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# On $s$ -convex extrema for $t$ -monotone distributions, with applications

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## Abstract

Our purpose in this paper is to provide a new and unifying approach to construct  $s$ -convex stochastic extrema for distributions that are known to be  $t$ -monotone. A key point is that such a monotonicity property on the distributions can be removed by having recourse to the stationary-excess operator and its iterates. Both discrete and continuous cases are investigated. Several extremal distributions under monotonicity conditions are obtained in explicit form. Their use is illustrated with some applications to insurance.

*MSC:* primary 60E15, 62P05; secondary 60E10

*Keywords:* Insurance risks;  $s$ -convex stochastic orders; Extremal distributions;  $t$ -monotone distributions; Stationary-excess operator; Discrete and continuous versions.

## 1 Introduction

The problem of stochastic extrema (with respect to some ordering) inside a particular set of distributions has a wide field of applications. This is especially true in insurance and finance when the extremal risks represent the less and more dangerous risks (in a specific sense) and can thus provide useful information on the possible risk scenarios. Applications arise, for instance, in life insurance, ruin theory and portfolio analysis.

The present work falls in this framework. In short, our purpose is to derive *convex-type* extremal distributions among *nonincreasing-type* probability distributions with bounded non-negative support. Both discrete and continuous distributions will be examined, with special emphasis on the less standard discrete case.

The class of stochastic orders under consideration is the class of  $s$ -convex orders, where  $s \in \mathbb{N}_0$  ( $\mathbb{N}_0 = \{1, 2, \dots\}$ ). For  $s = 1$ , this order corresponds to the classical stochastic dominance. For  $s = 2$ , it is the well-known convex order and corresponds, in actuarial sciences, to the stop-loss order with fixed mean. For an arbitrary  $s$ , the  $s$ -convex order compares the  $s$ -th right-tail distribution functions of random variables that have the same first  $s - 1$  moments. For discrete distributions, this class of orders was studied in Lefèvre and Picard (1993), Fishburn and Lavalley (1995), Lefèvre and Utev (1996), Denuit and Lefèvre (1997) and Denuit et al. (1999a), (1999c), among others. The case, more traditional, of real-valued random variables

was investigated by Rolski (1976), Levy (1992), Denuit et al. (1998), (1999b) and in many other works. For a theory on stochastic orders, the reader is referred to the books by Goovaerts et al. (1990), Kaas et al. (1994), Ross (1996), Müller and Stoyan (2002) and Shaked and Shanthikumar (2007).

The  $s$ -convex optimization problem corresponds to a traditional moment problem; see, e.g., Hoeffding (1955), Karlin and Studden (1966), Kemperman (1968), Utev (1985), Prékopa (1990) and Hürlimann (1999). In the discrete case, explicit extremal distributions are known for  $s = 1, 2, 3, 4$  and were derived by Denuit and Lefèvre (1997), Denuit et al. (1999c) and Courtois et al. (2006). In the real case, explicit extrema were obtained by Jansen et al. (1986), De Vylder (1996) and Denuit et al. (1998), (1999b), *inter alia*.

As already pointed out in several papers, the  $s$ -convex extrema can be improved when the distributions of interest are known to be nonincreasing (and, more generally, unimodal); see, e.g., Denuit et al. (1998), (1999b), (1999c). Such a distributional assumption is often met in actuarial theory with the number or the severity of risks in insurance or finance.

Motivated by these situations, we want to investigate the  $s$ -convex optimization problem within the subset of  $t$ -monotone probability distributions, where  $t \in \mathbb{N}_0$ . For  $t = 1$ , this property corresponds to the usual nonincreasingness of the distributions, and for  $t = 2$ , to both nonincreasingness and convexity of the distributions. For an arbitrary  $t$ , that property means that in the discrete (resp. continuous) case, the first  $t$  differences of the probability mass function (resp. derivatives of the density function) are assumed to be of alternating signs with the sequence  $\{-, +, \dots, (-1)^t\}$ .

Our key result states that a problem of  $s$ -convex optimization among  $t$ -monotone distributions is (almost) equivalent, up to a remarkable transform, to a problem of  $s + t$ -convex optimization without monotonicity constraints. To establish this equivalence, we propose a new and unifying approach that has recourse to the stationary-excess operator and its iterates.

More precisely, the paper is organized as follows. In Section 2, we present the stationary-excess operator, under its usual definition and in a non-standard version specific to discrete distributions. In Section 3, we show that this operator essentially transforms any  $s + 1$ -convex order to an  $s$ -convex order for nonincreasing distributions. The result is extended in Section 4 where the  $t$ -th iterate of the operator allows us to transform an  $s + t$ -convex order in an  $s$ -convex order for  $t$ -monotone distributions. In Section 5, we use this property, together with known convex extrema (recalled in the Appendix), to derive several explicit convex extrema for nonincreasing, possibly convex, distributions. Finally, the interest of these extrema is illustrated in Section 6 with some applications to insurance.

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## 2 Stationary-excess operator

The stationary-excess operator, denoted by  $H$ , has an important role in renewal theory and survival analysis (Cox (1972)). In its classical version (see Definition 2.3 below), this operator is built for any non-negative random variable  $X$  with distribution function  $F_X$  and mean

$E(X) > 0$ . It is worth recalling that in a renewal process, if an interval between points is of distribution function  $F_X$ , then the associated stationary-excess mapping  $H(F_X)$  gives the distribution function of the interval to the next point from an arbitrary time in equilibrium. Hereafter, we start by introducing a similar operator  $H$  that is specific to discrete distributions.

## 2.1 Discrete version

Let us assume that  $X$  is a *discrete non-negative* random variable with probability mass function  $P_X = \{P(X = j), j \in \mathbb{N}\}$  and mean  $E(X) > 0$  ( $\mathbb{N} = \{0, 1, \dots\}$ ). Obviously, the classical stationary-excess operator may be applied here too. As explained by Whitt (1985), however, it is more appropriate to work with a discrete version that is directly applicable to discrete-time renewal processes. We propose to adopt the following definition.

**Definition 2.1** *A discrete stationary-excess operator  $H$  maps any such random variable  $X$  to an associated discrete non-negative random variable  $X_H$  whose probability mass function  $H(P_X)$  is defined by*

$$H(P_X)(j) \equiv P(X_H = j) = \frac{P(X \geq j+1)}{E(X)}, \quad j \in \mathbb{N}. \quad (2.1)$$

Let us notice that a similar, but different, operator was investigated by Whitt (1985) for a *discrete positive* random variable  $X$  (i.e. valued on  $\mathbb{N}_0$ ). Specifically, the associated random variable  $X_H$  is of probability mass function defined by

$$P(X_H = j) = \frac{P(X \geq j)}{E(X)}, \quad j \in \mathbb{N}_0, \quad (2.2)$$

which thus slightly differs from our definition (2.1).

Contrary to (2.2),  $H$  defined by (2.1) does not yield a one-to-one correspondence on the set of probability measures on  $\mathbb{N}$ . Indeed, it is directly checked that  $X_H$  and  $(\nu X)_H$  are equidistributed if  $\nu$  is an indicator independent of  $X$ . Nevertheless, when  $E(X)$  is fixed,  $H$  gives a one-to-one correspondence since

$$P(X = 0) = 1 - E(X) P(X_H = 0), \quad \text{and} \quad (2.3)$$

$$P(X = j) = E(X) [P(X_H = j-1) - P(X_H = j)], \quad j = 1, 2, \dots \quad (2.4)$$

The definition (2.1) has the advantage to lead to a simple relationship between the binomial moments of  $X_H$  and  $X$ . We make the convention  $\binom{x}{y}$  if  $x < y$ .

**Lemma 2.2**

$$E\binom{X_H}{i} = \frac{1}{E(X)} E\binom{X}{i+1}, \quad i \in \mathbb{N}. \quad (2.5)$$

*Proof.* To show this, we consider the iterated right-tail distribution functions of  $X$ :

$$\bar{F}_0(X, j) = P(X = j), \quad j \in \mathbb{N}, \quad (2.6)$$

and for  $i = 0, 1, \dots$ ,

$$\bar{F}_{i+1}(X, j) = \sum_{k=j}^{\infty} \bar{F}_i(X, k), \quad j \in \mathbb{N}. \quad (2.7)$$

As proved in Lefèvre and Utev (1996), an equivalent expression for (2.6), (2.7) is

$$\bar{F}_{i+1}(X, j) = E \binom{X - j + i}{i}, \quad i, j \in \mathbb{N}. \quad (2.8)$$

Let us turn to the iterated right-tail distribution functions of  $X_H$ . By (2.1), we have

$$\bar{F}_0(X_H, j) = P(X_H = j) = \frac{\bar{F}_1(X, j+1)}{E(X)}, \quad j \in \mathbb{N}.$$

Arguing by induction, we then find from (2.7) that

$$\bar{F}_{i+1}(X_H, j) = \frac{\bar{F}_{i+2}(X, j+1)}{E(X)}, \quad i, j \in \mathbb{N}. \quad (2.9)$$

Finally, taking  $j = i$  in (2.8) and (2.9), we obtain the identity (2.5).  $\diamond$

## 2.2 Continuous version

Let  $X$  be any *non-negative* random variable with distribution function  $F_X$  and mean  $E(X) > 0$ . The usual stationary-exces operator is defined as follows (Cox (1972)). For easiness, it will be named *continuous* subsequently.

**Definition 2.3** *A continuous stationary-excess operator  $H$  maps any such random variable  $X$  to an associated non-negative random variable  $X_H$  whose distribution function  $H(F_X)$  is defined by*

$$H(F_X)(x) \equiv P(X_H \leq x) = \frac{1}{E(X)} \int_0^x [1 - F_X(y)] dy, \quad x \geq 0. \quad (2.10)$$

An equivalent expression for (2.10) is

$$P(X_H > x) = \frac{1}{E(X)} \int_x^{\infty} P(X > y) dy, \quad x \geq 0. \quad (2.11)$$

This definition of  $H$  guarantees a one-to-one correspondence. Indeed, denoting the density function of  $X_H$  by  $d_{X_H}$ , one has

$$P(X > x) = E(X) d_{X_H}(x), \quad x \geq 0, \quad (2.12)$$

and  $E(X)$  then follows as a consequence.

The moments of  $X_H$  and  $X$  are linked through a simple identity, known but rederived below for comparison with Lemma 2.2.

**Lemma 2.4**

$$E(X_H^i) = \frac{E(X^{i+1})}{(i+1)E(X)}, \quad i \in \mathbb{N}. \quad (2.13)$$

*Proof.* The result is related to the  $i$ -th stop-loss transform of  $X$ , defined as

$$\Pi^{(i)}(X, x) = E[(X - x)_+^i], \quad x \geq 0; \quad (2.14)$$

see, e.g., Cheng and Pai (2003). Indeed, considering again the iterated right-tail distribution functions of  $X$ , i.e.

$$\bar{F}_1(X, x) = P(X > x), \quad x \geq 0, \quad (2.15)$$

and for  $i = 1, 2, \dots$ ,

$$\bar{F}_{i+1}(X, x) = \int_x^\infty \bar{F}_i(X, y) dy, \quad x \geq 0, \quad (2.16)$$

one can directly see that

$$\bar{F}_{i+1}(X, x) = \frac{\Pi^{(i)}(X, x)}{i!}, \quad x \geq 0, \quad i \in \mathbb{N}. \quad (2.17)$$

Now, for the transformed random variable  $X_H$ , (2.15) and (2.11) yield

$$\begin{aligned} \bar{F}_1(X_H, x) = P(X_H > x) &= \frac{1}{E(X)} \int_x^\infty [1 - F_X(y)] dy \\ &= \frac{\bar{F}_2(X, x)}{E(X)}, \quad x \geq 0, \end{aligned}$$

so that by (2.16) and using induction, we get

$$\bar{F}_{i+1}(X_H, x) = \frac{\bar{F}_{i+2}(X, x)}{E(X)}, \quad x \geq 0, \quad i \in \mathbb{N}. \quad (2.18)$$

From (2.18) and (2.17), (2.14), we deduce that

$$E[(X_H - x)_+^i] = \frac{E[(X - x)_+^{i+1}]}{(i+1)E(X)}, \quad x \geq 0,$$

which yields (2.13) when  $x = 0$ .  $\diamond$

### 3 $H$ -transform and convex orderings

To begin with, we give a brief presentation of the  $s$ -convex stochastic orders and the associated moment spaces. Applying the stationary-excess operator, we then identify the image of these moment spaces and the corresponding stochastic orders. Both discrete and continuous cases will be examined.

### 3.1 Discrete problem

For any  $s \in \mathbb{N}_0$ , let  $\mathcal{F}_s$  denote the set of  $s$ -convex real functions  $f$  defined on  $\mathbb{N}$ , i.e.

$$\mathcal{F}_s = \{f : \Delta^s f(j) \geq 0, j \in \mathbb{N}\}, \quad (3.1)$$

$\Delta$  being the usual forward difference operator [ $\Delta f(j) = f(j+1) - f(j)$ ,  $j \in \mathbb{N}$ ], and  $\Delta^s$  the  $s$ -th iterate of  $\Delta$ .

**Definition 3.1** *Given any two probability mass functions  $P_1$  and  $P_2$  on  $\mathbb{N}$ , one says that  $P_1$  is smaller than  $P_2$  in the  $s$ -convex sense, written  $P_1 \preceq_s P_2$ , when*

$$\langle f, P_1 \rangle \equiv E_{P_1}(f) \leq E_{P_2}(f) \equiv \langle f, P_2 \rangle \quad \text{for all functions } f \in \mathcal{F}_s. \quad (3.2)$$

As the functions  $f(j) = j^i$  and  $f(j) = -j^i$ ,  $j \in \mathbb{N}$ , belong to  $\mathcal{F}_s$  for all  $i = 1, \dots, s-1$ , the ordering  $P_1 \preceq_s P_2$  necessarily implies that  $P_1$  and  $P_2$  have the same  $s-1$  first moments. In fact, if  $X_1$  and  $X_2$  are two random variables with probability mass functions  $P_1$  and  $P_2$ , then an equivalent characterization of (3.2), written as  $X_1 \preceq_s X_2$ , is that

$$\begin{cases} E(X_1^i) = E(X_2^i), & i = 1, \dots, s-1, \text{ and} \\ \bar{F}_s(X_1, j) \leq \bar{F}_s(X_2, j), & j \geq s. \end{cases} \quad (3.3)$$

$$(3.4)$$

Now, let us consider the set of probability mass functions on  $\mathbb{N}$  with prescribed first  $s$  moments, where  $s \in \mathbb{N}_0$ . For the moment, it will be more convenient to work on the basis of the combinatorial moments. So, we introduce a set

$$\mathcal{B}_{s+1} = \left\{ P_X : E\binom{X}{i} = c_i, i = 1, \dots, s \right\}, \quad (3.5)$$

where  $c_1, \dots, c_s$  represent  $s$  given (admissible) constants. By (3.3), one knows that the distributions in the set  $\mathcal{B}_{s+1}$  are susceptible to be ordered with respect to  $\preceq_{s+1}$ , the  $(s+1)$ -convex order.

Let us take the discrete stationary-excess operator  $H$  defined by (2.1). We have seen before that  $H$  yields a one-to-one correspondence on the set of distributions with fixed mean. Thus, this property holds true on the previous set  $\mathcal{B}_{s+1}$ . Let us then apply the operator  $H$  to  $\mathcal{B}_{s+1}$ , i.e. consider the set  $H(\mathcal{B}_{s+1})$ . On this set, it is natural to define a transformed order  $\preceq_H$  as follows.

**Definition 3.2** *Given any two probability mass functions  $Q_1$  and  $Q_2$  in  $H(\mathcal{B}_{s+1})$ , one says that  $Q_1 \preceq_H Q_2$  when*

$$H^{-1}(Q_1) \equiv P_1 \preceq_{s+1} P_2 \equiv H^{-1}(Q_2). \quad (3.6)$$

We are now in a position to identify the set  $H(\mathcal{B}_{s+1})$  and the transformed order  $\preceq_H$ . Let  $Q_Y = \{Q_Y(j), j \in \mathbb{N}\}$  denote the probability mass function of a discrete random variable  $Y$ .

### Proposition 3.3

$$H(\mathcal{B}_{s+1}) = \left\{ Q_Y : E\binom{Y}{i} = \frac{c_{i+1}}{c_1}, i = 1, \dots, s-1, \text{ and} \right. \quad (3.7)$$

$$\left. \Delta Q_Y(j) \leq 0, j \in \mathbb{N}, \text{ with } c_1 Q_Y(0) \leq 1 \right\}, \quad (3.8)$$

and the order  $\preceq_H$  corresponds to the  $s$ -convex order  $\preceq_s$ .

*Proof.* Let us first obtain  $H(\mathcal{B}_{s+1})$ . From (2.5) and (3.5), we find that the  $s-1$  first binomial moments of  $Y$  are provided by the formula (3.7). The remaining restrictions stipulated in (3.8) are straightforward from the relations (2.4) and (2.3).

Let us identify  $\preceq_H$ . By virtue of (3.2) and (3.6),  $Q_1 \preceq_H Q_2$  when

$$\langle f, H^{-1}(Q_1) \rangle \leq \langle f, H^{-1}(Q_2) \rangle \text{ for all } f \in \mathcal{F}_{s+1}. \quad (3.9)$$

Given any  $Q = Q_Y$  in  $H(\mathcal{B}_{s+1})$ , the expansion of  $\langle f, H^{-1}(Q) \rangle$  yields, after insertion of (2.3), (2.4),

$$\begin{aligned} \langle f, H^{-1}(Q) \rangle &= \sum_{j=0}^{\infty} f(j) [H^{-1}(Q)](j) \\ &= f(0) [1 - c_1 P(Y=0)] + c_1 \sum_{j=1}^{\infty} f(j) [P(Y=j) - P(Y=j-1)] \\ &= f(0) + c_1 \sum_{j=1}^{\infty} P(Y=j) [f(j+1) - f(j)]. \end{aligned}$$

Thus,  $\langle f, H^{-1}(Q) \rangle$  can be rewritten as

$$\langle f, H^{-1}(Q) \rangle = \langle (\delta_0 + c_1 \Delta)(f), Q \rangle, \quad (3.10)$$

where  $\delta_0$  is the operator such that  $\delta_0(f) = f(0)$ . In fact, the equivalence

$$\langle f, H^{-1}(Q) \rangle = \langle [(H^{-1})^*](f), Q \rangle \quad (3.11)$$

holds in general if  $(H^{-1})^*$  is the conjugate operator of  $H^{-1}$ ; thus, in our case,

$$(H^{-1})^* = \delta_0 + c_1 \Delta. \quad (3.12)$$

Substituting (3.10) in (3.9), we find that  $Q_1 \preceq_H Q_2$  when

$$\langle \Delta(f), Q_1 \rangle \leq \langle \Delta(f), Q_2 \rangle \text{ for all } f \in \mathcal{F}_{s+1}. \quad (3.13)$$

As  $\Delta^{s+1} = \Delta^s(\Delta)$ , the condition (3.13) means by (3.2) that  $Q_1 \preceq_s Q_2$ , hence the orders  $\preceq_H$  and  $\preceq_s$  are quite identical.  $\diamond$

Note that the image set  $H(\mathcal{B}_{s+1})$  contains  $s-1$  moment relations and the nonincreasingness property, as well as an additional technical constraint. Reciprocally, returning to  $\mathcal{B}_{s+1}$  through the operator  $H^{-1}$  has the effect of removing the nonincreasingness condition. To put it speaking, if for instance, all the combinatorial moments are equal to 1, then Proposition 3.3 yields the following simple result.



**Corollary 3.4** *The  $s$ -convex extremal distributions in the set*

$$\left\{ Q_Y : E\binom{Y}{i} = 1, \ i = 1, \dots, s-1, \quad \text{and} \quad \Delta Q_Y(j) \leq 0, \ j \in \mathbb{N} \right\} \quad (3.14)$$

*correspond to the  $H$ -transform of the  $s+1$ -convex extremal distributions in the set*

$$\left\{ P_X : E\binom{X}{i} = 1, \ i = 1, \dots, s \right\}. \quad (3.15)$$

### 3.2 Continuous problem

A similar approach is applicable to random variables valued in  $\mathbb{R}^+$ . Given any  $s \in \mathbb{N}_0$ , the set  $\mathcal{F}_s$  of  $s$ -convex real functions  $f$  is defined as

$$\mathcal{F}_s = \{f : \mathcal{D}^s f(x) \geq 0, \ x \geq 0\}, \quad (3.16)$$

$\mathcal{D}$  being the usual derivative operator (to be compared with (3.1)). For two probability distributions  $P_1$  and  $P_2$  on  $\mathbb{R}^+$ ,  $P_1 \preceq_s P_2$  when the condition (3.2) is again satisfied. Here too, an equivalent characterization is given by the conditions (3.3), (3.4).

We still work on the set of distributions with prescribed first  $s$  moments, i.e.

$$\mathcal{B}_{s+1} = \{P_X : E(X^i) = \mu_i, \ i = 1, \dots, s\}. \quad (3.17)$$

Let us apply to  $\mathcal{B}_{s+1}$  the continuous stationary-excess operator  $H$  defined by (2.10). This yields a one-to-one correspondence with the image set  $H(\mathcal{B}_{s+1})$ . A transformed order  $\preceq_H$  may be defined on  $H(\mathcal{B}_{s+1})$  exactly as by (3.6). The continuous analogue of Proposition 3.3 is then the following. Let  $Q_Y$  denote the probability distribution of a continuous random variable  $Y$  with density function  $\{d_Y(y), \ y \geq 0\}$ .

**Proposition 3.5**

$$H(\mathcal{B}_{s+1}) = \left\{ Q_Y : E(Y^i) = \frac{\mu_{i+1}}{(i+1)\mu_1}, \ i = 1, \dots, s-1, \quad \text{and} \right. \quad (3.18)$$

$$\left. \mathcal{D}d_Y(y) \leq 0, \ y \geq 0, \quad \text{with} \quad \mu_1 d_Y(0) \leq 1 \right\}, \quad (3.19)$$

*and the order  $\preceq_H$  corresponds to the continuous  $s$ -convex order  $\preceq_s$ .*

*Proof.* The derivation of the set  $H(\mathcal{B}_{s+1})$  is straightforward from (2.12) and (2.13). To identify the order  $\preceq_H$ , we have still recourse to the conjugate operator of  $H^{-1}$ . From (2.12), one finds that

$$(H^{-1})^* = \delta_0 + \mu_1 \mathcal{D}, \quad (3.20)$$

in place of (3.12). Thus,  $\preceq_H$  corresponds to the order  $\preceq_s$ .  $\diamond$

## 4 Extension to $H^t$ -transforms

At this point, a natural step is to apply successively the operator  $H$  and then investigate the effects on the  $s$ -convex orders and the associated moment spaces. So, given  $t \in \mathbb{N}_0$ , let  $H^t$  denote the  $t$ -th iterate of  $H$ . The construction of  $H^t$  is simple, although not totally immediate.

## 4.1 Discrete problem

Given  $s \in \mathbb{N}_0$ , we start by introducing, similarly to (3.5), a set  $\mathcal{B}_{s+t}$  of all probability mass functions  $P_X$  on  $\mathbb{N}$  with prescribed  $s+t-1$  first combinatorial moments:

$$\mathcal{B}_{s+t} = \left\{ P_X : E\binom{X}{i} = c_i, i = 1, \dots, s+t-1 \right\}. \quad (4.1)$$

This set may be ordered with respect to the  $(s+t)$  convex order.

Now, at step 1, consider the set  $D_1 = D_1^{(c_1)}$  of all probability mass functions on  $\mathbb{N}$  with fixed mean  $c_1$ . By construction, the operator  $H \equiv H_1 = H_1^{(c_1)}$  is a one-to-one mapping from  $D_1$  into a new set  $H_1(D_1) \equiv D_2$ . By (3.7), the mean in  $D_2$  is equal to  $c_2/c_1$ ; put  $D_2 = D_2^{(c_2/c_1)}$ . At step 2, the operator  $H_2 = H_2^{(c_2/c_1)}$  maps  $D_2$  into  $H_2(D_2) \equiv D_3$ , and again by (3.7),  $D_3 = D_3^{(c_3/c_2)}$ . Finally, at step  $t$ ,  $H_t = H_t^{(c_t/c_{t-1})}$  maps  $D_t$  into  $H_t(D_t) \equiv D_{t+1}$ . The operator  $H^t$  is then the product of the successive operators  $H_1, \dots, H_t$ , i.e.

$$H^t = H_t \circ \dots \circ H_2 \circ H_1 : D_1 \xrightarrow{H_1} D_2 \xrightarrow{H_2} \dots \xrightarrow{H_{t-1}} D_t \xrightarrow{H_t} D_{t+1}. \quad (4.2)$$

Let us apply  $H^t$  to the set to any random variable  $X$  in  $\mathcal{B}_{s+t}$ . Note that the moments  $c_1, \dots, c_t$  are well fixed within  $\mathcal{B}_{s+t}$ . From (2.1) and (4.2), we see that  $X$  is mapped to an associated discrete non-negative random variable  $X_{H^t}$  defined as follows.

**Definition 4.1**  $X_{H^t}$  has a probability mass function  $H^t(P_X)$  given by

$$H^t(P_X)(j) \equiv P(X_{H^t} = j) = \frac{\bar{F}_t(X, j+t)}{c_t}, \quad j \in \mathbb{N}. \quad (4.3)$$

By (2.8), an equivalent expression is

$$P(X_{H^t} = j) = \frac{E\binom{X-j-1}{t-1}}{c_t}, \quad j \in \mathbb{N}. \quad (4.4)$$

We underline that each  $H_k$  being one-to-one from  $D_k$  to  $D_{k+1}$ ,  $H^t$  provides a one-to-one correspondence on  $\mathcal{B}_{s+t}$ . Furthermore, let us introduce, as with (3.6), an order  $\preceq_H^t$  on  $H^t(\mathcal{B}_{s+t})$ .

**Definition 4.2** Two probability mass functions  $Q_1, Q_2 \in H^t(\mathcal{B}_{s+t})$  satisfy  $Q_1 \preceq_{H^t} Q_2$  when

$$(H^t)^{-1}(Q_1) \preceq_{s+t} (H^t)^{-1}(Q_2). \quad (4.5)$$

We are now in a position to generalize Proposition 3.3 to that transform by  $H^t$ . The previous notation  $Q_Y$  will be adopted here too, and it is convenient to put  $c_0 = 1$ .

**Proposition 4.3**

$$H^t(\mathcal{B}_{s+t}) = \left\{ Q_Y : E\binom{Y}{i} = \frac{c_{i+t}}{c_t}, i = 1, \dots, s-1, \quad \text{and} \right. \quad (4.6)$$

$$(-1)^k \Delta^k Q_Y(j) \geq 0, j \in \mathbb{N}, k = 1, \dots, t, \quad \text{with} \quad (4.7)$$

$$\left. (-1)^{t-k} c_t Q_Y(0) \leq \sum_{l=k-1}^{t-1} (-1)^{t-k+1} c_l, k = 1, \dots, t \right\}, \quad (4.8)$$

and the order  $\preceq_{H^t}$  corresponds to the  $s$ -convex order  $\preceq_s$ .

*Proof.* The key tool is the conjugate operator of  $(H^t)^{-1}$ , built from the conjugate operators of  $(H_1)^{-1}, \dots, (H_t)^{-1}$ . By arguing as with (3.12), we see that

$$[(H_k)^{-1}]^* = \delta_0 + (c_k/c_{k-1})\Delta, \quad k = 1, \dots, t.$$

From (4.2), we then obtain

$$\begin{aligned} [(H^t)^{-1}]^* &= [(H_1)^{-1}]^* \circ [(H_2)^{-1}]^* \circ \dots \circ [(H_t)^{-1}]^* \\ &= [c_1\Delta + \delta_0] \circ [(c_2/c_1)\Delta + \delta_0] \circ \dots \circ [(c_t/c_{t-1})\Delta + \delta_0]. \end{aligned} \quad (4.9)$$

Note that, for  $t = 2$  for example,

$$[a\Delta + \delta_0] \circ [b\Delta + \delta_0] = ab\Delta^2 + \delta_0 b\Delta + \delta_0,$$

as  $\Delta\delta_0 = 0$  and  $\delta_0^2 = \delta_0$ . So, we get from (4.9) that

$$\begin{aligned} [(H^t)^{-1}]^* &= \sum_{k=0}^t \delta_0^k \prod_{l=k+1}^t [(c_l/c_{l-1})\Delta] \\ &= c_t\Delta^t + \delta_0 \sum_{k=1}^t (c_t/c_k)\Delta^{t-k}. \end{aligned} \quad (4.10)$$

Applying (4.10) to any function  $f$  on  $\mathbb{N}$  thus gives

$$[(H^t)^{-1}]^*(f) = c_t\Delta^t f + \sum_{k=1}^t (c_t/c_k) (\Delta^{t-k}f)(0). \quad (4.11)$$

Let us first identify the order  $\preceq_{H^t}$ . By (4.5) and (3.11),  $Q_1 \preceq_{H^t} Q_2$  when

$$< [(H^{-1})^*](f), Q_1 > \leq < [(H^{-1})^*](f), Q_2 > \quad \text{for all } f \in \mathcal{F}_{s+t}.$$

From (4.11), this means that

$$< \Delta^t f, Q_1 > \leq < \Delta^t f, Q_2 > \quad \text{for all } f \in \mathcal{F}_{s+t}, \quad (4.12)$$

which shows the equivalence with the order  $\preceq_s$ .

We now determine the set  $H^t(\mathcal{B}_{s+t})$ . Let us choose the function  $f(j) = \binom{j}{i+t}$ ,  $j \in \mathbb{N}$ , for any fixed  $i, t \in \mathbb{N}$ . Since

$$\Delta^t \binom{j}{i+t} = \binom{j}{i}, \quad j \in \mathbb{N},$$

we have by (4.11) that

$$[(H^t)^{-1}]^* \binom{j}{i+t} = c_t \binom{j}{i}, \quad j \in \mathbb{N}. \quad (4.13)$$

Thus, applying (3.11), we deduce from (4.13) the relations

$$E \binom{X}{i+t} = c_t E \binom{X_{H^t}}{i}, \quad i \in \mathbb{N}, \quad (4.14)$$

which generalizes (2.5). The binomial moments in (4.6) then follow. Moreover, from (4.2) and using (3.8), we find that a probability mass function  $Q_Y$  necessarily satisfies the restrictions (4.7). This can also be seen from (4.4) as for  $k = 1, \dots, t-1$ ,

$$\Delta^k E \binom{X-j-1}{t-1} = (-1)^k E \binom{X-j-1-k}{t-1-k}, \quad j \in \mathbb{N}.$$

So,  $Q_Y$  is a  $t$ -monotone function in the sense given in the Introduction. Finally, the  $t$  conditions (4.8) guarantee that the  $t$  successive transforms of  $P_X$  are true probability mass functions. Indeed, writing  $X_{H^k} = H_k(X_{H^{k-1}})$ ,  $k \geq 1$ , with  $X_{H^0} = X$ , one gets (see (2.3))

$$P(X_{H^{k-1}} = 0) = 1 - (c_k/c_{k-1})P(X_{H^k} = 0),$$

that is

$$c_{k-1}P(X_{H^{k-1}} = 0) = c_{k-1} - c_k P(X_{H^k} = 0), \quad k = 1, \dots, t. \quad (4.15)$$

Note that for each  $k$ ,  $P(X_{H^{k-1}} = 0) \leq 1$  but the non-negativity is not guaranteed so far. By iterating (4.15)  $t-k$  times, we then obtain

$$c_{k-1}P(X_{H^{k-1}} = 0) = \sum_{l=k-1}^{t-1} (-1)^{l-k+1} c_l + (-1)^{t-k+1} c_t P(X_{H^t} = 0), \quad k = 1, \dots, t.$$

Thus, the constraints  $P(X_{H^{k-1}} = 0) \geq 0$  provide the restrictions (4.8).  $\diamond$

Observe that the image set  $H^t(\mathcal{B}_{s+t})$  contains  $s-1$  moment relations and the  $t$ -monotonicity property, as well as  $t$  other technical conditions. Going back to  $\mathcal{B}_{s+t}$  through the operator  $H^{-t}$  would allow us to remove the monotonicity condition.

## 4.2 Continuous problem

For  $s, t \in \mathbb{N}_0$ , consider now the set of distributions

$$\mathcal{B}_{s+t} = \{P_X : E(X^i) = \mu_i, \quad i = 1, \dots, s+t-1\}, \quad (4.16)$$

We construct the operator  $H^t$ , the  $t$ -th iterate of the operator  $H$  defined by (2.10), by following a similar argument as for (4.2). Let  $D_1 = D_1^{(\mu_1)}$  the set of all continuous distributions on  $\mathbb{R}^+$  with fixed mean  $\mu_1$ . Then,  $H^t$  is defined as  $H^t = H_t \circ \dots \circ H_2 \circ H_1$  where, for each  $k = 1, \dots, t$ ,  $H_k$  is an operator mapping the set  $D_k$  into the set  $H_k(D_k) = D_{k+1}$ . From (2.13), one easily checks the mean in  $D_{k+1}$  is equal to  $\mu_k/k\mu_{k-1}$ . So,  $H_k = H_k^{(\mu_k/k\mu_{k-1})}$  and it provides a one-to-one correspondence.

By definition of  $H^t$  and using (2.11), we see that any random variable  $X$  in  $\mathcal{B}_{s+t}$  is mapped to an associated non-negative random variable  $X_{H^t}$  whose survival function is given by

$$P(X_{H^t} > x) = \frac{t! \bar{F}_{t+1}(X, x)}{\mu_t}, \quad x \geq 0. \quad (4.17)$$

By (2.14) and (2.17), it can be rewritten as

$$P(X_{H^t} > x) = \frac{E[(X-x)_+^t]}{\mu_t}, \quad x \geq 0, \quad (4.18)$$

which generalizes (2.11). Here too,  $H^t$  is one-to-one on  $\mathcal{B}_{s+t}$ . A transformed order  $\preceq_{H^t}$  is then defined on  $H^t(\mathcal{B}_{s+t})$  by stipulating again the condition (4.5).

**Proposition 4.4**

$$H^t(\mathcal{B}_{s+t}) = \left\{ Q_Y : E(Y^i) = \frac{\mu_{i+t}}{\binom{i+t}{t} \mu_t}, \quad i = 1, \dots, s-1, \quad \text{and} \right. \quad (4.19)$$

$$(-1)^k \mathcal{D}^{(k)} d_Y(y) \geq 0, \quad y \geq 0, \quad k = 1, \dots, t, \quad \text{with} \quad (4.20)$$

$$(-1)^{t-k} \mathcal{D}^{(t-k)} d_Y(0) \leq \frac{t! \mu_{k-1}}{(k-1)! \mu_t}, \quad k = 1, \dots, t \left. \right\}, \quad (4.21)$$

and the order  $\preceq_{H^t}$  corresponds to the  $s$ -convex order  $\preceq_s$ .

*Proof.* Working with conjugate operators, one gets (see (3.20)) that

$$[(H_k)^{-1}]^* = \delta_0 + (\mu_k/k\mu_{k-1})\mathcal{D}, \quad k = 1, \dots, t.$$

Therefore,

$$\begin{aligned} [(H^t)^{-1}]^* &= [\mu_1\mathcal{D} + \delta_0] \circ [(\mu_2/2\mu_1)\mathcal{D} + \delta_0] \circ \dots \circ [(\mu_t/t\mu_{t-1})\mathcal{D} + \delta_0] \\ &= \frac{\mu_t}{t!} \mathcal{D}^t + \delta_0 \sum_{k=1}^t \frac{k! \mu_t}{t! \mu_k} \mathcal{D}^{t-k}. \end{aligned} \quad (4.22)$$

First, consider the function  $f(x) = x^{i+t}$ ,  $x \in \mathbb{R}^+$ , for fixed  $i, t \in \mathbb{N}$ . Applying (4.22) to this function and using (3.11), we find that

$$E(X^{i+t}) = \binom{i+t}{t} \mu_t E(X_{H^t}^i), \quad i \in \mathbb{N},$$

hence (4.19). Now, as for (2.12), one has

$$P(X_{H^{k-1}} > x) = (\mu_k/k\mu_{k-1}) d_{H^k}(x), \quad k = 1, \dots, t, \quad (4.23)$$

which gives by differentiation

$$\mu_{k-1} d_{H^{k-1}}(x) = (-1/k) \mathcal{D}\{\mu_k d_{H^k}(x)\}, \quad k = 1, \dots, t. \quad (4.24)$$

By iterating (4.24)  $t - k$  more times, we get

$$\mu_{k-1} d_{H^{k-1}}(x) = \frac{\mu_t}{k \dots t} (-1)^{t-k+1} \mathcal{D}^{(t-k+1)} d_{H^t}(x), \quad k = 1, \dots, t. \quad (4.25)$$

Thus, the conditions  $d_{H^{k-1}} \geq 0$  in (4.25) yield the constraints (4.20). Moreover, (4.23) implies that

$$\mu_k d_{H^k}(0) \leq k\mu_{k-1}, \quad k = 1, \dots, t,$$

and inserting (4.25) with  $x = 0$  and  $k$  instead of  $k - 1$  then leads to the restrictions (4.21).  $\diamond$

## 5 Convex extrema under monotonicity conditions

A further interest of Propositions 4.3 and 4.4 is to point out that the  $s$ -convex extrema for  $t$ -monotone distributions may be constructed from the simple  $s + t$ -convex extrema (i.e. without monotonicity restriction).

Hereafter, we are going to derive the explicit expression of a few extrema for monotone distributions that are defined on a bounded non-negative support. In the discrete case, the random variables are valued in a set  $\{0, \dots, n\}$ ,  $n \in \mathbb{N}_0$ ; in the continuous case, the random variables are valued on an interval  $[0, b]$ ,  $b > 0$ . We first obtain the  $s = 1, 2, 3$ -convex extrema when  $t = 1$  (nonincreasing distributions). These were found earlier by Denuit et al. (1999b) in the discrete case and Denuit et al. (1998) in the continuous case. The method of proof used there, however, is quite different as it relies on a Khinchine representation for unimodal distributions. Then, we examine the  $s = 1, 2$ -convex extrema when  $t = 2$  (convex nonincreasing distributions). These are seen to be the same as when  $t = 1$  except for the minimum in the discrete case.

Let us introduce the class  $\mathcal{B}_{s,t}(n)$  of all discrete random variables  $Y$  that have prescribed first  $s - 1$  moments  $\nu_i = E(Y^i)$ ,  $i = 1, \dots, s - 1$ , and possess a  $t$ -monotone probability mass function  $Q_Y$ , i.e. for  $k = 1, \dots, t$ ,  $(-1)^k \Delta^k Q_Y(j) \geq 0$ ,  $j = 0, \dots, n - k$ . The  $s$ -convex extrema in this set are denoted by  $Y_{\min}^{(s,t)}(n)$  and  $Y_{\max}^{(s,t)}(n)$ . Similarly,  $\mathcal{B}_{s,t}(b)$  is the class of all continuous random variables  $Y$  with prescribed first  $s - 1$  moments  $\nu_i$  and a  $t$ -monotone density function  $d_Y$ , i.e. for  $k = 1, \dots, t$ ,  $(-1)^k \mathcal{D}^{(k)} d_Y(y) \geq 0$ ,  $y \in [0, b]$ . The  $s$ -convex extrema in the set are written  $Y_{\min}^{(s,t)}(b)$  and  $Y_{\max}^{(s,t)}(b)$ .

For the discrete case, we observe that by the definition (2.1) of  $H$ , a random variable  $Y$  valued on  $\{0, \dots, n\}$  is obtained as the transform of a random variable  $X$  that is valued on  $\{0, \dots, n + 1\}$ . This remark will be used in the sequel.

Let us recall that the usual  $s$ -convex extrema (i.e. when  $t = 0$ , inside  $\mathcal{B}_s(n)$  or  $\mathcal{B}_s(b)$ ) are explicitly known for  $s = 1, 2, 3, 4$ . They are given in the Appendix and denoted by  $X_{\min}^{(s)}(n)$ ,  $X_{\max}^{(s)}(n)$  and  $X_{\min}^{(s)}(b)$ ,  $X_{\max}^{(s)}(b)$ .

**(i) 1-convex extrema for nonincreasing distributions.** This is the problem of extrema when  $s = 1$  and  $t = 1$ . Let us write  $\mathcal{U}(n)$  [ $\mathcal{U}(b)$ ] for a uniform distribution on  $\{0, \dots, n\}$  ( $[0, b]$ ).

**Corollary 5.1** *For the discrete [continuous] case (inside  $\mathcal{B}_{1,1}(n)$  [ $\mathcal{B}_{1,1}(b)$ ]),*

$$Y_{\min}^{(1,1)}(n) [Y_{\min}^{(1,1)}(b)] = 0 \quad a.s., \quad (5.1)$$

and

$$Y_{\max}^{(1,1)}(n) [Y_{\max}^{(1,1)}(b)] =_d \mathcal{U}(n) [\mathcal{U}(b)]. \quad (5.2)$$

*Proof.* For the discrete case, consider the set  $H[\mathcal{B}_2(n + 1)]$  defined by (3.7), (3.8), with  $n + 1$  substituted for  $n$  (see the remark above). We want to keep only the restriction of nonincreasing distributions (i.e.  $\Delta Q_Y(j) \leq 0$ ,  $j = 0, \dots, n - 1$ ). Note that as  $s = 1$ , there is no moment specification. Moreover, the condition  $c_1 Q_Y(0) \leq 1$  will be satisfied by choosing  $c_1 = 1$ , for instance. Now, by Proposition 3.3, the associated 1-convex extrema are provided by the  $H$ -transform (2.1) of the 2-convex extrema in  $\mathcal{B}_2(n + 1)$ . Using (7.3) and (7.4) where  $\xi = 0$ , we

then get

$$P[Y_{\min}^{(1,1)}(n) = j] = P[X_{\min}^{(2)}(n+1) \geq j+1] = \delta_{j,0}, \quad j = 0, \dots, n,$$

where  $\delta_{j,0} = 1$  (resp. 0) if  $j = 0$  (resp.  $> 0$ ), and

$$P[Y_{\max}^{(1,1)}(n) = j] = P[X_{\max}^{(2)}(n+1) \geq j+1] = 1/(n+1), \quad j = 0, \dots, n,$$

hence the extrema specified above.

For the continuous case,  $H[\mathcal{B}_2(b)]$  defined by (3.18), (3.19) contains now an undesired condition  $\mu_1 d_Y(0) \leq 1$ . It can be satisfied, however, by choosing  $\mu_1 \rightarrow 0$ . Then, applying the operator  $H$  given by (2.10) to the 2-convex minimum (7.5) gives

$$P[Y_{\min}^{(1,1)}(b) \leq x] = \frac{1}{\mu_1} \int_0^x [1 - F_{X_{\min}^{(2)}(b)}(y)] dy = \frac{x}{\mu_1}, \quad 0 \leq x \leq \mu_1,$$

i.e.  $Y_{\min}^{(1,1)}(b)$  is uniform on  $[0, \mu_1]$ . As  $\mu_1$  tends to 0, this minimum degenerates to the single point 0. Using the 2-convex maximum (7.6), we find that

$$P[Y_{\max}^{(1,1)}(b) \leq x] = \frac{1}{\mu_1} \int_0^x [1 - F_{X_{\max}^{(2)}(b)}(y)] dy = \frac{x}{b}, \quad 0 \leq x \leq b,$$

(independently of  $\mu_1$ ), i.e.  $Y_{\max}^{(1,1)}(b)$  is uniform on  $[0, b]$  as announced.  $\diamond$

**(ii) 2-convex extrema for nonincreasing distributions.** This time, we work with  $s = 2$  and  $t = 1$ . Denote by  $\nu_1 = E(Y)$  the fixed mean in  $\mathcal{B}_{2,1}(n)$  (and  $H[\mathcal{B}_3(n+1)]$ ) or  $\mathcal{B}_{2,1}(b)$  (and  $H[\mathcal{B}_3(b)]$ ).

**Corollary 5.2** *For the discrete case (inside  $\mathcal{B}_{2,1}(n)$ ), put  $\tilde{\xi}$  for the integer in  $[0, n-1]$  such that  $\tilde{\xi} < 2\nu_1 \leq \tilde{\xi} + 1$ . Then,*

$$Y_{\min}^{(2,1)}(n) = \begin{cases} 0, \dots, \tilde{\xi} & \text{with equal probabilities } 2(\tilde{\xi} + 1 - \nu_1)/(\tilde{\xi} + 1)(\tilde{\xi} + 2), \\ \tilde{\xi} + 1 & \text{with probability } (2\nu_1 - \tilde{\xi})/(\tilde{\xi} + 2), \end{cases} \quad (5.3)$$

and

$$Y_{\max}^{(2,1)}(n) = \begin{cases} 0 & \text{with probability } 1 - 2\nu_1/(n+1), \\ 1, \dots, n & \text{with equal probabilities } 2\nu_1/n(n+1). \end{cases} \quad (5.4)$$

For the continuous case (inside  $\mathcal{B}_{2,1}(b)$ ),

$$Y_{\min}^{(2,1)}(b) =_d \mathcal{U}(2\nu_1), \quad (5.5)$$

and putting  $\mathcal{I}_0$  for a distribution degenerated in 0,

$$Y_{\max}^{(2,1)}(b) =_d (1 - 2\nu_1/b) \mathcal{I}_0 + (2\nu_1/b) \mathcal{U}(b). \quad (5.6)$$

*Proof.* As indicated in the Appendix, the unconstrained 3-convex extrema are given in terms of the moments  $\mu_i = E(X^i)$ ,  $i = 1, 2$ . Let us start with the discrete case. By (3.7),  $\nu_1 = c_2/c_1$  which implies that  $\mu_2 = \mu_1(2\nu_1 + 1)$ . For  $X_{min}^{(3)}(n+1)$  given by (7.7), we have  $\xi_1 - 1 < 2\nu_1 \leq \xi_1$ , i.e.  $\tilde{\xi} < 2\nu_1 \leq \tilde{\xi} + 1$  by putting  $\tilde{\xi} = \xi_1 - 1$ . Moreover,

$$X_{min}^{(3)}(n+1) = \begin{cases} 0 & \text{with probability } p_1 = 1 - p_2 - p_3, \\ \tilde{\xi} + 1 & \text{with probability } p_2 = \mu_1(\tilde{\xi} - 2\nu_1 + 1)/(\tilde{\xi} + 1), \\ \tilde{\xi} + 2 & \text{with probability } p_3 = \mu_1(2\nu_1 - \tilde{\xi})/(\tilde{\xi} + 2). \end{cases}$$

For  $X_{max}^{(3)}(n+1)$  given by (7.8), we have  $\xi_2 < (n - 2\nu)/[(n+1)/\mu_1 - 1] \leq \xi_2 + 1$ , so that  $\xi_2 = 0$  as  $\mu_1 \rightarrow 0$ . Then,

$$X_{max}^{(3)}(n+1) = \begin{cases} 0 & \text{with probability } p_1 = 1 - p_2 - p_3, \\ 1 & \text{with probability } p_2 = \mu_1(n - 2\nu_1)/n, \\ n+1 & \text{with probability } p_3 = \mu_1 2\nu_1/n(n+1). \end{cases}$$

Applying the discrete operator  $H$  then leads to the extrema (5.3) and (5.4).

Let us now discuss the continuous case. By (3.18),  $\nu_1 = \mu_2/2\mu_1$ . We may let  $\mu_1 \rightarrow 0$  and choose  $\mu_2 = 2\mu_1\nu_1$  accordingly. For the minimum, we get from (7.9) that

$$P[Y_{min}^{(2,1)}(b) \leq x] = (\mu_1/\mu_2)x, \quad 0 \leq x \leq \mu_2/\mu_1,$$

i.e. the result (5.5).

For the maximum, (7.10) gives as  $\mu_1 \rightarrow 0$

$$X_{max}^{(3)}(b) = \begin{cases} \mu_1(1 - 2\nu_1/b) & \text{with probability } 1 - 2\nu_1\mu_1/b^2, \\ b & \text{with probability } 2\nu_1\mu_1/b^2, \end{cases}$$

so that

$$P[Y_{max}^{(2,1)}(b) \leq x] = 1 - 2\nu_1/b + 2\nu_1x/b^2, \quad 0 \leq x \leq b,$$

i.e. the maximum (5.6).  $\diamond$

**(iii) 3-convex extrema for nonincreasing distributions.** We have now  $s = 3$  and  $t = 1$ . Let  $\nu_1 = E(Y)$  and  $\nu_2 = E(Y^2)$  be the first two moments in  $\mathcal{B}_{3,1}(n)$  (and  $H[\mathcal{B}_4(n+1)]$ ) or  $\mathcal{B}_{3,1}(b)$  (and  $H[\mathcal{B}_4(b)]$ ).

**Corollary 5.3** *For the discrete case (inside  $\mathcal{B}_{3,1}(n)$ ), put  $\tilde{\xi}_1$  and  $\tilde{\xi}_2$  for the integers in  $[0, n-1]$  such that*

$$\tilde{\xi}_1 < (3\nu_2 - \nu_1)/2\nu_1 \leq \tilde{\xi}_1 + 1, \quad \text{and} \quad \tilde{\xi}_2 < (2n\nu_1 - 3\nu_2 + \nu_1)/(n - 2\nu_1) \leq \tilde{\xi}_2 + 1.$$

*Then,*

$$Y_{min}^{(3,1)}(n) = \begin{cases} 0 & \text{with probability } \frac{q_3}{\tilde{\xi}_1+2} + \frac{q_2}{\tilde{\xi}_1+1} + q_1, \\ 1, \dots, \tilde{\xi}_1 & \text{with equal probabilities } \frac{q_3}{\tilde{\xi}_1+2} + \frac{q_2}{\tilde{\xi}_1+1}, \\ \tilde{\xi}_1 + 1 & \text{with probability } \frac{q_3}{\tilde{\xi}_1+2}, \end{cases} \quad (5.7)$$



where  $q_1 + q_2 + q_3 = 1$  and

$$q_2 = \frac{-3\nu_2 + 3\nu_1 + 2\tilde{\xi}_1\nu_1}{\tilde{\xi}_1}, \quad q_3 = \frac{3\nu_2 - \nu_1 - 2\tilde{\xi}_1\nu_1}{\tilde{\xi}_1 + 1},$$

and

$$Y_{max}^{(3,1)}(n) = \begin{cases} 0, \dots, \tilde{\xi}_2 & \text{with equal probabilities } \frac{\chi_3}{n+1} + \frac{\chi_2}{\tilde{\xi}_2+2} + \frac{\chi_1}{\tilde{\xi}_2+1}, \\ \tilde{\xi}_2 + 1 & \text{with probability } \frac{\chi_3}{n+1} + \frac{\chi_2}{\tilde{\xi}_2+2}, \\ \tilde{\xi}_2 + 2, \dots, n & \text{with equal probabilities } \frac{\chi_3}{n+1}, \end{cases} \quad (5.8)$$

where  $\chi_1 + \chi_2 + \chi_3 = 1$  and

$$\chi_1 = \frac{(\tilde{\xi}_2 + 1)(n - 2\nu_1) + 3\nu_2 - \nu_1 - 2n\nu_1}{n - \tilde{\xi}_2}, \quad \chi_2 = \frac{2\nu_1(\tilde{\xi}_2 + n) - 3\nu_2 + \nu_1 - n\tilde{\xi}_2}{n - 1 - \tilde{\xi}_2}.$$

For the continuous case (inside  $\mathcal{B}_{3,1}(b)$ ),

$$Y_{min}^{(3,1)}(b) =_d \frac{3\nu_2 - 4\nu_1^2}{3\nu_2} \mathcal{I}_0 + \frac{4\nu_1^2}{3\nu_2} \mathcal{U}\left(\frac{3\nu_2}{2\nu_1}\right), \quad (5.9)$$

and

$$Y_{max}^{(3,1)}(b) =_d \frac{3\nu_2 - 4\nu_1^2}{b^2 - 4b\nu_1 + 3\nu_2} \mathcal{U}(b) + \frac{b^2 - 4b\nu_1 + 4\nu_1^2}{b^2 - 4b\nu_1 + 3\nu_2} \mathcal{U}\left(\frac{2b\nu_1 - 3\nu_2}{b - 2\nu_1}\right). \quad (5.10)$$

*Proof.* In the discrete case, we have, by (3.7),  $\nu_1 = c_2/c_1$  which yields  $\mu_2 = \mu_1(2\nu_1 + 1)$ , and  $(\nu_2 - \nu_1)/2 = c_3/c_2$  which implies that  $\mu_3 = \mu_1(3\nu_2 + 3\nu_1 + 1)$ . Applying (2.1) to  $X_{min}^{(4)}(n+1)$  given by (7.11) yields

$$Y_{min}^{(3,1)}(n) = \begin{cases} 0, \dots, \theta - 1 & \text{with equal probabilities } 1/\mu_1, \\ \theta & \text{with probability } (p_2 + p_3 + p_4)/\mu_1, \\ \theta + 1, \dots, \eta - 1 & \text{with equal probabilities } (p_3 + p_4)/\mu_1, \\ \eta & \text{with probability } p_4/\mu_1. \end{cases}$$

Let us choose  $\mu_1 \rightarrow 0$ . To avoid absurdity, we must have  $\theta = 0$ . Moreover, we then find, after some simplifications, that

$$\begin{cases} p_2/\mu_1 &= (3\nu_2 + \nu_1 - 4\nu_1\eta + \eta(\eta - 1))/\eta(\eta + 1), \\ p_3/\mu_1 &= (-3\nu_2 + \nu_1 + 2\nu_1\eta)/\eta(\eta - 1), \\ p_4/\mu_1 &= (3\nu_2 + \nu_1 - 2\nu_1\eta)/\eta(\eta - 1). \end{cases}$$

The condition  $p_4 > 0$  implies that  $\eta < 1 + (3\nu_2 - \nu_1)/2\nu_1$ , and  $p_3 \geq 0$  yields  $\eta \geq (3\nu_2 - \nu_1)/2\nu_1$ . Putting  $\eta = 1 + \tilde{\xi}_1$ , we then obtain for  $\tilde{\xi}_1$  the value announced above.

Note that the condition  $p_2 \geq 0$  is always verified. Indeed, this condition means that  $3\nu_2 - 3\nu_1 - 4\tilde{\xi}_1\nu_1 + \tilde{\xi}_1(\tilde{\xi}_1 + 1) \geq 0$ . It can be rewritten as  $\tilde{\nu}_2 - (2\tilde{\xi}_1 + 1)\tilde{\nu}_1 + \tilde{\xi}_1(\tilde{\xi}_1 + 1) \geq 0$  where  $\tilde{\nu}_1 = 2\nu_1$  and  $\tilde{\nu}_2 = 3\nu_2 - \nu_1$ . This is verified if  $\tilde{\nu}_1$  and  $\tilde{\nu}_2$  are the first two moments of some random variable on  $\{0, \dots, n\}$ , since  $i^2 - (2\tilde{\xi}_1 + 1)i + \tilde{\xi}_1(\tilde{\xi}_1 + 1) \geq 0$  for all  $0 \leq i \leq n$ . In fact,

the random variable  $Z$  in Denuit et al. (1999c) has precisely its first two moments equal to these  $\tilde{\nu}_1$  and  $\tilde{\nu}_2$ ; see formula (5.7) in that paper. So, we finally get the above result (5.7).

Consider  $X_{max}^{(4)}(n+1)$  given by (7.12). First, we observe that  $(\mu_2 n - \mu_3)/(\mu_1 n - \mu_2) = 1 + (2n\nu_1 - 3\nu_2 + \nu_1)/(n - 2\nu_1)$ ; thus,  $\zeta = \tilde{\xi}_2 + 1$  where  $\tilde{\xi}_2$  is defined through (5.8) above. Applying (2.1), we then get that

$$Y_{max}^{(3,1)}(n) = \begin{cases} 0, \dots, \tilde{\xi}_2 & \text{with equal probabilities } (p_2 + p_3 + p_4)/\mu_1, \\ \tilde{\xi}_2 + 1 & \text{with probability } (p_3 + p_4)/\mu_1, \\ \tilde{\xi}_2 + 2, \dots, n & \text{with equal probabilities } p_4/\mu_1. \end{cases}$$

Finally, one can check that  $p_2/\mu_1 = \chi_1/(\tilde{\xi}_2 + 1)$ ,  $p_3/\mu_1 = \chi_2/(\tilde{\xi}_2 + 2)$  and  $p_4/\mu_1 = \chi_3/(n + 1)$ , hence the result (5.8).

In the continuous case, (3.18) gives  $\nu_1 = \mu_2/2\mu_1$  and  $\nu_2 = \mu_3/3\mu_1$ . So, we choose  $\mu_2 = 2\mu_1\nu_1$  and  $\mu_3 = 3\mu_1\nu_2$ , with  $\mu_1 \rightarrow 0$ . For the minimum, we see from (7.13) that

$$X_{min}^{(4)}(b) = \begin{cases} r_- \rightarrow 0 & \text{with probability } (r_+ - \mu_1)/(r_+ - r_-) \rightarrow 1, \\ r_+ \rightarrow 3\nu_2/\nu_1 & \text{with probability } (\mu_1 - r_-)/(r_+ - r_-) \rightarrow 0. \end{cases}$$

Moreover, one can check that

$$r_-/\mu_1 \rightarrow (3\nu_2 - 4\nu_1^2)/3\nu_2,$$

implying also that  $(\mu_1 - r_-)/(r_+ - r_-)\mu_1 \rightarrow (4\nu_1^2/3\nu_2)(2\nu_1/3\nu_2)$ . All this then leads to the result (5.9) for  $Y_{min}^{(3,1)}(b)$ .

For the maximum, we have by (7.14) that

$$X_{max}^{(4)}(b) = \begin{cases} 0 & \text{with probability } p_1 = 1 - p_2 - p_3, \\ \frac{2b\nu_1 - 3\nu_2}{b - 2\nu_1} & \text{with probability } p_2 = \mu_1 \frac{b - 2\nu_1}{2b\nu_1 - 3\nu_2} \frac{(b - 2\nu_1)^2}{b^2 - 4b\nu_1 + 3\nu_2}, \\ b & \text{with probability } p_3 = \mu_1 \frac{1}{b} \frac{3\nu_2 - 4\nu_1^2}{b^2 - 4b\nu_1 + 3\nu_2}. \end{cases}$$

The result (5.10) for  $Y_{max}^{(3,1)}(b)$  then easily follows.  $\diamond$

**(iv) 1-convex extrema for nonincreasing convex distributions.** Here,  $s = 1$  and  $t = 2$ . We observe that the 1-convex extrema obtained before for nonincreasing distributions are convex. As a consequence, they are extremal too for nonincreasing convex distributions.

**(v) 2-convex extrema for nonincreasing convex distributions.** This time,  $s = 2$  and  $t = 2$ . The 2-convex extrema for nonincreasing distributions are again convex, except the minimum in the discrete case. Thus, apart this case which is discussed below, they remain extremal for nonincreasing convex distributions. Let  $\nu_1 = E(Y)$  be the fixed mean in the set  $\mathcal{B}_{2,2}(n)$  (and  $H^2[\mathcal{B}_4(n+1)]$ ).

**Corollary 5.4** *For the minimum in the discrete case (inside  $\mathcal{B}_{2,2}(n)$ ), put  $\tilde{\eta}$  for the integer in  $[2, n]$  such that  $\tilde{\eta} < 2 + 3\nu_1 \leq \tilde{\eta} + 1$ . Then,*

$$Y_{min}^{(2,2)}(n) = j \quad \text{with probability} \quad (\tilde{\eta} - j - 1)\pi_1 + (\tilde{\eta} - j)\pi_2, \quad j = 0, \dots, \tilde{\eta} - 1, \quad (5.11)$$

where

$$\pi_1 = \frac{2(\tilde{\eta} - 1 - 3\nu_1)}{\tilde{\eta}(\tilde{\eta} - 1)}, \quad \text{and} \quad \pi_2 = \frac{2(2 + 3\nu_1 - \tilde{\eta})}{\tilde{\eta}(\tilde{\eta} + 1)}.$$

*Proof.* In (4.8), the two constraints (for  $k = 1, 2$ ) yield the condition  $0 \leq c_1 - c_2 Q(0) \leq 1$ . From now, we assume that  $\mu_1 \rightarrow 0$ . To satisfy the previous condition, one may choose, for instance,  $E(X^2) = 2\mu_1$ . Moreover, by (4.6),  $\nu_1 = c_3/c_2$ , i.e.  $3(1 + \nu_1) = E(X^3 - X)/E(X^2 - X)$ . Given our choice of  $E(X^2)$ , one then gets  $E(X^3) = (4 + 3\nu_1)\mu_1$ . Finally, as  $\mu_1 \rightarrow 0$ , we use the approximations  $\mu_2 \simeq E(X^2)$  and  $\mu_3 \simeq E(X^3)$ .

Consider the 4-convex minimum given by (7.11). Under the first three moments above, one sees that  $\theta = 0$  and the minimum then becomes

$$X_{min}^{(4)}(n+1) = \begin{cases} 0 & \text{with probability } p_1 = 1 - (\eta^2 - \eta + 3\nu_1 + 1)\mu_1/\eta(\eta + 1), \\ 1 & \text{with probability } p_2 = (\eta^2 - 3\eta + 2 + 3\nu_1)\mu_1/\eta(\eta - 1), \\ \eta & \text{with probability } p_3 = (\eta - 1 - 3\nu_1)\mu_1/\eta(\eta - 1), \\ \eta + 1 & \text{with probability } p_4 = (2 + 3\nu_1 - \eta)\mu_1/\eta(\eta + 1). \end{cases}$$

Furthermore,  $p_1 \geq 0$  (as  $\mu_1 \rightarrow 0$ ),  $p_3 \geq 0$  means  $1 + 3\nu_1 \leq \eta$ , and  $p_4 > 0$  requires  $\eta < 2 + 3\nu_1$ , which implies  $p_2 > 0$  (since  $\eta \geq 2$ ). All this leads to the announced value  $\tilde{\eta}$ .

By (4.4), the operator  $H^2$  provides the distribution  $P(Y = j) = [P(X = j + 2) + 2P(X = j + 3) + \dots + (n - j)P(X = n + 1)]/c_2$ ,  $j = 0, \dots, n$ . Applied to  $X_{min}^{(4)}(n + 1)$ , it gives, as  $\mu_1 \rightarrow 0$ ,  $P[Y_{min}^{(2,2)}(n) = \tilde{\eta} - 1] = p_4/c_2$ ,  $P[Y_{min}^{(2,2)}(n) = \tilde{\eta} - 2] = (p_3 + 2p_4)/c_2$ ,  $P[Y_{min}^{(2,2)}(n) = \tilde{\eta} - 3] = (2p_3 + 3p_4)/c_2$  and in general,

$$P[Y_{min}^{(2,2)}(n) = j] = (\tilde{\eta} - j - 1)p_3/c_2 + (\tilde{\eta} - j)p_4/c_2, \quad j = 0, \dots, \tilde{\eta} - 1.$$

By construction one knows that  $c_2 = \mu_1/2$ , thus  $p_3/c_2 = \pi_1$  and  $p_4/c_2 = \pi_2$  as stated in (5.11).  $\diamond$

## 6 Some numerical illustrations

We present hereafter three applications, somewhat non-standard, in insurance and biostatistics. More traditional illustrations in ruin theory and life insurance (as in Denuit and Lefèvre (1997) and Denuit et al. (1998), (1999b)) could be considered too.

**Solvency Capital Requirements for large claims.** The new European regulation, Solvency II, to be in force in October 2012, imposes insurance companies to hold enough capital to deal with unfavorable events. The Directive was voted in April 2009 but many implementation measures are still being discussed currently. Some stakeholders recommend the creation of a so-called large loss module in order to cover claim amounts that are larger than attritional ones (but not of catastrophic type). First, for each risk, some buffer capital called Solvency Capital Requirement (SCR) has to be held. Then, at the company or group level, some aggregation formula takes diversification into account and defines the Basic Solvency Capital Requirement that the company must satisfy as

$$\sqrt{\sum_i \sum_j \rho_{ij} SCR_i SCR_j},$$

where  $SCR_i$  represents the Solvency Capital Requirement for the business line  $i$  and  $\rho_{ij}$  is a correlation-type parameter (without precise statistical meaning) between the lines  $i$  and  $j$ .

For each risk, the Solvency Capital Requirement is generally defined as the difference between a certain Value-at-Risk of the random loss and the expected loss. In most cases, a simplification is used and the SCR is defined by  $SCR = q\sigma$ , where  $\sigma$  is the standard error of the random loss and  $q > 0$  is called a quantile factor. So,  $q = 3$  is usually chosen for claim amounts with a moderate tail distribution (as it is in many Solvency II modules). For heavy-tailed risks, a more relevant value is  $q = 5$  or  $6$ ; hereafter, we will take  $q = 6$ . A collective risk model is often adopted to describe the occurrence of large claims. Thus, the number of claims is a counting random variable  $N$ , and the large claim amounts are independent identically distributed random variables (distributed as  $W$ , say), independently of  $N$ . By the variance decomposition formula, the variance of the large loss aggregated claim amount is given by

$$\sigma^2 = \text{Var}(W)E(N) + [E(W)]^2 \text{Var}(N).$$

Various specialized softwares or internal models provide an estimation of the distribution of  $W$  and the mean  $E(N)$ . As only limited information is available,  $N$  is usually assumed to have a Poisson distribution. For example, with respect to some French data, one could use a Poisson distribution with parameter  $\lambda(C_{27}) = 0.37$  for the business line  $C_{27}$  (Drought and earthquake) and  $\lambda(C_{35}) = 0.69$  for the business line  $C_{35}$  (Construction - damages to building). The Poisson assumption is partially satisfying as, in practice,  $N$  is usually over-dispersed ( $\text{Var}(N)$  is significantly larger than  $E(N)$ ) and only a bounded number of large claims are susceptible to occur (practitioners consider that observing more than a certain number  $n$  claims corresponds to a catastrophe). So, for the two previous business lines, one could set, for instance,  $n(C_{27}) = 10$  and  $n(C_{35}) = 20$ .

A property of the Poisson distribution (with parameter  $\lambda$ ) is that its probability mass function is nonincreasing convex if  $\lambda \leq 2 - \sqrt{2} \simeq 0.5858$ , and nonincreasing but not convex if  $2 - \sqrt{2} < \lambda \leq 1$ . Note that for our illustration,  $\lambda(C_{27}) = 0.37 < 2 - \sqrt{2} < \lambda(C_{35}) = 0.69 < 1$ . In fact, this shape for a counting distribution is quite frequent for standard (non-catastrophic) risks. Instead of a Poisson distribution, one could prefer to use a distribution with the same mean that is nonincreasing (convex) and has a bounded support  $\{0, \dots, n\}$ . The choice of such a distribution, however, is not an easy task. It is thus interesting to dispose of upper and lower bounds for the Solvency Capital Requirement.

The bounds for  $SCR = q\sigma$  are simply given by

$$SCR(N_{min}^{(2,2)}) = q\sqrt{\text{Var}(W)E(N) + [E(W)]^2 \text{Var}(N_{min}^{(2,2)})},$$

and

$$SCR(N_{max}^{(2,2)}) = q\sqrt{\text{Var}(W)E(N) + [E(W)]^2 \text{Var}(N_{max}^{(2,2)})}.$$

These provide a prudent version of the SCR and allow us to situate any particular choice for the distribution of  $N$  (within the class of interest). Let us fix the mean  $\nu_1 = E(N) = \lambda$ . From (7.3), (7.4), one gets (writing  $N^{(2,0)} \equiv N^{(2)}$ )

$$\begin{aligned} \text{Var}(N_{min}^{(2,0)}) &= \nu_1 - \xi(1 - 2\nu_1 + \xi) - \nu_1^2, \\ \text{Var}(N_{max}^{(2,0)}) &= n\nu_1 - \nu_1^2. \end{aligned}$$

From (5.3), (5.4),

$$\begin{aligned}\text{Var}(N_{min}^{(2,1)}) &= [-\tilde{\xi}^3 + \tilde{\xi}^2(-3 + 4\nu_1) + \tilde{\xi}(-2 + 11\nu_1 - 3\nu_1^2) + 6\nu_1(1 - \nu_1)]/3(\tilde{\xi} + 2), \\ \text{Var}(N_{max}^{(2,1)}) &= \nu_1(2n + 1)/3 - \nu_1^2,\end{aligned}$$

and from (5.11) with the remark before Corollary 5.4,

$$\begin{aligned}\text{Var}(N_{min}^{(2,2)}) &= (\tilde{\eta} - 1)(6\nu_1 + 2 - \tilde{\eta})/6 - \nu_1^2, \\ \text{Var}(N_{max}^{(2,2)}) &= \text{Var}(N_{max}^{(2,1)}),\end{aligned}$$

(as  $1 + 2^2 + \dots + n^2 = n(n+1)(2n+1)/6$  and  $1 + 2^3 + \dots + n^3 = n^2(n+1)^2/4$ ).

For  $n(C_{27}) = 10$ ,  $\lambda(C_{27}) = 0.37$ , one has  $\xi = 0$ ,  $\tilde{\xi} = 0$ ,  $\tilde{\eta} = 3$  and  $\text{Var}(N_{min}^{(2,1)}) = \text{Var}(N_{min}^{(2,0)})$ . For  $n(C_{35}) = 20$ ,  $\lambda(C_{35}) = 0.69$ , then  $\xi = 0$ ,  $\tilde{\xi} = 1$ ,  $\tilde{\eta} = 4$  and  $\text{Var}(N_{min}^{(2,1)}) > \text{Var}(N_{min}^{(2,0)})$ . Concerning the associated claim amounts, let us choose, for instance,  $E[W(C_{27})] = 1000$ ,  $\text{Var}[W(C_{27})] = 2500^2$  and  $E[W(C_{35})] = 2000$ ,  $\text{Var}[W(C_{35})] = 7000^2$  (in thousands of euros). We present some numerical results in Tables 1 and 2 below. Table 1 shows that the Poisson case is much closer to the lower bounds than the upper bounds. This means that the Poisson assumption may lead to serious underestimations if one uses it by default. Besides, the upper bound is significantly improved when the distribution is known to be nonincreasing. The \* indicates that the corresponding Poisson distribution is not convex; so, the bound given there is questionable if the Poisson distribution shape is taken for granted. From Table 2, one observes that, as expected, the choice of the maximal value  $n$  influences considerably the value of the variance of  $N$ ; its effect on the value of the SCR is, however, less important by comparison. Note that the lower bound does not depend on  $n$ .

**Residual lifetime at high ages.** Yearly mortality rates at high ages (larger than 100) are difficult to estimate. Nowadays, mortality rates at age 100 in France are close to 0.36. Until age 105, they slightly increase at a rate of around 0.015 per year; they tend to become stable after age 105. There is still some uncertainty about the pattern of mortality rates at high ages. Concerning, however, the residual lifetime floor at age 100, denoted by  $T_{100}$ , its distribution is observed to have a nonincreasing convex form whatever the pattern of mortality rates. This is shown in Figure 1 (resp. Figure 2) where the annual mortality rates increase of 0.015 -Hypothesis H1- (resp. remain constant -Hypothesis H2-); the life expectancy is then 1.68 (resp. 1.86) years. Note that at a lower age of 95, the mortality rate is of 0.20 with an increase of 0.03 per year up to age 105 say, after which the rate is taken constant -Hypothesis H3-; then, the residual lifetime floor  $T_{95}$  has a nonincreasing distribution which is no longer convex (see Figure 3).

Provided the residual lifetime floor has a nonincreasing (convex) distribution, its variance can be bounded by using again the previous extrema. Considering  $T_{100}$ , we choose  $n = 15$  as the probability to live more than 115 years is extremely small (positive of course). Results are presented in Table 3 for both Hypotheses 1 and 2. We observe that the lower bounds are not very far from the empirical case, especially when information on the shape of the distribution is provided.

**Number of asthma exacerbations.** Asthma exacerbation is another term for an asthma attack in which the bronchial tubes through which air flows to the lungs suddenly tighten

	Line $C_{27}$	Line $C_{35}$
$\lambda$	$\lambda(C_{27}) = 0.37$	$\lambda(C_{35}) = 0.69$
$n$	$n(C_{27}) = 10$	$n(C_{35}) = 20$
$\text{Var}(N_{min}^{(2,0)})$	0.2331	0.2139
$\text{Var}(N_{min}^{(2,1)})$	0.2331	0.4672
$\text{Var}(N_{min}^{(2,2)})$	0.2698	0.5939*
$\text{Var}(N)$ (Poisson case)	0.37	0.69
$\text{Var}(N_{max}^{(2,2)})$	2.4531	8.9539*
$\text{Var}(N_{max}^{(2,1)})$	2.4531	8.9539
$\text{Var}(N_{max}^{(2,0)})$	3.5631	13.3239
$SCR(N_{min}^{(2,0)})$	9573.0	35326.5
$SCR(N_{min}^{(2,1)})$	9573.0	35839.0
$SCR(N_{min}^{(2,2)})$	9641.7	36092.7*
$SCR$ (Poisson case)	9827.0	36283.9
$SCR(N_{max}^{(2,2)})$	13098.2	50065.2*
$SCR(N_{max}^{(2,1)})$	13098.2	50065.2
$SCR(N_{max}^{(2,0)})$	14543.8	55998.2

Table 1: Bounds on  $\text{Var}(N)$  and  $SCR$  for  $C_{27}$  and  $C_{35}$  when  $s = 2$  and  $t = 0, 1, 2$ .

$n(C_{27})$	$\text{Var}(N_{max}^{(2,2)})$	$SCR(N_{max}^{(2,2)})$
5	1.220	11276.6
10	2.453	13098.2
20	4.920	16135.7
30	7.386	18685.9
40	9.853	20927.5

Table 2: Bounds on  $\text{Var}(N)$  and  $SCR$  for  $C_{27}$  in function of  $n$  when  $s = t = 2$ .

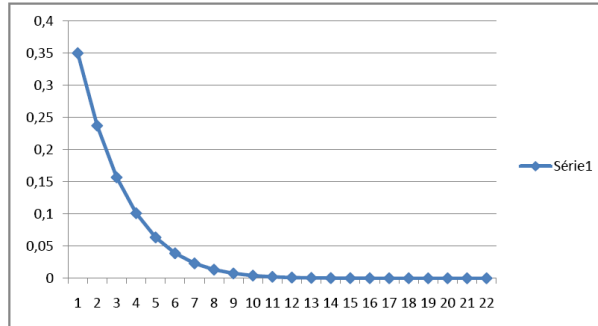


Figure 1: Probability mass function of  $T_{100}$  under  $H1$ .

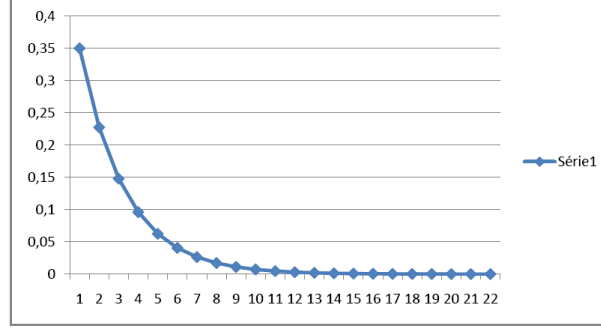


Figure 2: Probability mass function of  $T_{100}$  under H2.

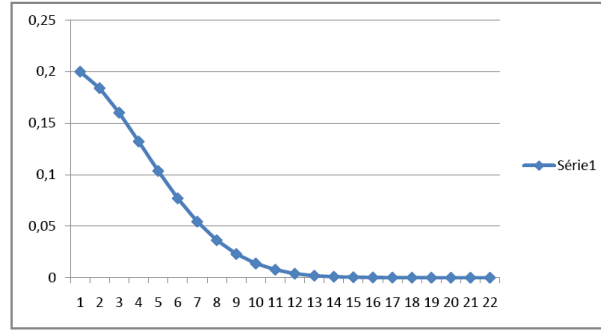


Figure 3: Probability mass function of  $T_{95}$  under H3.

Hypothesis	H1	H2
$E(T_{100})$	1.68	1.86
$\text{Var}(T_{100,min}^{(2,0)})$	0.22	0.22
$\text{Var}(T_{100,min}^{(2,1)})$	1.58	1.83
$\text{Var}(T_{100,min}^{(2,2)})$	2.26	2.69
$\text{Var}(T_{100})$ (empirical case)	3.77	5.27
$\text{Var}(T_{100,max}^{(2,2)})$	14.54	15.73
$\text{Var}(T_{100,max}^{(2,1)})$	14.54	15.73
$\text{Var}(T_{100,max}^{(2,0)})$	22.38	24.39

Table 3: Bounds on  $\text{Var}(T_{100})$  when  $s = 2$  and  $t = 0, 1, 2$ .

and become constricted. This makes it extremely difficult to breathe, resulting in an asthma exacerbation. There are many things that can trigger an asthma exacerbation and a quick treatment is necessary to prevent a medical emergency. A study involving 202 patients during one year has given the following empirical distribution (Table 4) for the number  $N$  of asthma exacerbations. This distribution is nonincreasing but is easily seen to be non-convex. There is almost no overdispersion as  $E(N) \simeq 1.1832$  and  $Var(N) \simeq 1.1852$ .

Given the observed mean of  $N$ , we computed the probability mass functions of the extrema for arbitrary or nonincreasing distributions valued on  $\{0, \dots, n = 5\}$  (using (7.3),(7.4) and (5.3),(5.4), respectively). The results are presented in Table 5. Moreover, bounds for the variance of  $N$  are provided in Table 6: for  $n = 5$ , the bounds are rather narrow and for  $n = 10$ , the upper bound becomes much higher (as expected).

Values $j$ of $N$	Frequencies	Proportions
0	67	0.33168
1	60	0.29703
2	53	0.26238
3	16	0.07921
4	5	0.02475
5	1	0.00495

Table 4: Empirical distribution for  $N$ .

$j$	$N_{min}^{(2,0)}$	$N_{max}^{(2,0)}$	$N_{min}^{(2,1)}$	$N_{max}^{(2,1)}$
0	0	0.7634	0.3028	0.60560
1	0.8168	0	0.3028	0.07888
2	0.1832	0	0.3028	0.07888
3	0	0	0.0916	0.07888
4	0	0	0	0.07888
5	0	0.2360	0	0.07888

Table 5: Probability mass functions of the extrema when  $s = 2$  and  $t = 0, 1$ , with  $n = 5$ .

	$n = 5$	$n = 10$
$Var(N_{min}^{(2,0)})$	0.1496	0.1496
$Var(N_{min}^{(2,1)})$	0.6942	0.6942
$Var(N)$ (empirical case)	1.1852	1.1852
$Var(N_{max}^{(2,1)})$	2.9384	6.8823
$Var(N_{max}^{(2,0)})$	5.9259	11.8519

Table 6: Bounds on  $Var(N)$  for  $s = 2$  and  $t = 0, 1$ , with  $n = 5$  or 10.



## 7 Appendix: convex extrema

The  $s$ -convex extrema are known under explicit form when  $s = 1, 2, 3, 4$  (see the references given in the Introduction). They are recalled below, for discrete random variables valued in a set  $\{0, \dots, n\}$ ,  $n \in \mathbb{N}$ , and for real-valued random variables with support in an interval  $[0, b]$ ,  $b > 0$ . We denote by  $\mathcal{B}_s(n)$  and  $\mathcal{B}_s(b)$  the classes of all such discrete and real-valued random variables  $X$  that have prescribed first  $s - 1$  moments  $\mu_i = E(X^i)$ ,  $i = 1, \dots, s - 1$ .

**1-convex extrema.** For the discrete [real-valued] case (inside  $\mathcal{B}_1(n)$  [ $\mathcal{B}_1(b)$ ]),

$$X_{min}^{(2)}(n) = 0 \quad [X_{min}^{(2)}(b) = 0] \quad \text{a.s.}, \quad (7.1)$$

and

$$X_{max}^{(2)}(n) = n \quad [X_{max}^{(2)}(b) = b] \quad \text{a.s.} \quad (7.2)$$

**2-convex extrema.** For the discrete case (inside  $\mathcal{B}_2(n)$ ), let  $\xi$  be the integer in  $[0, n - 1]$  such that  $\xi < \mu_1 \leq \xi + 1$ . Then, the minimum is given by

$$X_{min}^{(2)}(n) = \begin{cases} \xi & \text{with probability } \xi + 1 - \mu_1, \\ \xi + 1 & \text{with probability } \mu_1 - \xi. \end{cases} \quad (7.3)$$

and the maximum by

$$X_{max}^{(2)}(n) = \begin{cases} 0 & \text{with probability } 1 - \mu_1/n, \\ n & \text{with probability } \mu_1/n. \end{cases} \quad (7.4)$$

For the real-valued case (inside  $\mathcal{B}_2(b)$ ), the extrema are

$$X_{min}^{(2)}(b) = \mu_1 \quad \text{a.s.}, \quad (7.5)$$

and

$$X_{max}^{(2)}(b) = \begin{cases} 0 & \text{with probability } 1 - \mu_1/b, \\ b & \text{with probability } \mu_1/b. \end{cases} \quad (7.6)$$

**3-convex extrema.** For the discrete case (inside  $\mathcal{B}_3(n)$ ), let  $\xi_1$  and  $\xi_2$  be the integers in  $[0, n - 1]$  such that

$$\xi_1 < \mu_2/\mu_1 \leq \xi_1 + 1, \quad \text{and} \quad \xi_2 < (n\mu_1 - \mu_2)/(n - \mu_1) \leq \xi_2 + 1.$$

Then,

$$X_{min}^{(3)}(n) = \begin{cases} 0 & \text{with probability } p_1 = 1 - p_2 - p_3, \\ \xi_1 & \text{with probability } p_2 = [(\xi_1 + 1)\mu_1 - \mu_2]/\xi_1, \\ \xi_1 + 1 & \text{with probability } p_3 = (\mu_2 - \xi_1\mu_1)/(1 + \xi_1). \end{cases} \quad (7.7)$$

and

$$X_{max}^{(3)}(n) = \begin{cases} \xi_2 & \text{with probability } p_1 = 1 - p_2 - p_3, \\ \xi_2 + 1 & \text{with probability } p_2 = \frac{(n + \xi_2)\mu_1 - \mu_2 - n\xi_2}{n - 1 - \xi_2}, \\ n & \text{with probability } p_3 = \frac{(1 + \xi_2)(\xi_2 - \mu_1) + \mu_2 - \mu_1\xi_2}{(n - \xi_2)(n - 1 - \xi_2)}. \end{cases} \quad (7.8)$$

For the real-valued case (inside  $\mathcal{B}_3(b)$ ), the extrema are

$$X_{min}^{(3)}(b) = \begin{cases} 0 & \text{with probability } (\mu_2 - \mu_1^2)/\mu_2, \\ \mu_2/\mu_1 & \text{with probability } \mu_1^2/\mu_2, \end{cases} \quad (7.9)$$

and

$$X_{max}^{(3)}(b) = \begin{cases} \frac{b\mu_1 - \mu_2}{b - \mu_1} & \text{with probability } \frac{(b - \mu_1)^2}{(b - \mu_1)^2 + \mu_2 - \mu_1^2}, \\ b & \text{with probability } \frac{\mu_2 - \mu_1^2}{(b - \mu_1)^2 + \mu_2 - \mu_1^2}. \end{cases} \quad (7.10)$$

**4-convex extrema.** For the discrete case (inside  $\mathcal{B}_4(n)$ ), let  $\theta$  and  $\eta$  be integers in  $[0, n-1]$  with  $\theta + 1 < \eta$  such that  $\alpha(\theta, \eta) \geq 0$  where

$$\alpha(\theta, \eta) \equiv -\mu_3 + \mu_2(\theta + 2 + 2\eta) - \mu_1[(\theta + 1)\eta + (\theta + 1)(\eta + 1) + \eta(\eta + 1)] + (\theta + 1)\eta(\eta + 1),$$

as well as  $\alpha(\eta, \theta)$ ,  $-\alpha(\theta - 1, \eta)$  and  $-\alpha(\eta - 1, \theta) \geq 0$ . Then,

$$X_{min}^{(4)}(n) = \begin{cases} \theta & \text{with probability } p_1 = \alpha(\theta, \eta)/(\eta - \theta)(\eta - \theta + 1), \\ \theta + 1 & \text{with probability } p_2 = -\alpha(\theta - 1, \eta)/(\eta - \theta)(\eta - \theta - 1), \\ \eta & \text{with probability } p_3 = \alpha(\eta, \theta)/(\eta - \theta)(\eta - \theta - 1), \\ \eta + 1 & \text{with probability } p_4 = -\alpha(\eta - 1, \theta)/(\eta - \theta)(\eta - \theta + 1). \end{cases} \quad (7.11)$$

Let  $\zeta$  be the integer in  $[1, n-2]$  such that

$$\zeta < (\mu_2 n - \mu_3)/(\mu_1 n - \mu_2) \leq \zeta + 1.$$

Then,

$$X_{max}^{(4)}(n) = \begin{cases} 0 & \text{with probability } p_1 = 1 - p_2 - p_3 - p_4, \\ \zeta & \text{with probability } p_2 = \frac{n\mu_1(\zeta+1) - \mu_2(\zeta+1+n) + \mu_3}{\zeta(n-\zeta)}, \\ \zeta + 1 & \text{with probability } p_3 = \frac{\mu_2(\zeta+n) - n\mu_1\zeta - \mu_3}{(\zeta+1)(n-\zeta-1)}, \\ n & \text{with probability } p_4 = \frac{\mu_3 - \mu_2(2\zeta+1) + \mu_1\zeta(\zeta+1)}{n(n-\zeta)(n-\zeta-1)}. \end{cases} \quad (7.12)$$

For the real-valued case (inside  $\mathcal{B}_4(b)$ ), putting

$$r_- \text{ (resp. } r_+) = \frac{\mu_3 - \mu_1\mu_2 - (+) [(\mu_3 - \mu_1\mu_2)^2 - 4(\mu_2 - \mu_1^2)(\mu_1\mu_3 - \mu_2^2)]^{1/2}}{2(\mu_2 - \mu_1^2)},$$

then,

$$X_{min}^{(4)}(b) = \begin{cases} r_- & \text{with probability } (r_+ - \mu_1)/(r_+ - r_-), \\ r_+ & \text{with probability } (\mu_1 - r_-)/(r_+ - r_-), \end{cases} \quad (7.13)$$

and

$$X_{max}^{(4)}(b) = \begin{cases} 0 & \text{with probability } p_1 = 1 - p_2 - p_3, \\ \frac{b\mu_2 - \mu_3}{b\mu_1 - \mu_2} & \text{with probability } p_2 = \frac{(b\mu_1 - \mu_2)^3}{(b\mu_2 - \mu_3)(b^2\mu_1 - 2b\mu_2 + \mu_3)}, \\ b & \text{with probability } p_3 = \frac{\mu_1\mu_3 - \mu_2^2}{b(b^2\mu_1 - 2b\mu_2 + \mu_3)}. \end{cases} \quad (7.14)$$

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