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CANONICAL BASES AND AFFINE HECKE ALGEBRAS OF TYPE D

P. Shan, M. Varagnolo, E. Vasserot

Abstract. We prove a conjecture of Miemietz and Kashiwara on canonical bases and branching rules of affine Hecke algebras of type D. The proof is similar to the proof of the type B case in [VV].

INTRODUCTION

Let $f$ be the negative part of the quantized enveloping algebra of type $A^{(1)}$. Lusztig’s description of the canonical basis of $f$ implies that this basis can be naturally identified with the set of isomorphism classes of simple objects of a category of modules of the affine Hecke algebras of type $A$. This identification was mentioned in [G], and was used in [A]. More precisely, there is a linear isomorphism between $f$ and the Grothendieck group of finite dimensional modules of the affine Hecke algebras of type $A$, and it is proved in [A] that the induction/restriction functors for affine Hecke algebras are given by the action of the Chevalley generators and their transposed operators with respect to some symmetric bilinear form on $f$.

The branching rules for affine Hecke algebras of type $B$ have been investigated quite recently, see [E], [EK1,2,3], [M] and [VV]. In particular, in [E], [EK1,2,3] an analogue of Ariki’s construction is conjectured and studied for affine Hecke algebras of type $B$. Here $f$ is replaced by a module $\theta V(\lambda)$ over an algebra $\theta B$. More precisely it is conjectured there that $\theta V(\lambda)$ admits a canonical basis which is naturally identified with the set of isomorphism classes of simple objects of a category of modules of the affine Hecke algebras of type $B$. Further, in this identification the branching rules of the affine Hecke algebras of type $B$ should be given by the $\theta B$-action on $\theta V(\lambda)$. This conjecture has been proved [VV]. It uses both the geometric picture introduced in [E] (to prove part of the conjecture) and a new kind of graded algebras similar to the KLR algebras from [KL], [R].

A similar description of the branching rules for affine Hecke algebras of type $D$ has also been conjectured in [KM]. In this case $f$ is replaced by another module $\circ V$ over the algebra $\theta B$ (the same algebra as in the type $B$ case). The purpose of this paper is to prove the type $D$ case. The method of the proof is the same as in [VV].

First we introduce a family of graded algebras $\circ R_m$ for $m$ a non negative integer. They can be viewed as the Ext-algebras of some complex of constructible sheaves naturally attached to the Lie algebra of the group $SO(2m)$, see Remark 2.8. This complex enters in the Kazhdan-Lusztig classification of the simple modules of the affine Hecke algebra of the group $Spin(2m)$. Then we identify $\circ V$ with the sum of the Grothendieck groups of the graded algebras $\circ R_m$.

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The plan of the paper is the following. In Section 1 we introduce a graded algebra $^\circ \mathcal{R}(\Gamma)_\nu$. It is associated with a quiver $\Gamma$ with an involution $\theta$ and with a dimension vector $\nu$. In Section 2 we consider a particular choice of pair $(\Gamma, \theta)$. The graded algebras $^\circ \mathcal{R}(\Gamma)_\nu$ associated with this pair $(\Gamma, \theta)$ are denoted by the symbol $^\circ \mathcal{R}_m$. Next we introduce the affine Hecke algebra of type $D$, more precisely the affine Hecke algebra associated with the group $SO(2m)$, and we prove that it is Morita equivalent to $^\circ \mathcal{R}_m$. In Section 3 we categorify the module $^\circ \mathcal{V}$ from [KM] using the graded algebras $^\circ \mathcal{R}_m$, see Theorem 3.28. The main result of the paper is Theorem 3.33.

0. Notation

0.1. Graded modules over graded algebras. Let $k$ be an algebraically closed field of characteristic 0. By a graded $k$-algebra $R = \bigoplus_d R_d$ we’ll always mean a $\mathbb{Z}$-graded associative $k$-algebra. Let $R\text{-mod}$ be the category of finitely generated graded $R$-modules, $R\text{-fmod}$ be the full subcategory of finite-dimensional graded modules and $R\text{-proj}$ be the full subcategory of projective objects. Unless specified otherwise all modules are left modules. We’ll abbreviate $K(R) = [R\text{-proj}]$, $G(R) = [R\text{-fmod}]$.

Here $[\mathcal{C}]$ denotes the Grothendieck group of an exact category $\mathcal{C}$. Assume that the $k$-vector spaces $R_d$ are finite dimensional for each $d$. Then $K(R)$ is a free Abelian group with a basis formed by the isomorphism classes of the indecomposable objects in $R\text{-proj}$, and $G(R)$ is a free Abelian group with a basis formed by the isomorphism classes of the simple objects in $R\text{-fmod}$. Given an object $M$ of $R\text{-proj}$ or $R\text{-fmod}$ let $[M]$ denote its class in $K(R)$, $G(R)$ respectively. When there is no risk of confusion we abbreviate $M = [M]$. We’ll write $[M : N]$ for the composition multiplicity of the $R$-module $N$ in the $R$-module $M$. Consider the ring $A = \mathbb{Z}[v, v^{-1}]$. If the grading of $R$ is bounded below then the $A$-modules $K(R)$, $G(R)$ are free. Here $A$ acts on $G(R)$, $K(R)$ as follows

$$vM = M[1], \quad v^{-1}M = M[-1].$$

For any $M, N$ in $R\text{-mod}$ let

$$\text{hom}_R(M, N) = \bigoplus_d \text{Hom}_R(M, N[d])$$

be the $\mathbb{Z}$-graded $k$-vector space of all $R$-module homomorphisms. If $R = k$ we’ll omit the subscript $R$ in hom’s and in tensor products. For any graded $k$-vector space $M = \bigoplus_d M_d$ we’ll write

$$\text{gdim}(M) = \sum_d v^d \text{dim}(M_d),$$

where dim is the dimension over $k$. 
0.2. Quivers with involutions. Recall that a quiver $Γ$ is a tuple $(I, H, h \mapsto h', h \mapsto h'')$ where $I$ is the set of vertices, $H$ is the set of arrows and for each $h \in H$ the vertices $h', h'' \in I$ are the origin and the goal of $h$ respectively. Note that the set $I$ may be infinite. We’ll assume that no arrow may join a vertex to itself. For each $i, j \in I$ we write

$$H_{i,j} = \{ h \in H; h' = i, h'' = j \}.$$ 

We’ll abbreviate $i \rightarrow j$ if $H_{i,j} \neq \emptyset$. Let $h_{i,j}$ be the number of elements in $H_{i,j}$ and set

$$i \cdot j = -h_{i,j} - h_{j,i}, \quad i \cdot i = 2, \quad i \neq j.$$ 

An involution $θ$ on $Γ$ is a pair of involutions on $I$ and $H$, both denoted by $θ$, such that the following properties hold for each $h$ in $H$

- $θ(h)' = θ(h''')$ and $θ(h)'' = θ(h')$,
- $θ(h') = h''$ iff $θ(h) = h$.

We’ll always assume that $θ$ has no fixed points in $I$, i.e., there is no $i \in I$ such that $θ(i) = i$. To simplify we’ll say that $θ$ has no fixed point. Let

$$θNI = \{ ν = \sum_i ν_i i \in NI : νθ(i) = ν_i, \forall i \}. $$

For any $ν \in θNI$ set $|ν| = \sum_i ν_i$. It is an even integer. Write $|ν| = 2m$ with $m \in \mathbb{N}$. We’ll denote by $θ^ν$ the set of sequences

$$i = (i_{1-m}, \ldots, i_{m-1}, i_m)$$

of elements in $I$ such that $θ(i_j) = i_{j-1}$ and $\sum_k i_k = ν$. For any such sequence $i$ we’ll abbreviate $θ(i) = (θ(i_{1-m}), \ldots, θ(i_{m-1}), θ(i_m))$. Finally, we set

$$θ^νm = \bigcup_ν θ^ν, \quad ν \in θNI, \quad |ν| = 2m.$$ 

0.3. The wreath product. Given a positive integer $m$, let $S_m$ be the symmetric group, and $\mathbb{Z}_2 = \{-1, 1\}$. Consider the wreath product $W_m = S_m \wr \mathbb{Z}_2$. Write $s_1, \ldots, s_{m-1}$ for the simple reflections in $S_m$. For each $l = 1, 2, \ldots, m$ let $ε_l \in (\mathbb{Z}_2)^m$ be $-1$ placed at the $l$-th position. There is a unique action of $W_m$ on the set $\{1 - m, \ldots, m - 1, m\}$ such that $S_m$ permutes $1, 2, \ldots, m$ and such that $ε_l$ fixes $k$ if $k \neq l, 1 - l$ and switches $l$ and $1 - l$. The group $W_m$ acts also on $θ^ν$. Indeed, view a sequence $i$ as the map

$$\{1 - m, \ldots, m - 1, m\} \rightarrow I, \quad l \mapsto i_l.$$ 

Then we set $w(i) = i \circ w^{-1}$ for $w \in W_m$. For each $ν$ we fix once for all a sequence

$$i_ε = (i_{1-m}, \ldots, i_m) \in θ^ν.$$ 

Let $W_ε$ be the centralizer of $i_ε$ in $W_m$. Then there is a bijection

$$W_ε \backslash W_m \rightarrow θ^ν, \quad W_ε w \mapsto w^{-1}(i_ε).$$ 

Now, assume that $m > 1$. We set $s_0 = ε_1 s_1 ε_1$. Let $θW_m$ be the subgroup of $W_m$ generated by $s_0, \ldots, s_{m-1}$. We’ll regard it as a Weyl group of type $D_m$ such that $s_0, \ldots, s_{m-1}$ are the simple reflections. Note that $W_ε$ is a subgroup of $θW_m$. Indeed, if $W_ε \not\subset θW_m$ there should exist $l$ such that $ε_l$ belongs to $W_ε$. This would imply that $i_l = θ(i_l)$, contradicting the fact that $θ$ has no fixed point. Therefore $θ^ν$ decomposes into two $θW_m$-orbits. We’ll denote them by $θ^ν_+$ and $θ^ν_-$. For $m = 1$ we set $θW_1 = \{ ε \}$ and we choose again $θ^ν_+$ and $θ^ν_-$ in a obvious way.
1. The graded $k$-algebra $\mathcal{R}(\Sigma)$

Fix a quiver $\Sigma$ with set of vertices $I$ and set of arrows $H$. Fix an involution $\theta$ on $\Sigma$. Assume that $\Sigma$ has no 1-loops and that $\theta$ has no fixed points. Fix a dimension vector $\nu \neq 0$ in $\mathbb{N}I$. Set $|\nu| = 2m$.

1.1. Definition of the graded $k$-algebra $\mathcal{R}(\Sigma)$ Assume that $m > 1$. We define a graded $k$-algebra $\mathcal{R}(\Sigma)$ with 1 generated by $1_\Sigma$, $\sigma_k$, with $i \in \mathcal{I}$, $l = 1, 2, \ldots, m$, $k = 0, 1, \ldots, m - 1$ modulo the following defining relations

(a) $1_\Sigma 1_\Sigma = \delta_{I,1} 1_\Sigma$, $\sigma_k 1_\Sigma = 1_{s_k} \sigma_k$, $\sigma_k 1_\Sigma = 1_{s_k} \sigma_k$,

(b) $\sigma_k \sigma_k' = \sigma_k \sigma_k$,

(c) $\sigma_k^2 1_\Sigma = Q_{i_k, i_k} (\sigma_k(k), \sigma_k(k)) 1_\Sigma$,

(d) $\sigma_k \sigma_k = \sigma_k \sigma_k$ if $1 \leq k < k' - 1 < m - 1$ or $0 = k < k' \neq 2$,

(e) $(\sigma_k(k) \sigma_k(k) - \sigma_k \sigma_k(k) \sigma_k) 1_\Sigma =
\begin{cases} 
Q_{i_k, i_k} (\sigma_k(k), \sigma_k(k)) 1_\Sigma = Q_{i_k, i_k} (\sigma_k(k), \sigma_k(k) + 1) \\
0
\end{cases}

\text{if } i_k = i_{s_k(k) + 1},

\text{else},

(f) $(\sigma_k \sigma_k - \sigma_{s_k(k)} \sigma_k) 1_\Sigma =
\begin{cases} 
-1_\Sigma & \text{if } l = k, i_k = i_{s_k(k)}, \\
1_\Sigma & \text{if } l = s_k(k), i_k = i_{s_k(k)}, \\
0 & \text{else}
\end{cases}

Here we have set $\sigma_{1-I} = -\sigma_I$

(1.1) $Q_{i,j}(u, v) = \begin{cases} 
(-1)^{h_{i,j}(u - v)^{-i-j}} & \text{if } i \neq j, \\
0 & \text{else}
\end{cases}$

If $m = 0$ we set $\mathcal{R}(\Sigma) = k \oplus k$. If $m = 1$ then we have $\nu = i + \theta(i)$ for some $i \in I$. Write $i = i \theta(i)$, and

$\mathcal{R}(\Sigma) = k[\sigma_1] 1_\Sigma \oplus k[\sigma_1] 1_{\theta(1)}$.

We’ll abbreviate $\sigma_{1-k} = \sigma_{1-k}$ and $\sigma_{1-I} = \sigma_{1-1}$. The grading on $\mathcal{R}(\Sigma)$ is the trivial one. For $m \geq 1$ the grading on $\mathcal{R}(\Sigma)$ is given by the following rules:

$\deg(1_\Sigma) = 0$,

$\deg(\sigma_{1-k}) = 2$,

$\deg(\sigma_{1-k} \sigma_{1-k}) = -i_k \cdot s_{k}(k)$.

We define $\omega$ to be the unique involution of the graded $k$-algebra $\mathcal{R}(\Sigma)$ which fixes $1_\Sigma$, $\sigma_k$, $\sigma_k$. We set $\omega$ to be identity on $\mathcal{R}(\Sigma)$.

1.2. Relation with the graded $k$-algebra $\mathcal{R}(\Sigma)$ Assume a family of graded $k$-algebra $\mathcal{R}(\Sigma)_\lambda$ was introduced in [VV, sec. 5.1], for $\lambda$ an arbitrary dimension vector in $\mathbb{N}I$. Here we’ll only consider the special case $\lambda = 0$, and we abbreviate $\mathcal{R}(\Sigma) = \mathcal{R}(\Sigma)_0$. Recall that if $\nu \neq 0$ then $\mathcal{R}(\Sigma)_\nu$ is the graded $k$-algebra with 1 generated
by $1_i$, $x_l$, $\sigma_k$, $\pi_1$, with $i \in \Omega^\nu$, $l = 1, 2, \ldots, m$, $k = 1, \ldots, m - 1$ such that $1_i$, $x_l$, and $\sigma_k$ satisfy the same relations as before and

$$\pi_1^2 = 1, \quad \pi_1 1_i \pi_1 = 1_{\epsilon_{1i}}, \quad \pi_1 x_l \pi_1 = \kappa_{x_1(l)},$$

$$(\pi_1 1_i)^2 = (\sigma_1 1_i)^2, \quad \pi_1 1_i \pi_1 = \sigma_k \text{ if } k \neq 1.$$  

If $\nu = 0$ then $\theta \mathcal{R}(\Gamma)_0 = k$. The grading is given by setting $\deg(1_i)$, $\deg(x_l)$, and $\deg(\sigma_k, 1)$ to be as before and $\deg(\pi_1 1_i) = 0$. In the rest of Section 1 we'll assume $m > 0$. Then there is a canonical inclusion of graded $k$-algebras

$$\theta \mathcal{R}(\Gamma)_0 \subset \theta \mathcal{R}(\Gamma)_0$$

such that $1_i, x_l, \sigma_k \mapsto 1_i, x_l, \sigma_k$ for $i \in \Omega^\nu$, $l = 1, \ldots, m$, $k = 1, \ldots, m - 1$ and such that $\sigma_0 \mapsto \pi_1 \sigma_1 \pi_1$. From now on we'll write $\sigma_0 = \pi_1 \sigma_1 \pi_1$ whenever $m > 1$. The assignment $x \mapsto \pi_1 x \pi_1$ defines an involution of the graded $k$-algebra $\theta \mathcal{R}(\Gamma)_0$ which normalizes $\theta \mathcal{R}(\Gamma)_0$. Thus it yields an involution

$$\gamma : \theta \mathcal{R}(\Gamma)_0 \to \theta \mathcal{R}(\Gamma)_0.$$  

Let $(\gamma)$ be the group of two elements generated by $\gamma$. The smash product $\theta \mathcal{R}(\Gamma)_0 \rtimes (\gamma)$ is a graded $k$-algebra such that $\deg(\gamma) = 0$. There is an unique isomorphism of graded $k$-algebras

$$\theta \mathcal{R}(\Gamma)_0 \rtimes (\gamma) \to \theta \mathcal{R}(\Gamma)_0$$

which is identity on $\theta \mathcal{R}(\Gamma)_0$ and which takes $\gamma$ to $\pi_1$.  

1.3. The polynomial representation and the PBW theorem. For any $i$ in $\Omega^\nu$, let $\theta F_i$ be the subalgebra of $\theta \mathcal{R}(\Gamma)_0$ generated by $1_i$ and $x_l$ with $l = 1, 2, \ldots, m$. It is a polynomial algebra. Let

$$\theta F_\nu = \bigoplus_{i \in \Omega^\nu} \theta F_i.$$  

The group $W_m$ acts on $\theta F_\nu$ via $w(x_l) = x_{w(l)}$ for any $w \in W_m$. Consider the fixed points set

$$\mathcal{S}_\nu = (\theta F_\nu)^{W_m}.$$  

Regard $\theta \mathcal{R}(\Gamma)_0$ and $\text{End}(\theta F_\nu)$ as $\theta F_\nu$-algebras via the left multiplication. In [VV, prop. 5.4] is given an injective graded $\theta F_\nu$-algebra morphism $\theta \mathcal{R}(\Gamma)_0 \to \text{End}(\theta F_\nu)$. It restricts via (1.2) to an injective graded $\theta F_\nu$-algebra morphism

$$\theta \mathcal{R}(\Gamma)_\nu \to \text{End}(\theta F_\nu).$$

Next, recall that $\mathcal{W}_m$ is the Weyl group of type $D_m$ with simple reflections $s_0, \ldots, s_{m-1}$. For each $w \in \mathcal{W}_m$, we choose a reduced decomposition $\dot{w}$ of $w$. It has the following form

$$w = s_{k_1} s_{k_2} \cdots s_{k_r}, \quad 0 \leq k_1, k_2, \ldots, k_r \leq m - 1.$$  

We define an element $\sigma_{\dot{w}}$ in $\theta \mathcal{R}(\Gamma)_\nu$ by

$$\sigma_{\dot{w}} = \sum_1 1_i \sigma_{\dot{w}}, \quad 1_i \sigma_{\dot{w}} = \begin{cases} 1_i & \text{if } r = 0 \\ 1_i \sigma_{k_1} s_{k_2} \cdots s_{k_r} & \text{else.} \end{cases}$$

Observe that the element $\sigma_{\dot{w}}$ may depend on the choice of the reduced decomposition $\dot{w}$.  


1.4. Proposition. The $k$-algebra $\mathcal{R}(\Gamma)_\nu$ is a free (left or right) $\mathcal{F}_\nu$-module with basis $\{\sigma_w; w \in \mathcal{W}_m\}$. Its rank is $2^{m-1}m!$. The operator $1\sigma_w$ is homogeneous and its degree is independent of the choice of the reduced decomposition $\bar{w}$.

Proof: The proof is the same as in [VV, prop. 5.5]. First, we filter the algebra $\mathcal{R}(\Gamma)_\nu$ with $l_1$, $\bar{x}_l$, $i_k$ in degree 0 and $\sigma_k$ in degree 1. The Nil Hecke algebra of type $D_m$ is the $k$-algebra $\mathcal{N}\mathcal{H}_{m}$ generated by $\sigma_0, \sigma_1, \ldots, \sigma_{m-1}$ with relations
\[
\sigma_k\sigma_{k'} = \sigma_{k'}\sigma_k \text{ if } 1 \leq k < k' - 1 < m - 1 \text{ or } 0 = k < k' \neq 2,
\]
\[
\sigma_{s_k(k)}\sigma_k\sigma_{s_k(k)} = \sigma_k\sigma_{s_k(k)}\sigma_k, \quad \sigma_k^2 = 0.
\]

We can form the semidirect product $\mathcal{F}_\nu \ltimes \mathcal{N}\mathcal{H}_{m}$, which is generated by $l_1, \bar{\varphi}, \bar{\sigma}_k$ with the relations above and
\[
\sigma_k\bar{x}_l = \bar{x}_{s_k(l)}\sigma_k, \quad \bar{x}_l\sigma_k = \bar{x}_{l'}\sigma_k'.
\]

One proves as in [VV, prop. 5.5] that the map
\[
\mathcal{F}_\nu \ltimes \mathcal{N}\mathcal{H}_{m} \rightarrow \text{gr}(\mathcal{R}(\Gamma)_\nu), \quad 1_i \mapsto 1_i, \quad \bar{x}_l \mapsto \bar{x}_l, \quad \bar{\sigma}_k \mapsto \sigma_k.
\]
is an isomorphism of $k$-algebras.

Let $\mathcal{F}_\nu' = \bigoplus_{i} \mathcal{F}_i'$, where $\mathcal{F}_i'$ is the localization of the ring $\mathcal{F}_i$ with respect to the multiplicative system generated by
\[
\{x_{l,l} \pm x_{1,l'}; 1 \leq l \neq l' \leq m\} \cup \{x_{l,l}; l = 1, 2, \ldots, m\}.
\]

1.5. Corollary. The inclusion $\mathcal{R}(\Gamma)_\nu \subset \text{End}(\mathcal{F}_\nu)$ yields an isomorphism of $\mathcal{F}_\nu'$-algebras $\mathcal{F}_\nu' \otimes \mathcal{F}_\nu \rightarrow \mathcal{F}_\nu' \ltimes \mathcal{N}\mathcal{H}_{m}$, such that for each $i$ and each $l = 1, 2, \ldots, m, k = 0, 1, 2, \ldots, m - 1$ we have
\[
1_i \mapsto 1_i, \quad x_{l,l} \mapsto x_{l,l}.
\]

\[
\sigma_{1,k} \mapsto \begin{cases} (x_k - x_{s_k(k)})^{-1}(s_k - 1)1_i & \text{if } i_k = s_k(k), \\ (x_k - x_{s_k(k)})^{h_{s_k(k)}^{-1}k} s_k1_i & \text{if } i_k \neq s_k(k). \end{cases}
\]

Proof: Follows from [VV, cor. 5.6] and Proposition 1.4.

Restricting the $\mathcal{F}_\nu'$-action on $\mathcal{R}(\Gamma)_\nu$ to the $k$-subalgebra $\mathcal{S}_\nu$ we get a structure of graded $\mathcal{S}_\nu$-algebra on $\mathcal{R}(\Gamma)_\nu$.

1.6. Proposition. (a) $\mathcal{S}_\nu$ is isomorphic to the center of $\mathcal{R}(\Gamma)_\nu$.

(b) $\mathcal{R}(\Gamma)_\nu$ is a free graded module over $\mathcal{S}_\nu$ of rank $(2^{m-1}m!)^2$.

Proof: Part (a) follows from Corollary 1.5. Part (b) follows from (a) and Proposition 1.4.
2. AFFINE HECKE ALGEBRAS OF TYPE D

2.1. Affine Hecke algebras of type D. Fix $p$ in $k^\times$. For any integer $m \geq 0$ we define the extended affine Hecke algebra $H_m$ of type $D_m$ as follows. If $m > 1$ then $H_m$ is the $k$-algebra with 1 generated by

$$T_k, \quad X_i^{\pm 1}, \quad k = 0, 1, \ldots, m - 1, \quad l = 1, 2, \ldots, m$$

satisfying the following defining relations:

(a) $X_lX_{l'} = X_{l'}X_l$,

(b) $T_kT_{s_k(k)}T_k = T_{s_k(k)}T_k T_{s_k(k)}$, $T_kT_{k'} = T_{k'}T_k$ if $1 \leq k < k'$ or $k = 0$, $k' \neq 2$,

(c) $(T_k - p)(T_k + p^{-1}) = 0$,

(d) $T_0X_1^{-1}T_0 = X_2, \quad T_kX_kT_k = X_{s_k(k)}$ if $k \neq 0, \quad T_kX_l = X_lT_k$ if $k \neq 0, \quad l - 1$ or $k = 0, \quad l \neq 1, 2$.

Finally, we set $H_0 = k \oplus k$ and $H_1 = k[X_1^{\pm 1}]$.

2.2. Remarks. (a) The extended affine Hecke algebra $H^B_m$ of type $B_m$ with parameters $p, q \in k^\times$ such that $q = 1$ is generated by $P, \quad T_k, \quad X_i^{\pm 1}, \quad k = 1, \ldots, m - 1, \quad l = 1, \ldots, m$ such that $T_k, \quad X_i^{\pm 1}$ satisfy the relations as above and

$$P^2 = 1, \quad (PT_1)^2 = (T_1P)^2, \quad PT_k = T_kP$$

if $k \neq 1$,

$$PX_1^{-1}P = X_1, \quad PX_l = X_lP$$

if $l \neq 1$.

See e.g., [VV, sec. 6.1]. There is an obvious $k$-algebra embedding $H_m \subset H^B_m$. Let $\gamma$ denote also the involution $H_m \to H_m, \quad a \mapsto PaP$. We have a canonical isomorphism of $k$-algebras

$$H_m \cong \langle \gamma \rangle \cong H^B_m.$$ 

Compare Section 1.2.

(b) Given a connected reductive group $G$ we call affine Hecke algebra of $G$ the Hecke algebra of the extended affine Weyl group $W \times P$, where $W$ is the Weyl group of $(G, T)$, $P$ is the group of characters of $T$, and $T$ is a maximal torus of $G$. Then $H_m$ is the affine Hecke algebra of the group $SO(2m)$. Let $H^c_m$ be the affine Hecke algebra of the group $Spin(2m)$. It is generated by $H_m$ and an element $X_0$ such that

$$X_0^2 = X_1X_2 \ldots X_m, \quad T_kX_0 = X_0T_k$$

if $k \neq 0$, \quad $T_0X_0X_1^{-1}X_2^{-1}T_0 = X_0$.

Thus $H_m$ is the fixed point subset of the $k$-algebra automorphism of $H^c_m$ taking $T_k, X_l$ to $T_k, (-1)^{l_0}aX_l$ for all $k, l$. Therefore, if $p$ is not a root of 1 the simple $H_m$-modules can be recovered from the Kazhdan-Lusztig classification of the simple $H^c_m$-modules via Clifford theory, see e.g., [Re].

2.3. Intertwiners and blocks of $H_m$. We define

$$A = k[X_1^{\pm 1}, X_2^{\pm 1}, \ldots, X_m^{\pm 1}], \quad A' = A[\Sigma^{-1}], \quad H'_m = A' \otimes A H_m,$$
where $\Sigma$ is the multiplicative set generated by

$$1 - X_i X_{l'}^{-1}, \quad 1 - p^2 X_i^{l+1} X_{l'}^{-1}, \quad l \neq l'.$$

For $k = 0, \ldots, m-1$ the intertwiner $\varphi_k$ is the element of $H'_m$ given by the following formulas

$$(2.1) \quad \varphi_k - 1 = \frac{X_k - X_{s_k(k)}}{pX_k - p^{-1}X_{s_k(k)}} (T_k - p).$$

The group $\cong \mathbb{Z}$ acts on $A'$ as follows

$$(s \in \cong \mathbb{Z})(X_1, \ldots, X_m) = a(X_1, \ldots, X_{k+1}, X_k, \ldots, X_m) \text{ if } k \neq 0,$$

$$(s \in \cong \mathbb{Z})(X_1, \ldots, X_m) = a(X_1^{-1}, X_2^{-1}, \ldots, X_m).$$

There is an isomorphism of $\mathfrak{A}'$-algebras

$$\mathfrak{A}' \times \cong \mathbb{Z} \cong \mathbb{H}'_m, \quad s_k \mapsto \varphi_k.$$
2.6. Theorem. We have an equivalence of categories
\[ \mathcal{R}_{m} \rightarrow \mathcal{H}_{m}, \quad M \mapsto M \]
which is given by

(a) \( X_{l} \) acts on \( 1_{m} \) by \( i_{l}^{-1} f(x_{l}) \) for each \( l = 1, 2, \ldots, m \).

(b) if \( m > 1 \) then \( T_{k} \) acts on \( 1_{m} \) as follows for each \( k = 0, 1, \ldots, m - 1 \):

\[
\begin{align*}
\frac{(pf(x_{k}) - p^{-1}f(x_{k}))}{f(x_{k}) - f(x_{k})} & = \sigma_{k} + p & \text{if } i_{x_{k}}(k) = i_{k}, \\
\frac{f(x_{k}) - f(x_{k})}{f(x_{k}) - f(x_{k})} & = \sigma_{k} + \frac{(p^{-2} - 1)f(x_{k})}{pf(x_{k}) - p^{-1}f(x_{k})} & \text{if } i_{x_{k}}(k) = p^{2}i_{k}, \\
\frac{p_{i_{k}}f(x_{k}) - p^{-1}i_{k}f(x_{k})}{i_{k}f(x_{k}) - i_{k}f(x_{k})} & = \sigma_{k} + \frac{(p^{-1} - p)i_{k}f(x_{k})}{i_{k}f(x_{k})} & \text{if } i_{x_{k}}(k) \neq i_{k}, p^{2}i_{k}.
\end{align*}
\]

Proof: This follows from [VV, thm. 6.5] by Section 1.2 and Remark 2.2(a). One can also prove it by reproducing the arguments in loc. cit. by using (1.5) and (2.1).

\[ \square \]

2.7. Corollary. There is an equivalence of categories
\[ \Psi : \mathcal{R}_{m} \rightarrow \mathcal{F}_{m}, \quad M \mapsto M. \]

2.8. Remarks. (a) Let \( g \) be the Lie algebra of \( G = SO(2m) \). Fix a maximal torus \( T \subset G \). The group of characters of \( T \) is the lattice \( \bigoplus_{l=1}^{m} \mathbb{Z} \varepsilon_{l} \), with Bourbaki’s notation. Fix a dimension vector \( \nu \in \mathbb{N}^{m} \). Recall the sequence \( \nu_{c} = (i_{1}, \ldots, i_{m-1}, i_{m}) \) from Section 0.3. Let \( g \in T \) be the element such that \( \varepsilon_{l}(g) = i_{l}^{-1} \) for each \( l = 1, 2, \ldots, m \). Recall also the notation \( \mathcal{E}_{\nu}, \mathcal{H}, \mathcal{G}_{\nu} \) from [VV]. Then \( \mathcal{E} \) is an object of \( \mathcal{E}_{\nu}, \mathcal{H}, \mathcal{G}_{\nu} \) is the centralizer of \( g \) in \( G \), and
\[
\mathcal{E}_{\nu} = g_{g,p}, \quad g_{g,p} = \{ x \in g^{*} : \text{ad}_{g}(x) = p^{2}x \}.
\]

Let \( F_{g} \) be the set of all Borel Lie subalgebras of \( g \) fixed by the adjoint action of \( g \). It is a non connected manifold whose connected components are labelled by \( \mathcal{E}_{\nu}^{+} \). In Section 3.14 we’ll introduce two central idempotents \( 1_{\nu, +}, 1_{\nu, -} \) of \( \mathcal{R}_{\nu} \). This yields a graded \( k \)-algebra decomposition
\[ \mathcal{R}_{\nu} = \mathcal{R}_{\nu, 1_{\nu, +}} \oplus \mathcal{R}_{\nu, 1_{\nu, -}} \]
By [VV, thm. 5.8] the graded \( k \)-algebra \( \mathcal{R}_{\nu, 1_{\nu, +}} \) is isomorphic to
\[ \text{Ext}_{\mathcal{R}_{\nu}}^{\ast}(\mathcal{L}_{g,p}, \mathcal{L}_{g,p}), \]
where \( \mathcal{L}_{g,p} \) is the direct image of the constant perverse sheaf by the projection
\[
\{(b, x) \in F_{g} \times g_{g,p} : x \in b \} \rightarrow g_{g,p}, \quad (b, x) \mapsto x.
\]
The complex \( \mathcal{L}_{g,p} \) has been extensively studied by Lusztig, see e.g., [L1], [L2]. We hope to come back to this elsewhere.

(b) The results in Section 2.5 hold true if \( k \) is any field. Set \( f(x) = 1 + x \) for instance.
2.9. Induction and restriction of $H_m$-modules. For $i \in I$ we define functors

\[ E_i : H_{m+1}\text{-fMod}_I \to H_m\text{-fMod}_I, \]
\[ F_i : H_m\text{-fMod}_I \to H_{m+1}\text{-fMod}_I, \]

where $E_i M \subset M$ is the generalized $i^{-1}$-eigenspace of the $X_{m+1}$-action, and where

\[ F_i M = \text{Ind}_{H_{m+1}}^{H_m} (M \otimes k_i). \]

Here $k_i$ is the 1-dimensional representation of $k[\omega]_{m+1}$ defined by $\omega x \mapsto i^{-1}$.

3. Global bases of $^G\mathbb{V}$ and projective graded $^G\mathbb{R}$-modules

3.1. The Grothendieck groups of $^G\mathbb{R}_m$. The graded $k$-algebra $^G\mathbb{R}_m$ is free of finite rank over its center by Proposition 1.6, a commutative graded $k$-subalgebra. Therefore any simple object of $^G\mathbb{R}_m\text{-mod}$ is finite-dimensional and there is a finite number of isomorphism classes of simple modules in $^G\mathbb{R}_m\text{-mod}$. The Abelian group $G(\mathbb{R}_m)$ is free with a basis formed by the classes of the simple objects of $^G\mathbb{R}_m\text{-mod}$. The Abelian group $K(\mathbb{R}_m)$ is free with a basis formed by the classes of the indecomposable projective objects. Both $G(\mathbb{R}_m)$ and $K(\mathbb{R}_m)$ are free $A$-modules, where $v$ shifts the grading by 1. We consider the following $A$-modules

\[ ^G\mathbb{K}_I = \bigoplus_{m \geq 0} ^G\mathbb{K}_{I,m}, \quad ^G\mathbb{K}_{I,m} = K(\mathbb{R}_m), \]
\[ ^G\mathbb{G}_I = \bigoplus_{m \geq 0} ^G\mathbb{G}_{I,m}, \quad ^G\mathbb{G}_{I,m} = G(\mathbb{R}_m). \]

We’ll also abbreviate

\[ ^G\mathbb{K}_{I,*} = \bigoplus_{m > 0} ^G\mathbb{K}_{I,m}, \quad ^G\mathbb{G}_{I,*} = \bigoplus_{m > 0} ^G\mathbb{G}_{I,m}. \]

From now on, to unburden the notation we may abbreviate $\mathbb{R} = ^G\mathbb{R}_m$, hoping it will not create any confusion. For any $M, N$ in $^G\mathbb{R}\text{-mod}$ we set

\[ (M : N) = \text{gdim}(M^\omega \otimes \mathbb{R} N), \quad \langle M : N \rangle = \text{gdim} \text{hom}_{\mathbb{R}}(M, N), \]

where $\omega$ is the involution defined in Section 1.1. The Cartan pairing is the perfect $A$-bilinear form

\[ ^G\mathbb{K}_I \times ^G\mathbb{G}_I \to A, \quad (P, M) \mapsto \langle P : M \rangle. \]

First, we concentrate on the $A$-module $^G\mathbb{G}_I$. Consider the duality

\[ ^G\mathbb{R}\text{-fmod} \to ^G\mathbb{R}\text{-fmod}, \quad M \mapsto M^\flat = \text{hom}(M, k), \]

with the action and the grading given by

\[ (xf)(m) = f(\omega(x)m), \quad (M^\flat)_d = \text{Hom}(M_{-d}, k). \]

This duality functor yields an $A$-antilinear map

\[ ^G\mathbb{G}_I \to ^G\mathbb{G}_I, \quad M \mapsto M^\flat. \]

Let $^GB$ denote the set of isomorphism classes of simple objects of $^G\mathbb{R}\text{-fMod}_0$. We can now define the upper global basis of $^G\mathbb{G}_I$ as follows. The proof is given in Section 3.21.
3.2. Proposition/Definition. For each \( b \) in \( ^\circ B \) there is a unique selfdual irreducible graded \( ^\circ R \)-module \( ^\circ G_{up}(b) \) which is isomorphic to \( b \) as a (non graded) \( ^\circ R \)-module. We set \( ^\circ G_{up}(0) = 0 \) and \( ^\circ G_{up} = \{ ^\circ G_{up}(b) ; b \in ^\circ B \} \). Hence \( ^\circ G_{up} \) is a \( A \)-basis of \( ^\circ G_I \).

Now, we concentrate on the \( A \)-module \( ^\circ K_I \). We equip \( ^\circ K_I \) with the symmetric \( A \)-bilinear form

\[
(3.1) \quad ^\circ K_I \times ^\circ K_I \rightarrow A, \quad (M, N) \mapsto (M : N).
\]

Consider the duality

\[
^\circ R-proj \rightarrow ^\circ R-proj, \quad P \mapsto P^\sharp = \text{hom}^R(P, ^\circ R),
\]

with the action and the grading given by

\[
(xf)(p) = f(p)\omega(x), \quad (P^\sharp)_d = \text{Hom}^R(P[-d], ^\circ R).
\]

This duality functor yields an \( A \)-antilinear map

\[
^\circ K_I \rightarrow ^\circ K_I, \quad P \mapsto P^\sharp.
\]

Set \( K = \mathbb{Q}(v) \). Let \( K \rightarrow K, f \mapsto \bar{f} \) be the unique involution such that \( \bar{v} = v^{-1} \).

3.3. Definition. For each \( b \) in \( ^\circ B \) let \( ^\circ G_{low}(b) \) be the unique indecomposable graded module in \( ^\circ R-proj \) whose top is isomorphic to \( ^\circ G_{up}(b) \). We set \( ^\circ G_{low}(0) = 0 \) and \( ^\circ G_{low} = \{ ^\circ G_{low}(b) ; b \in ^\circ B \} \), a \( A \)-basis of \( ^\circ K_I \).

3.4. Proposition. (a) We have \( ( ^\circ G_{low}(b) : ^\circ G_{up}(b') ) = \delta_{b,b'} \) for each \( b, b' \) in \( ^\circ B \).

(b) We have \( (P^\sharp : M) = (P : M^\sharp) \) for each \( P, M \).

(c) We have \( ^\circ G_{low}(b)^\sharp = ^\circ G_{low}(b) \) for each \( b \) in \( ^\circ B \).

The proof is the same as in [VV, prop. 8.4].

3.5. Example. Set \( \nu = i + \theta(i) \) and \( i = i\theta(i) \). Consider the graded \( ^\circ R_i \)-modules

\[
^\circ R_i = ^\circ R l_i = l_i^\circ R, \quad ^\circ L_i = \text{top}(^\circ R_I).
\]

The global bases are given by

\[
^\circ G_i^{low} = \{ ^\circ R_i, ^\circ R_{\theta(i)} \}, \quad ^\circ G_i^{up} = \{ ^\circ L_i, ^\circ L_{\theta(i)} \}.
\]

For \( m = 0 \) we have \( ^\circ R_0 = k \oplus k \). Set \( \phi_+ = k \oplus 0 \) and \( \phi_- = 0 \oplus k \). We have

\[
^\circ G_0^{low} = ^\circ G_0^{up} = \{ \phi_+, \phi_- \}.
\]
3.6. Definition of the operators $e_i, f_i, e'_i, f'_i$. In this section we’ll always assume $m > 0$ unless specified otherwise. First, let us introduce the following notation. Let $D_{m,1}$ be the set of minimal representative in $\circ W_{m+1}$ of the cosets in $\circ W_m \backslash \circ W_{m+1}$. Write

$$D_{m,1;m,1} = D_{m,1} \cap (D_{m,1})^{-1}.$$ 

For each element $w$ of $D_{m,1;m,1}$ we set

$$W(w) = \circ W_m \cap w(\circ W_m)w^{-1}.$$ 

Let $R_1$ be the $k$-algebra generated by elements $1_i, \kappa_i, i \in I$, satisfying the defining relations $1_i 1_i' = \delta_{i,i'} 1_i$ and $\kappa_i = 1_i \kappa_i 1_i$. We equip $R_1$ with the grading such that $\deg(1_i) = 0$ and $\deg(\kappa_i) = 2$. Let

$$R_i = 1_i R_1 = R_1 1_i, \quad L_i = \text{top}(R_i) = R_i/(\kappa_i).$$ 

Then $R_i$ is a graded projective $R_1$-module and $L_i$ is simple. We abbreviate

$$\theta R_{m,1} = \theta R_m \otimes R_1, \quad \circ R_{m,1} = \circ R_m \otimes R_1.$$ 

There is an unique inclusion of graded $k$-algebras

$$\theta R_{m,1} \hookrightarrow \theta R_{m+1},$$

$$1_i \otimes 1_i \mapsto 1_{i'},$$

$$1_i \otimes \kappa_i \mapsto \kappa_i^1, i \mapsto \kappa_i^2, i \mapsto \kappa_i^3, i \mapsto \kappa_i^4, i \mapsto \sigma_i,$$

where, given $i \in I^m$ and $i \in I$, we have set $i' = \theta(i)i$, a sequence in $\theta I^{m+1}$. This inclusion restricts to an inclusion $\circ R_{m,1} \subset \circ R_{m+1}$.

3.7. Lemma. The graded $\circ R_{m,1}$-module $\circ R_{m+1}$ is free of rank $2(m+1)$.

Proof: For each $w$ in $D_{m,1}$ we have the element $\sigma_w$ in $\circ R_{m+1}$ defined in (1.5). Using filtered/graded arguments it is easy to see that

$$\circ R_{m+1} = \bigoplus_{w \in D_{m,1}} \circ R_{m,1} \sigma_w.$$ 

We define a triple of adjoint functors $(\psi_!, \psi^*, \psi_*)$ where

$$\psi^*: \circ R_{m+1} \text{-mod} \rightarrow \circ R_m \text{-mod} \times R_1 \text{-mod}.$$
is the restriction and $\psi_1, \psi_2$ are given by

$$
\psi_1 : \begin{cases}
\circ R_m\text{-mod} \times R_1\text{-mod} &\to \circ R_{m+1}\text{-mod}, \\
(M, M') &\mapsto \circ R_{m+1} \otimes_{\circ R_m} (M \otimes M'), \\
\psi_2 : \begin{cases}
\circ R_m\text{-mod} &\to \circ R_{m+1}\text{-mod}, \\
(M, M') &\mapsto \text{hom}_{\circ R_m}(\circ R_{m+1}, M \otimes M').
\end{cases}
\end{cases}
$$

First, note that the functors $\psi_1, \psi_2, \psi_3$ commute with the shift of the grading. Next, the functor $\psi_3$ is exact, and it takes finite dimensional graded modules to finite dimensional ones. The right graded $\circ R_{m,1}$-module $\circ R_{m+1}$ is free of finite rank. Thus $\psi_1$ is exact, and it takes finite dimensional graded modules to finite dimensional ones. The left graded $\circ R_{m,1}$-module $\circ R_{m+1}$ is also free of finite rank. Thus the functor $\psi_2$ is exact, and it takes finite dimensional graded modules to finite dimensional ones. Further $\psi_1$ and $\psi_2$ take projective graded modules to projective ones, because they are left adjoint to the exact functors $\psi^*, \psi_*$ respectively. To summarize, the functors $\psi_1, \psi_2, \psi_3$ are exact and take finite dimensional graded modules to finite dimensional ones, and the functors $\psi_1, \psi_2$ take projective graded modules to projective ones.

For any graded $\circ R_m$-module $M$ we write

$$
f_i(M) = \circ R_{m+1}1_{m,i} \otimes_{\circ R_m} M, \tag{3.3}
e_i(M) = \circ R_{m-1} \otimes_{\circ R_{m-1,1}} 1_{m-1,i} M.
$$

Let us explain these formulas. The symbols $1_{m,i}$ and $1_{m-1,i}$ are given by

$$
1_{m-1,i} M = \bigoplus_i 1_{\theta(i) i} M, \quad i \in \theta jm-1.
$$

Note that $f_i(M)$ is a graded $\circ R_{m+1}$-module, while $e_i(M)$ is a graded $\circ R_{m-1}$-module. The tensor product in the definition of $e_i(M)$ is relative to the graded $k$-algebra homomorphism

$$
\circ R_{m-1,1} \to \circ R_{m-1} \otimes R_1 \to \circ R_{m-1} \otimes R_1 \to \circ R_{m-1} \otimes L = \circ R_{m-1}.
$$

In other words, let $e_i'(M)$ be the graded $\circ R_{m-1}$-module obtained by taking the direct summand $1_{m-1,i} M$ and restricting it to $\circ R_{m-1}$. Observe that if $M$ is finitely generated then $e_i'(M)$ may not lie in $\circ R_{m-1}\text{-mod}$. To remedy this, since $e_i'(M)$ affords a $\circ R_{m-1} \otimes R$-action we consider the graded $\circ R_{m-1}$-module

$$
e_i(M) = e_i'(M)/\circ \delta e_i'(M).
$$

3.8. Definition. The functors $e_i, f_i$ preserve the category $\circ R\text{-proj}$, yielding $A$-linear operators on $\circ K_I$ which act on $\circ K_I$, by the formula (3.3) and satisfy also

$$
f_i(\delta_+) = \circ R_{\theta(i)}, \quad f_i(\delta_-) = \circ R_{\theta(i)}, \quad e_i(\circ R_{\theta(j)}) = \delta_{i,j} \delta_+ \delta_\theta(j) \phi_+.
$$

Let $e_i, f_i$ denote also the $A$-linear operators on $\circ G_I$ which are the transpose of $f_i, e_i$ with respect to the Cartan pairing.

Note that the symbols $e_i(M), f_i(M)$ have a different meaning if $M$ is viewed as an element of $\circ K_I$ or if $M$ is viewed as an element of $\circ G_I$. We hope this will not create any confusion. The proof of the following lemma is the same as in [VV, lem. 8.9].
3.9. Lemma. (a) The operators $e_i, f_i$ on $^G_\mathfrak{I}$ are given by

$$e_i(M) = 1_{m-1,i}M, \quad f_i(M) = \text{hom}_{R_m}(^R_{m+1}, M \otimes L_i), \quad M \in ^R_m\text{-fmod}.$$ 

(b) For each $M \in ^R_m\text{-mod}, M' \in ^R_{m+1}\text{-mod}$ we have

$$(e'_i(M') : M) = (M' : f_i(M)).$$

(c) We have $f_i(P)\equiv f_i(P^\theta)$ for each $P \in ^R\text{-proj}.$

(d) We have $e_i(M)\equiv e_i(M^\theta)$ for each $M \in ^R\text{-fmod}.$

3.10. Induction of $H_m\text{-modules versus induction of } ^R_m\text{-modules.}$ Recall the functors $E_i, F_i$ on $H\text{-fMod}_I$ defined in (2.2). We have also the functors

$$\text{for} : ^R_m\text{-fmod} \to ^R_m\text{-fMod}_0, \quad \Psi : ^R_m\text{-fMod}_0 \to H_m\text{-fMod}_I,$$

where $\text{for}$ is the forgetting of the grading. Finally we define functors

$$E_i : ^R_m\text{-fMod}_0 \to ^R_{m-1}\text{-fMod}_0, \quad E_i M = 1_{m-1,i}M,$$

$$F_i : ^R_m\text{-fMod}_0 \to ^R_{m+1}\text{-fMod}_0, \quad F_i M = \psi(M, L_i) \quad (3.4)$$

3.11. Proposition. There are canonical isomorphisms of functors

$$E_i \circ \Psi = \Psi \circ E_i, \quad F_i \circ \Psi = \Psi \circ F_i, \quad E_i \circ \text{for} = \text{for} \circ e_i, \quad F_i \circ \text{for} = \text{for} \circ f_{\theta(i)}.$$

Proof: The proof is the same as in [VV, prop. 8.17].

3.12. Proposition. (a) The functor $\Psi$ yields an isomorphism of Abelian groups

$$\bigoplus_{m \geq 0} [^R_m\text{-fMod}_0] = \bigoplus_{m \geq 0} [H_m\text{-fMod}_I].$$

The functors $E_i, F_i$ yield endomorphisms of both sides which are intertwined by $\Psi.$

(b) The functor $\text{for}$ factors to a group isomorphism

$$^G_\mathfrak{I}/(v-1) = \bigoplus_{m \geq 0} [^R_m\text{-fMod}_0].$$

Proof: Claim (a) follows from Corollary 2.7 and Proposition 3.11. Claim (b) follows from Proposition 3.2.
3.13. Type D versus type B. We can compare the previous constructions with their analogues in type B. Let $^\theta K$, $^\theta B$, $^\theta G_{\text{low}}$, etc, denote the type B analogues of $^\theta K$, $^\theta B$, $^\theta G_{\text{low}}$, etc. See [VV] for details. We'll use the same notation for the functors $\psi^*$, $\psi_*$, $\psi_i$, $f_i$, etc, on the type B side and on the type D side. Fix $m > 0$ and $\nu \in \theta N$ such that $|\nu| = 2m$. The inclusion of graded $k$-algebras $^\theta R_\nu \subset ^\theta R_\nu$ in (1.2) yields a restriction functor

$$\text{res}: ^\theta R_\nu \text{-mod} \to ^\circ R_\nu \text{-mod}$$

and an induction functor

$$\text{ind}: ^\circ R_\nu \text{-mod} \to ^\theta R_\nu \text{-mod}, \quad M \mapsto ^\theta R_\nu \otimes_{^\circ R_\nu} M.$$  

Both functors are exact, they map finite dimensional graded modules to finite dimensional ones, and they map projective graded modules to projective ones. Thus, they yield morphisms of $A$-modules

$$\text{res}: ^\theta K_{I,m} \to ^\circ K_{I,m}, \quad \text{res}: ^\theta G_{I,m} \to ^\circ G_{I,m},$$

$$\text{ind}: ^\circ K_{I,m} \to ^\theta K_{I,m}, \quad \text{ind}: ^\circ G_{I,m} \to ^\theta G_{I,m}.$$  

Moreover, for any $P \in ^\theta K_{I,m}$ and any $L \in ^\circ G_{I,m}$ we have

$$\text{res}(P^\gamma) = (\text{res} P)^\gamma, \quad \text{ind}(P^\gamma) = (\text{ind} P)^\gamma$$

$$\text{res}(L^\gamma) = (\text{res} L)^\gamma, \quad \text{ind}(L^\gamma) = (\text{ind} L)^\gamma.$$  

Note also that ind and res are left and right adjoint functors, because

$$^\theta R_\nu \otimes_{^\circ R_\nu} M = \text{hom}_{^\circ R_\nu}(^\theta R_\nu, M)$$

as graded $^\theta R_\nu$-modules.

3.14. Definition. For any graded $^\circ R_\nu$-module $M$ we define the graded $^\circ R_\nu$-module $M^\gamma$ with the same underlying graded $k$-vector space as $M$ such that the action of $^\circ R_\nu$ is twisted by $\gamma$, i.e., the graded $k$-algebra $^\circ R_\nu$ acts on $M^\gamma$ by $a \cdot m = \gamma(a)m$ for $a \in ^\circ R_\nu$ and $m \in M$. Note that $(M^\gamma)^\gamma = M$, and that $M^\gamma$ is simple (resp. projective, indecomposable) if $M$ has the same property.

For any graded $^\circ R_m$-module $M$ we have canonical isomorphisms of $^\circ R$-modules

$$(f_i(M))^\gamma = f_i(M^\gamma), \quad (e_i(M))^\gamma = e_i(M^\gamma).$$

The first isomorphism is given by

$$^\circ R_{m+1,l,m,i} \otimes_{^\circ R_m} M \to ^\circ R_{m+1,l,m,i} \otimes_{^\circ R_m} M, \quad a \otimes m \mapsto \gamma(a) \otimes m.$$  

The second one is the identity map on the vector space $1_{m,i} M$.

Recall that $^\theta T^\nu$ is the disjoint union of $^\theta T^\nu_+$ and $^\theta T^\nu_-$. We set

$$1_{\nu,+} = \sum_{i \in ^\theta T^\nu_+} 1_i, \quad 1_{\nu,-} = \sum_{i \in ^\theta T^\nu_-} 1_i.$$
3.15. Lemma. Let $M$ be a graded $\mathcal{R}_\nu$-module.

(a) Both $1_{\nu,+}$ and $1_{\nu,-}$ are central idempotents in $\mathcal{R}_\nu$. We have $1_{\nu,+} = \gamma(1_{\nu,-})$.
(b) There is a decomposition of graded $\mathcal{R}_\nu$-modules $M = 1_{\nu,+}M \oplus 1_{\nu,-}M$.
(c) We have a canonical isomorphism of $\mathcal{R}_\nu$-modules $\text{res} \circ \text{ind}(M) = M \oplus M^\gamma$.
(d) If there exists $a \in \{+,-\}$ such that $1_{\nu,-a}M = 0$, then there are canonical
  isomorphisms of graded $\mathcal{R}_\nu$-modules
  
  \[
  M = 1_{\nu,a}M, \quad 0 = 1_{\nu,a}M^\gamma, \quad M^\gamma = 1_{\nu,-a}M^\gamma.
  \]

Proof: Part (a) follows from Proposition 1.6 and the equality $\varepsilon_1(\theta I^\nu_\nu) = \theta \mu^\nu_\nu$. Part (b) follows from (a), (c) is given by definition, and (d) follows from (a), (b). \hfill \qed

Now, let $m$ and $\nu$ be as before. Given $i \in I$, we set $\nu' = \nu + i + \theta(i)$. There is an obvious inclusion $W_m \subset W_{m+1}$. Thus the group $W_m$ acts on $\theta I^\nu_\nu$, and the map

\[
\theta I^\nu_\nu \rightarrow \theta I^\nu_{\nu'}, \quad i \mapsto \theta(i)ii
\]

is $W_m$-equivariant. Thus there is $a_i \in \{+,-\}$ such that the image of $\theta I^\nu_{\nu'}$ is contained in $\theta I^\nu_{a_i}$, and the image of $\theta I^\nu_{-a_i}$ is contained in $\theta I^\nu_{-a_i}$.

3.16. Lemma. Let $M$ be a graded $\mathcal{R}_\nu$-module such that $1_{\nu,-a}M = 0$, with $a = +, -$. Then we have

\[
1_{\nu',-a,a}f_i(M) = 0, \quad 1_{\nu',a,a}f_{\theta(i)}(M) = 0.
\]

Proof: We have

\[
1_{\nu',-a,a}f_i(M) = 1_{\nu',-a,a}\mathcal{R}_\nu\mathcal{R}_\nu 1_{\nu,i} \oplus \mathcal{R}_\nu M
\]

\[
= \mathcal{R}_\nu 1_{\nu',-a,a}1_{\nu,i}1_{\nu,a} \oplus \mathcal{R}_\nu M.
\]

Here we have identified $1_{\nu,a}$ with the image of $(1_{\nu,a}, 1)$ via the inclusion (3.2). The definition of this inclusion and that of $a_i$ yield that

\[
1_{\nu',a,a}1_{\nu,a} = 1_{\nu,a}, \quad 1_{\nu',-a,a}1_{\nu,a}1_{\nu,a} = 0.
\]

The first equality follows. Next, note that for any $i \in \theta I^\nu_\nu$, the sequences $\theta(i)i$ and $ii\theta(i) = \varepsilon_{m+1}(\theta(i)i)$ always belong to different $\mathcal{R}_\nu$-$W_{m+1}$-orbits. Thus, we have $a_{\theta(i)} = -a_i$. So the second equality follows from the first. \hfill \qed

Consider the following diagram

\[
\begin{array}{ccc}
\mathcal{R}_\nu\text{-mod} \times \mathcal{R}_\nu\text{-mod} & \xrightarrow{\psi} & \mathcal{R}_\nu\text{-mod} \\
\text{res} \times \text{id} & \downarrow & \text{ind} \\
\mathcal{R}_\nu\text{-mod} \times \mathcal{R}_\nu\text{-mod} & \xrightarrow{\psi^*} & \mathcal{R}_\nu\text{-mod}.
\end{array}
\]
3.17. Lemma. There are canonical isomorphisms of functors

\[ \text{ind} \circ \psi_! = \psi_! \circ (\text{ind} \times \text{id}), \quad \psi^* \circ \text{ind} = (\text{ind} \times \text{id}) \circ \psi^*, \quad \text{ind} \circ \psi_* = \psi_* \circ (\text{ind} \times \text{id}), \]
\[ \text{res} \circ \psi_! = \psi_! \circ (\text{res} \times \text{id}), \quad \psi^* \circ \text{res} = (\text{res} \times \text{id}) \circ \psi^*, \quad \text{res} \circ \psi_* = \psi_* \circ (\text{res} \times \text{id}). \]

Proof: The functor \( \text{ind} \) is left and right adjoint to \( \text{res} \). Therefore it is enough to prove the first two isomorphisms in the first line. The isomorphism

\[ \text{ind} \circ \psi_! = \psi_! \circ (\text{ind} \times \text{id}) \]
comes from the associativity of the induction. Let us prove that

\[ \psi^* \circ \text{ind} = (\text{ind} \times \text{id}) \circ \psi^*. \]

For any \( M \in {^\nu \mathcal{R}}_{\mathcal{I}} \)-mod, the obvious inclusion \( {^\nu \mathcal{R}}_{\mathcal{I}} \otimes R_i \subset {^\nu \mathcal{R}}_{\mathcal{I}'} \) yields a map

\[ (\text{ind} \times \text{id}) \psi^*(M) = (\text{ind} \times \text{id}) \psi^*(M) \rightarrow \psi^*({^\nu \mathcal{R}}_{\mathcal{I}} \otimes \text{res}) \psi^*(M). \]

Combining it with the obvious map \( \psi^* \), we get a morphism of \( {^\nu \mathcal{R}}_{\mathcal{I}} \)-modules

\[ (\text{ind} \times \text{id}) \psi^*(M) \rightarrow \psi^* \text{ind}(M). \]

We need to show that it is bijective. This is obvious because at the level of vector spaces, the map above is given by

\[ M \oplus (\pi_{1,\nu} \otimes M) \rightarrow M \oplus (\pi_{1,\nu} \otimes M), \quad m + \pi_{1,\nu} \otimes n \mapsto m + \pi_{1,\nu} \otimes n. \]

Here \( \pi_{1,\nu} \) and \( \pi_{1,\nu'} \) denote the element \( \pi_1 \) in \( {^\nu \mathcal{R}}_{\mathcal{I}} \) and \( {^\nu \mathcal{R}}_{\mathcal{I}'} \) respectively.

\[ \square \]

3.18. Corollary. (a) The operators \( e_i, f_i \) on \( {^\nu \mathcal{K}}_{\mathcal{I},+} \) and on \( {^\nu \mathcal{K}}_{\mathcal{I},+} \) are intertwined by the maps \( \text{ind}, \text{res} \), i.e., we have

\[ e_i \circ \text{ind} = \text{ind} \circ e_i, \quad f_i \circ \text{ind} = \text{ind} \circ f_i, \quad e_i \circ \text{res} = \text{res} \circ e_i, \quad f_i \circ \text{res} = \text{res} \circ f_i. \]

(b) The same result holds for the operators \( e_i, f_i \) on \( {^\nu \mathcal{G}}_{\mathcal{I},+} \) and on \( {^\nu \mathcal{G}}_{\mathcal{I},+} \).

3.19. Now, we concentrate on non graded irreducible modules. First, let

\[ \text{Res} : {^\nu \mathcal{R}}_{\mathcal{I}}\text{-Mod} \rightarrow {^\nu \mathcal{R}}_{\mathcal{I}}\text{-Mod}, \quad \text{Ind} : {^\nu \mathcal{R}}_{\mathcal{I}}\text{-Mod} \rightarrow {^\nu \mathcal{R}}_{\mathcal{I}}\text{-Mod} \]

be the (non graded) restriction and induction functors. We have

\[ \text{for} \circ \text{res} = \text{Res} \circ \text{for}, \quad \text{for} \circ \text{ind} = \text{Ind} \circ \text{for}. \]
3.23. Lemma. Let $L, L'$ be irreducible $^\vartheta \mathfrak R_v$-modules.
(a) The $^\vartheta \mathfrak R_v$-modules $L$ and $L'$ are not isomorphic.
(b) $\text{Ind}(L)$ is an irreducible $^\vartheta \mathfrak R_v$-module, and every irreducible $^\vartheta \mathfrak R_v$-module is obtained in this way.
(c) $\text{Ind}(L) \simeq \text{Ind}(L')$ iff $L' \simeq L$ or $L' = 0$.

Proof: For any $^\vartheta \mathfrak R_v$-module $M \neq 0$, both $1_{\nu,+}M$ and $1_{\nu,-}M$ are nonzero. Indeed, we have $M = 1_{\nu,+}M \oplus 1_{\nu,-}M$, and we may suppose that $1_{\nu,+}M \neq 0$. By Lemma 3.15(a), $1_{\nu,-}M$ is selfdual. This proves existence part of the proposition. To prove the uniqueness, suppose that $M$ is another module with the same properties. Then $\text{ind}(M)$ is isomorphic to $M$ as a non graded $^\vartheta \mathfrak R_v$-module. Parts (b), (c) follow from (a) by Clifford theory, see e.g., [RR, appendix].

We can now prove Proposition 3.2.

3.21. Proof of Proposition 3.2. Let $b \in \mathfrak B$. We may suppose that $b = 1_{\nu,+}b$. By Lemma 3.20(b) the module $\% = \text{Ind}(b)$ lies in $\mathfrak B$. By [VV, prop. 8.2] there exists a unique selfdual irreducible graded $^\vartheta \mathfrak R$-module $\mathcal G^{\text{up}}(\%)$ which is isomorphic to $\%$ as a non graded module. Set

$$\mathcal G^{\text{up}}(\%) = 1_{\nu,+}\text{res}(\mathcal G^{\text{up}}(\%)).$$

By Lemma 3.15(d) we have $\mathcal G^{\text{up}}(\%) = b$ as a non graded $^\vartheta \mathfrak R$-module, and by (3.5) it is selfdual. This proves existence part of the proposition. To prove the uniqueness, suppose that $M$ is another module with the same properties. Then $\text{ind}(M)$ is a selfdual graded $^\vartheta \mathfrak R$-module which is isomorphic to $\%$ as a non graded $^\vartheta \mathfrak R$-module. Thus we have $\text{ind}(M) = \mathcal G^{\text{up}}(\%)$ by loc. cit. By Lemma 3.15(d) we have also

$$M = 1_{\nu,+}\text{res}(\mathcal G^{\text{up}}(\%)).$$

So $M$ is isomorphic to $\mathcal G^{\text{up}}(\%)$.

3.22. The crystal operators on $^\vartheta \mathfrak G_I$ and $^\vartheta \mathfrak B$. Fix a vertex $i$ in $I$. For each irreducible graded $^\vartheta \mathfrak R_m$-module $M$ we define

$$\tilde e_i(M) = \text{soc}(e_i(M)), \quad \tilde f_i(M) = \text{top}\psi_i(M, L_i), \quad e_i(M) = \max\{n \geq 0; e_i^n(M) \neq 0\}.$$

3.23. Lemma. Let $M$ be an irreducible graded $^\vartheta \mathfrak R$-module such that $e_i(M), f_i(M)$ belong to $^\vartheta \mathfrak G_{I,i}$. We have

$$\text{ind}(\tilde e_i(M)) = e_i(\text{ind}(M)), \quad \text{ind}(\tilde f_i(M)) = f_i(\text{ind}(M)), \quad e_i(M) = e_i(\text{ind}(M)).$$

In particular, $\tilde e_i(M)$ is irreducible or zero and $\tilde f_i(M)$ is irreducible.
Proof: By Corollary 3.18 we have $\text{ind}(e_i(M)) = e_i(\text{ind}(M))$. Thus, since ind is an exact functor we have $\text{ind}(\tilde{e}_i(M)) \subset e_i(\text{ind}(M))$. Since ind is an additive functor, by Lemma 3.20(b) we have indeed

$$\text{ind}(\tilde{e}_i(M)) \subset e_i(\text{ind}(M)).$$

Note that ind$(M)$ is irreducible by Lemma 3.20(b), thus $e_i(\text{ind}(M))$ is irreducible by [VV, prop. 8.21]. Since ind$(\tilde{e}_i(M))$ is nonzero, the inclusion is an isomorphism. The fact that ind$(\tilde{e}_i(M))$ is irreducible implies in particular that $\tilde{e}_i(M)$ is simple. The proof of the second isomorphism is similar. The third equality is obvious.

Similarly, for each irreducible $^6R$-module $b$ in $^6B$ we define

$$\tilde{E}_i(b) = \text{soc}(E_i(b)), \quad \tilde{F}_i(b) = \text{top}(F_i(b)), \quad \varepsilon_i(b) = \max\{n \geq 0; E_i^n(b) \neq 0\}.$$

Hence we have

$$\text{for } o \tilde{e}_i = \tilde{E}_i \text{ for }, \quad \text{for } o \tilde{f}_i = \tilde{F}_i \text{ for }, \quad \varepsilon_i = \varepsilon_i \circ o.$$

3.24. Proposition. For each $b, b'$ in $^6B$ we have

(a) $\tilde{F}_i(b) \in \tilde{E}_i(b)$,

(b) $\tilde{E}_i(b) \in \tilde{E}_i(0)$,

(c) $\tilde{F}_i(b) = b' \iff \tilde{E}_i(b') = b$,

(d) $\varepsilon_i(b) = \max\{n \geq 0; \tilde{E}_i^n(b) \neq 0\}$,

(e) $\varepsilon_i(\tilde{F}_i(b)) = \varepsilon_i(b) + 1$,

(f) if $\tilde{E}_i(b) = 0$ for all $i$ then $b = \phi_i$.

Proof: Part (c) follows from adjunction. The other parts follow from [VV, prop. 3.14] and Lemma 3.23.

3.25. Remark. For any $b \in \tilde{E}_i(b)$ and any $i \neq j$ we have $\tilde{F}_i(b) \neq \tilde{F}_j(b)$. This is obvious if $j \neq \theta(i)$. Because in this case $\tilde{F}_i(b)$ and $\tilde{F}_j(b)$ are $^6R_{\theta(i)}$-modules for different $\nu$. Now, consider the case $j = \theta(i)$. We may suppose that $\tilde{F}_i(b) = 1_{\nu} + \tilde{F}_i(b)$ for certain $\nu$. Then by Lemma 3.16 we have $1_{\nu} + \tilde{F}_{\theta(i)}(b) = 0$. In particular $\tilde{F}_i(b)$ is not isomorphic to $\tilde{F}_{\theta(i)}(b)$.

3.26. The algebra $^6B$ and the $^6B$-module $^6V$. Following [EK1,2,3] we define a $\mathcal{K}$-algebra $^6B$ as follows.

3.27. Definition. Let $^6B$ be the $\mathcal{K}$-algebra generated by $e_i, f_i$ and invertible elements $t_i, i \in I$, satisfying the following defining relations

(a) $t_it_j = t_jt_i$ and $t_{\theta(i)} = t_i$ for all $i, j$,

(b) $t_i f_i = v_i^{\theta(i)j} f_j$ and $t_i f_j = v_i^{j-\theta(i)j} f_j$ for all $i, j$,

(c) $e_i f_j = v_i^{j-\theta(i)j} f_j e_i + \delta_{i,j} + \delta_{\theta(i),j} t_i$ for all $i, j$.
Here and below we use the following notation

\[ \theta^{(a)} = \frac{\theta^a}{(a)!}, \quad (a) = \sum_{l=1}^a q^{a+1-2l}, \quad (a)! = \prod_{l=1}^m (l). \]

We can now construct a representation of \( \mathfrak{B} \) as follows. By base change, the operators \( e_i, f_i \) in Definition 3.8 yield \( \mathcal{K} \)-linear operators on the \( \mathcal{K} \)-vector space

\[ \mathfrak{V} = \mathcal{K} \otimes \mathfrak{A} \mathfrak{K}_I. \]

We equip \( \mathfrak{V} \) with the \( \mathcal{K} \)-bilinear form given by

\[ (M : N)_{\mathcal{K}E} = (1 - v^2)^m \sum M : N, \quad \forall M, N \in \mathfrak{R}_m \text{-proj}. \]

**3.28. Theorem.** (a) The operators \( e_i, f_i \) define a representation of \( \mathfrak{B} \) on \( \mathfrak{V} \).

The \( \mathfrak{B} \)-module \( \mathfrak{V} \) is generated by linearly independent vectors \( \phi_+ \) and \( \phi_- \) such that for each \( i \) we have

\[ e_i \phi_{\pm} = 0, \quad f_i \phi_{\pm} = \phi_{\mp}, \quad \{ x \in \mathfrak{V}; e_j x = 0, \forall j \} = k \phi_+ \oplus k \phi_- . \]

(b) The symmetric bilinear form on \( \mathfrak{V} \) is non-degenerate. We have \( (\phi_a : \phi_{a'})_{\mathcal{K}E} = \delta_{a,a'} \) for \( a, a' = +, - \), and \( (e_i x : y) = (x : f_j y)_{\mathcal{K}E} \) for \( i \in I \) and \( x, y \in \mathfrak{V} \).

**Proof:** For each \( i \) in \( I \) we define the \( \mathfrak{A} \)-linear operator \( t_i \) on \( \mathfrak{V} \) by setting

\[ t_i \phi_{\pm} = \phi_{\mp} \quad \text{and} \quad t_i P = v^{-\nu(i+\theta(i))} P, \quad \forall P \in \mathfrak{R}_P \text{-proj}. \]

We must prove that the operators \( e_i, f_i \), and \( t_i \) satisfy the relations of \( \mathfrak{B} \). The relations (a), (b) are obvious. The relation (d) is standard. It remains to check (c). For this we need a version of the Mackey’s induction-restriction theorem. Note that for \( m > 1 \) we have

\[ D_{m,1; m,1} = \{ e, s_m, \varepsilon_{m+1}\varepsilon_1 \} , \]

\[ W(e) = \mathfrak{W}_m, \quad W(s_m) = \mathfrak{W}_{m-1}, \quad W(\varepsilon_{m+1}\varepsilon_1) = \mathfrak{W}_m. \]

Recall also that for \( m = 1 \) we have set \( \mathfrak{W}_1 = \{ e \} \).

**3.29. Lemma.** Fix \( i, j \) in \( I \). Let \( \nu, \mu \in \mathfrak{M} \) be such that \( \nu + i + \theta(i) = \mu + j + \theta(j) \).

Put \( |\nu| = |\mu| = 2m \). The graded \((\mathfrak{R}_m, \mathfrak{R}_m)\)-bimodule \( \mathfrak{R}_{m+1}^{\mathfrak{R}_{m+1} \mathfrak{R}_{m+1}} \) has a filtration by graded bimodules whose associated graded is isomorphic to

\[ \delta_{i,j} (\mathfrak{R}_\nu \otimes \mathfrak{R}_\mu) \otimes \delta_{\theta(i),j} ((\mathfrak{R}_\nu)^\gamma \otimes \mathfrak{R}_{\theta(i)}) [d'] \otimes A[d], \]

where \( A \) is equal to

\[ (\mathfrak{R}_m \mathfrak{R}_{m+1} \mathfrak{R}_i) \otimes \mathfrak{V}_i (\mathfrak{R}_m \mathfrak{R}_{m+1} \mathfrak{R}_i) \quad \text{if} \quad m > 1, \]

\[ (\mathfrak{R}_{\theta(i)} \otimes \mathfrak{R}_i) \otimes \mathfrak{R}_i \mathfrak{R}_i \otimes \mathfrak{R}_i (\mathfrak{R}_\theta(i) \otimes \mathfrak{R}_j) \oplus (\mathfrak{R}_j \otimes \mathfrak{R}_i) \otimes \mathfrak{R}_i \mathfrak{R}_i \otimes \mathfrak{R}_i \quad \text{if} \quad m = 1. \]
Here we have set \( \nu' = \nu - j - \theta(j), \) \( R' = \mathcal{O}_{m-1,1} \otimes \mathbb{R}_1, \) \( i = i\theta(i), \) \( j = j\theta(j), \) \( d = -i \cdot j, \) and \( d' = -\nu \cdot (i + \theta(i))/2. \)

The proof is standard and is left to the reader. Now, recall that for \( m > 1 \) we have

\[
f_j(P) = \mathcal{O}_{m+1,m} \otimes \mathcal{O}_{m,1} \otimes \mathbb{R}_1, \quad e'_i(P) = 1_{m-1,i},
\]

where \( 1_{m-1,i} \) is regarded as a \( \mathcal{O}_{m-1} \)-module. Therefore we have

\[
\begin{align*}
\epsilon'_i f_j(P) &= 1_{m,i} \mathcal{O}_{m+1,m} \otimes \mathcal{O}_{m,1} \otimes \mathbb{R}_1, \\
f_j \epsilon'_i(P) &= \mathcal{O}_{m,m-1} \otimes \mathcal{O}_{m-1,1} \otimes \mathbb{R}_1.
\end{align*}
\]

Thereupon, up to some filtration we have the following identities

- \( \epsilon'_i f_j(P) = P \otimes \mathbb{R}_i + f_i e'_i(P) [-2], \)
- \( \epsilon'_i f_i(\theta(i)) = P \otimes \mathbb{R}_{\theta(i)} [-\nu \cdot (i + \theta(i))/2] + f_{\theta(i)} e'_i(P) [-i \cdot \theta(i)], \)
- \( \epsilon'_i f_j(P) = f_j e'_i(P) [-i \cdot j] \) if \( i \neq j, \theta(j). \)

These identities also hold for \( m = 1 \) and \( P = \mathcal{O}_{\theta(i)} \) for any \( i \in I. \) The first claim of part (a) follows because \( \mathbb{R}_i = \mathbb{R}_i \otimes [2]. \) The fact that \( \mathcal{O} \mathbb{V} \) is generated by \( \phi \pm \) is a corollary of Proposition 3.31 below. Part (b) of the theorem follows from [KM, prop. 2.2(ii)] and Lemma 3.9(b).

\[ \square \]

3.30. Remarks. (a) The \( \mathcal{O} \mathbb{B}-\)module \( \mathcal{O} \mathbb{V} \) is the same as the \( \mathcal{O} \mathbb{B}-\)module \( \mathcal{O} \mathbb{V}_0 \) from [KM, prop. 2.2]. The involution \( \sigma : \mathcal{O} \mathbb{V} \rightarrow \mathcal{O} \mathbb{V} \) in [KM, rem. 2.5(ii)] is given by \( \sigma(P) = P^\nu. \) It yields an involution of \( \mathcal{O} \mathbb{B} \) in the obvious way. Note that Lemma 3.20(a) yields \( \sigma(b) \neq b \) for any \( b \in \mathcal{O} \mathbb{B}. \)

(b) Let \( \mathcal{O} \mathbb{V} \) be the \( \mathcal{O} \mathbb{B}-\)module \( \mathcal{O} \mathbb{A} \mathcal{O} \mathcal{O}_\mathcal{K} \) and let \( \phi \) be the class of the trivial \( \mathcal{O} \mathcal{R}_\mathcal{O}-\)module \( \mathcal{K}, \) see [VV, thm. 8.30]. We have an inclusion of \( \mathcal{O} \mathbb{B}-\)modules

\[
\mathcal{O} \mathcal{V} \rightarrow \mathcal{O} \mathbb{V}, \quad \phi \mapsto \phi_+ + \phi_-, \quad P \mapsto \text{res}(P).
\]

3.31. Proposition. For any \( b \in \mathcal{O} \mathbb{B} \) the following holds

\[
\begin{align*}
(f_i(\mathcal{O} \mathcal{G}^{\text{low}}(b))) &= (\varepsilon_i(b) + 1) \mathcal{O} \mathcal{G}^{\text{low}}(\mathcal{F}_i b) + \sum_{b'} f_{b,b'} \mathcal{O} \mathcal{G}^{\text{low}}(b'), \\
b' &\in \mathcal{O} \mathbb{B}, \quad \varepsilon_i(b') > \varepsilon_i(b) + 1, \quad f_{b,b'} \in \nu^{2-\varepsilon_i(b')} \mathbb{Z}[v], \\
\epsilon_i(\mathcal{O} \mathcal{G}^{\text{low}}(b)) &= v^{1-\varepsilon_i(b)} \mathcal{O} \mathcal{G}^{\text{low}}(\mathcal{E}_i b) + \sum_{b'} \epsilon_{b,b'} \mathcal{O} \mathcal{G}^{\text{low}}(b'), \\
b' &\in \mathcal{O} \mathbb{B}, \quad \varepsilon_i(b') \geq \varepsilon_i(b), \quad \epsilon_{b,b'} \in \nu^{1-\varepsilon_i(b')} \mathbb{Z}[v].
\end{align*}
\]

Proof: We prove part (a), the proof for (b) is similar. If \( \mathcal{O} \mathcal{G}^{\text{low}}(b) = \phi_\pm \) this is obvious. So we assume that \( \mathcal{O} \mathcal{G}^{\text{low}}(b) \) is a \( \mathcal{O} \mathcal{R}_m \)-module for \( m \geq 1. \) Fix \( \nu \in \mathcal{O} \mathcal{M} \)}
such that \( f_i(^\circ G \text{low}(b)) \) is a \(^\circ R_a\)-module. We’ll abbreviate \( 1_{\nu,a} = 1_a \) for \( a \in \{+, -\} \). Since \(^\circ G \text{low}(b)\) is indecomposable, it fulfills the condition of Lemma 3.16. So there exists \( a \in \{+, -\} \) such that \( 1_{-a} f_i(^\circ G \text{low}(b)) = 0 \). Thus, by Lemma 3.15(c), (d) and Corollary 3.18 we have

\[
f_i(^\circ G \text{low}(b)) = 1_a \text{res ind} f_i(^\circ G \text{low}(b)) = 1_a \text{res} f_i \text{ind}(^\circ G \text{low}(b)).
\]

Note that \( \theta_b = \text{Ind}(b) \) belongs to \(^\circ B\) by Lemma 3.20(b). Hence (3.5) yields

\[
\text{ind}(^\circ G \text{low}(b)) = \theta_{^\circ G \text{low}(\theta_b)}.
\]

We deduce that

\[
f_i(^\circ G \text{low}(b)) = 1_a \text{res} f_i(^\circ G \text{low}(\theta_b)).
\]

Now, write

\[
f_i(^\circ G \text{low}(\theta_b)) = \sum f_{\theta_b, \theta_b'} \theta_{^\circ G \text{low}(\theta_b')}, \quad \theta_b' \in \theta B.
\]

Then we have

\[
f_i(^\circ G \text{low}(b)) = \sum f_{\theta_b, \theta_b'} 1_a \text{res}(^\circ G \text{low}(\theta_b')).
\]

For any \( \theta_b' \in \theta B \) the \(^\circ R\)-module \( 1_a \text{Res}(\theta_b') \) belongs to \(^\circ B\). Thus we have

\[
1_a \text{res}(^\circ G \text{low}(\theta_b')) = ^\circ G \text{low}(1_a \text{Res}(\theta_b')).
\]

If \( \theta_b' \neq \theta_b'' \) then \( 1_a \text{Res}(\theta_b') \neq 1_a \text{Res}(\theta_b'') \), because \( \theta_b' = \text{Ind}(1_a \text{Res}(\theta_b')) \). Thus

\[
f_i(^\circ G \text{low}(b)) = \sum f_{\theta_b, \theta_b'} ^\circ G \text{low}(1_a \text{Res}(\theta_b')),
\]

and this is the expansion of the lhs in the lower global basis. Finally, we have

\[
\varepsilon_i(1_a \text{Res}(\theta_b')) = \varepsilon_i(\theta_b')
\]

by Lemma 3.23. So part (a) follows from [VV, prop. 10.11(b), 10.16].

\[\square\]

3.32. **The global bases of \(^\circ V\).** Since the operators \( e_i, f_i \) on \(^\circ V\) satisfy the relations \( e_i f_i = v^{-2} f_i e_i + 1 \), we can define the modified root operators \( \tilde{e}_i, \tilde{f}_i \) on the \(^\circ B\)-module \(^\circ V\) as follows. For each \( u \) in \(^\circ V\) we write

\[
u = \sum_{n \geq 0} f_i^{(n)} u_n \quad \text{with} \quad e_i u_n = 0,
\]

\[
\tilde{e}_i(u) = \sum_{n \geq 1} f_i^{(n-1)} u_n, \quad \tilde{f}_i(u) = \sum_{n \geq 0} f_i^{(n+1)} u_n.
\]

Let \( R \subset K \) be the set of functions which are regular at \( v = 0 \). Let \(^\circ L\) be the \( R\)-submodule of \(^\circ V\) spanned by the elements \( \tilde{f}_{i_1} \ldots \tilde{f}_{i_l}(\phi_{l \pm}) \) with \( l \geq 0, i_1, \ldots, i_l \in I \). The following is the main result of the paper.
3.33. Theorem. (a) We have
\[ \mathfrak{g} = \bigoplus_{b \in \mathbb{Z}^B} \mathcal{R}G^\text{low}(b), \quad \mathfrak{e}_i(\mathfrak{g}) \subset \mathfrak{g}, \quad \mathfrak{f}_i(\mathfrak{g}) \subset \mathfrak{g}, \]
\[ \mathfrak{e}_i\mathfrak{g}^\text{low}(b) = \mathfrak{g}^\text{low}(\tilde{E}_i(b)) \mod \mathfrak{v}^\text{low} \mathfrak{g}, \quad \mathfrak{f}_i\mathfrak{g}^\text{low}(b) = \mathfrak{g}^\text{low}(\tilde{F}_i(b)) \mod \mathfrak{v}^\text{low} \mathfrak{g}. \]

(b) The assignment \( b \mapsto \mathfrak{g}^\text{low}(b) \mod \mathfrak{v}^\text{low} \mathfrak{g} \) yields a bijection from \( \mathfrak{v}^\text{low} \mathfrak{g} \) to the subset of \( \mathfrak{g}/\mathfrak{v}^\text{low} \mathfrak{g} \) consisting of the \( \tilde{E}_1, \ldots, \tilde{E}_i(\phi_k) \)'s. Further \( \mathfrak{g}^\text{low}(b) \) is the unique element \( x \in \mathfrak{v}^\text{low} \mathfrak{g} \) such that \( x^2 = x \) and \( x = \mathfrak{g}^\text{low}(b) \mod \mathfrak{v}^\text{low} \mathfrak{g} \).

(c) For each \( b, b' \in \mathfrak{v}^\text{low} \mathfrak{g} \) let \( E_{i,b,b'}, F_{i,b,b'} \in A \) be the coefficients of \( \mathfrak{g}^\text{low}(b) \) in \( e_{\phi(i)}(\mathfrak{g}^\text{low}(b)), f_{\phi(i)}(\mathfrak{g}^\text{low}(b)) \) respectively. Then we have
\[ E_{i,b,b'}|_{v=1} = [F_i \Psi_{\text{for}}(\mathfrak{g}^\text{up}(b')) : \Psi_{\text{for}}(\mathfrak{g}^\text{up}(b))], \]
\[ F_{i,b,b'}|_{v=1} = [E_i \Psi_{\text{for}}(\mathfrak{g}^\text{up}(b')) : \Psi_{\text{for}}(\mathfrak{g}^\text{up}(b))]. \]

Proof: Part (a) follows from [EK3, thm. 4.1, cor. 4.4], [E, Section 2.3], and Proposition 3.31. The first claim in (b) follows from (a). The second one is obvious. Part (c) follows from Proposition 3.11. More precisely, by duality we can regard \( E_{i,b,b'}, F_{i,b,b'} \) as the coefficients of \( \mathfrak{g}^\text{up}(b) \) in \( f_{\phi(i)}(\mathfrak{g}^\text{up}(b')), e_{\phi(i)}(\mathfrak{g}^\text{up}(b')) \) respectively. Therefore, by Proposition 3.11 we can regard \( E_{i,b,b'}|_{v=1}, F_{i,b,b'}|_{v=1} \) as the coefficients of \( \Psi_{\text{for}}(\mathfrak{g}^\text{up}(b')) \) in \( F_i \Psi_{\text{for}}(\mathfrak{g}^\text{up}(b')) \) and \( E_i \Psi_{\text{for}}(\mathfrak{g}^\text{up}(b')) \) respectively.

\[ \blacksquare \]

References


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