcal{L}, L \subset L'} e^{2i\pi\langle \bar{m}_{L'}, \cdot \rangle} P_L^{\mathcal{L}} \quad (134)$$

so for all $L_1, L_2 \in \mathcal{L}'$,

$$P_{L_1}^{\mathcal{L}'}HP_{L_2}^{\mathcal{L}'}e^{2i\pi\langle \bar{m}_{L_1} - \bar{m}_{L_2}, \cdot \rangle} = \sum_{L'_1 \subset L_1, L'_2 \subset L_2} P_{L'_1}^{\mathcal{L}}HP_{L'_2}^{\mathcal{L}}e^{2i\pi\langle m_{L'_1} - m_{L'_2}, \cdot \rangle} e^{2i\pi\langle \bar{m}_{L_1} - m_{L'_1} - (\bar{m}_{L_2} - m_{L'_2}), \cdot \rangle} \quad (135)$$

which is continuous on \mathbb{T}^d . Moreover, let $L_0 \in \mathcal{L}'$, then

$$P_{L_0}^{\mathcal{L}'}HP_{L_0}^{\mathcal{L}'} = \sum_{L, L' \in \mathcal{L}, L \subset L_0, L' \subset L_0} P_L^{\mathcal{L}}HP_{L'}^{\mathcal{L}} \quad (136)$$

which is continuous on \mathbb{T}^d . Thus H has nice periodicity properties with respect to \mathcal{L}' .

By definition, \mathcal{L}' is A -invariant. Moreover, assumption (131) implies

$$A' - A = \sum_{L' \in \mathcal{L}'} P_{L'}^{\mathcal{L}'}(A' - A)P_{L'}^{\mathcal{L}'}$$

so it also implies that \mathcal{L}' is $A' - A$ -invariant. Thus, \mathcal{L}' is A' -invariant and so it is an A' -decomposition. \square

Here is a standard lemma on the estimate of the rest of the Fourier series for an analytic function.

Lemma 4.3 *Let $H \in C_r^\omega(2\mathbb{T}^d, gl(n, \mathbb{C}))$. Soit $N \in \mathbb{N}$ and H^N the truncation of H at order N . Then for all $r' < r$,*

$$|H - H^N|_{r'} \leq \frac{CN^d}{(r - r')^{d+1}} |H|_r e^{-2\pi N(r-r')} \quad (137)$$

where C only depends on d .

Proof: It is a simple computation. Since

$$H - H^N = \sum_{|m| > N} \hat{H}(m) e^{2i\pi \langle m, \cdot \rangle} \quad (138)$$

then

$$\begin{aligned} |H - H^N|_{r'} &\leq \sum_{|m| > N} \|\hat{H}(m)\| e^{2\pi|m|r'} \leq |H|_r \sum_{|m| > N} e^{-2\pi|m|(r-r')} \\ &\leq C|H|_r \sum_{M > N} M^d e^{-2\pi M(r-r')} \leq C|H|_r \frac{N^d}{(r - r')^{d+1}} e^{-2\pi N(r-r')} \quad \square \end{aligned} \quad (139)$$

4.2 Inductive lemma

Proposition 4.4 *Let*

- $\tilde{\epsilon} > 0, \tilde{r} \leq 1, \tilde{r}' \in [\frac{\tilde{r}}{2}, \tilde{r}], \kappa' > 0, \tilde{N} \in \mathbb{N}, \gamma \geq n(n+1), C > 0;$
- $\tilde{F} \in C_{\tilde{r}}^\omega(2\mathbb{T}^d, \mathcal{G}), \tilde{A} \in \mathcal{G},$
- \mathcal{L} an $(\tilde{A}, \kappa', \gamma)$ -decomposition.

There exists a constant $C'' > 0$ depending only on τ, n such that if

1. \tilde{A} has $DC_\omega^{\tilde{N}}(\kappa', \tau)$ spectrum;
- 2.

$$\|\hat{\tilde{F}}(0)\| \leq \tilde{\epsilon} \leq C'' \left(\frac{C^\tau \kappa'}{1 + \|\tilde{A}\|} \right)^{2n} \quad (140)$$

and

$$\tilde{N} \leq \frac{|\log \tilde{\epsilon}|}{C} \quad (141)$$

3. \tilde{F} has nice periodicity properties with respect to \mathcal{L}

then there exist

- $C' \in \mathbb{R}$ depending only on n, d, κ, τ ,
- $D \in \mathbb{N}$ depending only on n, d, τ ,
- $X \in C_{\tilde{r}'}^\omega(2\mathbb{T}^d, \mathcal{G})$,
- $A' \in \mathcal{G}$
- an $(A', \frac{3\kappa'}{4}, \gamma)$ -decomposition \mathcal{L}'

satisfying the following properties:

1. A' has $DC_\omega^{\tilde{N}}(\frac{3\kappa'}{4}, \tau)$ spectrum,
2. $\|A' - \tilde{A}\| \leq \tilde{\epsilon}$;
3. the map $F' \in C_{\tilde{r}'}^\omega(2\mathbb{T}^d, \mathcal{G})$ defined by

$$\forall \theta \in 2\mathbb{T}^d, \partial_\omega e^{X(\theta)} = (\tilde{A} + \tilde{F}(\theta))e^{X(\theta)} - e^{X(\theta)}(A' + F'(\theta)) \quad (142)$$

has nice periodicity properties with respect to \mathcal{L}'

4. If Φ is trivial with respect to \mathcal{L} ,
then

$$|\Phi^{-1}X\Phi|_{\tilde{r}'} \leq C' \left(\frac{1 + \|\tilde{A}_{\mathcal{N}}\|}{\kappa'(\tilde{r} - \tilde{r}')} \right)^{D\gamma} |\Phi^{-1}\tilde{F}\Phi|_{\tilde{r}} \quad (143)$$

5. and if Φ is trivial with respect to \mathcal{L} ,

$$\begin{aligned} |\Phi^{-1}F'\Phi|_{\tilde{r}'} &\leq C' \left(\frac{1 + \|\tilde{A}_{\mathcal{N}}\|}{\kappa'(\tilde{r} - \tilde{r}')} \right)^{D\gamma} e^{|\Phi^{-1}X\Phi|_{\tilde{r}'}} |\Phi^{-1}\tilde{F}\Phi|_{\tilde{r}} \\ & \quad (|\Phi|_{\tilde{r}}^2 |\Phi^{-1}|_{\tilde{r}}^2 \tilde{N}^d e^{-2\pi\tilde{N}(\tilde{r} - \tilde{r}')} + |\Phi^{-1}\tilde{F}\Phi|_{\tilde{r}'} (1 + e^{|\Phi^{-1}X\Phi|_{\tilde{r}'}})) \end{aligned} \quad (144)$$

Moreover, if \tilde{F} is continuous on \mathbb{T}^d , then so are X and F' .

Proof: By assumption, \tilde{F} has nice periodicity properties with respect to \mathcal{L} and some family (m_L) and \tilde{A} has $DC_\omega^{\tilde{N}}(\kappa', \tau)$ spectrum, so one can apply Proposition 3.2. Let $X \in C_{\tilde{r}'}^\omega(2\mathbb{T}^d, \mathcal{G})$ be a solution of

$$\forall \theta \in 2\mathbb{T}^d, \partial_\omega X(\theta) = [\tilde{A}, X(\theta)] + \tilde{F}^{\tilde{N}}(\theta) - \hat{\tilde{F}}(0) \quad (145)$$

satisfying the conclusion of Proposition 3.2.

Let $A' := \tilde{A} + \hat{F}(0)$. Then $A' \in \mathcal{G}$ and $\|\tilde{A} - A'\| = \|\hat{F}(0)\|$, whence property 2.

Moreover, let c be the constant given by Lemma 4.1, and assume $C''' \leq c$. Assumptions (140) and (141) make it possible to apply Lemma 4.1 and infer that A' has $DC_{\omega}^{\tilde{N}}(\frac{3\kappa'}{4}, \tau)$ spectrum, whence property 1.

Let $F' \in C_{\tau'}^{\omega}(2\mathbb{T}^d, \mathcal{G})$ the map defined in (142). Then

$$F' = e^{-X}(\tilde{F} - \tilde{F}^{\tilde{N}}) + e^{-X}\tilde{F}(e^X - Id) + (e^{-X} - Id)\hat{F}(0) - e^{-X} \sum_{k \geq 2} \frac{1}{k!} \sum_{l=0}^{k-1} X^l (\tilde{F}^{\tilde{N}} - \hat{F}(0)) X^{k-1-l} \quad (146)$$

We shall apply Lemma 4.2 with $A = \tilde{A}$ and $G = F'$, in order to get property 3. The map F' has nice periodicity properties with respect to \mathcal{L} and some family (m_L) since X and \tilde{F} have them. Moreover, as \tilde{F} has nice periodicity properties with respect to \mathcal{L} ,

$$P_L^{\mathcal{L}} \hat{F}(0) P_{L'}^{\mathcal{L}} \neq 0 \Rightarrow P_L^{\mathcal{L}} \tilde{F} P_{L'}^{\mathcal{L}} \in C^0(\mathbb{T}^d) \quad (147)$$

and since

$$P_L^{\mathcal{L}} \tilde{F} P_{L'}^{\mathcal{L}} \in C^0(\mathbb{T}^d) \Rightarrow m_L - m_{L'} \in \mathbb{Z}^d \Rightarrow P_L^{\mathcal{L}} F' P_{L'}^{\mathcal{L}} \in C^0(\mathbb{T}^d) \quad (148)$$

then assumption (131) of Lemma 4.2 is fulfilled. By Lemma 4.2, F' has therefore nice periodicity properties with respect to an A' -decomposition \mathcal{L}' which is less fine than \mathcal{L} , so \mathcal{L}' is an $(\tilde{A}, \kappa', \gamma)$ -decomposition. As it is an $(\tilde{A}, \kappa', \gamma)$ -decomposition, each subspace $L \in \mathcal{L}'$ satisfies

$$\|P_L^{\mathcal{L}'}\| \leq C_0 \left(\frac{1 + \|\tilde{A}_{\mathcal{N}}\|}{\kappa'} \right)^{\gamma} \quad (149)$$

thus

$$\|P_L^{\mathcal{L}'}\| \leq C_0 \left(\frac{1 + \|A'_{\mathcal{N}}\| + 2\tilde{\epsilon}}{\kappa'} \right)^{\gamma} \leq C_0 \left(\frac{1 + \|A'_{\mathcal{N}}\|}{\frac{3\kappa'}{4}} \right)^{\gamma} \quad (150)$$

and so \mathcal{L}' is an $(A', \frac{3\kappa'}{4}, \gamma)$ -decomposition, whence property 3.

Property 4 is given by Proposition 3.2.

- By Lemma 4.3,

$$|\tilde{F} - \tilde{F}^{\tilde{N}}|_{\tilde{r}'} \leq C_1 \tilde{N}^d |\tilde{F}|_{\tilde{r}} \frac{e^{-2\pi\tilde{N}(\tilde{r}-\tilde{r}')}}{(\tilde{r}-\tilde{r}')^{d+1}} \quad (151)$$

where C_1 only depends on d . By (146), (96) and Lemma 4.3, it is true that

$$\begin{aligned} |\Phi^{-1}F'\Phi|_{\tilde{r}'} &\leq C' \left(\frac{1 + \|\tilde{A}_{\mathcal{N}}\|}{\kappa'(\tilde{r}-\tilde{r}')} \right)^{D\gamma} e^{|\Phi^{-1}X\Phi|_{\tilde{r}'}} |\Phi^{-1}\tilde{F}\Phi|_{\tilde{r}} (|\Phi|_{\tilde{r}}^2 |\Phi^{-1}|_{\tilde{r}}^2 \tilde{N}^d e^{-2\pi\tilde{N}(\tilde{r}-\tilde{r}')} \\ &\quad + |\Phi^{-1}\tilde{F}\Phi|_{\tilde{r}'} (1 + e^{|\Phi^{-1}X\Phi|_{\tilde{r}'}})) \end{aligned} \quad (152)$$

where C' only depends on n, d, κ, τ and D only depends on n, d, τ , whence property 5. \square

5 Inductive lemma with renormalization

5.1 Statement of the inductive lemma

Proposition 5.1 *Let*

- $A \in \mathcal{G}$,
- $r \leq \frac{1}{2}, r' \in [\frac{95}{96}r, r], \gamma \geq n(n+1)$,
- $\bar{A}, \bar{F} \in C_r^\omega(2\mathbb{T}^d, \mathcal{G}), \Psi \in C_r^\omega(2\mathbb{T}^d, G)$,
- $|\bar{F}|_r = \epsilon$,
-

$$R = \frac{1}{(r-r')^8} 80^4 \left(\frac{1}{2}n(n-1) + 1 \right)^2 \quad (153)$$

•

$$N = \frac{1}{2\pi r} |\log \epsilon| \quad (154)$$

$$\text{Let } \kappa'' = \frac{\kappa}{n(8R^{\frac{1}{2}n(n-1)+1}N)^\tau}.$$

There exists $\tilde{C}' > 0$ only depending on n, d, κ, τ and $D_1 \in \mathbb{N}$ only depending on n, d, τ such that if

1.

$$\epsilon \leq \tilde{C}' \left(\frac{\kappa''(r-r')}{\|A\|+1} \right)^{D_1\gamma} \quad (155)$$

2. \bar{A} is reducible to A by Ψ ,

3. $\Psi^{-1}\bar{F}\Psi$ has nice periodicity properties with respect to an (A, κ'', γ) -decomposition \mathcal{L} ,
4. $|\Psi|_r \leq (\frac{1}{\epsilon})^{r-r'}$ and $|\Psi^{-1}|_r \leq (\frac{1}{\epsilon})^{r-r'}$,

then there exists

- $\bar{N} \in [N, R^{\frac{1}{2}n(n-1)}N]$,
- $Z' \in C_{r'}^\omega(2\mathbb{T}^d, G)$,
- $\bar{A}', \bar{F}' \in C_{r'}^\omega(2\mathbb{T}^d, \mathcal{G})$,
- $\Psi' \in C_r^\omega(2\mathbb{T}^d, G)$,
- $A' \in \mathcal{G}$

satisfying the following properties:

1. \bar{A}' is reducible by Ψ' to A' ,
2. the function $(\Psi')^{-1}\bar{F}'\Psi'$ has nice periodicity properties with respect to an $(A', \frac{3\kappa''}{4C_0}, 2\gamma)$ -decomposition \mathcal{L}'
3. $|\Psi'|_r \leq (\frac{1}{\epsilon})^{(r-r')+\frac{1}{96}}e^{4\pi r\bar{N}}$ and $|(\Psi')^{-1}|_r \leq (\frac{1}{\epsilon})^{(r-r')+\frac{1}{96}}e^{4\pi r\bar{N}}$,
4. A' has $DC_\omega^{R\bar{N}}(\frac{3}{4}\kappa'', \tau)$ spectrum,
- 5.

$$\partial_\omega Z' = (\bar{A} + \bar{F})Z' - Z'(\bar{A}' + \bar{F}') \quad (156)$$

$$6. \|A'\| \leq \|A\| + \epsilon^{\frac{23}{24}} + 4\pi\bar{N};$$

7.

$$|Z' - Id|_{r'} \leq \frac{1}{\tilde{C}'} \left(\frac{(1 + \|A\|)|\log \epsilon|}{r - r'} \right)^{D_1\gamma} \epsilon^{1-4(r-r')} \quad (157)$$

and so does $(Z')^{-1} - Id$,

$$8. |\Psi^{-1}\bar{F}'\Psi|_{r'} \leq \epsilon^{\frac{3}{2}},$$

9. the function $\Psi'^{-1}\Psi$ is trivial with respect to $\mathcal{L}_{A, \kappa''}$,

10. for all $s' \geq 0$,

$$|\Psi'^{-1}\Psi|_{s'} \leq C \left(\frac{1 + \|A\|}{\kappa''} \right)^{n(n+1)} e^{4\pi\bar{N}s'}; \quad |\Psi^{-1}\Psi'|_{s'} \leq C \left(\frac{1 + \|A\|}{\kappa''} \right)^{n(n+1)} e^{4\pi\bar{N}s'} \quad (158)$$

where C only depends on n .

Moreover, in dimension 2, if \bar{A}, \bar{F} are continuous on \mathbb{T}^d , if assumption 3 is replaced by 3' for all function H continuous on \mathbb{T}^d , $\Psi H \Psi^{-1}$ is continuous on \mathbb{T}^d

then Z', \bar{A}', \bar{F}' are continuous on \mathbb{T}^d and property 2 is replaced by

2' for every function H continuous on \mathbb{T}^d , $\Psi' H (\Psi')^{-1}$ is continuous on \mathbb{T}^d .

Finally, if $\mathcal{G} = gl(n, \mathbb{C})$ or $u(n)$ and if \bar{A}, \bar{F}, Ψ are continuous on \mathbb{T}^d , then $Z', \bar{A}', \bar{F}', \Psi'$ are continuous on \mathbb{T}^d .

5.2 Algebraic aspects of the proof

5.2.1 General case

Let \bar{N} be given by Proposition 2.3 and Φ a renormalization of A of order R, \bar{N} . Let $\tilde{A} \in \mathcal{G}$ be such that

$$\forall \theta \in 2\mathbb{T}^d, \partial_\omega \Phi(\theta) = A\Phi(\theta) - \Phi(\theta)\tilde{A} \quad (159)$$

The matrix \tilde{A} thus has $DC_\omega^{R\bar{N}}(\kappa'', \tau)$ spectrum. Let $\Psi' = \Psi\Phi$ and $\tilde{F} := (\Psi')^{-1}\bar{F}\Psi'$. By assumption, $\Psi^{-1}\bar{F}\Psi$ has nice periodicity properties with respect to an (A, κ'', γ) -decomposition \mathcal{L} and some family (m_L) . Moreover Φ is trivial with respect to $\mathcal{L}_{A, \kappa''}$. Since \mathcal{L} and $\mathcal{L}_{A, \kappa''}$ are A -decompositions, one can define an A -decomposition $\bar{\mathcal{L}}$ in the following way:

$$L \in \bar{\mathcal{L}} \Leftrightarrow \exists L_1 \in \mathcal{L}, L_2 \in \mathcal{L}_{A, \kappa''} \mid L = L_1 \cap L_2 \quad (160)$$

$\bar{\mathcal{L}}$ is an $(A, \frac{\kappa''}{C_0}, 2\gamma)$ -decomposition since \mathcal{L} and $\mathcal{L}_{A, \kappa''}$ are (A, κ'', γ) -decompositions and so

$$\|P_L^{\bar{\mathcal{L}}}\| = \|P_{L_1}^{\mathcal{L}} P_{L_2}^{\mathcal{L}_{A, \kappa''}}\| \leq C_0^2 \left(\frac{1 + \|A_{\mathcal{N}}\|}{\kappa''} \right)^{2\gamma} \quad (161)$$

Moreover \tilde{F} has nice periodicity properties with respect to $\bar{\mathcal{L}}$. Since $\bar{\mathcal{L}}$ is an $(A, \frac{\kappa''}{C_0}, 2\gamma)$ -decomposition, it is also an $(\tilde{A}, \frac{\kappa''}{C_0}, 2\gamma)$ -decomposition (because the nilpotent parts of A and \tilde{A} coincide).

Moreover,

$$\|\hat{\tilde{F}}(0)\| \leq |\tilde{F}|_0 \leq |\Phi|_0 |\Phi^{-1}|_0 |\Psi|_0 |\Psi^{-1}|_0 |\bar{F}|_0 \quad (162)$$

Now by (65), for all $s' \geq 0$,

$$|\Phi|_{s'} \leq C_0 \left(\frac{1 + \|A_{\mathcal{N}}\|}{\kappa''} \right)^{n(n+1)} e^{4\pi\bar{N}s'} \quad (163)$$

and so does Φ^{-1} (whence property 10). Thus

$$\|\hat{\tilde{F}}(0)\| \leq \epsilon^{1-2(r-r')} C_0^2 \left(\frac{1 + \|A_{\mathcal{M}}\|}{\kappa''} \right)^{2n(n+1)} \quad (164)$$

whence, if $\tilde{C}' \leq C_0^{96}$ et $D_1\gamma \geq 96n(n+1)$,

$$\|\hat{\tilde{F}}(0)\| \leq \epsilon^{1-2(r-r')-\frac{1}{48}} \quad (165)$$

Apply Proposition 4.4 with

$$\tilde{\epsilon} = \epsilon^{1-2(r-r')-\frac{1}{48}}, \tilde{r} = r, \tilde{r}' = r', \kappa' = \frac{\kappa''}{C_0}, \tilde{N} = R\bar{N}, C = \frac{2\pi r}{R^{\frac{1}{2}n(n-1)+1}}, \mathcal{L} = \bar{\mathcal{L}} \quad (166)$$

Let C'' be given by Proposition 4.4 (depending only on n and τ). Assumption (155), which implies (140) with

$$\tilde{C}' \leq C''^4 \left(\frac{C}{(r-r')^{4n(n-1)+9}} \right)^{8n\tau}, \quad D_1\gamma \geq 64n(n(n-1)+2)\tau \quad (167)$$

(note that $\frac{C}{(r-r')^{4n(n-1)+9}}$ has a lower bound which is independent of $r-r'$), the expression (154) which implies (141), the nice periodicity properties of \tilde{F} and the fact that \tilde{A} has $DC_{\omega}^{R\bar{N}}(\kappa'', \tau)$ spectrum, make it possible to apply Proposition 4.4 to obtain functions $X \in C_{r'}^{\omega}(2\mathbb{T}^d, \mathcal{G})$, $F' \in C_{r'}^{\omega}(2\mathbb{T}^d, G)$, and a matrix $A' \in \mathcal{G}$ such that

- A' has $DC_{\omega}^{R\bar{N}}(\frac{3}{4}(\frac{\kappa''}{C_0}), \tau)$ spectrum (whence property 4),
- $\|A' - \tilde{A}\| \leq \epsilon^{\frac{23}{24}}$, which implies

$$\|A' - A\| \leq \|A' - \tilde{A}\| + \|A - \tilde{A}\| \leq \epsilon^{\frac{23}{24}} + 4\pi\bar{N} \quad (168)$$

whence property 6,

- $\partial_{\omega} e^X = (\tilde{A} + \tilde{F})e^X - e^X(A' + F')$,
- F' has nice periodicity properties with respect to an $(A', \frac{3\kappa''}{4C_0}, 2\gamma)$ -decomposition \mathcal{L}'
- and since Φ is trivial with respect to $\bar{\mathcal{L}}$, for some C' depending only on n, d, κ, τ and some D depending only on n, d, τ ,

$$|\Phi X \Phi^{-1}|_{r'} \leq C' \left(\frac{C_0(1 + \|A_{\mathcal{M}}\|)}{\kappa''(r-r')} \right)^{D\gamma} |\Phi \tilde{F} \Phi^{-1}|_r \quad (143)$$

and

$$\begin{aligned}
|\Phi F' \Phi^{-1}|_{r'} &\leq C' \left(\frac{C_0(1 + \|A_{\mathcal{N}}\|)}{\kappa''(r - r')} \right)^{D\gamma} e^{|\Phi X \Phi^{-1}|_{r'}} |\Phi \tilde{F} \Phi^{-1}|_r (|\Phi|_r^2 |\Phi^{-1}|_r^2 (R\bar{N})^d e^{-2\pi R\bar{N}(r-r')}) \\
&\quad + |\Phi \tilde{F} \Phi^{-1}|_{r'} (1 + e^{|\Phi X \Phi^{-1}|_{r'}})
\end{aligned} \tag{169}$$

Let $\bar{F}' = \Psi \Phi F' (\Psi \Phi)^{-1}$ (which satisfies property 2) and let $\bar{A}' \in C_r^\omega(2\mathbb{T}^d, \mathcal{G})$ such that

$$\partial_\omega \Psi \Phi = \bar{A}' \Psi \Phi - \Psi \Phi A' \tag{170}$$

which is exactly property 1. The function $Z' := \Psi \Phi e^X (\Psi \Phi)^{-1}$ is solution of

$$\partial_\omega Z' = (\bar{A} + \bar{F}) Z' - Z' (\bar{A}' + \bar{F}') \tag{171}$$

whence property 5.

5.2.2 Case of dimension 2

In dimension 2, since by assumption $\Psi^{-1} \bar{F} \Psi$ is continuous on \mathbb{T}^d , and by the remark made in section 2.2, then \tilde{F}, X et F' are continuous on \mathbb{T}^d .

Thus the functions $\Phi F' \Phi^{-1}$, $\Phi \hat{\tilde{F}}(0) \Phi^{-1}$ and $\Phi X \Phi^{-1}$ are continuous on \mathbb{T}^d , and by assumption on Ψ , then $\Psi \Phi F' (\Psi \Phi)^{-1}$, $\Psi \Phi \hat{\tilde{F}}(0) (\Psi \Phi)^{-1}$ and $\Psi \Phi X (\Psi \Phi)^{-1}$ are thus continuous on \mathbb{T}^d and finally $\bar{A}' = \bar{A} + \Psi \Phi \hat{\tilde{F}}(0) (\Psi \Phi)^{-1}$ is continuous on \mathbb{T}^d .

It only remains to show that for every function H continuous on \mathbb{T}^d , the function $(\Psi \Phi)^{-1} H \Psi \Phi$ is continuous on \mathbb{T}^d . Now

$$(\Psi \Phi)^{-1} H \Psi \Phi = \Phi^{-1} \Psi^{-1} H \Psi \Phi$$

By assumption, $\Psi^{-1} H \Psi$ is continuous on \mathbb{T}^d , and again using 2.2, so is $\Phi^{-1} \Psi^{-1} H \Psi \Phi$.

5.3 Estimates

- The estimate (163), the assumption (155) and the fact that $\|A_{\mathcal{N}}\| \leq \|A\|$, imply that

$$|\Phi|_r \leq \epsilon^{-\frac{1}{96}} e^{4\pi \bar{N} r} \tag{172}$$

thus

$$|\Psi \Phi|_r \leq |\Psi|_r |\Phi|_r \leq \epsilon^{-(r-r') - \frac{1}{96}} e^{4\pi r \bar{N}} \tag{173}$$

whence property 3.

- By Proposition 4.4,

$$|\Phi F' \Phi^{-1}|_{r'} \leq C' \left(\frac{C_0(1 + \|A_{\mathcal{N}}\|)}{\kappa''(r - r')} \right)^{D\gamma} e^{|\Phi X \Phi^{-1}|_{r'}} |\Phi \tilde{F} \Phi^{-1}|_r ((R\bar{N})^d e^{16\pi r \bar{N}} e^{-2\pi R \bar{N}(r-r')}) + |\Phi \tilde{F} \Phi^{-1}|_{r'} (1 + e^{|\Phi X \Phi^{-1}|_{r'}}) \quad (169)$$

for some C' depending only on n, d, κ, τ and some D depending only on n, d, τ . Now

$$|\Phi \tilde{F} \Phi^{-1}|_r \leq |\Psi|_r |\Psi^{-1}|_r |\bar{F}|_r \leq \epsilon^{1-2(r-r')} \quad (174)$$

and by property 4 of Proposition 4.4,

$$|\Phi X \Phi^{-1}|_{r'} \leq C' \left(\frac{C_0(1 + \|A_{\mathcal{N}}\|)}{\kappa''(r - r')} \right)^{D\gamma} |\Phi \tilde{F} \Phi^{-1}|_r \quad (143)$$

so, by (155), if D_1 is great enough as a function of n, γ, D , then

$$|\Phi X \Phi^{-1}|_{r'} \leq \epsilon^{-\frac{1}{96}} |\Phi \tilde{F} \Phi^{-1}|_{r'} \leq \epsilon^{\frac{7}{8}} \quad (175)$$

so

$$e^{|\Phi X \Phi^{-1}|_{r'}} \leq 2 \quad (176)$$

Moreover

$$e^{16\pi r \bar{N} - 2\pi R \bar{N}(r-r')} = e^{2\pi \bar{N}(8r - R(r-r'))} \leq \epsilon^{100} \quad (177)$$

so by assumption (155),

$$|\Phi F' \Phi^{-1}|_{r'} \leq \epsilon^{-\frac{1}{96}} \epsilon^{1-2(r-r')} ((R\bar{N})^d \epsilon^{100} + \epsilon^{1-2(r-r')}) \quad (178)$$

There exists a constant c_d which only depends on d such that if $\epsilon \leq c_d$, then

$$|\log \epsilon|^d \leq \epsilon^{-1} \quad (179)$$

and in this case there exists c_2, D_2 which only depend on n, d, τ such that

$$|\Phi F' \Phi^{-1}|_{r'} \leq \epsilon^{-\frac{1}{96}} \epsilon^{1-2(r-r')} \left(\frac{C_2}{(r-r')^{D_2}} \epsilon^{99} + \epsilon^{1-2(r-r')} \right) \quad (180)$$

thus if \tilde{C}' is small enough and D_1 big enough (as a function of c_2, D_2),

$$|\Phi F' \Phi^{-1}|_{r'} \leq \epsilon^{2-4(r-r')-\frac{1}{96}} \quad (181)$$

whence 8.

• We shall estimate $|\Psi \Phi e^X (\Psi \Phi)^{-1} - Id|_{r'}$. The estimate (143) implies

$$|\Phi e^X \Phi^{-1} - Id|_{r'} \leq C'' \left(\frac{(1 + \|A_{\mathcal{N}}\|) R^{\frac{1}{2}(n(n-1)+1)\tau} N^\tau}{r - r'} \right)^{D\gamma} |\Phi \tilde{F} \Phi^{-1}|_r \quad (182)$$

for some C'' only depending on n, d, κ, τ , so

$$|\Psi \Phi e^X (\Psi \Phi)^{-1} - Id|_{r'} \leq C_3 \left(\frac{(1 + \|A_{\mathcal{N}}\|) |\log \epsilon|}{r - r'} \right)^{D'_1 \gamma} |\bar{F}|_r \left(\frac{1}{\epsilon} \right)^{4(r-r')} \quad (183)$$

for some C_3 depending only on n, d, κ, τ and D'_1 depending only on n, d, τ . The same estimate holds for $|\Psi \Phi e^{-X} (\Psi \Phi)^{-1} - Id|_{r'}$, so property 7 holds with $\tilde{C}' \leq \frac{1}{C_3}$, $D_1 \geq 2D'_1$. \square

5.4 Inductive step

Now we are able to state the whole inductive step. In the following we will denote

$$\left\{ \begin{array}{l} N(r, \epsilon) = \frac{1}{2\pi r} |\log \epsilon| \\ R(r, r') = \frac{1}{(r-r')^8} 80^4 \left(\frac{1}{2} n(n-1) + 1 \right)^2 \\ \kappa''(r, r', \epsilon) = \frac{\kappa}{n(8R(r, r')^{\frac{1}{2}n(n-1)+1} N(r, \epsilon))^\tau} \end{array} \right. \quad (184)$$

These functions coincide with functions R, N, κ'' defined in Proposition 5.1.

5.4.1 Statement

Proposition 5.2 *Let*

- $A \in \mathcal{G}$,
- $r \leq \frac{1}{2}, r'' \in [\frac{95}{96}r, r], \gamma \geq n(n+1)$,
- $\bar{A}, \bar{F} \in C_r^\omega(2\mathbb{T}^d, \mathcal{G})$ and $\Psi \in C_r^\omega(2\mathbb{T}^d, G)$,
- $\epsilon = |\bar{F}|_r$,

There exists $C' > 0$ depending only on $n, d, \kappa, \tau, \gamma$ and there exists $D_3 \in \mathbb{N}$ depending only on n, d, τ such that if

1. \bar{A} is reducible to A by Ψ ,
2. $\Psi^{-1}\bar{F}\Psi$ has nice periodicity properties with respect to an $(A, \kappa''(r, r''), \epsilon, \gamma)$ -decomposition \mathcal{L}
- 3.

$$\epsilon \leq \frac{C'}{(\|A\| + 1)^{D_3\gamma}} (r - r'')^{D_3\gamma} \quad (185)$$

$$4. |\Psi|_r \leq \left(\frac{1}{\epsilon}\right)^{-\frac{1}{2}(r-r'')} \text{ et } |\Psi^{-1}|_r \leq \left(\frac{1}{\epsilon}\right)^{-\frac{1}{2}(r-r'')},$$

then there exist

- $\epsilon' \leq \epsilon^{100}$;
- $Z' \in C_{r''}^\omega(2\mathbb{T}^d, G)$,
- $\bar{A}', \bar{F}' \in C_{r''}^\omega(2\mathbb{T}^d, \mathcal{G})$,
- $\Psi' \in C_r^\omega(2\mathbb{T}^d, G)$,
- $A' \in \mathcal{G}$

satisfying the following properties:

1. \bar{A}' is reducible by Ψ' to A' ,
2. the map $\Psi'^{-1}\bar{F}'\Psi'$ has nice periodicity properties with respect to an $(A', \kappa''(r'', r'' - \frac{r-r''}{2}, \epsilon'), 2\gamma)$ -decomposition \mathcal{L}'
3. $|\bar{F}'|_{r''} \leq \epsilon'$,
4. $|\Psi'|_{r''} \leq \left(\frac{1}{\epsilon'}\right)^{\frac{1}{4}(r-r'')} \text{ and } |\Psi'^{-1}|_{r''} \leq \left(\frac{1}{\epsilon'}\right)^{\frac{1}{4}(r-r'')}$,
5. $\|A'\| \leq \|A\| + |\log \epsilon| \left(\frac{1}{r-r''}\right)^{D_3}$;
- 6.

$$\partial_\omega Z' = (\bar{A} + \bar{F})Z' - Z'(\bar{A}' + \bar{F}') \quad (186)$$

7.

$$|Z' - Id|_{r''} \leq \frac{1}{C'} \left(\frac{(1 + \|A\|)|\log \epsilon|}{r - r''} \right)^{D_3\gamma} \epsilon^{1-4(r-r'')} \quad (187)$$

and so does $(Z')^{-1} - Id$.

Moreover, in dimension 2, if \bar{A}, \bar{F} are continuous on \mathbb{T}^d , and if assumption 2 is replaced by

2'. Ψ is such that for all function H continuous on \mathbb{T}^d , $\Psi H \Psi^{-1}$ is continuous on \mathbb{T}^d ,

then Z', \bar{A}', \bar{F}' are continuous on \mathbb{T}^d and property 2 is replaced by

2'. Ψ' is such that for every function H continuous on \mathbb{T}^d , $\Psi' H \Psi'^{-1}$ is continuous on \mathbb{T}^d .

Finally, if $\mathcal{G} = gl(n, \mathbb{C})$ or $u(n)$ and if \bar{A}, \bar{F}, Ψ are continuous on \mathbb{T}^d , then $Z', \bar{A}', \bar{F}', \Psi'$ are continuous on \mathbb{T}^d .

The proof will be made in two steps: the first step is to apply Proposition 5.1 to reduce the perturbation when there are resonances. The second step is to iterate Proposition 4.4 as many times as possible using the fact that resonances, once removed, do not reappear immediately.

5.4.2 First step: removing the resonances

Apply Proposition 5.1 with $r' = \frac{r+r''}{2}$. Let $R = R(r, r')$; $N = N(r, \epsilon)$; $\kappa'' = \kappa''(r, r', \epsilon)$ and let \tilde{C}', D_1 as in Proposition 5.1. Let c depending only on D_1, γ, τ such that if $\epsilon \leq c$, then

$$\epsilon^{\frac{1}{2}} |\log \epsilon|^{D_1 \gamma \tau} \leq 1 \quad (188)$$

Assumption (185) with C' small enough as a function of $\tilde{C}', D_1, n, \kappa, \tau, \gamma$ and D_3 big enough as a function of D_1, γ, n , implies that

$$\epsilon \leq \tilde{C}' \left(\frac{\kappa(r - r')}{n(8R^{\frac{1}{2}n(n-1)+1}N)^\tau (\|A\| + 1)} \right)^{D_1 \gamma} \quad (189)$$

so the assumption (155) of Proposition 5.1:

$$\epsilon \leq \tilde{C}' \left(\frac{\kappa''(r - r')}{\|A\| + 1} \right)^{D_1 \gamma} \quad (155)$$

holds true. Thus it is possible to apply Proposition 5.1 to get

- $\bar{N} \in [N, R^{\frac{1}{2}n(n-1)}N]$,
- $Z_1, \Psi' \in C_{r'}^\omega(2\mathbb{T}^d, G)$,
- $A_1 \in \mathcal{G}$
- $\bar{A}_1 \in C_{r'}^\omega(2\mathbb{T}^d, \mathcal{G})$

- and $F_1 = (\Psi')^{-1} \bar{F}_1 \Psi'$

such that

1. \bar{A}_1 is reducible to A_1 by Ψ'
2. F_1 has nice periodicity properties with respect to an $(A_1, \frac{3\kappa''}{4C_0}, 2\gamma)$ -decomposition \mathcal{L}_1
3. $|\Psi'|_{r'} \leq \epsilon^{-(r-r')-\frac{1}{96}} e^{4\pi r \bar{N}}$ and $|\Psi'^{-1}|_{r'} \leq \epsilon^{-(r-r')-\frac{1}{96}} e^{4\pi r \bar{N}}$
4. A_1 has $DC_\omega^{R\bar{N}}(\frac{3}{4}\kappa'', \tau)$ spectrum,
5. $\partial_\omega Z_1 = (\bar{A} + \bar{F})Z_1 - Z_1(\bar{A}_1 + \bar{F}_1)$,
6. $\|A_1\| \leq \|A\| + \epsilon^{\frac{23}{24}} + 4\pi \bar{N}$,
- 7.

$$|Z_1 - Id|_{r'} \leq \frac{1}{\bar{C}'} \left(\frac{(1 + \|A_{\mathcal{N}}\|) |\log \epsilon|}{r - r'} \right)^{D_1 \gamma} \epsilon^{1-4(r-r')} \quad (190)$$

and so does $|Z_1^{-1} - Id|_{r'}$,

8.

$$|\Psi^{-1} \bar{F}_1 \Psi|_{r'} \leq \epsilon^{\frac{3}{2}} \quad (191)$$

9. $\Psi'^{-1} \Psi$ is trivial with respect to $\mathcal{L}_{A, \kappa''}$,
10. and for every $s' \geq 0$,

$$|\Psi'^{-1} \Psi|_{s'} \leq C_n \left(\frac{1 + \|A_{\mathcal{N}}\|}{\kappa''} \right)^{n(n+1)} e^{4\pi \bar{N} s'} \quad (192)$$

and so does $|\Psi^{-1} \Psi'|_{s'}$, where C_n only depends on n .

5.4.3 Second step: iteration far from resonances

Let l such that

$$\epsilon^{(\frac{4}{3})^{l+1}} \leq e^{-2\pi(r-r'') \sqrt[4]{R\bar{N}}} := \epsilon' \leq \epsilon^{(\frac{4}{3})^l} \quad (193)$$

Define the sequence $\epsilon_j = \epsilon^{(\frac{3}{2})^j - \frac{1}{48}}$. We shall iterate $l - 1$ times Proposition 4.4, starting with $j = 2$, with

- $\tilde{\epsilon} = \epsilon_{j-1}$,

- $C = \left(\frac{r-r''}{160(\frac{1}{2}n(n-1)+1)} \right)^{8(\frac{1}{2}n(n-1)+1)}$
- $\tilde{r} = r_{j-2} = \frac{r+r''}{2} - (j-2)\frac{r-r''}{2l}$,
- $\tilde{r}' = r_{j-1} = \frac{r+r''}{2} - (j-1)\frac{r-r''}{2l}$,
- $\kappa' = \left(\frac{3}{4}\right)^{j-1} \frac{\kappa''}{C_0}$,
- $\tilde{N} = R\bar{N}$,
- $\tilde{F} = F_{j-1}$,
- $\tilde{A} = A_{j-1}$,
- $\Phi = \Psi^{-1}\Psi'$,
- $\mathcal{L} = \mathcal{L}_1$,

Note that for every j ,

$$\epsilon_j \leq C'' \left(\frac{C^\tau \left(\frac{3}{4}\right)^j \frac{\kappa''}{C_0}}{1 + \|A_1\| + \sum_{l=1}^{j-1} \epsilon_l} \right)^{2n} \quad (194)$$

Estimates (191) and (192) imply

$$\|\hat{F}_1(0)\| \leq \|F_1\|_0 \leq \|\Psi'^{-1}\Psi\|_0 \|\Psi^{-1}\Psi'\|_0 \|\Psi^{-1}\bar{F}_1\Psi\|_0 \leq C_n^2 \left(\frac{1 + \|A_{\mathcal{N}}\|}{\kappa''} \right)^{2n(n+1)} \epsilon^{\frac{3}{2}} \leq \epsilon^{\frac{3}{2} - \frac{1}{48}} \quad (195)$$

Moreover, A_1 has $DC_\omega^{R\bar{N}}(\frac{3}{4}\kappa'', \tau)$ spectrum and F_1 has nice periodicity properties with respect to \mathcal{L} . Let C'' be the constant given by Proposition 4.4. By assumption on ϵ , with C' depending only on n, d, κ, τ and D_3 depending only on n, τ , one has

$$\tilde{\epsilon} \leq \frac{C''}{(1 + \|A_1\|)^{2n}} \left(\frac{3\kappa''}{4C_0} \right)^{2n} C^{2n\tau} \quad (196)$$

Moreover,

$$R\bar{N} \leq R^{n_0+1}N \leq \frac{1}{C} |\log \epsilon| \quad (197)$$

so the assumptions (140) and (141) of Proposition 4.4 hold with $\tilde{F} = F_1, \kappa' = \kappa'', \tilde{N} = R\bar{N}$.

Fix j and assume A_{j-1} has $DC_\omega^{\tilde{N}}(\kappa', \tau)$ spectrum, F_{j-1} has nice periodicity properties with respect to an $(A_{j-1}, (\frac{3}{4})^{j-1} \frac{\kappa''}{C_0}, 2\gamma)$ -decomposition,

$$\|\hat{F}_{j-1}(0)\| \leq \epsilon_{j-1} \quad (198)$$

and

$$CR\bar{N} \leq |\log \epsilon_{j-1}| \quad (199)$$

One obtains functions F_j, X_j and a matrix A_j such that

1. A_j has $DC^{R\bar{N}}((\frac{3}{4})^j \frac{\kappa''}{C_0}, \tau)$ spectrum,
2. $\|A_j\| \leq \|A_{j-1}\| + \epsilon_{j-1}$,
- 3.

$$\partial_\omega e^{X_j} = (A_{j-1} + F_{j-1})e^{X_j} - e^{X_j}(A_j + F_j) \quad (200)$$

and F_j has nice periodicity properties with respect to an $(A_j, (\frac{3}{4})^j \frac{\kappa''}{C_0}, 2\gamma)$ -decomposition

- 4.

$$|\Psi^{-1}\Psi'X_j\Psi'^{-1}\Psi|_{r_{j-1}} \leq C' \left(\frac{1 + \|(A_{j-1})\mathcal{N}\|}{\kappa''(r_{j-2} - r_{j-1})} \right)^{D\gamma} |\Psi^{-1}\Psi'F_j\Psi'^{-1}\Psi|_{r_{j-1}} \quad (201)$$

for some C' depending only on n, d, κ, τ and some D depending only on n, d, τ ,

5. and

$$\begin{aligned} |\Psi^{-1}\Psi'F_j\Psi'^{-1}\Psi|_{r_{j-1}} &\leq C' \left(\frac{1 + \|(A_{j-1})\mathcal{N}\|}{\kappa''(r_{j-2} - r_{j-1})} \right)^{D\gamma} e^{|\Psi^{-1}\Psi'X_{j-1}\Psi'^{-1}\Psi|_{r_{j-2}}} |\Psi^{-1}\Psi'F_{j-1}\Psi'^{-1}\Psi|_{r_{j-2}} \\ &\quad (|\Psi'^{-1}\Psi|_{r_{j-2}}^4 (R\bar{N})^d e^{-2\pi R\bar{N}(r_{j-2}-r_{j-1})} \\ &\quad + (1 + 2e^{|\Psi^{-1}\Psi'X_{j-1}\Psi'^{-1}\Psi|_{r_{j-2}}}) |\Psi^{-1}\Psi'F_{j-1}\Psi'^{-1}\Psi|_{r_{j-2}}) \end{aligned} \quad (202)$$

We shall bound $\|\hat{F}_j(0)\|$ to iterate Proposition 4.4. Estimates (191) and (185) imply

$$e^{|\Psi^{-1}\Psi'X_{j-1}\Psi'^{-1}\Psi|_{r_{j-2}}} \leq 2$$

so

$$\begin{aligned} |\Psi^{-1}\Psi'F_j\Psi'^{-1}\Psi|_{r_{j-1}} &\leq 3C' \left(\frac{1 + \|(A_{j-1})\mathcal{N}\|}{\kappa''(r_{j-2} - r_{j-1})} \right)^{D\gamma} |\Psi^{-1}\Psi'F_{j-1}\Psi'^{-1}\Psi|_{r_{j-2}} \\ &\quad (|\Psi'^{-1}\Psi|_{r_{j-2}}^4 (R\bar{N})^d e^{-2\pi R\bar{N}(r_{j-2}-r_{j-1})} + |\Psi^{-1}\Psi'F_{j-1}\Psi'^{-1}\Psi|_{r_{j-2}}) \end{aligned} \quad (203)$$

and since $r_{j-2} - r_{j-1} = \frac{r-r''}{2l}$,

$$\begin{aligned} |\Psi^{-1}\Psi'F_j\Psi'^{-1}\Psi|_{r_{j-1}} &\leq |\Psi^{-1}\Psi'F_{j-1}\Psi'^{-1}\Psi|_{r_{j-2}}^{\frac{3}{4}} (|\Psi'^{-1}\Psi|_{r_{j-2}}^4 (R\bar{N})^d e^{-2\pi\frac{R\bar{N}(r-r'')}{2l}} + |\Psi^{-1}\Psi'F_{j-1}\Psi'^{-1}\Psi|_{r_{j-2}}) \\ &\leq |\Psi^{-1}\Psi'F_{j-1}\Psi'^{-1}\Psi|_{r_{j-2}}^{\frac{3}{4}} (|\Psi'^{-1}\Psi|_{r_{j-2}}^4 (R\bar{N})^d \epsilon'^{\frac{3}{2l}} + |\Psi^{-1}\Psi'F_{j-1}\Psi'^{-1}\Psi|_{r_{j-2}}) \end{aligned} \quad (204)$$

Now l is bounded by

$$l \leq 8\left(\frac{1}{2}n(n-1) + 1\right)\sqrt[4]{R} \quad (205)$$

Moreover

$$|\Psi'^{-1}\Psi|_{r_{j-2}} \leq C_n \left(\frac{1 + \|A\|}{\kappa''}\right)^{n(n+1)} e^{4\pi\bar{N}r_{j-2}} \leq C_n \left(\frac{1 + \|A\|}{\kappa''}\right)^{n(n+1)} \epsilon'^{\frac{2r_{j-2}}{(r-r'')\sqrt[4]{R}}} \quad (206)$$

and so

$$\begin{aligned} |\Psi^{-1}\Psi'F_j\Psi'^{-1}\Psi|_{r_{j-1}} &\leq |\Psi^{-1}\Psi'F_{j-1}\Psi'^{-1}\Psi|_{r_{j-2}}^{\frac{3}{4}} (\epsilon' + |\Psi^{-1}\Psi'F_{j-1}\Psi'^{-1}\Psi|_{r_{j-2}}) \\ &\leq |\Psi^{-1}\Psi'F_{j-1}\Psi'^{-1}\Psi|_{r_{j-2}}^{\frac{3}{2}} \end{aligned} \quad (207)$$

By a simple induction, for every j ,

$$|\Psi^{-1}\Psi'F_j\Psi'^{-1}\Psi|_{r_{j-1}} \leq |\Psi^{-1}\Psi'F_1\Psi'^{-1}\Psi|_{r_0}^{\left(\frac{3}{2}\right)^{j-1}} \leq \epsilon^{\left(\frac{3}{2}\right)^j} \quad (208)$$

Finally

$$\|\hat{F}_j(0)\| \leq |\Psi^{-1}\Psi'F_j\Psi'^{-1}\Psi|_{r_{j-1}} \|\Psi^{-1}\Psi'\|_0 \|\Psi'^{-1}\Psi\|_0 \leq \epsilon_j \quad (209)$$

so it is possible to iterate Proposition 4.4.

5.4.4 Conclusion

After $l-1$ steps,

$$|\Psi^{-1}\Psi'F_{l+1}\Psi'^{-1}\Psi|_{r_l} \leq \epsilon'^{\frac{l}{16}} \quad (210)$$

Let $Z = e^{X_2} \dots e^{X_{l+1}} \in C_{r'}^\omega(2\mathbb{T}^d, G)$, $A' = A_{l+1}$, $F' = F_{l+1}$. Then

$$\partial_\omega Z = (A_1 + F_1)Z - Z(A' + F') \quad (211)$$

and

$$\|A'\| \leq \|A_1\| + \sum_{j=1}^l \|\hat{F}_j(0)\| + 4\pi\bar{N} \leq \|A\| + |\log \epsilon| \left(\frac{1}{r-r'} \right)^{D_4} \quad (212)$$

for D_4 great enough depending only on n , whence property 5.

• To prove that \mathcal{L}_{l+1} is indeed an $(A_{l+1}, \kappa''(r'', r - \frac{r-r''}{2}, \epsilon'), 2\gamma)$ -decomposition, it is enough to show that

$$\kappa''(r'', r'' - \frac{r-r''}{2}, \epsilon') \leq \left(\frac{3}{4}\right)^{l+1} \frac{\kappa''}{C_0} \quad (213)$$

which comes from the definition of the function κ'' .

• Let us prove property 4. It is true that

$$|\Psi'|_{r''} \leq \epsilon^{-\frac{1}{2}(r-r'')} \epsilon^{-\frac{1}{96}} e^{4\pi r \bar{N}} \leq \epsilon^{-\frac{1}{2}(r-r'')} \epsilon^{-\frac{1}{96}} \epsilon'^{-\frac{r-r''}{200}} \quad (214)$$

and property 4 comes from it, since

$$\epsilon = \epsilon'^{\frac{|\log \epsilon|}{2\pi \sqrt[4]{R\bar{N}}(r-r'')}} \quad (215)$$

• Moreover,

$$|\Psi' F' \Psi'^{-1}|_{r''} \leq |\Psi|_r |\Psi^{-1}|_r |\Psi^{-1} \Psi' F' \Psi'^{-1} \Psi|_{r''} \leq \epsilon' \quad (216)$$

whence 3. Let $Z' = Z_1 \Psi' Z \Psi'^{-1}$, $\bar{F}' = \Psi' F' \Psi^{-1}$ (which satisfies property 2) and \bar{A} such that

$$\partial_\omega \Psi' = \bar{A}' \Psi' - \Psi' A' \quad (217)$$

Then

$$\partial_\omega Z' = (\bar{A}_1 + \bar{F}_1) Z' - Z' (\bar{A}' + \bar{F}') \quad (218)$$

whence 6 and 1, so by (190),

$$\begin{aligned} |Z' - Id|_{r''} &\leq |Z_1 - Id|_{r_1} + |\Psi|_r |\Psi^{-1}|_r \sum_j |\Psi^{-1} \Psi' X_j \Psi'^{-1} \Psi|_{r_j} \\ &\leq \frac{1}{\tilde{C}'} \left(\frac{l(1 + \|A_{\mathcal{N}}\|) |\log \epsilon|}{r-r''} \right)^{D_1 \gamma} \left(\frac{1}{\epsilon} \right)^{4(r-r'')} \left(\epsilon + \sum_j |\Psi^{-1} \Psi' F_j \Psi'^{-1} \Psi|_{r_j} \right) \end{aligned} \quad (219)$$

and by (191) and (208),

$$|Z' - Id|_{r''} \leq \frac{2}{\tilde{C}'} \left(\frac{l(1 + \|A_{\mathcal{N}}\|) |\log \epsilon|}{r - r''} \right)^{D_1 \gamma} \left(\frac{1}{\epsilon} \right)^{4(r-r'')} \epsilon \quad (220)$$

whence property 7 with $D_3 \gamma \geq 2D_1 \gamma$ if $C' \leq \frac{\tilde{C}'}{2(l(r-r''))^{D_1 \gamma}}$, since $l(r - r'')$ has a bound which is independent of $r - r''$. \square

This proposition is the inductive step which can be iterated as a whole. It is necessary to obtain an ϵ' which is much smaller than ϵ so as to control $|\Psi'|_{r'}$ as a function of ϵ' and make sure that the output be similar to the input.

5.5 Main theorem

5.5.1 Numerical lemma

First let us give a lemma which will enable us to iterate Proposition 5.2.

Lemma 5.3 *Let $C' \leq 1, b_0 > 0, r \leq \frac{1}{2}$ and $r' \in [\frac{95}{96}r, r[$. Let $D_5, \gamma_0 \in \mathbb{N}$. There exists C depending only on C', D_5, γ_0 such that for all $\epsilon \leq C \left(\frac{r-r'}{b_0+1} \right)^{2\gamma_0 D_5}$, choosing a sequence (ϵ_k) such that for all k ,*

$$\epsilon_k \leq \epsilon_{k-1}^{100} < 1 \quad (221)$$

and letting for all k

$$\left\{ \begin{array}{l} \gamma_k = 2^k \gamma_0 \\ r_k = r' + \frac{r-r'}{2^k} \\ b_k := b_{k-1} + |\log \epsilon_{k-1}| \left(\frac{2^k}{r-r'} \right)^{D_5} \end{array} \right. \quad (222)$$

then for every $k \in \mathbb{N}$,

$$|\log \epsilon_k|^{2D_5 \gamma_k} \leq \epsilon_k^{-\frac{1}{4}} \quad (223)$$

and

$$a_k := \left(\frac{b_k + 1}{r_k - r_{k+1}} \right)^{D_5 \gamma_k} \epsilon_k \leq C' \quad (224)$$

Proof: Let us first prove (223). It is equivalent to

$$2^{k+3} D_5 \gamma_0 \leq \frac{|\log \epsilon_k|}{\log |\log \epsilon_k|} \quad (225)$$

The function $t \mapsto \frac{|\log t|}{\log |\log t|}$ is decreasing for $t \in]0, e^{-\frac{1}{e}}]$ so it is enough to show that

$$2^{k+3} D_5 \gamma_0 \leq \frac{100^k |\log \epsilon|}{k \log 100 + \log |\log \epsilon|} \quad (226)$$

which is true if we choose C as a function of D_5, γ_0 .

• For all k ,

$$\begin{aligned} a_{k+1} &= \left(\frac{(b_{k+1} + 1) 2^{k+2}}{r - r'} \right)^{D_5 \gamma_{k+1}} \epsilon_{k+1} \\ &\leq \left(\frac{(b_0 + (k + 1) |\log \epsilon_k|) 2^{k+2}}{r - r'} \right)^{2D_5 \gamma_{k+1}} \frac{\epsilon_{k+1}}{\epsilon_k} a_k \end{aligned} \quad (227)$$

so by (223),

$$a_{k+1} \leq \left(\frac{(b_0 + 1)}{r - r'} \right)^{\gamma_0 16^{k+1} D_5} \epsilon^{100^k \cdot 98} a_k \quad (228)$$

thus, if ϵ is also smaller than $\left(\frac{r-r'}{b_0+1}\right)^{16\gamma_0 D_5}$, then $a_{k+1} \leq a_k$. If ϵ is also small enough to satisfy

$$a_0 = \left(\frac{b_0 + 1}{r - r'} \right)^{D_5 \gamma_0} \epsilon \leq C' \quad (229)$$

for instance

$$\left(\frac{b_0 + 1}{r - r'} \right)^{D_5 \gamma_0} \epsilon^{\frac{3}{4}} \leq C' \quad (230)$$

then (224) is true for all k . \square

Lemma 5.3 implies that assumption (185) of Proposition 5.2 holds for all k with $\epsilon \leq \epsilon_k$, $\|A\| = b_k$, $r = r_k$ and $r'' = r_{k+1}$.

As a consequence, one gets the main result, of which we will give various formulations.

5.5.2 Almost reducibility

Theorem 5.4 *Let $r \leq \frac{1}{2}$, $A \in \mathcal{G}$ and $F \in C_r^\omega(2\mathbb{T}^d, \mathcal{G})$ with nice periodicity properties with respect to \mathcal{L}_A . Let*

$$r' \in \left[\frac{95}{96} r, r[\right.$$

There exists D_7 depending only on n, d, τ, κ, A such that if

$$|F|_r \leq \epsilon'_0(r, r') = \left(\frac{r - r'}{\|A\| + 1} \right)^{D_7}$$

then for any $\epsilon \leq \epsilon'_0$, there exists

- $Z_\epsilon \in C_{r'}^\omega(2\mathbb{T}^d, G)$,
- $A_\epsilon \in \mathcal{G}$,
- $\bar{A}_\epsilon, \bar{F}_\epsilon \in C_{r'}^\omega(2\mathbb{T}^d, \mathcal{G})$,

such that

1. \bar{A}_ϵ is reducible to A_ϵ ,
2. $|\bar{F}_\epsilon|_{r'} \leq \epsilon$
3. for every $\theta \in 2\mathbb{T}^d$,

$$\partial_\omega Z_\epsilon(\theta) = (A + F(\theta))Z_\epsilon(\theta) - Z_\epsilon(\theta)(\bar{A}_\epsilon(\theta) + \bar{F}_\epsilon(\theta))$$

4.

$$|Z_\epsilon - Id|_{r'} \leq 2^{D_7} \epsilon_0^{\frac{1}{4} - 4(r - r')}$$

and so does $Z_\epsilon^{-1} - Id$,

5. $Z_\epsilon, \partial_\omega Z_\epsilon$ are bounded in $C_{r'}^\omega(2\mathbb{T}^d, gl(n, \mathbb{C}))$ uniformly in ϵ .

Moreover, in dimension 2 or if $\mathcal{G} = gl(n, \mathbb{C})$ or $u(n)$, if F is continuous on \mathbb{T}^d , then $\bar{A}_\epsilon, \bar{F}_\epsilon$ and Z_ϵ are continuous on \mathbb{T}^d . If \mathcal{G} is $o(n)$ or $u(n)$, then D_7 does not depend on A .

Proof: The proof will be made by induction as follows. Let $r'' = \frac{r+r'}{2}$. Let $R(r, r''), N(r, \epsilon), \kappa''(r, r'', \epsilon)$ be as in (184). There exists $\gamma_0 \in \mathbb{N}$ depending only on n, d, τ, κ, A , such that \mathcal{L}_A is an (A, κ, γ_0) -decomposition (one can assume $\gamma_0 \geq n(n+1)$). Let C', D_3 be as in Proposition 5.2. Let $D_5 = 2D_3$. Let C be as in Lemma 5.3 and D_7 such that

$$\epsilon'_0 := \left(\frac{r - r''}{\|A\| + 1} \right)^{D_7} \leq C \left(\frac{r - r''}{\|A\| + 1} \right)^{4\gamma_0 D_5} \quad (231)$$

Before carrying on with the proof, note that if \mathcal{G} is $o(n)$ or $u(n)$, then \mathcal{L}_A is a unitary decomposition, therefore it is an $(A, \kappa, 0)$ -decomposition, so one can take $\gamma_0 = n(n+1)$ and then γ_0, D_3, D_5 and D_7 do not depend on A . For all $k \in \mathbb{N}$, let

$$\begin{cases} r_k = r'' + \frac{r-r''}{2^k}, \\ b_0 = \|A\|, \\ b_{k+1} = \|A\| + \sum_{j \leq k} \frac{|\log \epsilon_j|}{(r_{j-1} - r_j)^{D_5}} \end{cases} \quad (232)$$

where (ϵ_j) will be defined by induction in the following. Suppose that $|F|_r \leq \epsilon'_0$. Apply Proposition 5.2 a first time: there exist functions

- $Z_1 \in C_{r_1}^\omega(2\mathbb{T}^d, G)$,
- $\bar{A}_1, \bar{F}_1 \in C_{r_1}^\omega(2\mathbb{T}^d, \mathcal{G})$,
- $A_1 \in \mathcal{G}$,
- $\Psi_0 \in C_{r_1}^\omega(2\mathbb{T}^d, G)$

and a real number $\epsilon_1 \leq |F|_r^{100}$ such that

1. \bar{A}_1 is reducible to A_1 by Ψ_0 .
2. $\Psi_0^{-1}\bar{F}_1\Psi_0$ has nice periodicity properties with respect to an $(A_1, \kappa''(r_1, r_2, \epsilon_1), 2\gamma_0)$ -decomposition,
3. $|\bar{F}_1|_{r_1} \leq \epsilon_1$,
4. $|\Psi_0|_{r_1} \leq \epsilon_1^{-\frac{1}{2}(r_1-r_2)}$ and $|\Psi_0^{-1}|_{r_1} \leq \epsilon_1^{-\frac{1}{2}(r_1-r_2)}$,
5. $\|A_1\| \leq b_1$;
6. for all $\theta \in 2\mathbb{T}^d$,

$$\partial_\omega(Z_1(\theta)) = (A + F(\theta))Z_1(\theta) - Z_1(\theta)(\bar{A}_1(\theta) + \bar{F}_1(\theta))$$

7.

$$|Z_1 - Id|_{r_1} \leq \frac{1}{C'} \left(\frac{(1 + \|A\|) |\log \epsilon_0|}{r_0 - r_1} \right)^{D_{3\gamma_0}} \epsilon_0^{1-4(r_0-r_1)}$$

which implies, using Lemma 5.3, that

$$|Z_1 - Id|_{r_1} \leq \frac{1}{C'} \epsilon_0^{\frac{1}{4}-4(r_0-r_1)}$$

and so does $Z_1^{-1} - Id$.

- Let $k \geq 1$. Let
 - $\bar{A}_k \in C^\omega(2\mathbb{T}^d, \mathcal{G})$,
 - $A_k \in \mathcal{G}$,
 - $\bar{F}_k \in C_{r_k}^\omega(2\mathbb{T}^d, \mathcal{G})$
 - $\Psi_{k-1} \in C_{r_k}^\omega(2\mathbb{T}^d, G)$
 - $\epsilon_k \leq |F|^{100^k}$

such that

- \bar{A}_k is reducible to $A_k \in \mathcal{G}$ by Ψ_{k-1} ,

- $\Psi_{k-1}^{-1} \bar{F}_k \Psi_{k-1}$ has nice periodicity properties with respect to an $(A_k, \kappa''(r_k, r_{k+1}, \epsilon_k), 2^k \gamma_0)$ -decomposition,
- $|\bar{F}_k|_{r_k} \leq \epsilon_k$,
- $|\Psi_{k-1}|_r \leq \epsilon_k^{-\frac{1}{2}(r_k - r_{k+1})}$ et $|\Psi_{k-1}^{-1}|_r \leq \epsilon_k^{-\frac{1}{2}(r_k - r_{k+1})}$,

Lemma 5.3 says that

$$\left(\frac{b_k + 1}{r_k - r_{k+1}} \right)^{2^k D_3 \gamma_0} \epsilon_k \leq C' \quad (233)$$

Therefore one can apply again Proposition 5.2 to find

- $Z_{k+1} \in C_{r_{k+1}}^\omega(2\mathbb{T}^d, G)$,
- $A_{k+1} \in \mathcal{G}$,
- $\bar{A}_{k+1} \in C_r^\omega(2\mathbb{T}^d, \mathcal{G})$,
- $\Psi_k \in C_r^\omega(2\mathbb{T}^d, G)$,
- $\bar{F}_{k+1} \in C_{r_{k+1}}^\omega(2\mathbb{T}^d, \mathcal{G})$
- $\epsilon_{k+1} \leq |F|^{100^{k+1}}$

such that

1. \bar{A}_{k+1} is reducible to A_{k+1} by Ψ_k ,
2. $\Psi_k^{-1} \bar{F}_{k+1} \Psi_k$ has nice periodicity properties with respect to an $(A_{k+1}, \kappa''(r_{k+1}, r_{k+2}, \epsilon_{k+1}), 2^{k+1} \gamma_0)$ -decomposition,
3. $|\bar{F}_{k+1}|_{r_{k+1}} \leq \epsilon_{k+1}$,
4. $|\Psi_k|_r \leq \epsilon_{k+1}^{-\frac{1}{2}(r_{k+1} - r_{k+2})}$ and $|\Psi_k^{-1}|_r \leq \epsilon_{k+1}^{-\frac{1}{2}(r_{k+1} - r_{k+2})}$,
5. $\|A_{k+1}\| \leq b_{k+1}$,

6.

$$\partial_\omega Z_{k+1} = (\bar{A}_k + \bar{F}_k) Z_{k+1} - Z_{k+1} (\bar{A}_{k+1} + \bar{F}_{k+1})$$

7.

$$|Z_{k+1} - Id|_{r_{k+1}} \leq \frac{1}{C'} \left(\frac{(1 + \|A_k\|) |\log \epsilon_k|}{r_k - r_{k+1}} \right)^{2^k D_3 \gamma_0} \epsilon_k^{1-4(r_k - r_{k+1})}$$

which implies, using Lemma 5.3, that

$$|Z_{k+1} - Id|_{r_{k+1}} \leq \frac{1}{C'} \epsilon_k^{\frac{1}{4} - 4(r_k - r_{k+1})}$$

and so does $Z_{k+1}^{-1} - Id$.

- Let $\epsilon \leq \epsilon'_0$ and $k_\epsilon \in \mathbb{N}$ such that $|F|_r^{100k_\epsilon} \leq \epsilon$. Let

$$\begin{cases} Z_\epsilon = Z_1 \dots Z_{k_\epsilon} \\ \bar{A}_\epsilon = \bar{A}_{k_\epsilon} \\ \bar{F}_\epsilon = \bar{F}_{k_\epsilon} \end{cases} \quad (234)$$

then properties 1 and 2 hold. Thus for all $\theta \in 2\mathbb{T}^d$,

$$\partial_\omega Z_\epsilon(\theta) = (A + F(\theta))Z_\epsilon(\theta) - Z_\epsilon(\theta)(\bar{A}_\epsilon(\theta) + \bar{F}_\epsilon(\theta)) \quad (235)$$

whence property 3. Moreover, let $a_k := |Z_1 \dots Z_k - Id|_{r''}$, then

•

$$a_1 = |Z_1 - Id|_{r''} \leq \frac{1}{C'} \epsilon_0^{\frac{1}{4}-4(r_0-r_1)} \quad (236)$$

so

$$|Z_1|_{r''} \leq 1 + \frac{1}{C'} \epsilon_0^{\frac{1}{4}-4(r_0-r_1)} \quad (237)$$

- let $k \geq 2$ and assume that for all $j \leq k-1$,

$$|Z_1 \dots Z_j|_{r''} \leq 1 + \frac{3}{C'} \epsilon_0^{\frac{1}{4}-4(r_0-r_1)} \quad (238)$$

then

$$\begin{aligned} a_k &\leq |Z_k - Id|_{r''} |Z_1 \dots Z_{k-1}|_{r''} + a_{k-1} \\ &\leq a_1 + \frac{1}{C'} \sum_{j=1}^{k-1} |Z_1 \dots Z_j|_{r''} \epsilon_j^{\frac{1}{4}-4(r_j-r_{j+1})} \leq \frac{3}{C'} \epsilon_0^{\frac{1}{4}-4(r_0-r_1)} \end{aligned} \quad (239)$$

and

$$|Z_1 \dots Z_k|_{r''} \leq 1 + \frac{3}{C'} \epsilon_0^{\frac{1}{4}-4(r_0-r_1)} \quad (240)$$

whence property 4. This also implies that

$$|Z_\epsilon|_{r''} \leq 2 + \frac{3}{C'} \epsilon_0^{\frac{1}{4} - 4(r_0 - r_1)} \quad (241)$$

Moreover, by a Cauchy estimate,

$$|\partial_\omega Z_\epsilon|_{r'} \leq \frac{1}{r'' - r'} |Z_\epsilon|_{r''} \quad (242)$$

so 5 is true.

If \mathcal{G} is either $gl(n, \mathbb{C})$ or $u(n)$ or in dimension 2, if F is continuous on \mathbb{T}^d , each step will give functions $Z_{k+1}, A_{k+1}, \bar{A}_{k+1}, \bar{F}_{k+1}$ continuous on \mathbb{T}^d so, at the end of the process, the functions $Z_\epsilon, \bar{A}_\epsilon$ et \bar{F}_ϵ are continuous on \mathbb{T}^d . \square

This proves Theorem 0.1.

In general, almost reducibility does not imply reducibility. Reducibility happens if there are a finite number of renormalizations, or if the sequence (Ψ_k) given by Theorem 5.4 converges in $C_{r'}^\omega(2\mathbb{T}^d, G)$. In general, this sequence is not even bounded in $C_0^\omega(2\mathbb{T}^d, G)$. However, if the method above has been used to conjugate the system $A + F$ to a system $\bar{A}_\epsilon + \bar{F}_\epsilon$ where \bar{A}_ϵ is reducible by Ψ_ϵ to a constant A_ϵ , and where \bar{F}_ϵ is bounded by ϵ , one can also bound $\Psi_\epsilon^{-1} \bar{F}_\epsilon \Psi_\epsilon$.

Corollary 5.5 *Let $r \leq \frac{1}{2}$, $A \in \mathcal{G}$ and $F \in C_r^\omega(2\mathbb{T}^d, \mathcal{G})$ with nice periodicity properties with respect to \mathcal{L}_A . Let $r' \in [\frac{95}{96}r, r[$. There exists D_8 depending only on n, d, κ, τ, A such that if*

$$|F|_r \leq (r - r')^{D_8}$$

then there exists

- $Z \in C_{r'}^\omega(2\mathbb{T}^d, G)$,
- a family (A_l) of reducible functions in $C_{r'}^\omega(2\mathbb{T}^d, \mathcal{G})$
- and $A_\infty \in C_{r'}^\omega(2\mathbb{T}^d, \mathcal{G})$

such that

$$\partial_\omega Z(\theta) = (A + F(\theta))Z(\theta) - Z(\theta)A_\infty(\theta) \quad (243)$$

and

$$\lim_{l \rightarrow \infty} |A_l - A_\infty|_{r'} = 0 \quad (244)$$

Moreover, in dimension 2 or if $\mathcal{G} = gl(n, \mathbb{C})$ or $u(n)$, if F is continuous on \mathbb{T}^d , then Z , A_l and A_∞ are continuous on \mathbb{T}^d .

Proof: Let D_7 be as in Theorem 5.4 and D_8 such that

$$(r - r')^{D_8} \leq \left(\frac{r - r'}{1 + \|A\|} \right)^{D_7} \quad (245)$$

Let $Z_\epsilon \in C_{r'}^\omega(2\mathbb{T}^d, G)$, $A_\epsilon \in C_{r'}^\omega(2\mathbb{T}^d, \mathcal{G})$ be as in Theorem 5.4. Then Z_ϵ and $\partial_\omega Z_\epsilon$ remain bounded in $C_{r'}^\omega(2\mathbb{T}^d, G)$ when $\epsilon \rightarrow 0$. Let Z be the limit in $C_{r'}^\omega(2\mathbb{T}^d, G)$ of a subsequence $(Z_{\frac{1}{k_l}})$ of $(Z_{\frac{1}{k}})_{k \in \mathbb{N} \setminus \{0\}}$ and

$$A_\infty(\theta) := Z(\theta)^{-1}(A + F(\theta))Z(\theta) - Z(\theta)^{-1}\partial_\omega Z(\theta)$$

then

$$A_\infty \in C_{r'}^\omega(2\mathbb{T}^d, \mathcal{G}), \lim_{l \rightarrow \infty} |A_{\frac{1}{k_l}} - A_\infty|_{r'} = 0$$

and so equation (243) holds.

In dimension 2 or if $\mathcal{G} = gl(n, \mathbb{C})$ or $u(n)$, if F is continuous on \mathbb{T}^d , all functions that one has to consider are continuous on \mathbb{T}^d . \square

Remark: In Corollary 5.5, the function A_∞ is not reducible in general, it is only a limit of reducible functions.

5.5.3 A result of density of reducible cocycles

Corollary 5.6 *Let $0 < r' < r \leq \frac{1}{2}$, $A \in \mathcal{G}$ and $F \in C_r^\omega(\mathbb{T}^d, \mathcal{G})$. There exists ϵ'_0 depending only on $n, d, \tau, \kappa, A, r - r'$ such that if $|F - A|_r \leq \epsilon'_0$, then for all $\epsilon > 0$ there exists $H \in C_{r'}^\omega(2\mathbb{T}^d, \mathcal{G})$ such that $|F - H|_{r'} \leq \epsilon$ and H is reducible.*

Proof: Let D_7 be as in Theorem 5.4. Assume that

$$|F - A|_r \leq (r - r')^{D_7} =: \epsilon'_0 \quad (246)$$

Let $\epsilon > 0$. By Theorem 5.4, there exist $Z_\epsilon \in C_{r'}^\omega(2\mathbb{T}^d, G)$, $\bar{A}_\epsilon, \bar{F}_\epsilon \in C_{r'}^\omega(2\mathbb{T}^d, \mathcal{G})$ and $A_\epsilon \in \mathcal{G}$ such that

- \bar{A}_ϵ is reducible to A_ϵ ,
- $\partial_\omega Z_\epsilon = FZ_\epsilon - Z_\epsilon(\bar{A}_\epsilon + \bar{F}_\epsilon)$,
- $|Z_\epsilon|_{r'} \leq 2$, $|Z_\epsilon^{-1}|_{r'} \leq 2$,
- $|\bar{F}_\epsilon|_{r'} \leq \frac{\epsilon}{4}$.

Therefore

$$\partial_\omega Z_\epsilon = HZ_\epsilon - Z_\epsilon \bar{A}_\epsilon \quad (247)$$

where $H = F - Z_\epsilon \bar{F}_\epsilon Z_\epsilon^{-1}$ is reducible to A_ϵ and satisfies

$$|H - F|_{r'} \leq 4|\bar{F}_\epsilon|_{r'} \leq \epsilon \quad (248)$$

Corollary 5.7 *Let $0 < r' < r \leq \frac{1}{2}$, $A \in sl(2, \mathbb{R})$ and $F \in C_r^\omega(\mathbb{T}^d, sl(2, \mathbb{R}))$. There exists ϵ'_0 depending only on $n, d, \tau, \kappa, A, r - r'$ such that if $|F - A|_r \leq \epsilon'_0$, then for any $\epsilon > 0$ there exist $H \in C_{r'}^\omega(\mathbb{T}^d, sl(2, \mathbb{R}))$ such that $|F - H|_{r'} \leq \epsilon$ and H is reducible.*

Proof: Do the same construction as in Corollary 5.6. Theorem 5.4 gives functions $\bar{A}_\epsilon, \bar{F}_\epsilon, Z_\epsilon$ which are, in fact, continuous on \mathbb{T}^d . Thus H is continuous on \mathbb{T}^d . \square

Corollary 5.7 also holds with $gl(n, \mathbb{C})$ or $u(n)$ instead of $sl(2, \mathbb{R})$. This proves Theorem 0.2.

6 Appendix

Lemma 6.1 *Let \mathcal{G} be a Lie algebra and $A, F \in \mathcal{G}$ with $\|F\| \leq 1$. Let $\alpha_1(\lambda), \dots, \alpha_n(\lambda)$ be a continuous choice of the eigenvalues of $A + \lambda F$ as λ varies from 0 to 1. Then for all $1 \leq j \leq n$, there exists $1 \leq j' \leq n$ such that*

$$|\alpha_{j'}(\lambda) - \alpha_j(0)| \leq 2n\lambda^{\frac{1}{n}}(\|A\| + 1) \quad (249)$$

Proof: Fix $j \leq n$. Let $\lambda_0 > 0$. For every λ , let

$$A(\lambda) = A + \lambda F \quad (250)$$

and

$$f(\lambda) = \det(\alpha_j(\lambda_0)I - A(\lambda)) \quad (251)$$

Then $f(\lambda_0) = 0$ and for every λ ,

$$f(\lambda) = \det(\alpha_j(\lambda_0)I - A(\lambda)) = \prod_{j'} (\alpha_j(\lambda_0) - \alpha_{j'}(\lambda)) \quad (252)$$

so

$$\left| \prod_{j'} (\alpha_j(\lambda_0) - \alpha_{j'}(\lambda)) \right| = |f(\lambda_0) - f(\lambda)| \leq \sup_{\lambda''} |f'(\lambda'')| |\lambda_0 - \lambda| \quad (253)$$

and since

$$\begin{aligned}
|f'(\lambda'')| &= \left| \sum_{\sigma} \frac{d}{d\lambda''} \prod_k (\alpha_j(\lambda_0)I - A - \lambda''F)_{k,\sigma(k)} \right| \\
&\leq nn! [\|A(\lambda_0)\| + \|A(\lambda'')\|]^{n-1} \\
&\leq 2^{n-1} nn! [\|A\| + 1]^{n-1}
\end{aligned} \tag{254}$$

then there exists j' such that

$$| \alpha_j(\lambda_0) - \alpha_{j'}(\lambda) | \leq 2n | \lambda - \lambda_0 |^{\frac{1}{n}} [\|A\| + 1] \tag{255}$$

In particular, there exists j' such that

$$| \alpha_j(0) - \alpha_{j'}(\lambda) | \leq 2n | \lambda |^{\frac{1}{n}} [\|A\| + 1] \square \tag{256}$$

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