Canonical bases and affine Hecke algebras of type D
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Abstract. We prove a conjecture of Miemietz and Kashiwara on canonical bases and branching rules of affine Hecke algebras of type D. The proof is similar to the proof of the type B case in [VV].

Introduction

Let $f$ be the negative part of the quantized enveloping algebra of type $A^{(1)}$. Lusztig’s description of the canonical basis of $f$ implies that this basis can be naturally identified with the set of isomorphism classes of simple objects of a category of modules of the affine Hecke algebras of type A. This identification was mentioned in [G], and was used in [A]. More precisely, there is a linear isomorphism between $f$ and the Grothendieck group of finite dimensional modules of the affine Hecke algebras of type A, and it is proved in [A] that the induction/restriction functors for affine Hecke algebras are given by the action of the Chevalley generators and their transposed operators with respect to some symmetric bilinear form on $f$.

The branching rules for affine Hecke algebras of type B have been investigated quite recently, see [E], [EK1,2,3], [M] and [VV]. In particular, in [E], [EK1,2,3] an analogue of Ariki’s construction is conjectured and studied for affine Hecke algebras of type B. Here $f$ is replaced by a module $\theta V(\lambda)$ over an algebra $\theta B$. More precisely it is conjectured there that $\theta V(\lambda)$ admits a canonical basis which is naturally identified with the set of isomorphism classes of simple objects of a category of modules of the affine Hecke algebras of type B. Further, in this identification the branching rules of the affine Hecke algebras of type B should be given by the $\theta B$-action on $\theta V(\lambda)$. This conjecture has been proved [VV]. It uses both the geometric picture introduced in [E] (to prove part of the conjecture) and a new kind of graded algebras similar to the KLR algebras from [KL], [R].

A similar description of the branching rules for affine Hecke algebras of type D has also been conjectured in [KM]. In this case $f$ is replaced by another module $^\theta V$ over the algebra $^\theta B$ (the same algebra as in the type B case). The purpose of this paper is to prove the type D case. The method of the proof is the same as in [VV]. First we introduce a family of graded algebras $^\theta R_m$ for $m$ a non negative integer. They can be viewed as the Ext-algebras of some complex of constructible sheaves naturally attached to the Lie algebra of the group $SO(2m)$, see Remark 2.8. This complex enters in the Kazhdan-Lusztig classification of the simple modules of the affine Hecke algebra of the group $Spin(2m)$. Then we identify $^\theta V$ with the sum of the Grothendieck groups of the graded algebras $^\theta R_m$. 

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The plan of the paper is the following. In Section 1 we introduce a graded algebra \( \circ R(\Gamma)\nu \). It is associated with a quiver \( \Gamma \) with an involution \( \theta \) and with a dimension vector \( \nu \). In Section 2 we consider a particular choice of pair \( (\Gamma, \theta) \). The graded algebras \( \circ R(\Gamma)\nu \) associated with this pair \( (\Gamma, \theta) \) are denoted by the symbol \( \circ R\nu \).

Next we introduce the affine Hecke algebra of type D, more precisely the affine Hecke algebra associated with the group \( SO(2m) \), and we prove that it is Morita equivalent to \( \circ R\nu \). In Section 3 we categorify the module \( \circ V \) from [KM] using the graded algebras \( \circ R\nu \), see Theorem 3.28. The main result of the paper is Theorem 3.33.

0. Notation

0.1. Graded modules over graded algebras. Let \( k \) be an algebraically closed field of characteristic 0. By a graded \( k \)-algebra \( R = \bigoplus_d R_d \) we’ll always mean a \( \mathbb{Z} \)-graded associative \( k \)-algebra. Let \( R\text{-mod} \) be the category of finitely generated graded \( R \)-modules, \( R\text{-fmod} \) be the full subcategory of finite-dimensional graded modules and \( R\text{-proj} \) be the full subcategory of projective objects. Unless specified otherwise all modules are left modules. We’ll abbreviate

\[
K(R) = [R\text{-proj}], \quad G(R) = [R\text{-fmod}].
\]

Here \([\mathcal{C}]\) denotes the Grothendieck group of an exact category \( \mathcal{C} \). Assume that the \( k \)-vector spaces \( R_d \) are finite dimensional for each \( d \). Then \( K(R) \) is a free Abelian group with a basis formed by the isomorphism classes of the indecomposable objects in \( R\text{-proj} \), and \( G(R) \) is a free Abelian group with a basis formed by the isomorphism classes of the simple objects in \( R\text{-fmod} \). Given an object \( M \) of \( R\text{-proj} \) or \( R\text{-fmod} \) let \([M] \) denote its class in \( K(R) \), \( G(R) \) respectively. When there is no risk of confusion we abbreviate \( M = [M] \). We’ll write \([M : N]\) for the composition multiplicity of the \( R \)-module \( N \) in the \( R \)-module \( M \). Consider the ring \( \mathcal{A} = \mathbb{Z}[v, v^{-1}] \). If the grading of \( R \) is bounded below then the \( \mathcal{A} \)-modules \( K(R) \), \( G(R) \) are free. Here \( \mathcal{A} \) acts on \( G(R) \), \( K(R) \) as follows

\[
vM = M[1], \quad v^{-1}M = M[-1].
\]

For any \( M, N \) in \( R\text{-mod} \) let

\[
\text{hom}_R(M, N) = \bigoplus_d \text{Hom}_R(M, N[d])
\]

be the \( \mathbb{Z} \)-graded \( k \)-vector space of all \( R \)-module homomorphisms. If \( R = k \) we’ll omit the subscript \( R \) in hom’s and in tensor products. For any graded \( k \)-vector space \( M = \bigoplus_d M_d \) we’ll write

\[
g\text{dim}(M) = \sum_d v^d \text{dim}(M_d),
\]

where \( \text{dim} \) is the dimension over \( k \).
0.2. Quivers with involutions. Recall that a quiver $\Gamma$ is a tuple $(I, H, h \mapsto h', h \mapsto h'')$ where $I$ is the set of vertices, $H$ is the set of arrows and for each $h \in H$ the vertices $h', h'' \in I$ are the origin and the goal of $h$ respectively. Note that the set $I$ may be infinite. We'll assume that no arrow may join a vertex to itself. For each $i, j \in I$ we write $H_{i,j} = \{h \in H; h' = i, h'' = j\}$.

We'll abbreviate $i \rightarrow j$ if $H_{i,j} \neq \emptyset$. Let $h_{i,j}$ be the number of elements in $H_{i,j}$ and set

$$i \cdot j = -h_{i,j} - h_{j,i}, \quad i \cdot i = 2, \quad i \neq j.$$  

An involution $\theta$ on $\Gamma$ is a pair of involutions on $I$ and $H$, both denoted by $\theta$, such that the following properties hold for each $h$ in $H$

- $\theta(h) = \theta(h')$ and $\theta(h'') = \theta(h)$,
- $\theta(h) = h''$ iff $\theta(h) = h$.

We'll always assume that $\theta$ has no fixed points in $I$, i.e., there is no $i \in I$ such that $\theta(i) = i$. To simplify we'll say that $\theta$ has no fixed point. Let

$$\theta NI = \{\nu = \sum_i \nu_i \in NI : \nu_{\theta(i)} = \nu, \forall i\}.$$ 

For any $\nu \in \theta NI$ set $|\nu| = \sum_i \nu_i$. It is an even integer. Write $|\nu| = 2m$ with $m \in \mathbb{N}$. We'll denote by $\theta^{\nu}$ the set of sequences $i = (i_{1-m}, \ldots, i_{m-1}, i_m)$ of elements in $I$ such that $\theta(i_j) = i_{j-1}$ and $\sum_k i_k = \nu$. For any such sequence $i$ we'll abbreviate $\theta(i) = (\theta(i_{1-m}), \ldots, \theta(i_{m-1}), \theta(i_m))$. Finally, we set

$$\theta^m = \bigcup_{\nu \in \theta NI} \theta^{\nu}, \quad \nu \in \theta NI, \quad |\nu| = 2m.$$ 

0.3. The wreath product. Given a positive integer $m$, let $S_m$ be the symmetric group, and $\mathbb{Z}_2 = \{-1, 1\}$. Consider the wreath product $W_m = S_m \wr \mathbb{Z}_2$. Write $s_1, \ldots, s_{m-1}$ for the simple reflections in $S_m$. For each $l = 1, 2, \ldots, m$ let $\varepsilon_l \in (\mathbb{Z}_2)^m$ be $-1$ placed at the $l$-th position. There is a unique action of $W_m$ on the set $\{1 - m, \ldots, m - 1, m\}$ such that $S_m$ permutes $1, 2, \ldots, m$ and such that $\varepsilon_l$ fixes $k$ if $k \neq l, 1 - l$ and switches $l$ and $1 - l$. The group $W_m$ acts also on $\theta^{\nu}$. Indeed, view a sequence $i$ as the map

$$\{1 - m, \ldots, m - 1, m\} \rightarrow I, \quad l \mapsto i_l.$$ 

Then we set $w(i) = i \circ w^{-1}$ for $w \in W_m$. For each $\nu$ we fix once for all a sequence $i_\nu = (i_{1-m}, \ldots, i_m) \in \theta^{\nu}$.

Let $W_c$ be the centralizer of $i_\nu$ in $W_m$. Then there is a bijection

$$W_c \setminus W_m \rightarrow \theta^\nu, \quad W_c w \mapsto w^{-1}(i_\nu).$$ 

Now, assume that $m > 1$. We set $s_0 = \varepsilon_1 s_1 \varepsilon_1$. Let $\sigma W_m$ be the subgroup of $W_m$ generated by $s_0, \ldots, s_{m-1}$. We'll regard it as a Weyl group of type $D_m$ such that $s_0, \ldots, s_{m-1}$ are the simple reflections. Note that $W_c$ is a subgroup of $\sigma W_m$. Indeed, if $W_c \not\subseteq \sigma W_m$ there should exist $l$ such that $\varepsilon_l$ belongs to $W_c$. This would imply that $i_l = \theta(i_l)$, contradicting the fact that $\theta$ has no fixed point. Therefore $\theta^\nu$ decomposes into two $\sigma W_m$-orbits. We'll denote them by $\theta^\nu_+$ and $\theta^\nu_-$. For $m = 1$ we set $\sigma W_1 = \{e\}$ and we choose again $\theta^{\nu_+}$ and $\theta^{\nu_-}$ in a obvious way.
1. The graded $k$-algebra $^\circ R(\Gamma)_\nu$

Fix a quiver $\Gamma$ with set of vertices $I$ and set of arrows $H$. Fix an involution $\theta$ on $\Gamma$. Assume that $\Gamma$ has no 1-loops and that $\theta$ has no fixed points. Fix a dimension vector $\nu \neq 0$ in $^\circ N I$. Set $|\nu| = 2m$.

1.1. Definition of the graded $k$-algebra $^\circ R(\Gamma)_\nu$. Assume that $m > 1$. We define a graded $k$-algebra $^\circ R(\Gamma)_\nu$ with 1 generated by $1_i$, $\varkappa_i$, $\sigma_k$, with $i \in \nu I$, $l = 1, 2, \ldots, m$, $k = 0, 1, \ldots, m - 1$ modulo the following defining relations

(a) $1_i 1_j = \delta_{i,j} 1_i$, $\varkappa_i 1_1 = 1_{s_k} \varkappa_k$, $\varkappa_i 1_l = 1_l \varkappa_i$,
(b) $\varkappa_i \varkappa_j = \varkappa_j \varkappa_i$,
(c) $\sigma_k^2 1_i = Q_{i,s_k(k)}(\varkappa_{s_k(k)}, \varkappa_k) 1_i$,
(d) $\sigma_k \sigma_{k'} = \sigma_{k'} \sigma_k$ if $1 \leq k < k' < m - 1 = 0$ or $k < k' \neq 2$,
(e) $(\sigma_{s_k(k)} \sigma_k \sigma_{s_k(k)} - \sigma_k \sigma_{s_k(k)} \sigma_k) 1_i =
\begin{cases}
Q_{i,s_k(k)}(\varkappa_{s_k(k)}, \varkappa_k) - Q_{i,s_k(k)}(\varkappa_{s_k(k)}, \varkappa_k) & \text{if } i_k = i_{s_k(k)} + 1, \\
0 & \text{else},
\end{cases}
(f) (\sigma_k \varkappa_l - \varkappa_{s_k(l)} \sigma_k) 1_i =
\begin{cases}
-l_i & \text{if } l = k, i_k = i_{s_k(k)}, \\
l_i & \text{if } l = s_k(k), i_k = i_{s_k(k)}, \\
0 & \text{else}.
\end{cases}

Here we have set $\varkappa_{-l} = -\varkappa_l$ and

\begin{equation}
Q_{i,j}(u,v) = \begin{cases}
(-1)^{i,j}(u-v)^{-i-j} & \text{if } i \neq j, \\
0 & \text{else}.
\end{cases}
\end{equation}

If $m = 0$ we set $^\circ R(\Gamma)_0 = k \oplus k$. If $m = 1$ then we have $\nu = i + \theta(i)$ for some $i \in I$. Write $i = i \theta(i)$, and $^\circ R(\Gamma)_\nu = k[\varkappa_1] 1_l \oplus k[\varkappa_1] 1_l \theta(i)$. We’ll abbreviate $\varkappa_{1,l} = \varkappa_1 1_l$ and $\varkappa_{l,1} = \varkappa_1 1_l$. The grading on $^\circ R(\Gamma)_0$ is the trivial one. For $m \geq 1$ the grading on $^\circ R(\Gamma)_\nu$ is given by the following rules:

\begin{align*}
\deg(1_i) &= 0, \\
\deg(\varkappa_1 1_l) &= 2, \\
\deg(\varkappa_{1,l}) &= -i_k \cdot i_{s_k(k)}.
\end{align*}

We define $\omega$ to be the unique involution of the graded $k$-algebra $^\circ R(\Gamma)_\nu$ which fixes $1_i$, $\varkappa_i$, $\sigma_k$. We set $\omega$ to be identity on $^\circ R(\Gamma)_0$.

1.2. Relation with the graded $k$-algebra $^\circ R(\Gamma)_\nu$. A family of graded $k$-algebra $^\circ R(\Gamma)_{\lambda, \nu}$ was introduced in [VV, sec. 5.1], for $\lambda$ an arbitrary dimension vector in $\mathbb{N} I$. Here we’ll only consider the special case $\lambda = 0$, and we abbreviate $^\circ R(\Gamma)_\nu = ^\circ R(\Gamma)_{0, \nu}$. Recall that if $\nu \neq 0$ then $^\circ R(\Gamma)_\nu$ is the graded $k$-algebra with 1 generated
by $1_i$, $\kappa_l$, $\sigma_k$, $\pi_1$, with $i \in \mathcal{I}^{(l)}$, $l = 1, 2, \ldots, m$, $k = 1, \ldots, m - 1$ such that $1_i$, $\kappa_l$ and $\sigma_k$ satisfy the same relations as before and
\[
\begin{align*}
\pi_1^2 &= 1, \quad \pi_11_i\pi_1 = 1_{c_1}, \quad \pi_1\kappa_l\pi_1 = \kappa_{c_1(l)}, \\
(\pi_1\sigma_k)^2 &= (\sigma_1\pi_1)^2, \quad \pi_1\sigma_k\pi_1 = \sigma_k \text{ if } k \neq 1.
\end{align*}
\]
If $\nu = 0$ then $^\nu \mathcal{R}(\Gamma)_0 = k$. The grading is given by setting deg($1_i$), deg($\kappa_l$), deg($\sigma_k$) to be as before and deg($\pi_11_i$) = 0. In the rest of Section 1 we’ll assume $m > 0$. Then there is a canonical inclusion of graded $k$-algebras
\[
^\nu \mathcal{R}(\Gamma) \subset ^\nu \mathcal{R}(\Gamma)
\]
such that $1_i, \kappa_l, \nu \in g \mapsto 1_i, \kappa_l, \sigma_k$ for $i \in \mathcal{I}^{(l)}$, $l = 1, \ldots, m$, $k = 1, \ldots, m - 1$ and such that $\sigma_0 \mapsto \pi_1\sigma_1\pi_1$. From now on we’ll write $\sigma_0 = \pi_1\sigma_1\pi_1$ whenever $m > 1$. The assignment $x \mapsto \pi_1x\pi_1$ defines an involution of the graded $k$-algebra $^\nu \mathcal{R}(\Gamma)$ which normalizes $^\nu \mathcal{R}(\Gamma)$. Thus it yields an involution
\[
\gamma : ^\nu \mathcal{R}(\Gamma) \rightarrow ^\nu \mathcal{R}(\Gamma).
\]
Let $(\gamma)$ be the group of two elements generated by $\gamma$. The smash product $^\nu \mathcal{R}(\Gamma) \rtimes (\gamma)$ is a graded $k$-algebra such that deg$(\gamma) = 0$. There is an unique isomorphism of graded $k$-algebras
\[
^\nu \mathcal{R}(\Gamma) \rtimes (\gamma) \rightarrow ^\nu \mathcal{R}(\Gamma)
\]
which is identity on $^\nu \mathcal{R}(\Gamma)$ and which takes $\gamma$ to $\pi_1$.

1.3. The polynomial representation and the PBW theorem. For any $i$ in $\mathcal{I}^{(l)}$ let $^\nu \mathcal{F}_i$ be the subalgebra of $^\nu \mathcal{R}(\Gamma)$ generated by $1_i$ and $\kappa_{c_1(l)}$ with $l = 1, 2, \ldots, m$. It is a polynomial algebra. Let
\[
^\nu \mathcal{F} = \bigoplus_{i \in \mathcal{I}^{(l)}} ^\nu \mathcal{F}_i.
\]
The group $W_m$ acts on $^\nu \mathcal{F}$ via $w(\kappa_{c_1(l)}) = \kappa_{w(i)w(l)}$ for any $w \in W_m$. Consider the fixed points set
\[
^\nu S = (^\nu \mathcal{F})^W_m.
\]
Regard $^\nu \mathcal{R}(\Gamma)$ and End($^\nu \mathcal{F}$) as $^\nu \mathcal{F}$-algebras via the left multiplication. In [VV, prop. 5.4] is given an injective graded $^\nu \mathcal{F}$-algebra morphism $^\nu \mathcal{R}(\Gamma) \rightarrow $ End($^\nu \mathcal{F}$). It restricts via (1.2) to an injective graded $^\nu \mathcal{F}$-algebra morphism
\[
^\nu \mathcal{R}(\Gamma) \rightarrow $ End($^\nu \mathcal{F}$).
\]
Next, recall that $^\nu W_m$ is the Weyl group of type $D_m$ with simple reflections $s_0, \ldots, s_{m-1}$. For each $w$ in $^\nu W_m$ we choose a reduced decomposition $\hat{w}$ of $w$. It has the following form
\[
w = s_{k_1}s_{k_2}\cdots s_{k_r}, \quad 0 \leq k_1, k_2, \ldots, k_r \leq m - 1.
\]
We define an element $\sigma_\hat{w}$ in $^\nu \mathcal{R}(\Gamma)$ by
\[
\sigma_\hat{w} = \sum_1 1_i\sigma_\hat{w}, \quad 1_i\sigma_\hat{w} = \begin{cases} 1_i & \text{if } r = 0 \\
1_i\sigma_{k_1}\sigma_{k_2}\cdots\sigma_{k_r} & \text{else} \end{cases}
\]
Observe that the element $\sigma_\hat{w}$ may depend on the choice of the reduced decomposition $\hat{w}$. 
1.4. Proposition. The \( k \)-algebra \( ^{\circ}R(\Gamma)_\nu \) is a free (left or right) \( ^{\circ}F_\nu \)-module with basis \( \{ \sigma_w; w \in \mathcal{W}_m \} \). Its rank is \( 2^{m-1}m! \). The operator \( l_1 \sigma_w \) is homogeneous and its degree is independent of the choice of the reduced decomposition \( \dot{w} \).

Proof: The proof is the same as in [VV, prop. 5.5]. First, we filter the algebra \( ^{\circ}R(\Gamma)_\nu \) with \( l_1, \sigma_{1,l} \) in degree 0 and \( \sigma_{i,k} \) in degree 1. The Nil Hecke algebra of type \( D_m \) is the \( k \)-algebra \( ^{\circ}NH_m \) generated by \( \sigma_0, \sigma_1, \ldots, \sigma_{m-1} \) with relations

\[
\sigma_k \sigma_{k'} = \sigma_{k'} \sigma_k \quad \text{if} \; 1 \leq k < k' < m-1 \quad \text{or} \quad 0 = k < k' \neq 2,
\]

\[
\sigma_{s_k(k)} \sigma_{s_k(k)} = \sigma_k \sigma_{s_k(k)} \sigma_k, \quad \sigma_k^2 = 0.
\]

We can form the semidirect product \( ^{\circ}F_\nu \rtimes ^{\circ}NH_m \), which is generated by \( l_1, \sigma_{1}, \sigma_k \) with the relations above and

\[
\sigma_k \sigma_l = \sigma_{s_k(l)} \sigma_k, \quad \sigma_l \sigma_k = \sigma_k \sigma_l, \quad \sigma_k \sigma_{1} = \sigma_{1} \sigma_k.
\]

One proves as in [VV, prop. 5.5] that the map

\[
^{\circ}F_\nu \rtimes ^{\circ}NH_m \to \text{gr}(^{\circ}R(\Gamma)_\nu), \quad l_1 \mapsto l_1, \quad \sigma_l \mapsto \sigma_l, \quad \sigma_k \mapsto \sigma_k.
\]

is an isomorphism of \( k \)-algebras.

\[ \square \]

Let \( ^{\circ}F'_\nu = \bigoplus_i ^{\circ}F'_i \), where \( ^{\circ}F'_i \) is the localization of the ring \( ^{\circ}F_1 \) with respect to the multiplicative system generated by

\[
\{ \sigma_{1,l} \pm \sigma_{1,l'}; 1 \leq l \neq l' \leq m \} \cup \{ \sigma_{1,l}; l = 1, 2, \ldots, m \}.
\]

1.5. Corollary. The inclusion \( ^{\circ}R(\Gamma)_\nu \subseteq \text{End}(^{\circ}F_\nu) \) yields an isomorphism of \( ^{\circ}F'_\nu \)-algebras \( ^{\circ}F'_\nu \otimes ^{\circ}F_\nu \to ^{\circ}F'_\nu \rtimes ^{\circ}W_m \), such that for each \( i \) and each \( l = 1, 2, \ldots, m, k = 0, 1, 2, \ldots, m-1 \) we have

\[
l_1 \mapsto l_1, \quad \sigma_{1,l} \mapsto \sigma_{1,l}, \quad \sigma_{1,k} \mapsto \begin{cases} \sigma_k - \sigma_{s_k(k)}^{-1}(s_k - 1) l_1 & \text{if} \; i_k = i_{s_k(k)}, \\ \sigma_k - \sigma_{s_k(k)}^{-1} i_k s_k l_1 & \text{if} \; i_k \neq i_{s_k(k)}. \end{cases}
\]

Proof: Follows from [VV, cor. 5.6] and Proposition 1.4.

\[ \square \]

Restricting the \( ^{\circ}F_\nu \)-action on \( ^{\circ}R(\Gamma)_\nu \) to the \( k \)-subalgebra \( ^{\circ}S_\nu \) we get a structure of graded \( ^{\circ}S_\nu \)-algebra on \( ^{\circ}R(\Gamma)_\nu \).

1.6. Proposition. (a) \( ^{\circ}S_\nu \) is isomorphic to the center of \( ^{\circ}R(\Gamma)_\nu \).

(b) \( ^{\circ}R(\Gamma)_\nu \) is a free graded module over \( ^{\circ}S_\nu \) of rank \( 2^{m-1}m!^2 \).

Proof: Part (a) follows from Corollary 1.5. Part (b) follows from (a) and Proposition 1.4.

\[ \square \]
2. **Affine Hecke algebras of type D**

2.1. **Affine Hecke algebras of type D.** Fix $p$ in $k^\times$. For any integer $m \geq 0$ we define the extended affine Hecke algebra $H_m$ of type $D_m$ as follows. If $m > 1$ then $H_m$ is the $k$-algebra with 1 generated by

$$T_k, \quad X_i^{\pm 1}, \quad k = 0, 1, \ldots, m - 1, \quad l = 1, 2, \ldots, m$$

satisfying the following defining relations:

(a) $X_iX_{i'} = X_{i'}X_i$,

(b) $T_kT_{s_k(k)} = T_{s_k(k)}T_k$, $T_kT_{k'} = T_{k'}T_k$ if $1 \leq k < k' - 1$ or $k = 0$, $k' \neq 2$,

(c) $(T_k - p)(T_k + p^{-1}) = 0$,

(d) $T_kX_1T_k^{-1} = X_2$, $T_kX_lT_k = X_{s_k(k)}$ if $k \neq 0$, $T_kX_l = X_lT_k$ if $k \neq 0, l, l - 1$ or $k = 0, l \neq 1, 2$.

Finally, we set $H_0 = k \oplus k$ and $H_1 = k[X_1^{\pm 1}]$.

2.2. **Remarks.** (a) The extended affine Hecke algebra $H^B_m$ of type $B_m$ with parameters $p, q \in k^\times$ such that $q - 1$ is generated by $P$, $T_k$, $X_i^{\pm 1}$, $k = 1, \ldots, m - 1$, $l = 1, \ldots, m$ such that $T_k$, $X_i^{\pm 1}$ satisfy the relations as above and

$$P^2 = 1, \quad (PT_1)^2 = (T_1P)^2, \quad PT_k = T_kP$$

and $l \neq 1$.

See e.g., [VV, sec. 6.1]. There is an obvious $k$-algebra embedding $H_m \subset H^B_m$. Let $\gamma$ denote also the involution $H_m \to H_m$, $a \to PaP$. We have a canonical isomorphism of $k$-algebras

$$H_m \rtimes \langle \gamma \rangle \simeq H^B_m.$$  

Compare Section 1.2.

(b) Given a connected reductive group $G$ we call affine Hecke algebra of $G$ the Hecke algebra of the extended affine Weyl group $W \ltimes P$, where $W$ is the Weyl group of $(G, T)$, $P$ is the group of characters of $T$, and $T$ is a maximal torus of $G$. Then $H_m$ is the affine Hecke algebra of the group $SO(2m)$. Let $H^B_m$ be the affine Hecke algebra of the group $Spin(2m)$. It is generated by $H_m$ and an element $X_0$ such that

$$X_0^2 = X_1X_2 \ldots X_m, \quad T_kX_0 = X_0T_k$$

and $l \neq 1$. Thus $H_m$ is the fixed point subset of the $k$-algebra automorphism of $H^B_m$ taking $T_k, X_l$ to $T_k, (-1)^{X_0}X_l$ for all $k, l$. Therefore, if $p$ is not a root of 1 the simple $H_m$-modules can be recovered from the Kazhdan-Lusztig classification of the simple $H^B_m$-modules via Clifford theory, see e.g., [Re].

2.3. **Intertwiners and blocks of $H_m$.** We define

$$A = k[X_1^{\pm 1}, X_2^{\pm 1}, \ldots, X_m^{\pm 1}], \quad A' = A[\Sigma^{-1}], \quad H'_m = A' \otimes_A H_m,$$

where $\Sigma$ is a distinguished variable.
where $\Sigma$ is the multiplicative set generated by
\[ 1 - X_l X_{l'}^{-1}, \quad 1 - p^2 X_l^{\pm 1} X_{l'}^{\pm 1}, \quad l \neq l'. \]

For $k = 0, \ldots, m - 1$ the intertwiner $\varphi_k$ is the element of $H_m'$ given by the following formulas
\[
(2.1) \quad \varphi_k - 1 = \frac{X_k - X_{s_k(k)}}{pX_k - p^{-1}X_{s_k(k)}} (T_k - p).
\]

The group $°W_m$ acts on $A'$ as follows
\[
(s_k a)(X_1, \ldots, X_m) = a(X_1, \ldots, X_{k+1}, X_k, \ldots, X_m) \quad \text{if} \quad k \neq 0,
\]
\[
(s_0 a)(X_1, \ldots, X_m) = a(X_2^{-1}, X_1^{-1}, \ldots, X_m).
\]

There is an isomorphism of $A'$-algebras
\[
A' \times °W_m \to H_m', \quad s_k \mapsto \varphi_k.
\]

The semi-direct product group $\mathbb{Z} \times \mathbb{Z}/2 = \mathbb{Z} \times \{ \pm 1 \}$ acts on $k^\times$ by $(n, \varepsilon) : z \mapsto z^\varepsilon p^n$.

Given a $\mathbb{Z} \times \mathbb{Z}/2$-invariant subset $I$ of $k^\times$ we denote by $H_{m,I}$ the category of all $H_m$-modules such that the action of $X_1, X_2, \ldots, X_m$ is locally finite with eigenvalues in $I$. We associate to the set $I$, and to the element $p \in k^\times$ a quiver $\Gamma$ as follows. The set of vertices is $I$, and there is one arrow $p^2 i \to i$ whenever $i$ lies in $I$. We equip $\Gamma$ with an involution $\theta$ such that $\theta(i) = i^{-1}$ for each vertex $i$ and such that $\theta$ takes the arrow $p^2 i \to i$ to the arrow $i^{-1} \to p^{-2} i^{-1}$. We'll assume that the set $I$ does not contain $1$ nor $-1$ and that $p \neq 1, -1$. Thus the involution $\theta$ has no fixed points and no arrow may join a vertex of $\Gamma$ to itself.

2.4. Remark. We may assume that $I = \{p^n; n \in \mathbb{Z}_{\text{odd}} \}$. See the discussion in [KM]. Then $\Gamma$ is of type $A_{\infty}$ if $p$ has infinite order and $\Gamma$ is of type $A_{\varepsilon}^{(1)}$ if $p^2$ is a primitive $r$-th root of unity.

2.5. $H_m$-modules versus $°R_m$-modules. Assume that $m \geq 1$. We define the graded $k$-algebra
\[
(°R_{I,m})_{\nu} = °R_{I,m} = °R_{\nu} = °R(\Gamma)_{\nu}, \quad °R_m = °R(\Gamma),
\]
\[
°R_{I,m} = \bigoplus_{\nu} °R_{I,m} = °R_{I,m} = °R_{I,m} = °R_{I,m}.
\]

Note that $°R_{I,m}$ are the same as in [VV, sec. 6.4], with $\lambda = 0$. Note also that the $k$-algebra $°R_m$ may not have $1$, because the set $I$ may be infinite. We define $°R_{m,I}$ as the category of all (non-graded) $°R_m$-modules such that the elements $x_1, x_2, \ldots, x_m$ act locally nilpotently. Let $°R_{m,f}\text{Mod}_0$ and $H_{m,f}\text{Mod}_I$ be the full subcategories of finite dimensional modules in $°R_{m,I}$ and $H_{m,I}$ respectively. Fix a formal series $f(\varepsilon)$ in $k[[\varepsilon]]$ such that $f(\varepsilon) = 1 + \varepsilon$ modulo $(\varepsilon^2)$. 
2.6. Theorem. We have an equivalence of categories
\[ \mathbb{D}^\circ R_m \text{-Mod}_0 \to H_m \text{-Mod}_I, \quad M \mapsto M \]
which is given by

(a) \( X_l \) acts on \( 1_4 M \) by \( i_l^{-1} f(x_l) \) for each \( l = 1, 2, \ldots, m \),

(b) if \( m > 1 \) then \( T_k \) acts on \( 1_4 M \) as follows for each \( k = 0, 1, \ldots, m - 1 \),

\[ \frac{(pf(x_k) - p^{-1}f(x_{s_k}(k)))(x_k - x_{s_k}(k))}{f(x_k) - f(x_{s_k}(k))} \sigma_k + p \]

if \( i_{s_k}(k) = i_k \),

\[ \frac{f(x_k) - f(x_{s_k}(k))}{(p^{-1}f(x_k) - p^{-1}f(x_{s_k}(k)))(x_k - x_{s_k}(k))} \sigma_k + \frac{(p^{-2} - 1)f(x_{s_k}(k))}{pf(x_k) - p^{-1}f(x_{s_k}(k))} \]

if \( i_{s_k}(k) = p^2 i_k \),

\[ \frac{p_i k f(x_k) - p^{-1} i_{s_k}(k) f(x_{s_k}(k))}{i_k f(x_k) - i_{s_k}(k) f(x_{s_k}(k))} \sigma_k + \frac{(p^{-1} - p) i_k f(x_{s_k}(k))}{i_k f(x_k) - i_{s_k}(k) f(x_{s_k}(k))} \]

if \( i_{s_k}(k) \neq i_k, p^2 i_k \).

Proof: This follows from [VV, thm. 6.5] by Section 1.2 and Remark 2.2(a). One can also prove it by reproducing the arguments in loc. cit. by using (1.5) and (2.1).

\[ \square \]

2.7. Corollary. There is an equivalence of categories
\[ \Psi : \mathbb{D}^\circ R_m \text{-fMod}_0 \to H_m \text{-fMod}_I, \quad M \mapsto M. \]

2.8. Remarks. (a) Let \( g \) be the Lie algebra of \( G = SO(2m) \). Fix a maximal torus \( T \subset G \). The group of characters of \( T \) is the lattice \( \bigoplus_{l \in I} \mathbb{Z} \varepsilon_l \), with Bourbaki's notation. Fix a dimension vector \( \nu \in \mathbb{N}^I \). Recall the sequence \( I_\nu = (i_{-1}, \ldots, i_{m-1}, i_m) \) from Section 0.3. Let \( g \in T \) be the element such that \( \varepsilon_l(g) = i_l^{-1} \) for each \( l = 1, 2, \ldots, m \). Recall also the notation \( \theta \nu \nu, V, \theta \nu E \nu, \) and \( \theta \nu G \nu \) from [VV]. Then \( V \) is an object of \( \theta \nu \nu, G \nu = G_g \) is the centralizer of \( g \) in \( G \), and

\[ \theta \nu E \nu = g_{\nu, \nu}, \quad g_{\nu, \nu} = \{ x \in g; \text{ad}_g(x) = p^2 x \}. \]

Let \( F_g \) be the set of all Borel Lie subalgebras of \( g \) fixed by the adjoint action of \( g \). It is a non connected manifold whose connected components are labelled by \( \nu \nu \nu \nu \nu \nu \nu \). In Section 3.14 we'll introduce two central idempotents \( 1_{\nu, +}, 1_{\nu, -} \) of \( \mathbb{D}^\circ R \nu \). This yields a graded \( k \)-algebra decomposition

\[ \mathbb{D}^\circ R \nu = \mathbb{D}^\circ R \nu 1_{\nu, +} \oplus \mathbb{D}^\circ R \nu 1_{\nu, -}. \]

By [VV, thm. 5.8] the graded \( k \)-algebra \( \mathbb{D}^\circ R \nu 1_{\nu, +} \) is isomorphic to

\[ \text{Ext} \mathbb{D}^\circ R \nu (L_g, L_g), \]

where \( L_g \) is the direct image of the constant perverse sheaf by the projection

\[ \{ (b, x) \in F_g \times g_{\nu, \nu}; x \in b \} \to g_{\nu, \nu}, \quad (b, x) \mapsto x. \]

The complex \( L_g \) has been extensively studied by Lusztig, see e.g., [L1], [L2]. We hope to come back to this elsewhere.

(b) The results in Section 2.5 hold true if \( k \) is any field. Set \( f(x) = 1 + x \) for instance.
2.9. Induction and restriction of $H_m$-modules. For $i \in I$ we define functors
\[
\begin{align*}
E_i &: \text{H}_{m+1} \text{-Mod}_I \to \text{H}_m \text{-Mod}_I, \\
F_i &: \text{H}_m \text{-Mod}_I \to \text{H}_{m+1} \text{-Mod}_I,
\end{align*}
\]
where $E_i M \subset M$ is the generalized $i^{-1}$-eigenspace of the $X_{m+1}$-action, and where
\[
F_i M = \text{Ind}_{H_{m+1}}^{H_m} (M \otimes k).
\]
Here $k_i$ is the 1-dimensional representation of $k[X_{m+1}]$ defined by $X_m \mapsto i^{-1}$.

3. Global bases of $^\circ V$ and projective graded $^\circ R$-modules

3.1. The Grothendieck groups of $^\circ R_m$. The graded $k$-algebra $^\circ R_m$ is free of finite rank over its center by Proposition 1.6, a commutative graded $k$-subalgebra. Therefore any simple object of $^\circ R_m$-mod is finite-dimensional and there is a finite number of isomorphism classes of simple modules in $^\circ R_m$-mod. The Abelian group $G(^\circ R_m)$ is free with a basis formed by the classes of the simple objects of $^\circ R_m$-mod. The Abelian group $K(^\circ R_m)$ is free with a basis formed by the classes of the indecomposable projective objects. Both $G(^\circ R_m)$ and $K(^\circ R_m)$ are free $A$-modules, where $v$ shifts the grading by 1. We consider the following $A$-modules
\[
\begin{align*}
^\circ K_I &= \bigoplus_{m \geq 0} ^\circ K_{I,m}, \quad ^\circ K_{I,m} = K(^\circ R_m), \\
^\circ G_I &= \bigoplus_{m \geq 0} ^\circ G_{I,m}, \quad ^\circ G_{I,m} = G(^\circ R_m).
\end{align*}
\]
We’ll also abbreviate
\[
^\circ K_{I,*} = \bigoplus_{m > 0} ^\circ K_{I,m}, \quad ^\circ G_{I,*} = \bigoplus_{m > 0} ^\circ G_{I,m}.
\]
From now on, to unburden the notation we may abbreviate $^\circ R = ^\circ R_m$, hoping it will not create any confusion. For any $M, N$ in $^\circ R$-mod we set
\[
\langle M : N \rangle = \text{gdim}(M^\omega \otimes R N), \quad (M : N) = \text{gdim} \text{hom}_R(M, N),
\]
where $\omega$ is the involution defined in Section 1.1. The Cartan pairing is the perfect $A$-bilinear form
\[
^\circ K_I \times ^\circ G_I \to A, \quad (P, M) \mapsto \langle P : M \rangle.
\]
First, we concentrate on the $A$-module $^\circ G_I$. Consider the duality
\[
^\circ R$-$\text{fmod} \to ^\circ R$-$\text{fmod}, \quad M \mapsto M^\flat = \text{hom}(M, k),
\]
with the action and the grading given by
\[
(xf)(m) = f(\omega(x)m), \quad (M^\flat)_d = \text{Hom}(M_{-d}, k).
\]
This duality functor yields an $A$-antilinear map
\[
^\circ G_I \to ^\circ G_I, \quad M \mapsto M^\flat.
\]
Let $^\circ B$ denote the set of isomorphism classes of simple objects of $^\circ R$-$\text{fMod}_0$. We can now define the upper global basis of $^\circ G_I$ as follows. The proof is given in Section 3.21.
3.2. Proposition/Definition. For each \( b \) in \( °B \) there is a unique selfdual irreducible graded \( °R \)-module \( °G^{\text{up}}(b) \) which is isomorphic to \( b \) as a (non graded) \( °R \)-module. We set \( °G^{\text{up}}(0) = 0 \) and \( °G^{\text{up}} = \{ °G^{\text{up}}(b); b ∈ °B \} \). Hence \( °G^{\text{up}} \) is a \( A \)-basis of \( °G_I \).

Now, we concentrate on the \( A \)-module \( °K_I \). We equip \( °K_I \) with the symmetric \( A \)-bilinear form

\[
(3.1) \quad °K_I × °K_I → A, \quad (M, N) ↦ (M : N).
\]

Consider the duality

\[
°R\text{-proj} → °R\text{-proj}, \quad P ↦ P^♯ = \text{hom}_{°R}(P, °R),
\]

with the action and the grading given by

\[
(xf)(p) = f(p)ω(x), \quad (P^♯)_d = \text{Hom}_{°R}(P[-d], °R).
\]

This duality functor yields an \( A \)-antilinear map

\[
°K_I → °K_I, \quad P ↦ P^♯.
\]

Set \( K = \mathbb{Q}(v) \). Let \( K → K, f ↦ \bar{f} \) be the unique involution such that \( \bar{v} = v^{-1} \).

3.3. Definition. For each \( b \) in \( °B \) let \( °G^{\text{low}}(b) \) be the unique indecomposable graded module in \( °R\text{-proj} \) whose top is isomorphic to \( °G^{\text{up}}(b) \). We set \( °G^{\text{low}}(0) = 0 \) and \( °G^{\text{low}} = \{ °G^{\text{low}}(b); b ∈ °B \} \), a \( A \)-basis of \( °K_I \).

3.4. Proposition. (a) We have \( °G^{\text{low}}(b) : °G^{\text{up}}(b') = δ_{b,b'} \) for each \( b, b' \) in \( °B \).

(b) We have \( (P^♯ : M) = (P : M^♯) \) for each \( P, M \).

(c) We have \( °G^{\text{low}}(b)^♯ = °G^{\text{low}}(b) \) for each \( b \) in \( °B \).

The proof is the same as in [VV, prop. 8.4].

3.5. Example. Set \( ν = i + θ(i) \) and \( i = iθ(i) \). Consider the graded \( °R_ν \)-modules

\[
°R_i = °Rl_i = l_i°R, \quad °L_i = \text{top}(°R_i).
\]

The global bases are given by

\[
°G_ν^{\text{low}} = \{ °R_i, °R_{θ(i)} \}, \quad °G_ν^{\text{up}} = \{ °L_i, °L_{θ(i)} \}.
\]

For \( m = 0 \) we have \( °R_0 = k ⊕ k \). Set \( φ_+ = k ⊕ 0 \) and \( φ_- = 0 ⊕ k \). We have

\[
°G_0^{\text{low}} = °G_0^{\text{up}} = \{ φ_+, φ_- \}.
\]
Definition of the operators $e_i, f_i, e'_i, f'_i$. In this section we'll always assume $m > 0$ unless specified otherwise. First, let us introduce the following notation. Let $D_{m,1}$ be the set of minimal representative in $W_{m+1}$ of the cosets in $W_m \setminus W_{m+1}$. Write

\[ D_{m,1;m,1} = D_{m,1} \cap (D_{m,1})^{-1}. \]

For each element $w$ of $D_{m,1;m,1}$ we set

\[ W(w) = \text{wt}(w) \cap w(\text{wt}(w))^{-1}. \]

Let $R_1$ be the $k$-algebra generated by elements $1_i, \kappa_i, i \in I$, satisfying the defining relations $1_i 1_i' = \delta_{i,i'} 1_i$ and $\kappa_i 1_i = 1_i \kappa_i$. We equip $R_1$ with the grading such that $\deg(1_i) = 0$ and $\deg(\kappa_i) = 2$. Let

\[ R_i = 1_i R_1 = R_1 1_i, \quad L_i = \text{top}(R_i) = R_i/(\kappa_i). \]

Then $R_i$ is a graded projective $R_1$-module and $L_i$ is simple. We abbreviate

\[ R_{m,1} = R_m \otimes R_1, \quad R_{m,1} = R_m \otimes R_1. \]

There is an unique inclusion of graded $k$-algebras

\[
\begin{align*}
\theta R_{m,1} & \hookrightarrow \theta R_{m+1}, \\
1_i \otimes 1_i & \mapsto 1_i', \\
1_i \otimes x_i & \mapsto x_i', \\
\sigma_i \otimes 1_i & \mapsto \sigma_i', \\
\sigma_i \otimes \kappa & \mapsto \sigma_i',
\end{align*}
\]

(3.2)

where, given $i \in \theta I$, we have set $i' = \theta(i)i$, a sequence in $\theta I$. This inclusion restricts to an inclusion $\theta R_{m,1} \subset \theta R_{m+1}$.  

3.7. Lemma. The graded $R_{m,1}$-module $R_{m+1}$ is free of rank $2(m + 1)$.

Proof: For each $w$ in $D_{m,1}$ we have the element $\sigma_w$ in $R_{m+1}$ defined in (1.5). Using filtered/graded arguments it is easy to see that

\[ R_{m+1} \otimes m_{\theta w}. \]

We define a triple of adjoint functors $(\psi_!, \psi^*, \psi_*)$ where

\[ \psi^*: \quad \text{mod} \rightarrow \text{mod} \times \text{mod} \]
is the restriction and \( \psi, \psi^* \) are given by

\[
\psi_! : \begin{cases} 
\circ \mathbf{R}_m \text{-mod} \times \mathbf{R}_1 \text{-mod} \to \circ \mathbf{R}_{m+1} \text{-mod}, \\
(M, M') \mapsto \circ \mathbf{R}_{m+1} \otimes_{\circ \mathbf{R}_m} (M \otimes M'), 
\end{cases}
\]

\[
\psi_* : \begin{cases} 
\circ \mathbf{R}_m \text{-mod} \times \mathbf{R}_1 \text{-mod} \to \circ \mathbf{R}_{m+1} \text{-mod}, \\
(M, M') \mapsto \text{hom}_{\circ \mathbf{R}_m}(\circ \mathbf{R}_{m+1}, M \otimes M'). 
\end{cases}
\]

First, note that the functors \( \psi, \psi^* \) commute with the shift of the grading. Next, the functor \( \psi^* \) is exact, and it takes finite dimensional graded modules to finite dimensional ones. The right graded \( \circ \mathbf{R}_{m,1} \)-module \( \circ \mathbf{R}_{m+1} \) is free of finite rank. Thus \( \psi_! \) is exact, and it takes finite dimensional graded modules to finite dimensional ones. The left graded \( \circ \mathbf{R}_{m,1} \)-module \( \circ \mathbf{R}_{m+1} \) is also free of finite rank. Thus the functor \( \psi_* \) is exact, and it takes finite dimensional graded modules to finite dimensional ones. Further \( \psi_! \) and \( \psi_* \) take projective graded modules to projective ones, because they are left adjoint to the exact functors \( \psi^*, \psi_* \) respectively. To summarize, the functors \( \psi, \psi^*, \psi_* \) are exact and take finite dimensional graded modules to finite dimensional ones, and the functors \( \psi_!, \psi_* \) take projective graded modules to projective ones.

For any graded \( \circ \mathbf{R}_m \)-module \( M \) we write

\[
f_i(M) = \circ \mathbf{R}_{m+1} 1_{m,i} \otimes_{\circ \mathbf{R}_m} M,
\]

\[
e_i(M) = \circ \mathbf{R}_{m-1} \otimes_{\circ \mathbf{R}_{m-1}} 1_{m-1,i} M.
\]

Let us explain these formulas. The symbols \( 1_{m,i} \) and \( 1_{m-1,i} \) are given by

\[
1_{m-1,i} M = \bigoplus \limits_i 1_{\theta(i) |i} M, \quad i \in \theta^m_{m-1}.
\]

Note that \( f_i(M) \) is a graded \( \circ \mathbf{R}_{m+1} \)-module, while \( e_i(M) \) is a graded \( \circ \mathbf{R}_{m-1} \)-module. The tensor product in the definition of \( e_i(M) \) is relative to the graded \( k \)-algebra homomorphism

\[
\circ \mathbf{R}_{m-1,1} \to \circ \mathbf{R}_{m-1} \otimes \mathbf{R}_1 \to \circ \mathbf{R}_{m-1} \otimes \mathbf{R}_i \to \circ \mathbf{R}_{m-1} \otimes \mathbf{L}_i = \circ \mathbf{R}_{m-1}.
\]

In other words, let \( e'_i(M) \) be the graded \( \circ \mathbf{R}_{m-1} \)-module obtained by taking the direct summand \( 1_{m-1,i} M \) and restricting it to \( \circ \mathbf{R}_{m-1} \). Observe that if \( M \) is finitely generated then \( e'_i(M) \) may not lie in \( \circ \mathbf{R}_{m-1} \text{-mod} \). To remedy this, since \( e'_i(M) \) affords a \( \circ \mathbf{R}_{m-1} \otimes \mathbf{R}_i \)-action we consider the graded \( \circ \mathbf{R}_{m-1} \)-module

\[
e_i(M) = e'_i(M)/\langle \alpha e'_i(M) \rangle.
\]

3.8. Definition. The functors \( e_i, f_i \) preserve the category \( \circ \mathbf{R} \text{-proj} \), yielding \( \mathcal{A} \)-linear operators on \( \circ \mathbf{K}_I \) which act on \( \circ \mathbf{K}_{I,*} \) by the formula (3.3) and satisfy also

\[
f_i(\phi_+) = \circ \mathbf{R}_{\theta(i)+}, \quad f_i(\phi_-) = \circ \mathbf{R}_{\theta(i)-}, \quad e_i(\circ \mathbf{R}_{\theta(j)}) = \delta_{i,j} \phi_+ + \delta_{i,j} \theta(j) \phi_-.
\]

Let \( e_i, f_i \) denote also the \( \mathcal{A} \)-linear operators on \( \circ \mathbf{G}_I \) which are the transpose of \( f_i, e_i \) with respect to the Cartan pairing.

Note that the symbols \( e_i(M), f_i(M) \) have a different meaning if \( M \) is viewed as an element of \( \circ \mathbf{K}_I \) or if \( M \) is viewed as an element of \( \circ \mathbf{G}_I \). We hope this will not create any confusion. The proof of the following lemma is the same as in [VV, lem. 8.9].
3.9. Lemma. (a) The operators $e_i, f_i$ on $^6G_I$ are given by

$$e_i(M) = 1_{m-1,i}M, \quad f_i(M) = \text{hom}_{^6R_{m+1}}(^6R_{m+1}, M \otimes L_i), \quad M \in \mathbb{R}_m.$$ (b) For each $M \in \mathbb{R}_m$, $M' \in \mathbb{R}_{m+1}$ we have

$$(e'_i(M') : M) = (M' : f_i(M)).$$

(c) We have $f_i(P) = f_i(P')$ for each $P \in \mathbb{R}_{\text{proj}}$. (d) We have $e_i(M) = e_i(M')$ for each $M \in \mathbb{R}_{\text{fmod}}$.

3.10. Induction of $H_m$-modules versus induction of $^6\mathbb{R}_m$-modules. Recall the functors $E_i, F_i$ on $\text{H-fMod}_I$ defined in (2.2). We have also the functors

$$\text{for} : \mathbb{R}_m \to \mathbb{R}_{m-1}, \quad \Psi : \mathbb{R}_m \to H_m \text{-fMod}_I,$$

where $\text{for}$ is the forgetting of the grading. Finally we define functors

$$E_i : \mathbb{R}_m \to \mathbb{R}_{m-1}, \quad E_i = 1_{m-1,i}M,$$

$$F_i : \mathbb{R}_m \to \mathbb{R}_{m+1}, \quad F_i = \psi(M, L_i).$$  \hspace{1cm} (3.4)

3.11. Proposition. There are canonical isomorphisms of functors

$$E_i \circ \Psi = \Psi \circ E_i, \quad F_i \circ \Psi = \Psi \circ F_i, \quad \text{for} \circ E_i = \text{for} \circ e_i, \quad \text{for} \circ F_i = \text{for} \circ f_{\theta(i)}.$$

Proof: The proof is the same as in [VV, prop. 8.17]. \hfill \Box

3.12. Proposition. (a) The functor $\Psi$ yields an isomorphism of Abelian groups

$$\bigoplus_{m \geq 0} [^6\mathbb{R}_m] \cong \bigoplus_{m \geq 0} [H_m \text{-fMod}_I].$$

The functors $E_i, F_i$ yield endomorphisms of both sides which are intertwined by $\Psi$. (b) The functor $\text{for}$ factors to a group isomorphism

$$^6G_I/(v-1) = \bigoplus_{m \geq 0} [^6\mathbb{R}_m].$$

Proof: Claim (a) follows from Corollary 2.7 and Proposition 3.11. Claim (b) follows from Proposition 3.2. \hfill \Box
3.13. Type D versus type B. We can compare the previous constructions with their analogues in type B. Let $^{\theta}K$, $^{\theta}B$, $^{\theta}G_{\text{low}}$, etc, denote the type B analogues of $^{o}K$, $^{o}B$, $^{o}G_{\text{low}}$, etc. See [VV] for details. We’ll use the same notation for the functors $\psi^{*}, \psi_{*}, \psi_{!}, \epsilon$, $f$, etc, on the type B side and on the type D side. Fix $m > 0$ and $\nu \in {}^{\theta}N$ such that $|\nu| = 2m$. The inclusion of graded $k$-algebras $^{o}R_{\nu} \subset {}^{\theta}R_{\nu}$ in (1.2) yields a restriction functor

$$\text{res} : {}^{\theta}R_{\nu}-\text{mod} \to {}^{o}R_{\nu}-\text{mod}$$

and an induction functor

$$\text{ind} : {}^{o}R_{\nu}-\text{mod} \to {}^{\theta}R_{\nu}-\text{mod}, \quad M \mapsto {}^{\theta}R_{\nu} \otimes {}^{o}R_{\nu} M.$$ 

Both functors are exact, they map finite dimensional graded modules to finite dimensional ones, and they map projective graded modules to projective ones. Thus, they yield morphisms of $A$-modules

$$\text{res} : {}^{\theta}K_{I,m} \to {}^{o}K_{I,m}, \quad \text{res} : {}^{\theta}G_{I,m} \to {}^{o}G_{I,m}, \quad \text{ind} : {}^{o}K_{I,m} \to {}^{\theta}K_{I,m}, \quad \text{ind} : {}^{o}G_{I,m} \to {}^{\theta}G_{I,m}.$$ 

Moreover, for any $P \in {}^{\theta}K_{I,m}$ and any $L \in {}^{o}G_{I,m}$ we have

$$\text{res}(P^{\gamma}) = (\text{res} P)^{\gamma}, \quad \text{ind}(P^{\gamma}) = (\text{ind} P)^{\gamma}$$

$$\text{res}(L^{\gamma}) = (\text{res} L)^{\gamma}, \quad \text{ind}(L^{\gamma}) = (\text{ind} L)^{\gamma}. \quad (3.5)$$

Note also that $\text{ind}$ and $\text{res}$ are left and right adjoint functors, because

$$^{\theta}R_{\nu} \otimes {}^{o}R_{\nu} M = \text{hom}_{^{\theta}R_{\nu}}({}^{\theta}R_{\nu}, M)$$

as graded $^{\theta}R_{\nu}$-modules.

3.14. Definition. For any graded $^{o}R_{\nu}$-module $M$ we define the graded $^{\theta}R_{\nu}$-module $M^{\gamma}$ with the same underlying graded $k$-vector space as $M$ such that the action of $^{o}R_{\nu}$ is twisted by $\gamma$, i.e., the graded $k$-algebra $^{o}R_{\nu}$ acts on $M^{\gamma}$ by $a \cdot m = (\gamma(a)) m$ for $a \in {}^{o}R_{\nu}$ and $m \in M$. Note that $(M^{\gamma})^{\gamma} = M$, and that $M^{\gamma}$ is simple (resp. projective, indecomposable) if $M$ has the same property.

For any graded $^{o}R_{m}$-module $M$ we have canonical isomorphisms of $^{o}R$-modules

$$(f_{i}(M))^{\gamma} = f_{i}(M^{\gamma}) , \quad (e_{i}(M))^{\gamma} = e_{i}(M^{\gamma}).$$

The first isomorphism is given by

$${}^{o}R_{m+1} \otimes {}^{o}R_{m} M \to {}^{o}R_{m+1} \otimes {}^{o}R_{m} M, \quad a \otimes m \mapsto \gamma(a) \otimes m.$$ 

The second one is the identity map on the vector space $1_{m,1} M$.

Recall that $^{\theta}I_{\nu}$ is the disjoint union of $^{\theta}I_{\nu}^{+}$ and $^{\theta}I_{\nu}^{-}$. We set

$$1_{\nu,+} = \sum_{i \in {}^{\theta}I_{\nu}^{+}} 1_{i}, \quad 1_{\nu,-} = \sum_{i \in {}^{\theta}I_{\nu}^{-}} 1_{i}.$$
3.15. Lemma. Let $M$ be a graded $^\circ R$-module.
(a) Both $1_{\nu,+}$ and $1_{\nu,-}$ are central idempotents in $^\circ R$. We have $1_{\nu,+} = \gamma(1_{\nu,-})$.
(b) There is a decomposition of graded $^\circ R$-modules $M = 1_{\nu,+}M \oplus 1_{\nu,-}M$.
(c) We have a canonical isomorphism of $^\circ R$-modules $\text{res} \circ \text{ind}(M) = M \oplus M^\gamma$.
(d) If there exists $a \in \{+, -\}$ such that $1_{\nu,-a}M = 0$, then there are canonical isomorphisms of graded $^\circ R$-modules

\[ M = 1_{\nu,a}M, \quad 0 = 1_{\nu,a}M^\gamma, \quad M^\gamma = 1_{\nu,-a}M^\gamma. \]

Proof: Part (a) follows from Proposition 1.6 and the equality $\epsilon_1(\theta I_{\nu}^a) = \theta I_{\nu}^a$. Part (b) follows from (a), (c) is given by definition, and (d) follows from (a), (b).

Now, let $m$ and $\nu$ be as before. Given $i \in I$, we set $\nu' = \nu + i + \theta(i)$. There is an obvious inclusion $W_m \subset W_{m+1}$. Thus the group $W_m$ acts on $^\theta I_{\nu'}^a$, and the map

\[ ^\theta I_{\nu'}^a \rightarrow ^\theta I_{\nu}^a, \quad i \mapsto \theta(i)i \]

is $W_m$-equivariant. Thus there is $a_i \in \{+, -\}$ such that the image of $^\theta I_{\nu}^a$ is contained in $^\theta I_{\nu}^a$, and the image of $^\theta I_{\nu}^a$ is contained in $^\theta I_{\nu}^a$.

3.16. Lemma. Let $M$ be a graded $^\circ R$-module such that $1_{\nu,-a}M = 0$, with $a = +, -$. Then we have

\[ 1_{\nu'-a,a}f_i(M) = 0, \quad 1_{\nu',a,i}f_{\theta(i)}(M) = 0. \]

Proof: We have

\[ 1_{\nu',-a,a}f_i(M) = 1_{\nu'-a,a}^\circ R_{\nu'}^a1_{\nu,i} \otimes^a R_{\nu}^a M \]

\[ = ^\circ R_{\nu'}^a1_{\nu'-a,a}1_{\nu,i}1_{\nu,a} \otimes^a R_{\nu}^a M. \]

Here we have identified $1_{\nu,a}$ with the image of $(1_{\nu,a}, 1_i)$ via the inclusion (3.2). The definition of this inclusion and that of $a_i$ yield that

\[ 1_{\nu',a,a}1_{\nu,i}1_{\nu,a} = 1_{\nu,a}, \quad 1_{\nu',-a,a}1_{\nu,i}1_{\nu,a} = 0. \]

The first equality follows. Next, note that for any $i \in ^\theta I_{\nu}^a$, the sequences $\theta(i)i$ and $i\theta(i) = \varepsilon_{m+1}(\theta(i)i)$ always belong to different $^\circ W_{m+1}$-orbits. Thus, we have $a_{\theta(i)} = -a_i$. So the second equality follows from the first.

Consider the following diagram

\[
\begin{array}{ccc}
^\circ R_{\nu'}^a \text{-mod} \times R_{\nu}^a \text{-mod} & \overset{\psi}{\longrightarrow} & ^\circ R_{\nu'}^a \text{-mod} \\
\text{res} \times \text{id} & \overset{\psi}{\longrightarrow} & \text{ind} \times \text{id} \\
\text{ind} \times \text{id} & \overset{\psi}{\longrightarrow} & \text{res} \times \text{id} \\
^\theta R_{\nu'}^a \text{-mod} \times R_{\nu}^a \text{-mod} & \overset{\psi}{\longrightarrow} & ^\theta R_{\nu'}^a \text{-mod}.
\end{array}
\]
3.17. Lemma. There are canonical isomorphisms of functors

\[
\begin{align*}
\text{ind} \circ \psi &= \psi_1 \circ (\text{id} \times \text{id}), & \text{ind} \circ \psi &= \psi_1 \circ (\text{id} \times \text{id}), \\
\psi^* \circ \text{ind} &= (\text{id} \times \text{id}) \circ \psi^*, & \psi^* \circ \text{ind} &= (\text{id} \times \text{id}) \circ \psi^*, \\
\text{res} \circ \psi &= \psi_1 \circ (\text{res} \times \text{id}), & \text{res} \circ \psi &= \psi_1 \circ (\text{res} \times \text{id}), \\
\psi^* \circ \text{res} &= (\text{res} \times \text{id}) \circ \psi^*, & \psi^* \circ \text{res} &= (\text{res} \times \text{id}) \circ \psi^*, \\
\text{ind} \circ \psi^* &= \psi_1 \circ (\text{res} \times \text{id}), & \text{ind} \circ \psi^* &= \psi_1 \circ (\text{res} \times \text{id}).
\end{align*}
\]

Proof: The functor ind is left and right adjoint to res. Therefore it is enough to prove the first two isomorphisms in the first line. The isomorphism

\[
\text{ind} \circ \psi_! = \psi_! \circ (\text{ind} \times \text{id})
\]

comes from the associativity of the induction. Let us prove that \(\psi^* \circ \text{ind} = (\text{ind} \times \text{id}) \circ \psi^*\).

For any \(M \in \mathcal{R}_\nu\)-mod, the obvious inclusion \(\theta \mathcal{R}_\nu \otimes \mathcal{R}_i \subset \theta \mathcal{R}_\nu' \otimes \mathcal{R}_i\) yields a map

\[
(\text{ind} \times \text{id}) \psi_*(M) = (\theta \mathcal{R}_\nu \otimes \mathcal{R}_i) \otimes \mathcal{R}_i,\quad \psi^*(M) \rightarrow \psi^*(\theta \mathcal{R}_\nu' \otimes \mathcal{R}_i, M).
\]

Combining it with the obvious map \(\theta \mathcal{R}_\nu' \otimes \mathcal{R}_i, M \rightarrow \theta \mathcal{R}_\nu' \otimes \mathcal{R}_i, M\)

we get a morphism of \(\theta \mathcal{R}_\nu \otimes \mathcal{R}_i\)-modules

\[
(\text{ind} \times \text{id}) \psi^*(M) \rightarrow \psi^* \text{ind}(M).
\]

We need to show that it is bijective. This is obvious because at the level of vector spaces, the map above is given by

\[
M \oplus (\pi_{1,\nu} \otimes M) \rightarrow M \oplus (\pi_{1,\nu'} \otimes M), \quad m + \pi_{1,\nu} \otimes n \mapsto m + \pi_{1,\nu'} \otimes n.
\]

Here \(\pi_{1,\nu}\) and \(\pi_{1,\nu'}\) denote the element \(\pi_1\) in \(\theta \mathcal{R}_\nu\) and \(\theta \mathcal{R}_{\nu'}\) respectively.

\(\Box\)

3.18. Corollary. (a) The operators \(e_i, f_i\) on \(\theta \mathcal{K}_{I,+}\) and on \(\theta \mathcal{K}_{I,+}\) are intertwined by the maps ind, res, i.e., we have

\[
e_i \circ \text{ind} = \text{ind} \circ e_i, \quad f_i \circ \text{ind} = \text{ind} \circ f_i, \quad e_i \circ \text{res} = \text{res} \circ e_i, \quad f_i \circ \text{res} = \text{res} \circ f_i.
\]

(b) The same result holds for the operators \(e_i, f_i\) on \(\theta \mathcal{G}_{I,*}\) and on \(\theta \mathcal{G}_{I,*}\).

3.19. Now, we concentrate on non graded irreducible modules. First, let

\[
\text{Res} : \theta \mathcal{R}_\nu\text{-Mod} \rightarrow \mathcal{R}_\nu\text{-Mod}, \quad \text{Ind} : \mathcal{R}_\nu\text{-Mod} \rightarrow \theta \mathcal{R}_\nu\text{-Mod}
\]

be the (non graded) restriction and induction functors. We have

\[
\text{for} \circ \text{res} = \text{Res} \circ \text{for}, \quad \text{for} \circ \text{ind} = \text{Ind} \circ \text{for}.
\]
3.20. Lemma. Let $L, L'$ be irreducible $^\circ R_v$-modules.

(a) The $^\circ R_v$-modules $L$ and $L'$ are not isomorphic.

(b) $\text{Ind}(L)$ is an irreducible $^\circ R_v$-module, and every irreducible $^\circ R_v$-module is obtained in this way.

(c) $\text{Ind}(L) \simeq \text{Ind}(L')$ iff $L' \simeq L$ or $L' \simeq L^\gamma$.

Proof: For any $^\circ R_v$-module $M \neq 0$, both $1_{\nu,+}M$ and $1_{\nu,-}M$ are nonzero. Indeed, we have $M = 1_{\nu,+}M \oplus 1_{\nu,-}M$, and we may suppose that $1_{\nu,+}M \neq 0$. By Lemma 3.15(d) we have $\text{Ind} (1_{\nu,+}M) \neq 0$. Hence $1_{\nu,-}M \neq 0$.

Now, to prove part (a), suppose that $\phi : L \to L'$ is an isomorphism of $^\circ R_v$-modules. We can regard $\phi$ as a $\gamma$-antilinear map $L \to L$. Since $L$ is irreducible, by Schur’s lemma we may assume that $\phi^2 = \text{id}_L$. Then $L$ admits a $^\circ R_v$-module structure such that the $^\circ R_v$-action is as before and $\pi_1$ acts as $\phi$. Thus, by the discussion above, neither $1_{\nu,+}L$ nor $1_{\nu,-}L$ is zero. This contradicts the fact that $L$ is an irreducible $^\circ R_v$-module.

Parts (b), (c) follow from (a) by Clifford theory, see e.g., [RR, appendix].

We can now prove Proposition 3.2.

3.21. Proof of Proposition 3.2. Let $b \in \mathcal{B}$. We may suppose that $b = 1_{\nu,+}b$. By Lemma 3.20(b) the module $\mathcal{B} = \text{Ind}(b)$ lies in $^\circ B$. By [VV, prop. 8.2] there exists a unique selfdual irreducible graded $^\circ R$-module $^\circ G^{up}(\mathcal{B})$ which is isomorphic to $\mathcal{B}$ as a non graded module. Set

$$^\circ G^{up}(\mathcal{B}) = 1_{\nu,+}\text{res}(^\circ G^{up}(\mathcal{B})).$$

By Lemma 3.15(d) we have $^\circ G^{up}(\mathcal{B}) = b$ as a non graded $^\circ R$-module, and by (3.5) it is selfdual. This proves existence part of the proposition. To prove the uniqueness, suppose that $M$ is another module with the same properties. Then $\text{ind}(M)$ is a selfdual graded $^\circ R$-module which is isomorphic to $\mathcal{B}$ as a non graded $^\circ R$-module. Thus we have $\text{ind}(M) = ^\circ G^{up}(\mathcal{B})$ by loc. cit. By Lemma 3.15(d) we have also

$$M = 1_{\nu,+}\text{res}(^\circ G^{up}(\mathcal{B})).$$

So $M$ is isomorphic to $^\circ G^{up}(\mathcal{B})$.

3.22. The crystal operators on $^\circ G_I$ and $^\circ B$. Fix a vertex $i$ in $I$. For each irreducible graded $^\circ R_m$-module $M$ we define

$$\bar{e}_i(M) = \text{soc} (e_i(M)), \quad \bar{f}_i(M) = \text{top} \psi_i(M, L_i), \quad e_i(M) = \max \{n \geq 0; e_i^n(M) \neq 0\}.$$

3.23. Lemma. Let $M$ be an irreducible graded $^\circ R$-module such that $e_i(M), f_i(M)$ belong to $^\circ G_{I,*}$. We have

$$\text{ind}(\bar{e}_i(M)) = \bar{e}_i(\text{ind}(M)), \quad \text{ind}(\bar{f}_i(M)) = \bar{f}_i(\text{ind}(M)), \quad e_i(M) = e_i(\text{ind}(M)).$$

In particular, $\bar{e}_i(M)$ is irreducible or zero and $\bar{f}_i(M)$ is irreducible.
Proof: By Corollary 3.18 we have \( \text{ind}(e_i(M)) = e_i(\text{ind}(M)) \). Thus, since ind is an exact functor we have \( \text{ind}(\tilde{e}_i(M)) \subset e_i(\text{ind}(M)) \). Since ind is an additive functor, by Lemma 3.20(b) we have indeed
\[
\text{ind}(\tilde{e}_i(M)) \subset e_i(\text{ind}(M)).
\]
Note that \( \text{ind}(M) \) is irreducible by Lemma 3.20(b), thus \( e_i(\text{ind}(M)) \) is irreducible by [VV, prop. 8.21]. Since \( \text{ind}(\tilde{e}_i(M)) \) is nonzero, the inclusion is an isomorphism. The fact that \( \text{ind}(\tilde{e}_i(M)) \) is irreducible implies in particular that \( \tilde{e}_i(M) \) is simple. The proof of the second isomorphism is similar. The third equality is obvious.

\[
\square
\]

Similarly, for each irreducible \( ^\circ R \)-module \( b \) in \( ^\circ B \) we define
\[
\tilde{E}_i(b) = \text{soc}(E_i(b)), \quad \tilde{F}_i(b) = \text{top}(F_i(b)), \quad \varepsilon_i(b) = \max\{n \geq 0; \tilde{E}_i^n(b) \neq 0\}.
\]

Hence we have
\[
\text{for } \circ e_i = \tilde{E}_i \circ \text{for, for } \circ f_i = \tilde{F}_i \circ \text{for, } \varepsilon_i = \varepsilon_i \circ \text{for.}
\]

**3.24. Proposition.** For each \( b, b' \in ^\circ B \) we have

(a) \( \tilde{F}_i(b) \in ^\circ B \),
(b) \( \tilde{E}_i(b) \in ^\circ B \cup \{0\} \),
(c) \( \tilde{F}_i(b) = b' \iff \tilde{E}_i(b') = b \),
(d) \( \varepsilon_i(b) = \max\{n \geq 0; \tilde{E}_i^n(b) \neq 0\} \),
(e) \( \varepsilon_i(\tilde{F}_i(b)) = \varepsilon_i(b) + 1 \),
(f) \( \circ \tilde{E}_i(b) = 0 \) for all \( i \) then \( b = \phi_{\pm} \).

Proof: Part (c) follows from adjunction. The other parts follow from [VV, prop. 3.14] and Lemma 3.23.

\[
\square
\]

**3.25. Remark.** For any \( b \in ^\circ B \) and any \( i \neq j \) we have \( \tilde{F}_i(b) \neq \tilde{F}_j(b) \). This is obvious if \( j \neq \theta(i) \). Because in this case \( \tilde{F}_i(b) \) and \( \tilde{F}_j(b) \) are \( ^\circ R_i \)-modules for different \( \nu \). Now, consider the case \( j = \theta(i) \). We may suppose that \( \tilde{F}_i(b) = 1_{\nu, +} \tilde{F}_i(b) \) for certain \( \nu \). Then by Lemma 3.16 we have \( 1_{\nu, +} \tilde{F}_\theta(i)(b) = 0 \). In particular \( \tilde{F}_i(b) \) is not isomorphic to \( \tilde{F}_\theta(i)(b) \).

**3.26. The algebra \( ^\circ B \) and the \( ^\circ B \)-module \( ^\circ V \).** Following [EK1,2,3] we define a \( \mathcal{K} \)-algebra \( ^\circ B \) as follows.

**3.27. Definition.** Let \( ^\circ B \) be the \( \mathcal{K} \)-algebra generated by \( e_i, f_i \) and invertible elements \( t_i, i \in I \), satisfying the following defining relations

(a) \( t_it_j = t_jt_i \) and \( t_{\theta(i)} = t_i \) for all \( i, j \),
(b) \( t_ie_jt_i^{-1} = v^{i+j+\theta(i)}e_j \) and \( t_if_jt_i^{-1} = v^{-i-j-\theta(i)}f_j \) for all \( i, j \),
(c) \( e_if_j = v^{-i-j}f_j e_i + \delta_{i,j} + \delta_{\theta(i),\theta(j)}t_i \) for all \( i, j \),
(d) \[ \sum_{a+b=1-i-j} (-1)^a e_i^{(a)} f_j^{(b)} = \sum_{a+b=1-i-j} (-1)^a f_i^{(a)} f_j^{(b)} = 0 \text{ if } i \neq j. \]

Here and below we use the following notation

\[ \theta^{(a)} = \theta^a/(a!), \quad (a) = \sum_{l=1}^{a} \frac{a!}{l!(a-l)!}, \quad (a)! = \prod_{l=1}^{m} (l). \]

We can now construct a representation of \( \mathfrak{B} \) as follows. By base change, the operators \( e_i, f_i \) in Definition 3.8 yield \( K \)-linear operators on the \( K \)-vector space

\[ \mathfrak{g} \mathfrak{B} = K \otimes A \mathfrak{g} \mathfrak{K}_I. \]

We equip \( \mathfrak{g} \mathfrak{V} \) with the \( K \)-bilinear form given by

\[ (M : N)_{K,K} = (1 - \nu^2)^m (M : N), \quad \forall M, N \in \mathfrak{g} \mathfrak{B}_m \text{-proj}. \]

**3.28. Theorem.** (a) The operators \( e_i, f_i \) define a representation of \( \mathfrak{B} \) on \( \mathfrak{g} \mathfrak{V} \). The \( \mathfrak{B}_m \)-module \( \mathfrak{g} \mathfrak{V} \) is generated by linearly independent vectors \( \phi_+ \) and \( \phi_- \) such that for each \( i \) we have

\[ e_i \phi_\pm = 0, \quad t_i \phi_\pm = \phi_\mp, \quad \{ x \in \mathfrak{g} \mathfrak{V}; e_j x = 0, \forall j \} = k \phi_+ \oplus k \phi_. \]

(b) The symmetric bilinear form on \( \mathfrak{g} \mathfrak{V} \) is non-degenerate. We have \( (\phi_\alpha : \phi_{a'})_{\mathfrak{g} \mathfrak{B}} = \delta_{a,a'} \) for \( a, a' = +, - \), and \( (e_i x : y) = (x : f_j y)_{\mathfrak{g} \mathfrak{B}} \) for \( i \in I \) and \( x, y \in \mathfrak{g} \mathfrak{V} \).

**Proof:** For each \( i \) in \( I \) we define the \( A \)-linear operator \( t_i \) on \( \mathfrak{g} \mathfrak{K}_I \) by setting

\[ t_i \phi_\pm = \phi_\mp \quad \text{and} \quad t_i P = v^{-\nu(i+\theta(i))} P \nu, \quad \forall P \in \mathfrak{g} \mathfrak{B}_\nu \text{-proj}. \]

We must prove that the operators \( e_i, f_i \), and \( t_i \) satisfy the relations of \( \mathfrak{B} \). The relations \( (a), (b) \) are obvious. The relation \( (d) \) is standard. It remains to check \( (c) \). For this we need a version of the Mackey’s induction-restriction theorem. Note that for \( m > 1 \) we have

\[ D_{m, 1; m, 1} = \{ e, s_m, e_{m+1} \}, \]

\[ W(e) = \mathfrak{g} W_m, \quad W(s_m) = \mathfrak{g} W_{m-1}, \quad W(e_{m+1}) = \mathfrak{g} W_m. \]

Recall also that for \( m = 1 \) we have set \( \mathfrak{g} W_1 = \{ e \} \).

**3.29. Lemma.** Fix \( i, j \) in \( I \). Let \( \nu, \mu \in \mathbb{N} \) be such that \( \nu + i + \theta(i) = \mu + j + \theta(j) \).

Put \( |\nu| = |\mu| = 2m \). The graded \( (\mathfrak{g} \mathfrak{R}_m, \mathfrak{g} \mathfrak{R}_m) \)-bimodule \( 1_{\nu,i}^{\nu} \mathfrak{R}_m \otimes \mathfrak{g} \mathfrak{R}_{m+1}^{\mu,j} \) has a filtration by graded bimodules whose associated graded is isomorphic to

\[ \delta_{i,j} \left( (\mathfrak{g} \mathfrak{R}_\nu \otimes \mathfrak{R}_i) \oplus \delta_{i-j}(\mathfrak{g} \mathfrak{R}_\nu \otimes \mathfrak{R}_{\theta(j)})[d'] \oplus A[d], \right. \]

where \( A \) is equal to

\[
\begin{align*}
& (\mathfrak{g} \mathfrak{R}_m 1_{\nu,i}^{\nu} \otimes \mathfrak{R}_i) \otimes (1_{\nu,i}^{\nu} \mathfrak{g} \mathfrak{R}_m \otimes \mathfrak{R}_i) \quad \text{if } m > 1, \\
& (\mathfrak{g} \mathfrak{R}_\theta(i) \otimes \mathfrak{R}_j \otimes \mathfrak{g} \mathfrak{R}_{\theta(i)} \otimes \mathfrak{R}_j) \oplus (\mathfrak{g} \mathfrak{R}_j \otimes \mathfrak{R}_i \otimes \mathfrak{g} \mathfrak{R}_i \otimes \mathfrak{R}_j) \quad \text{if } m = 1.
\end{align*}
\]
Here we have set \( \nu' = \nu - j - \theta(j) \), \( R' = \mathcal{R}_{m-1,1} \otimes R_1 \), \( i = i\theta(i) \), \( j = j\theta(j) \),
\( d = -i \cdot j \), and \( d' = -\nu \cdot (i + \theta(i))/2 \).

The proof is standard and is left to the reader. Now, recall that for \( m > 1 \) we have
\[
f_j(P) = \mathcal{R}_{m+1,m,j} \otimes \mathcal{R}_{m-1} (P \otimes R_1),
\]
where \( \mathcal{R}_{m-1} \) is regarded as a \( \mathcal{R}_{m-1} \)-module. Therefore we have
\[
e'_j f_j(P) = \mathcal{R}_{m+1,m,j} \otimes \mathcal{R}_{m-1} (P \otimes R_1),
\]
\[
f_j e'_j(P) = \mathcal{R}_{m+1,m,j} \otimes \mathcal{R}_{m-1,1} (1_{m-1} \otimes P \otimes R_1).
\]

Therefore, up to some filtration we have the following identities
\[
\begin{align*}
& e'_j f_i(P) = P \otimes R_i + f_i e'_j(P)[-2], \\
& e'_j f_{\theta(i)}(P) = P_i \otimes R_{\theta(i)}[-\nu \cdot (i + \theta(i))/2] + f_{\theta(i)} e'_j(P)[-i \cdot \theta(i)], \\
& e'_j f_j(P) = f_j e'_j(P)[-i \cdot j] \text{ if } i \neq j, \theta(j).
\end{align*}
\]

These identities also hold for \( m = 1 \) and \( P = \mathcal{R}_{\theta(i),1} \) for any \( i \in I \). The first claim of part (a) follows because \( R_i = k \oplus R_i[2] \). The fact that \( \mathcal{V} \) is generated by \( \phi_\pm \) is a corollary of Proposition 3.31 below. Part (b) of the theorem follows from [KM, prop. 2.2(ii)] and Lemma 3.9(b).

\[
\square
\]

3.30. Remarks. (a) The \( \mathcal{B} \)-module \( \mathcal{V} \) is the same as the \( \mathcal{B} \)-module \( V_\theta \) from [KM, prop. 2.2]. The involution \( \sigma : \mathcal{V} \to \mathcal{V} \) in [KM, rem. 2.5(ii)] is given by \( \sigma(P) = P^\nu \). It yields an involution of \( \mathcal{B} \) in the obvious way. Note that Lemma 3.20(a) yields \( \sigma(b) \neq b \) for any \( b \in \mathcal{B} \).

(b) Let \( \mathcal{V} \) be the \( \mathcal{B} \)-module \( \mathcal{K} \otimes \mathcal{A} \mathcal{B} \) and let \( \phi \) be the class of the trivial \( \mathcal{B} \)-module \( k \), see [VV, thm. 8.30]. We have an inclusion of \( \mathcal{B} \)-modules
\[
\mathcal{V} \to \mathcal{V}, \quad \phi \mapsto \phi_+ \oplus \phi_-, \quad P \mapsto \text{res}(P).
\]

3.31. Proposition. For any \( b \in \mathcal{B} \) the following holds
\[
\begin{align*}
& \left\{ \begin{array}{l}
\sigma_i(G) = (\varepsilon_i(b) + 1) G \otimes (\hat{E} b) + \sum_{b'} f_{b,b'} G \otimes (b'), \\
\varepsilon_i(b') > \varepsilon_i(b) + 1, \quad f_{b,b'} \in v^{1-\varepsilon_i(b')}\mathbb{Z}[v],
\end{array} \right.
\end{align*}
\]
\[
(b) \left\{ \begin{array}{l}
\sigma_i(G) = v^{1-\varepsilon_i(b)} G \otimes (\hat{E} b) + \sum_{b'} e_{b,b'} G \otimes (b'), \\
\varepsilon_i(b') \geq \varepsilon_i(b), \quad e_{b,b'} \in v^{1-\varepsilon_i(b')}\mathbb{Z}[v].
\end{array} \right.
\]

Proof: We prove part (a), the proof for (b) is similar. If \( G \otimes (b) = \phi_\pm \) this is obvious. So we assume that \( G \otimes (b) \) is a \( \mathcal{R}_m \)-module for \( m \geq 1 \). Fix \( \nu \in \mathcal{M} \)
such that \( f_i(\mathcal{G}^{\text{low}}(b)) \) is a \( \mathcal{R}_a \)-module. We'll abbreviate \( 1_{\nu,a} = 1_a \) for \( a \in \{+, -\} \).

Since \( \mathcal{G}^{\text{low}}(b) \) is indecomposable, it fulfills the condition of Lemma 3.16. So there exists \( a \in \{+, -\} \) such that \( 1_{-a} f_i(\mathcal{G}^{\text{low}}(b)) = 0 \). Thus, by Lemma 3.15(c), (d) and Corollary 3.18 we have

\[
f_i(\mathcal{G}^{\text{low}}(b)) = 1_a \text{res ind} f_i(\mathcal{G}^{\text{low}}(b)) = 1_a \text{res ind}(\mathcal{G}^{\text{low}}(b)).
\]

Note that \( \theta b = \text{Ind}(b) \) belongs to \( \theta B \) by Lemma 3.20(b). Hence (3.5) yields

\[
\text{ind}(\mathcal{G}^{\text{low}}(b)) = \theta G^{\text{low}}(\theta b).
\]

We deduce that

\[
f_i(\mathcal{G}^{\text{low}}(b)) = 1_a \text{res} f_i(\theta G^{\text{low}}(\theta b)).
\]

Now, write

\[
f_i(\theta G^{\text{low}}(\theta b)) = \sum_{n \geq 0} f_i(n) u_n \text{ with } e_i u_n = 0,
\]

\[
\tilde{e}_i(u) = \sum_{n \geq 1} f_i(n-1) u_n, \quad \tilde{f}_i(u) = \sum_{n \geq 0} f_i(n+1) u_n.
\]

Let \( \mathcal{R} \subset \mathcal{K} \) be the set of functions which are regular at \( v = 0 \). Let \( \mathcal{L} \) be the \( \mathcal{R} \)-submodule of \( \mathcal{V} \) spanned by the elements \( \tilde{f}_i , \ldots, \tilde{f}_i(\phi_{\pm}) \) with \( l \geq 0, i_1, \ldots, i_l \in I \).

The following is the main result of the paper.
3.33. Theorem. (a) We have

\[ \mathcal{G}_{\text{low}}(b) = \bigoplus_{b \in \mathcal{B}} \mathcal{R} \mathcal{G}^{\text{low}}(b), \quad \tilde{e}_i(\mathcal{G}_{\text{low}}(b)) \subset \mathcal{G}_{\text{low}}, \quad \tilde{f}_i(\mathcal{G}_{\text{low}}(b)) \subset \mathcal{G}_{\text{low}}. \]

(b) The assignment \( b \mapsto \mathcal{G}_{\text{low}}(b) \mod \mathcal{G}_{\text{low}} \) yields a bijection from \( \mathcal{B} \) to the subset of \( \mathcal{G}^\mathcal{L}/\mathcal{G}_{\text{low}} \) consisting of the \( \tilde{f}_1, \ldots, \tilde{f}_i(\phi) \)'s. Further \( \mathcal{G}_{\text{low}}(b) \) is the unique element \( x \in \mathcal{G}^\mathcal{L} \) such that \( x^2 = x \) and \( x = \mathcal{G}_{\text{low}}(b) \mod \mathcal{G}_{\text{low}}. \)

(c) For each \( b, b' \) in \( \mathcal{B} \) let \( E_{i, b, b'}, F_{i, b, b'} \in \mathcal{A} \) be the coefficients of \( \mathcal{G}_{\text{low}}(b') \) in \( e_{\mathcal{R}^i}(\mathcal{G}_{\text{low}}(b)), f_{\mathcal{R}^i}(\mathcal{G}_{\text{low}}(b)) \) respectively. Then we have

\[
E_{i, b, b'}|_{v=1} = [F_i \Psi_{\text{for}}(\mathcal{G}^{\text{cop}}(b')) : \Psi_{\text{for}}(\mathcal{G}^{\text{cop}}(b))],
\]

\[
F_{i, b, b'}|_{v=1} = [E_i \Psi_{\text{for}}(\mathcal{G}^{\text{cop}}(b')) : \Psi_{\text{for}}(\mathcal{G}^{\text{cop}}(b))].
\]

Proof: Part (a) follows from [EK3, thm. 4.1, cor. 4.4], [E, Section 2.3], and Proposition 3.31. The first claim in (b) follows from (a). The second one is obvious. Part (c) follows from Proposition 3.11. More precisely, by duality we can regard \( E_{i, b, b'}, F_{i, b, b'} \) as the coefficients of \( \mathcal{G}^{\text{cop}}(b) \) in \( f_{\mathcal{R}^i}(\mathcal{G}^{\text{cop}}(b')) \) and \( e_{\mathcal{R}^i}(\mathcal{G}^{\text{cop}}(b')) \) respectively. Therefore, by Proposition 3.11 we can regard \( E_{i, b, b'}|_{v=1}, F_{i, b, b'}|_{v=1} \) as the coefficients of \( \Psi_{\text{for}}(\mathcal{G}^{\text{cop}}(b')) \) in \( F_i \Psi_{\text{for}}(\mathcal{G}^{\text{cop}}(b')) \) and \( E_i \Psi_{\text{for}}(\mathcal{G}^{\text{cop}}(b')) \) respectively.

\[ \square \]

References


