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Typing rule-based transformations over topological collections

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Abstract

Pattern-matching programming is an example of a rule-based programming style developed in functional languages. This programming style is intensively used in dialects of ML but is restricted to algebraic data-types.

This restriction limits the field of application. However, as shown by [9] at RULE'02, case-based function definitions can be extended to more general data structures called *topological collections*. We show in this paper that this extension retains the benefits of the typed discipline of the functional languages. More precisely, we show that topological collections and the rule-based definition of functions associated with them fit in a polytypic extension of mini-ML where type inference is still possible.

1 Introduction

Pattern-matching on algebraic data-types (ADT) allows the definition of functions by cases, a restricted form of rule based programming that is both relevant and powerful to specify function acting on ADTs. ML adopted a restricted form of pattern matching, where only the top-level structure of an ADT is matched against the pattern [15]. Examples of more expressive patterns are given, e.g., by the Mathematica language. However, both ML-like language or Mathematica are restricted to the handling of terms, that is, tree-shaped data structures (sets or multisets handled in Mathematica are represented by terms modulo associativity and commutativity).

In [9] and [8] a framework where pattern matching can be expressed uniformly on many different data structures is exhibited. They rely on the notion of topological collection which embeds a neighborhood relation over its elements. The neighborhood relation enables the definition of a general notion of path (a sequential specification of a sub-structure); a pattern is used to specify a path that selects

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an arbitrary sub-collection to be substituted. This leads to a general functional language where the pattern matching is not limited to ADTs.

We show in this paper that the topological collections bring a smooth extension of the Hindley-Milner type system [10][14] with some polytypism [12] and we suggest an extension of the Damas-Milner type inference algorithm that allows to find a type to programs expressed in an extension of mini-ML with topological collections and rule based transformations over them.

Section 2 gives a brief description of the topological collections and their transformation; section 3 gives an overview of types in this framework; the types are investigated in section 4 where the typing rules and the inference algorithm are given; several direct extensions of the language are discussed in section 5 and section 6 concludes this paper.

2 Topological Collections and Transformations

Topological collections are data structures corresponding conceptually to a mapping from a set of positions into a set of values such that there is a neighborhood relation over the positions. Two values of a collection are said to be neighbors if their positions are neighbors. The *sequence* is an example of topological collection where the elements have at most a *left* neighbor and a *right* neighbor. The NEWS *grid* which is a generalization of arrays of dimension 2 is another example where each element has at most four neighbors, considering a Von Neumann neighborhood [13].

The notion of neighborhood is a means to embed in the programming language the spatial locality of computations of programs.

Many other data structures can be seen from the topological point of view. For example the *set* and the *multi-set* (or *bag*) are topological collections where each element is neighbor of each other element (the set of positions of a set, is the set of the elements itself). See [7] for other examples of topological collections.

These data structures come with a rule based style of programming: a rule defines a local transformation by specifying some elements to be matched and the corresponding action. The topological disposition of the matched elements is expressed directly within the pattern of the rule. Thus a collection can be transformed by the simultaneous application of local transformations to non-intersecting matching sub-sets of the collection.

The MGS programming language described in [7] and [8] supplies the topological collections as first-class values and transformations as a means to describe rule based functions over collections. The language we work on in our paper is largely inspired by MGS although some features such as the possibility for a collection to contain elements of different types have been left out.

In the rest of this section we describe the handling of collections via rules in our restriction of MGS.

A *rule* is written $p \Rightarrow e$ where p is the pattern and e is the expression that will replace the instances of p . A *transformation* is a list of rules introduced by the keyword `trans`. The application of a transformation `trans [p1 => e1; p2 => e2]` to a

collection c consists in selecting a number of non-intersecting occurrences of p_1 in c such that there is no further possible occurrence; then replacing the selected parts by the appropriate elements calculated from e_1 ; then selecting a number of non-intersecting occurrences of p_2 and replacing them with the appropriate values.

The pattern can be a single element x or a single element satisfying a condition x/e where e is a boolean expression; it can also be a two elements pattern x, y such that y is a neighbor of x . Here the comma expresses the neighborhood relation and is not intended to express a tuple. The pattern $x/(x = 0), y/(y = 1), z/(z = 2)$ matches three values such that the first is a 0, the second is a 1, the third is a 2, the second is in the neighborhood of the first and the third is in the neighborhood of the second.

The right hand side of the rule is composed of an expression denoting the elements replacing the selected elements. In order to allow the replacement of parts by parts of different size, the value expressed in the right hand side of a rule must be a sequence. The elements of this sequence will substitute the matched elements. Thus we can consider rules replacing sub-parts constituted of a single element with several element, or sub-parts constituted of several elements with one element or even with no element, and so on.

A way of building a sequence is using the empty sequence `empty_seq` and the constructor `::`. The syntactic shortcut $[e]$ can be used to express $e :: \text{empty_seq}$.

2.1 Two examples

The following two examples show two programs acting respectively on sequences and sets.

Sorting a Sequence.

A kind of bubble-sort is immediate:

```
trans[ x, y/(y<x) => y :: x :: empty_seq ; x => [x] ]
```

This two rules transformation has to be applied on the sequence until a fixpoint is reached. The fixpoint is a sorted sequence.

This is not really the bubble-sort because the swapping of elements can happen at arbitrary places; hence an out-of-order element does not necessarily bubble to the top in the characteristic way.

We will see in section 4 that the rule $x => [x]$ is required.

Eratosthene's Sieve on a Set.

The idea is to apply the transformation on the set of the integers between 2 and n . The transformation replaces an x and an y such that x divides y by x . The iteration until a fixpoint of this transformation results in the set of the prime integers less than n .

```
trans [ x, y/(y mod x = 0) => [x] ; x => [x] ]
```

3 Typing the Collections and the Transformations

The type of a topological collection is described by two pieces of information: the type of the elements inside the collection and its organization. The former is called its *content type* and the latter its *topology* (see [11] for an example of separation between the shape and the data). For example, a set of integers and a set of strings do not have the same content type but have the same topology. Collection types will be denoted by $[\tau]\rho$ where τ is the content type and ρ is the topology. Thus a set of strings will have the type `[string]set`.

The usual notion of polymorphism of ML languages is provided on the content type. For example the *cardinal* function that returns the number of elements of a set would have the type $[\alpha]\text{set} \rightarrow \text{int}$ where α is a free type variable since it can be applied to a set irrespectively of the type of its elements. The nature of the content type does not affect the behavior of the cardinal function, therefore the polymorphism is said to be uniform on the content type.

Instead of providing different functions that count the number of elements for each topology, the language provides the function *size* with the type $[\alpha]\theta \rightarrow \text{int}$ where θ is a free topology variable. Functions that accept any kind of topology are said to be *polytypic* [12].

A way of handling collections is using polytypic operators and constant collections: the constructor operator `::` has the type $\alpha \rightarrow [\alpha]\theta \rightarrow [\alpha]\theta$; the destructors `oneof` and `rest` have the type $[\alpha]\theta \rightarrow \alpha$ and $[\alpha]\theta \rightarrow [\alpha]\theta$ and are such that for any collection c , `oneof(c)` and `rest(c)` make a partition of c (see [3]).

The constant collections are `empty_set`, `empty_seq` and so on.

Collections can also be handled with transformations. As seen in the previous section, transformations are functions on collections described by rewriting rules. This kind of function is introduced by the keyword `trans`. For example the function `trans [x=>[x]]` implements the identity over collections and has the type $[\alpha]\theta \rightarrow [\alpha]\theta$. It is the identity because it maps the identity to all the elements of the collection.

As we said, the right hand side of a rule must be a sequence because the pattern matched can be replaced by a different number of elements. On some topologies such as the *grid*, the pattern and the replacement sequence must have the same size. If the sizes are not compatible a *structural error* will be raised at execution time. These structural errors are not captured by our type system. See [7] for more details on the substitution process in the collections.

The *map* function can be expressed as follows:

```
fun f -> trans [ x => [f x] ]
```

and has the type $(\alpha \rightarrow \beta) \rightarrow [\alpha]\theta \rightarrow [\beta]\theta$.

Unlike in the original MGS language, a collection cannot contain elements of different types. We have chosen to set this restriction to allow to build an inference

| | |
|---|----------------------|
| $e ::= id \mid cte \mid (e, e) \mid \mathbf{fun} \ x \rightarrow e$ | $p ::= id$ |
| $\quad \mid ee \mid \mathbf{let} \ id = e \ \mathbf{in} \ e$ | $\quad \mid id/e$ |
| $\quad \mid \mathbf{trans} \ [l]$ | $\quad \mid id, p$ |
| $l ::= id \Rightarrow e$ | $\quad \mid id/e, p$ |
| $\quad \mid p \Rightarrow e ; l$ | |

Fig. 1. Syntax of the language

algorithm in the Damas-Milner style [5]. Allowing such heterogeneous collections would lead to a system with subsumption and union types that would need complex techniques to determine the types of a program.

4 The Language

In this section we first describe the syntax of the studied language. Then we describe the type verification rules and finally we give the type inference algorithm that computes the principal type of a program.

4.1 Syntax

Topological collections are values manipulated with constants, operators, functions and transformations, no new syntactic construction is needed.

For the transformation we have to enrich the syntax of mini-ML [4] as shown in figure 1.

The construction $p \Rightarrow e$ is called a *rule* and a transformation is a syntactic list of rules. In the construction id/e occurring in a pattern, e is called a *guard*.

The last rule of a transformation must be a variable for exhaustiveness purpose. Putting the rule $x \Rightarrow [x]$ in last position of a transformation expresses that all unmatched values are left unmodified. It is not possible to infer a relevant default case for a transformation. For example the rule $x \Rightarrow [x]$ cannot be the default case for a transformation of the type $[string]\theta \rightarrow [int]\theta$. Therefore the default case must be specified explicitly by the programmer. This explains the grammar for the list of rules l which enforces the presence of a last rule of the form $id \Rightarrow e$ matching every remaining element. The expression e in the right hand side provides the appropriate default value.

We will use some operators such as $::$ in an infix position but this syntax can be easily transformed into the one of figure 1. Operators are functional constants of the language.

4.2 The Type System

Types Algebra

We enrich the polymorphic type system of mini-ML with the topological collections. The collection type introduces a new kind of construction in types: the *topology*.

From a type point of view, transformations are just functions that act on topological collections without changing their topology, so no new construct is needed for them in the type algebra.

| | | |
|---------|-------------------------|---|
| Types : | $\tau ::= T$ | base type (<code>int</code> , <code>float</code> , <code>bool</code> , <code>string</code>) |
| | α | type variables |
| | $\tau \rightarrow \tau$ | functions |
| | $\tau \times \tau$ | tuples |
| | $[\tau]\rho$ | collections |

| | | |
|--------------|--------------|---|
| Topologies : | $\rho ::= R$ | base topology (<code>bag</code> , <code>set</code> , <code>seq</code> , <code>grid</code> , ...) |
| | θ | topology variables |

We give in appendix A the definitions of \mathcal{L}_t and \mathcal{L}_r which calculate the type variables and the topology variables occurring in a type.

Type Schemes

A type scheme is a type quantified over some type variables and some topology variables:

$$\sigma ::= \forall[\alpha_1, \dots, \alpha_n][\theta_1, \dots, \theta_m].\tau$$

A type τ is an instance of a type scheme $\sigma = \forall[\alpha_1, \dots, \alpha_n][\theta_1, \dots, \theta_m].\tau'$ and we write $\sigma \leq \tau$ if and only if there are some types τ_1, \dots, τ_n and some topologies ρ_1, \dots, ρ_m such that $\tau = \tau'[\alpha_1 \leftarrow \tau_1, \dots, \alpha_n \leftarrow \tau_n, \theta_1 \leftarrow \rho_1, \dots, \theta_m \leftarrow \rho_m]$.

In the following, an environment is a function from identifiers to type schemes.

TC is the function that gives the type scheme of the constants of the language. For example $TC(\cdot)$ is $\forall[\alpha][\theta].\alpha \rightarrow [\alpha]\theta \rightarrow [\alpha]\theta$.

\mathcal{L}_t and \mathcal{L}_r are extended to type schemes and calculate the free variables of a type scheme, that is the variables occurring in the type scheme which are not bound by the quantifier. For example if σ is $\forall[\alpha_1][\theta_1].[\alpha_1]\theta_1 \rightarrow [\alpha_2]\theta_2$ then $\mathcal{L}_t(\sigma)$ is α_2 and $\mathcal{L}_r(\sigma)$ is θ_2 .

Typing Rules

The typing rules are nearly the same as the Hindley-Milner rules [10][14]. The differences are that a rule has been added for the transformations and that the notions of instance and the *Gen* function have been adapted to the type algebra.

The *Gen* function transforms a type into a type scheme by quantifying the variables that are free in the type and that are not bound in the current environment.

The definition of Gen is the following:

$$Gen(\tau, \Gamma) = \forall[\alpha_1, \dots, \alpha_n][\theta_1, \dots, \theta_m].\tau \text{ with } \{\alpha_1, \dots, \alpha_n\} = \mathcal{L}_t(\tau) \setminus \mathcal{L}_t(\Gamma) \text{ and } \{\theta_1, \dots, \theta_m\} = \mathcal{L}_r(\tau) \setminus \mathcal{L}_r(\Gamma).$$

The typing rules are:

$$\frac{\Gamma(x) \leq \tau}{\Gamma \vdash x : \tau} (var - inst) \quad \frac{TC(c) \leq \tau}{\Gamma \vdash c : \tau} (const - inst)$$

$$\frac{\Gamma \cup \{x : \tau_1\} \vdash e : \tau_2}{\Gamma \vdash (\mathbf{fun} \ x \rightarrow e) : \tau_1 \rightarrow \tau_2} (fun) \quad \frac{\Gamma \vdash e_1 : \tau' \rightarrow \tau \quad \Gamma \vdash e_2 : \tau'}{\Gamma \vdash e_1 \ e_2 : \tau} (app)$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \cup \{x : Gen(\tau_1, \Gamma)\} \vdash e_2 : \tau_2}{\Gamma \vdash (\mathbf{let} \ x = e_1 \ \mathbf{in} \ e_2) : \tau_2} (let)$$

$$\frac{\frac{\Gamma \cup \{x_i^j : \tau\}_{(j \leq m_i)} \cup \{\mathbf{self} : [\tau]\rho\} \vdash e_i : [\tau']seq\}_{(i \leq n)}}{\Gamma \cup \{x_i^j : \tau\}_{(j \leq k)} \cup \{\mathbf{self} : [\tau]\rho\} \vdash e_i^k : \mathbf{bool}\}_{(i \leq n), (k \leq m_i)}}{\Gamma \vdash \mathbf{trans} \ [x_1^1/e_1^1, \dots, x_1^{m_1}/e_1^{m_1} \Rightarrow e_1; \dots; x_n^1/e_n^1, \dots, x_n^{m_n}/e_n^{m_n} \Rightarrow e_n] : [\tau]\rho \rightarrow [\tau']\rho} (trans)$$

In the (trans) rule, k_n is always equal to 1 and e_n^1 is always equal to **true**.

Inside a rule the **self** identifier refers to the collection the transformation is applied on.

The (trans) rule expresses that a transformation has the type $[\tau]\rho \rightarrow [\tau']\rho$ if when you suppose that all the x_i^j have the same type τ and that *self* has the type $[\tau]\rho$ it can be proven that the e_i^j are boolean values and that the e_i have the type $[\tau']seq$.

We can see that if **self** is not used in a transformation, this one will be polytypic since ρ will not be bound to any topology.

The following examples show a type verification on a polytypic transformation and on a non-polytypic one.

Polytypic Example

The following transformation can be proven to be an $[int]\theta \rightarrow [int]\theta$ function for any topology θ .

```
trans [ x, y/x>y => x :: y :: (x-y) :: empty_seq ; x => [x] ]
```

The proof is given in figure 2a where $\Gamma_0 = \{x : int; y : int; self : [int]\theta\}$, $\Gamma_1 = \{x : int; self : [int]\theta\}$ and with the following lemmas:

$$\frac{\Gamma \vdash e_1 : int \quad \Gamma \vdash e_2 : [int]seq}{\Gamma \vdash e_1 :: e_2 : [int]seq} \quad \frac{\Gamma \vdash e : \tau}{\Gamma \vdash [e] : [\tau]seq}$$

Non-Polytypic Example

The operator `is_left` acts as a predicate that returns *true* if the element is at the left extremity of the sequence. Thus it returns *false* if the element has a left

Fig. 2. Two examples of type verification

$$\begin{array}{c}
\frac{\dots \quad \frac{\Gamma_0(\mathbf{x}) \leq \mathit{int}}{\Gamma_0 \vdash \mathbf{x} : \mathit{int}} \quad \frac{\dots}{\Gamma_0 \vdash \mathbf{y} :: (\mathbf{x}-\mathbf{y}) :: \mathit{empty_seq} : [\mathit{int}]seq} \quad \frac{\Gamma_1(\mathbf{x}) \leq \mathit{int}}{\Gamma_1 \vdash \mathbf{x} : \mathit{int}}}{\frac{\Gamma_0 \vdash \mathbf{x} > \mathbf{y} : \mathit{bool} \quad \Gamma_0 \vdash \mathbf{x} :: \mathbf{y} :: (\mathbf{x}-\mathbf{y}) :: \mathit{empty_seq} : [\mathit{int}]seq \quad \Gamma_1 \vdash [\mathbf{x}] : [\mathit{int}]seq}{\vdash \mathit{trans} [\mathbf{x}, \mathbf{y} / \mathbf{x} > \mathbf{y} \Rightarrow \mathbf{x} :: \mathbf{y} :: (\mathbf{x}-\mathbf{y}) :: \mathit{empty_seq} ; \mathbf{x} \Rightarrow [\mathbf{x}]] : [\mathit{int}]\theta \rightarrow [\mathit{int}]\theta}}
\\
\\
\frac{\frac{TC(\mathit{not}) \leq \mathit{bool} \rightarrow \mathit{bool}}{\Gamma_2 \vdash \mathit{not} : \mathit{bool} \rightarrow \mathit{bool}} \quad \frac{\dots}{\Gamma_2 \vdash \mathit{not}(\mathit{is_left} \ \mathbf{x} \ \mathit{self}) : \mathit{bool}} \quad \frac{\dots}{\Gamma_2 \vdash \mathbf{x} + (\mathit{left} \ \mathbf{x} \ \mathit{self}) : \mathit{int}} \quad \frac{\Gamma_2(\mathbf{x}) \leq \mathit{int}}{\Gamma_2 \vdash \mathbf{x} : \mathit{int}}}{\frac{\Gamma_2 \vdash \mathit{not}(\mathit{is_left} \ \mathbf{x} \ \mathit{self}) : \mathit{bool} \quad \Gamma_2 \vdash [\mathbf{x} + (\mathit{left} \ \mathbf{x} \ \mathit{self})] : [\mathit{int}]seq \quad \Gamma_2 \vdash [\mathbf{x}] : [\mathit{int}]seq}{\vdash \mathit{trans} [\mathbf{x} / (\mathit{not}(\mathit{is_left} \ \mathbf{x} \ \mathit{self})) \Rightarrow [\mathbf{x} + (\mathit{left} \ \mathbf{x} \ \mathit{self})] ; \mathbf{x} \Rightarrow [\mathbf{x}]] : [\mathit{int}]seq \rightarrow [\mathit{int}]seq}}
\end{array}$$

(b)

neighbor. It can be used only within a transformation² and takes two arguments: the first is a pattern variable and the second is a collection. Similarly, the operator `left` takes a pattern variable x and a sequence s and returns the left neighbor of x in s .

Let us consider the following transformation:

```
trans [ x/(not (is_left x self)) => [x+(left x self)] ; x=>[x] ]
```

This transformation does not have the same effect as the following one:

```
trans [ l, x => (l :: l+x :: empty_seq) ; x=>[x] ]
```

because in the former, every element x of the sequence except the leftmost one will be replaced by the sum of itself and its left neighbor whereas in the latter, the l element will be replaced by itself and thus will not be increased. For example the former transformation applied to the sequence $(1::2::3::4::\text{empty_seq})$ results in $(1::3::5::7::\text{empty_seq})$ whereas the application of the latter transformation to the same sequence would result in $(1::4::3::7::\text{empty_seq})$.

The figure 2b where $\Gamma_2 = \{x : \text{int}; \text{self} : [\text{int}]seq\}$ proves that the first transformation has the type $[\text{int}]seq \rightarrow [\text{int}]seq$.

This transformation cannot be proven to have the type $[\text{int}]\rho \rightarrow [\text{int}]\rho$ if $\rho \neq seq$ because `left` and `is_left` act exclusively on sequences.

4.3 Type Inference

The typing rules given in section 4.2 are a means to verify that a program has a given type but this type is a parameter of the verification procedure. We now give the equivalent of the Damas-Milner type inference that enables the full automated type verification since it computes the principal type of a program. The resulting type is said to be principal because every type that can fit the program is an instance of this type.

The type inference algorithm is given after the unification procedure.

Unification

Unifying two types τ_1 and τ_2 consists in finding a substitution φ over the free variables of τ_1 and τ_2 called the unifier such that $\varphi(\tau_1) = \varphi(\tau_2)$.

A substitution is a most general unifier (mgu) for two types τ_1 and τ_2 if for any unifier φ_1 of τ_1 and τ_2 , there is a substitution φ_2 such that $\varphi = \varphi_2 \circ \varphi_1$.

We give the `mgu` function that computes the most general unifier of a set of pairs of types denoted by $\tau_1 = \tau_2$. This function is necessary to the type inference procedure. If `mgu` fails then there is no unifier for the given types.

The difference between our `mgu` and Damas and Milner's original `mgu` is the addition of the case for the collection types. Two collection types are unified by unifying their content types and their topologies. The substitution doing this unifi-

² The `is_left` operator is only available in transformations, where the identifiers introduced by the pattern are bound to a position in the collection. Allowing only such identifiers to be arguments of `is_left` allows to remove any ambiguity on the position denoted in the sequence, even if the position contains a value occurring several times.

cation is found as $\varphi_1 \circ \varphi_2$ where φ_2 unifies the topologies and φ_1 unifies the content types. The computation of φ_2 is made by the dedicated mgu_r function. This function fails when the two topologies are different base topologies since they cannot be unified. The substitution φ_2 is applied to the content types before computing φ_1 with mgu .

The standard cases of the definition of mgu are:

$$\begin{aligned} \text{mgu}(\emptyset) &= [] \\ \text{mgu}(\{\tau = \tau\} \cup C) &= \text{mgu}(C) \\ \text{mgu}(\{\alpha = \tau\} \cup C) \text{ (if } \alpha \text{ is not free in } \tau) &= \text{let } \varphi = [\alpha \leftarrow \tau] \text{ in } \text{mgu}(\varphi(C)) \circ \varphi \\ \text{mgu}(\{\tau = \alpha\} \cup C) \text{ (if } \alpha \text{ is not free in } \tau) &= \text{let } \varphi = [\alpha \leftarrow \tau] \text{ in } \text{mgu}(\varphi(C)) \circ \varphi \\ \text{mgu}(\{\tau_1 \rightarrow \tau_2 = \tau'_1 \rightarrow \tau'_2\} \cup C) &= \text{mgu}(\{\tau_1 = \tau'_1 ; \tau_2 = \tau'_2\} \cup C) \\ \text{mgu}(\{\tau_1 \times \tau_2 = \tau'_1 \times \tau'_2\} \cup C) &= \text{mgu}(\{\tau_1 = \tau'_1 ; \tau_2 = \tau'_2\} \cup C) \end{aligned}$$

The new case for the collections is:

$$\text{mgu}(\{[\tau]\rho = [\tau']\rho'\} \cup C) = \text{let } \varphi = \text{mgu}_r(\rho = \rho') \text{ in } \text{mgu}(\varphi(\{\tau = \tau'\} \cup C)) \circ \varphi$$

The unification of topologies is defined by:

$$\begin{aligned} \text{mgu}_r(\rho = \rho) &= [] \\ \text{mgu}_r(\theta = \rho) &= [\theta \leftarrow \rho] \\ \text{mgu}_r(\rho = \theta) &= [\theta \leftarrow \rho] \end{aligned}$$

Type Inference

The type reconstruction algorithm is nearly the same as the Damas-Milner one. The differences are that it uses specialized versions of mgu and Gen functions and that there is a new case for the transformations. It is described here in an imperative way: φ is the current substitution and V_t and V_r are sets of free type variables and topology variables.

The algorithm is given in figure 3.

The case for the transformations consists in unifying the types of all the pattern variables and unifying the types of the right hand side rules together and with a sequence collection type. These unifications have to be made with respect to the guards that are boolean values.

If W succeeds it computes the most general type of the program analyzed and this one can be run without type error. If it fails because of an mgu or an mgu_r failure then the program is ill-typed and might lead to a type error at execution time.

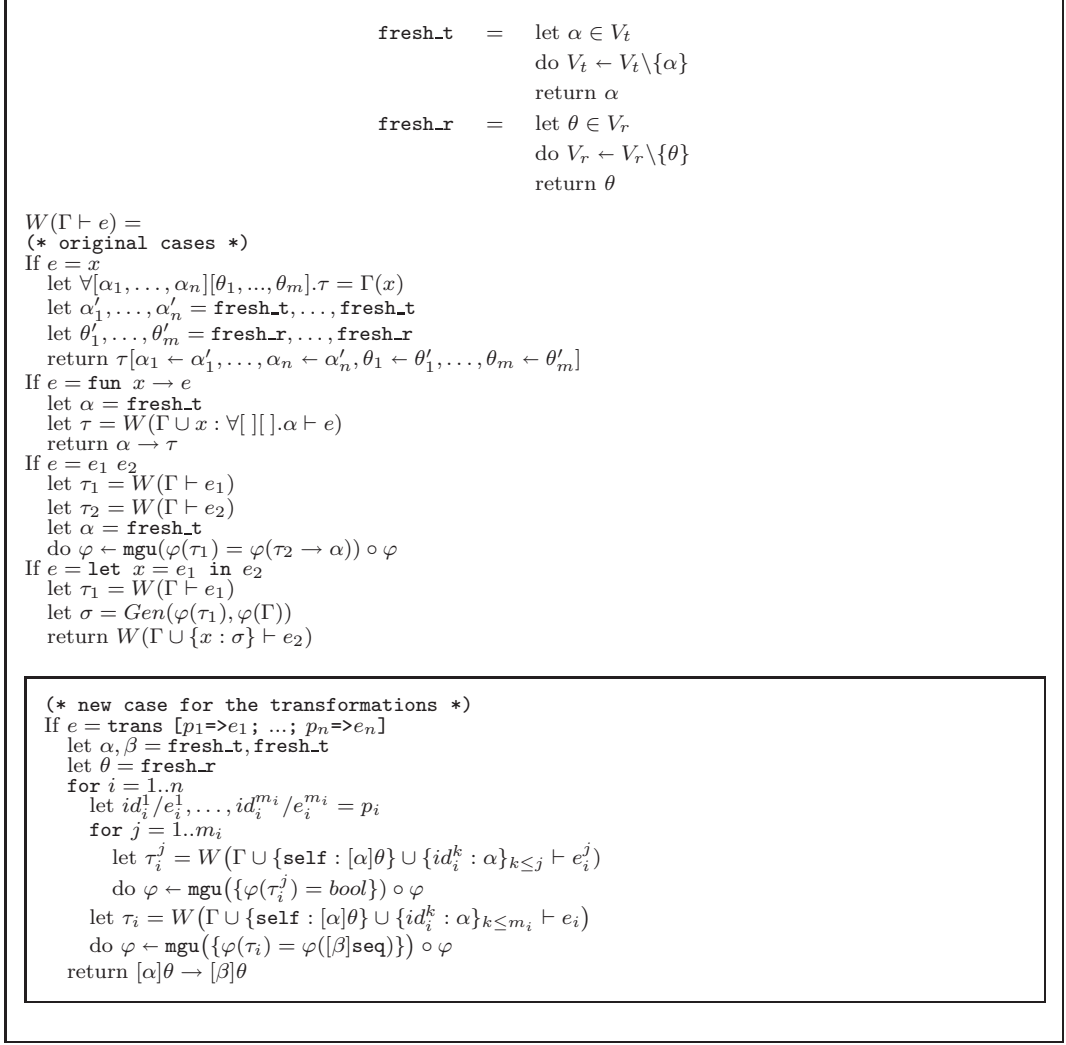


Fig. 3. Type inference algorithm

5 Extensions

5.1 Repetition in a Pattern

The *star* $*$ expressing an arbitrary repetition of a sub-pattern during the matching process has been introduced in [9]. The pattern $x/(x=0)$, $* \text{ as } y$, $z/(z=0)$ for example can match an arbitrary subcollection such that it contains two 0 and that there is a *path* between these 0. This means that one can reach the second 0 from the first one only by going from an element to one of its neighbors repetitively.

To take the star into account we modify the syntax of the patterns as follows:

$$p ::= q \mid q, p$$

$$q ::= id \mid * \text{ as } id$$

where q stands for elementary patterns.

We have not kept the guards in the elementary patterns in order to keep the

formulas readable but their addition does not lead to new problems.

The elements matched by the star are named and can be referred to as a sequence.

The star could have been considered as a repetition of a subpattern as in $(x, y/x=y)^*$ but we have chosen to restrict the star to the repetition of single elements for the sake of simplicity.

Before giving the new typing rule, we introduce a function which gives the type binding corresponding to an elementary pattern: $b(q, \tau)$ is such that $b(x, \tau) = (x : \tau)$ and $b(* \text{ as } x, \tau) = (x : [\tau]seq)$. This function is used in the *trans* typing rule which is modified as follows:

$$\frac{\{\Gamma \cup \{b(q_i^j, \tau)\}_{j \leq m_i} \cup \{\text{self} : [\tau]\rho\} \vdash e_i : [\tau']seq\}_{i \leq n}}{\Gamma \vdash \text{trans } [q_1^1, \dots, q_1^{m_1} \Rightarrow e_1; \dots; q_n^1, \dots, q_n^{m_n} \Rightarrow e_n] : [\tau]\rho \rightarrow [\tau']\rho} \text{ (trans')}$$

5.2 Directions in Patterns

In section 4.2 we saw the operator `left` that returns the left neighbor of an element in a sequence. In the framework of topological collections, a topology can supply several neighborhood operators. For example `left` and `right` are the neighborhood operators of the sequence and `north` and `east` are neighborhood operators of the grid. Neighborhood operators are also called *directions*.

A direction can be used to refine the patterns: the commas of the pattern can be substituted by a direction to restrict the accepted neighbors for the rest of the pattern. The substituting direction is surrounded with the symbols `|` and `>` to sketch a kind of arrow.

For example if d is a direction we can use the pattern $x \ |d> \ y$ which is a shortcut³ for $x, y/y=(d \ x \ \text{self})$. However, the pattern $x \ |d> \ y$ allows faster research of the instances of the pattern in the collection than $x, y/y=(d \ x \ \text{self})$.

The pattern $x \ |d> \ y$ can be typed as $x, y/y=(d \ x \ \text{self})$.

The Bead-Sort Example

The bead-sort is an original way of sorting positive integers presented by [2]. The sorting algorithm considers a column of numbers written in unary basis. Figure 4a shows the numbers 3, 2, 4 and 2 where the beads stand for the digits. The sorting is done by letting the beads fall down as shown on figure 4b.

The problem can be represented on a grid of booleans where *true* stands for a digit and *false* for the absence of digit as shown on figure 4c. The bead-sort is achieved by iterating the application of the following transformation until a fixpoint is reached:

```
trans [ x/x=false |north> y/y=true => y::x::empty_seq ; x=>[x] ]
```

The first rule of this transformation is expressed as

³ The expression $y=(d \ x \ \text{self})$ in a guard where y is a pattern variable and d is a direction tests that the values denoted are the same and that their positions in the collection are the same. See the MGS manual [6] for more details.

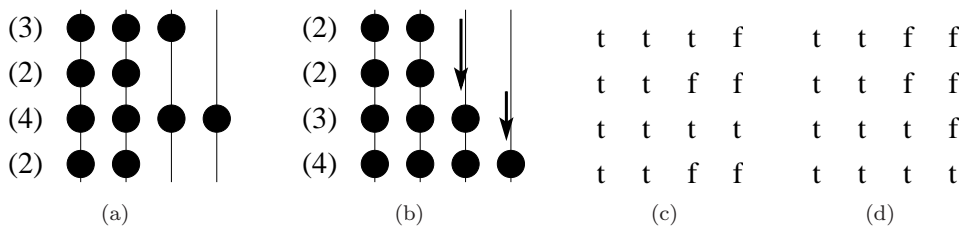


Fig. 4. The Bead-Sort

$$x/x=false , y/(y=true \ \&\& \ y=north \ x \ self) \Rightarrow y::x::empty_seq$$

in order to fit the type system. The result of W on this transformation is $[bool]grid \rightarrow [bool]grid$.

5.3 Strategies

As far as the rules application strategy guarantees that every element of the collection is matched (this is always possible since the last rule always matches) the type system is not affected.

For instance, the MGS language provides several strategies such as higher priority given to the first rules or random application of the rules.

6 Conclusion

Including the topological collections and pattern matching programming on these structures in the ML framework allows to bring together a powerful programming language with a rule programming framework common to several other languages.

Our algorithm has been tested on MGS programs and has been included in a prototype MGS compiler in order to achieve type-oriented optimizations on the produced code. We believe that the best pattern matching algorithms would be wasted on a dynamically typed language and thus a type inference algorithm is an important step in the development of an efficient compiler for rule based transformations.

However some restrictions on the MGS language had to be done in order to keep the simplicity of the Damas-Milner algorithm. We are currently working on a type inference system with union types [1] to account for heterogeneous collections supplied by the MGS language.

Finally, we said that an error could occur when a transformation tries to replace a subpart by a part of different shape on topologies as the grid which cannot get out of shape. Such errors are not type errors but some of them could be detected statically with a specific type based analysis. Some research such as [11] manage with this kind of error but the concerned languages do not provide the flexibility of the rule based transformations proposed here.

A Free Variables

The free variables of a type are the variables occurring in that type. \mathcal{L}_t computes the free type variables whereas \mathcal{L}_r computes the free topology variables.

$$\begin{array}{ll}
\mathcal{L}_t(T) & = \emptyset & \mathcal{L}_r(T) & = \emptyset \\
\mathcal{L}_t(\alpha) & = \{\alpha\} & \mathcal{L}_r(\alpha) & = \emptyset \\
\mathcal{L}_t(\tau_1 \rightarrow \tau_2) & = \mathcal{L}_t(\tau_1) \cup \mathcal{L}_t(\tau_2) & \mathcal{L}_r(\tau_1 \rightarrow \tau_2) & = \mathcal{L}_r(\tau_1) \cup \mathcal{L}_r(\tau_2) \\
\mathcal{L}_t(\tau_1 \times \tau_2) & = \mathcal{L}_t(\tau_1) \cup \mathcal{L}_t(\tau_2) & \mathcal{L}_r(\tau_1 \times \tau_2) & = \mathcal{L}_r(\tau_1) \cup \mathcal{L}_r(\tau_2) \\
\mathcal{L}_t([\tau]\rho) & = \mathcal{L}_t(\tau) & \mathcal{L}_r([\tau]\theta) & = \{\theta\} \cup \mathcal{L}_r(\tau) \\
& & \mathcal{L}_r([\tau]R) & = \mathcal{L}_r(\tau)
\end{array}$$

The free variables of a type scheme are the non-quantified variables occurring in it:

$$\begin{array}{l}
\mathcal{L}_t(\forall[\alpha_1, \dots, \alpha_n], [\theta_1, \dots, \theta_m].\tau) = \mathcal{L}_t(\tau) \setminus \{\alpha_1, \dots, \alpha_n\} \\
\mathcal{L}_r(\forall[\alpha_1, \dots, \alpha_n], [\theta_1, \dots, \theta_m].\tau) = \mathcal{L}_r(\tau) \setminus \{\theta_1, \dots, \theta_m\}
\end{array}$$

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