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A Radon-Nikodym derivative for almost subadditive set functions

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A Radon-Nikodym derivative for almost subadditive set functions

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Abstract

In classical measure theory, the Radon-Nikodym theorem states in a concise condition, namely domination, how a measure can be factorized by another (bounded) measure through a density function. Several approaches have been undertaken to see under which conditions an exact factorization can be obtained with set functions that are not σ -additive (for instance finitely additive set functions or submeasures). We provide a Radon-Nikodym type theorem with respect to a measure for almost subadditive set functions of bounded sum. The necessary and sufficient condition to guarantee a one-sided Radon-Nikodym derivative remains the standard domination condition for measures.

Keywords: Radon-Nikodym derivative, Choquet integral, subadditive set function.

1 Introduction

In classical measure theory, the Radon-Nikodym theorem states in a concise condition, namely domination, how a measure can be factorized by another (bounded) measure through a density function. Several approaches have been undertaken to see under which conditions an exact factorization can be obtained with set functions that are not σ -additive. Necessary and sufficient conditions are found in [1] for finitely additive set functions. Another direction has been taken in [7, 10] where continuity is kept but subadditivity is substituted for additivity. These approaches reveal how Hahn decomposition properties play a prominent

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rôle in order to exhibit exact Radon-Nikodym derivatives. Building set functions from existing ones can be naturally carried on through Choquet integrals (see [15]), i.e. an extension of Lebesgue's integral for set functions (see [4]). Another version for non-additive set functions without continuity is exposed in [8] where a local derivative can be shown to exist on every finite subalgebras. For different motivations stemming from potential theory, a concise note [9] presents a Radon-Nikodym theorem for a measure with respect a σ -submeasure. Our study is devoted to the reverse direction and we exhibit a one-sided Radon-Nikodym derivatives for a set function w.r.t. a measure. Our motivation comes from decision making under uncertainty theory and how information can be handle in the expected utility framework (see [6]). We shall consider set functions of bounded sum (see [5, 11]) that are not necessarily monotone and that satisfy almost subadditivity and semicontinuity from below. These conditions are weaker than the standard properties of monotonicity, subadditivity and continuity from below. For these set functions the necessary and sufficient condition to guarantee a one-sided Radon-Nikodym derivative remains the standard domination condition.

Next section introduces the class of almost subadditive semicontinuous from below set functions of bounded sum. The final section exposes some Radon-Nikodym theorems version for almost subadditive set functions. An appendix contains the technical material related to the existence of the Choquet integral for semicontinuous from below set functions of bounded sum.

2 Set functions of bounded sum

From now on (Ω, \mathcal{A}) denotes a measurable space.

$w : \mathcal{A} \rightarrow \mathbb{R}^+$ with $w(\emptyset) = 0$ is a *set function*.

w is *null hereditary*¹(0-H) if $\forall A, B \in \mathcal{A}, w(A) = 0 \Rightarrow w(B) = 0$ whenever $B \subset A$.

w is *monotone* if $\forall A, B \in \mathcal{A}, w(A) \leq w(B)$ whenever $A \subset B$.

A monotone set function is null hereditary.

w is *subadditive* if $\forall A, B \in \mathcal{A}, w(A \cup B) \leq w(A) + w(B)$ whenever $A \cap B = \emptyset$.

w is *superadditive* when the opposite inequality holds.

w is of *bounded sum* i.e., $w \in BS$ if

$$\|w\|_{BS} = \sup\{\sum_i w(P_i) : \{P_i\}_i \in \mathcal{P}\} < +\infty$$

where \mathcal{P} denotes the set of partitions: $\{P_i\}_i \in \mathcal{P}$ if $\sum_i P_i = \Omega$ i.e., $\cup_i P_i = \Omega$ and $P_i \cap P_j = \emptyset$ for $i \neq j$.

$\|\cdot\|_{BS}$ is the classical norm for additive set functions ([11]).

If w is of bounded sum then it is *exhaustive*. For any countable partition $\{A_n\}_n \subset \mathcal{A}$ we have $\|w\|_{BS} \geq \sum_n w(A_n)$, thus $\lim_n w(A_n) = 0$.

¹Otherwise stated, w is converse null-null additive, i.e. $w(A \cup B) = 0 \Rightarrow w(A) = w(B) = 0$, whenever A, B are disjoint sets.

We can define a (extended) set function \bar{w} in the following manner, $\forall A \in \mathcal{A}$,

$$\bar{w}(A) = \sup\left\{\sum_i w(P_i) : \sum_i P_i = A\right\} \in [0, \infty],$$

in particular $\bar{w}(\Omega) = \|w\|_{BS}$, $\bar{w}(\emptyset) = 0$. By construction \bar{w} is superadditive (see Theorem 3.1 in [11]) and $\bar{w} \geq w$.

2.1 Almost subadditive set functions

We introduce a class of set functions which is a kind of an intermediate between subadditive and null-additive set functions.

A set function is *almost subadditive* if

$$\forall A, B \in \mathcal{A}, AB = \emptyset, w(A \cup B) \leq \bar{w}(A) + w(B),$$

otherwise putted,

$$\forall A, \sup_{B:AB=\emptyset} w(A \cup B) - w(B) \leq \bar{w}(A).$$

Hence the marginal contribution of A to B is always smaller than the potential contribution of A standing alone. A subadditive set function is clearly almost subadditive². A monotone a.-subadditive set function is null-additive³.

EXAMPLE 1: Let $\Omega = \{1, 2, 3, 4\}$ be a four element set. Consider w defined by $w(i) = 3$, $w(ij) = 4$, $w(ijk) = 7$ and $w(\Omega) = 9$. Then w is almost subadditive since

$$w(ijk) - w(jk) = 3 = \bar{w}(i), \quad w(ij) - w(j) = 1 \leq \bar{w}(i), \quad w(\Omega) - w(jkl) = 2 \leq \bar{w}(i)$$

and

$$w(\Omega) - w(kl) = 5 \leq 6 = \bar{w}(ij), \quad w(ijk) - w(k) = 4 \leq \bar{w}(ij),$$

and

$$w(\Omega) - w(l) = 6 \leq 9 = \bar{w}(ijk)$$

But w is not subadditive since $w(ij) + w(kl) = 8 < 9 = w(\Omega)$.

Lemma 1 and 2 are restatements of Theorem 3.1 in [11].

Lemma 1 Let w be a set function. Then $w \in BS$ if and only if there exists a superadditive set function v such that $v \geq w$.

²In general, subadditive monotone set functions are not necessarily of bounded sum. For instance, assume \mathcal{A} infinite and consider the degenerate set function w defined by $w(A) = 1$ if $A \neq \emptyset$.

³A non-monotone 0-H a.-subadditive is null-null-additive i.e. $w(A) = w(B) = 0 \Rightarrow w(A \cup B) = 0$, whenever A, B are disjoint sets.

Proof: (Only if). Take $v = \bar{w}$.

(If). Let $\{P_i\}_i \in \mathcal{P}$ and $v \geq w$, then $\sum_i w(P_i) \leq \sum_i v(P_i) \leq v(\Omega)$ thus $\|w\|_{BS} \leq v(\Omega)$. \square

It can be shown that \bar{w} is the superadditive envelope of w .

Lemma 2 *Let w be a set function, $w \in BS$. Define w^{SA} in the following manner, for all $A \in \mathcal{A}$,*

$$w^{SA}(A) = \inf\{v(A) : v \geq w, v \text{ superadditive}\}$$

then $w^{SA} = \bar{w}$, and the infimum is attained.

Proof: (\leq) By definition of w^{SA} , since \bar{w} is superadditive.

(\geq) Let $v \geq w$, v be superadditive and take $\sum_i P_i = A$, we have $\sum_i w(P_i) \leq \sum_i v(P_i) \leq v(A)$ thus $\bar{w}(A) \leq v(A)$. Hence $\bar{w} \leq w^{SA}$. \square

In particular if there exists an additive set function P , $P \geq w$ then $w \in BS$. A sufficient condition for \bar{w} being additive is that w is a.-subadditive.

Lemma 3 (SEE THEOREM 3.2 IN PAP) *Let w be a set function, $w \in BS$ be a.-subadditive. Then \bar{w} is additive and $\bar{w} = \wedge_{\{P:P \geq w\}} P$. In particular, $w(\Omega) = \|w\|_{BS}$ if and only if w is additive.*

Proof: Let us prove that \bar{w} is subadditive. Take disjoint sets $A_1, A_2 \in \mathcal{A}$. For $\epsilon > 0$, there exists a partition $\{P_i\}_i \in \mathcal{P}$ such that

$$\begin{aligned} \bar{w}(A_1 \cup A_2) - \epsilon &< \sum_i w((A_1 \cup A_2)P_i) \\ &\leq \sum_i \bar{w}(A_1 P_i) + \sum_i w(A_2 P_i) \leq \bar{w}(A_1) + \sum_i w(A_2 P_i) \leq \bar{w}(A_1) + \bar{w}(A_2) \end{aligned}$$

since \bar{w} is superadditive.

As for the infimum condition, if $P \geq w$ then $P \geq \bar{w}$ and this infimum is attained for \bar{w} which is additive.

(If) It is immediate.

(Only if). For $A \in \mathcal{A}$ we have

$$w(\Omega) \leq w(A) + \bar{w}(A^c) \leq \bar{w}(A) + \bar{w}(A^c) \leq \bar{w}(\Omega) = \|w\|_{BS} = w(\Omega)$$

thus $w(A) = \bar{w}(A)$, which is additive. \square

We may state,

Sandwich Theorem: *Let w be an a.-subadditive set function and v a superadditive set function such that $v \geq w$. Then there exists an additive set function Q such that $v \geq Q \geq w$.*

Proof: Since $v \geq w$ and v is superadditive, w is of bounded sum. So according to Lemma 2 and 3, we can take $Q = \bar{w}$. \square

2.2 Continuity properties of set functions

We now come to continuity properties of a.-subadditive set functions. Let w be a set function.

w is *order-continuous* i.e., $w \downarrow \emptyset$, if $w(A_n) \rightarrow 0$ whenever $A_n \downarrow \emptyset$.

w is *semicontinuous from below* i.e., $w \uparrow_s$, if $\liminf_n w(A_n) \geq w(A)$ whenever $A_n \uparrow A$.

w is *continuous from below* i.e., $w \uparrow$, if $w(A_n) \rightarrow w(A)$ whenever $A_n \uparrow A$.

w is *monotonely-continuous* i.e., $w \downarrow \uparrow$, if $w(A_n) \rightarrow w(A)$ whenever $A_n \downarrow \uparrow A$.

w is σ -*continuous* if $w(A_n) \rightarrow w(A)$ whenever $A_n \rightarrow A^4$.

w is σ -*subadditive* if $\forall \{A_n\}_n \subset \mathcal{A}, \sum_n A_n = A \Rightarrow w(A) \leq \sum_n w(A_n)$.

For additive set functions, order-continuity, monotone-continuity and σ -continuity are all equivalent. Order-continuous additive set functions are generally termed *measures*. Semicontinuity from below implies continuity from below under monotonicity. According to Proposition 2.1 in [11], monotone continuity and σ -continuity coincide whenever w is monotone.

Proposition 1 *Let w be a set function, $w \in BS$ be a.-subadditive. The implications (i) \Rightarrow (ii) \iff (iii) hold,*

(i) *w is monotonely-continuous,*

(ii) *w is semicontinuous from below,*

(iii) *\bar{w} is σ -additive.*

Moreover, if w is monotone (iii) \Rightarrow (i) holds.

Proof: We shall prove (i) \Rightarrow (ii) \iff (iii) \Rightarrow (i).

(i) \Rightarrow (ii). By definition.

(ii) \Rightarrow (iii). \bar{w} being additive it suffices to prove that \bar{w} is continuous at Ω . Let $A_n \uparrow \Omega$, $\epsilon > 0$. There exist $B_1, \dots, B_K \in \mathcal{A}$ such that $\sum_k w(B_k) > \|w\|_{BS} - \epsilon/2$. As $A_n \uparrow \Omega$ we also have $A_n \cap B_k \uparrow B_k$ and since $w \uparrow_s$, for n large enough, it holds for all k

$$w(A_n \cap B_k) > w(B_k) - \epsilon/2K$$

hence

$$\bar{w}(A_n) \geq \sum_k w(A_n \cap B_k) > \sum_k w(B_k) - \epsilon/2 > \|w\|_{BS} - \epsilon.$$

(iii) \Rightarrow (ii). Let $A_n \uparrow A$. Since w is a.-subadditive it holds, $w(A) \leq \bar{w}(A \setminus A_n) + w(A_n)$. And by σ -additivity of \bar{w} , $w(A) \leq \liminf_n w(A_n)$.

(iii) \Rightarrow (i). Since (iii) \iff (ii) and w is monotone, w is continuous from below. As for continuity from above.

Let $A_n \downarrow A$. By almost subadditivity and monotonicity we get, $w(A) \leq w(A_n) \leq w(A) + \bar{w}(A_n \setminus A)$. Taking the limit as n goes to ∞ , $w(A) \leq \lim_n w(A_n) \leq w(A)$. \square

We may precise the Sandwich Theorem,

⁴ $A_n \rightarrow A$ if $\liminf A_n = \limsup_n A_n = A$.

Sandwich Theorem: Let w be an a -subadditive semicontinuous from below set function and v a superadditive set function such that $v \geq w$. Then there exists a σ -additive set function Q such that $v \geq Q \geq w$.

Proposition 2 Let w be a set function, $w \in BS$ be subadditive. The following statements are equivalent,

- (i) w is semicontinuous from below,
- (ii) w is order-continuous,
- (iii) w is σ -subadditive.

Proof: (i) \Rightarrow (iii). Let $\{A_n\}_n \subset \mathcal{A}, \sum_n A_n = A$. By subadditivity we have, $w(\sum_{k=1}^n A_k) \leq \sum_{k=1}^n w(A_k)$. Thus, by semicontinuity from below, it comes,

$$w\left(\sum_n A_n\right) \leq \liminf_n w\left(\sum_{k=1}^n A_k\right) \leq \liminf_n \sum_{k=1}^n w(A_k) = \sum_n w(A_n).$$

(iii) \Rightarrow (ii). Let $A_n \downarrow \emptyset$. Put $B_n = A_n \setminus A_{n+1}$. Since w is of bounded sum, we have $\sum_n w(B_n) \leq \|w\|_{BS} < \infty$. By σ -subadditivity, $w(A_n) \leq \sum_{k \geq n} w(B_k) \downarrow 0$.
(ii) \Rightarrow (i). Let $A_n \uparrow A$. By subadditivity, $w(A) \leq w(A_n) + w(A \setminus A_n)$. Thus by order-continuity, $w(A) \leq \liminf_n w(A_n)$. \square

Remark: An alternative way to prove (i) \Rightarrow (ii) if w is monotone can be used. On one hand w is exhaustive since $w \in BS$. On the other hand μ is σ -subadditive since it is continuous from below and subadditive. Now from Theorem 2.8 (iii) in [11], an exhaustive and σ -subadditive monotone set function is order-continuous

We may precise the Sandwich Theorem for subadditive set functions,

Sandwich Theorem (continuation): Let w be subadditive and v be superadditive set functions such that $v \geq w$. Then, there exists a σ -additive set function Q such that $v \geq Q \geq w$ if and only if w is order-continuous.

In light of the previous results we obtain,

Jordan decomposition Theorem: Let w be subadditive. Then, $w \in BS$ if and only if there are superadditive set functions s^+, s^- such that $w = s^+ - s^-$. Where, w is order-continuous if and only if s^+, s^- are order-continuous. Moreover, $(\bar{w}, \bar{w} - w)$ is an optimal decomposition i.e.

$$w = s^+ - s^- \Rightarrow \bar{w} \leq s^+ \text{ and } \bar{w} - w \leq s^-.$$

Proof: (If). Since $0 \leq w = s^+ - s^- \leq s^+$, w is of bounded sum according to Lemma 1. It is immediate to check that w is order-continuous whenever s^+, s^- are order-continuous.

(Only if). Since w is subadditive, by Lemma 3, \bar{w} is additive, thus superadditive and $\bar{w} - w$ is superadditive. If w is order-continuous then by Proposition 1 and 2, \bar{w} is order-continuous, hence $\bar{w} - w$ too.

(Moreover). By Lemma 2, \bar{w} is minimal, thus $\bar{w} \leq s^+$ hence $\bar{w} - w \leq s^-$. \square

We provide simple ways to produce a.-subadditive set functions of finite sum.

EXAMPLE 2: Let P_1, P_2 be measures, define $w = \max\{P_1, P_2\}$.

EXAMPLE 3: Let P be a non-atomic probability measure on \mathcal{A} and $f : [0, 1] \rightarrow \mathbb{R}^+$ be a function with $f(0) = 0$ that admits limits from the left. Put $w = f \circ P$. If the derivative of f at 0 is finite i.e., $f'(0) < \infty$, and for all $0 \leq q \leq p \leq 1$, $f(p) - f(q) \leq f'(0)(p - q)$, then w is an a.-subadditive set function of finite sum and semicontinuous from below⁵. And w is 0-H whenever $f(p) > 0$ for $p > 0$. Moreover $f'(0)P = \bar{w}$.

Proof: For $q = 0$ we have $f'(0)p \geq f(p)$ for all $p \in [0, 1]$. We have $f'(0)P(A) \geq f(P(A)) = w(A)$ for all $A \in \mathcal{A}$, thus $f'(0)P(A) \geq \bar{w}(A)$ and w is of bounded sum.

For the opposite inequality. For $A \in \mathcal{A}$ and $n \in \mathbb{N}$, since P is non-atomic there are disjoint subsets $\{A_i\}_{i=1}^n$ such that $P(A_i) = \frac{1}{n}P(A)$ and $\sum_{i=1}^n A_i = A$. So,

$$\begin{aligned} \bar{w}(A) &\geq \sum_{i=1}^n f(P(A_i)) = \sum_{i=1}^n f\left(\frac{1}{n}P(A)\right) \\ &= P(A) \frac{f\left(\frac{1}{n}P(A)\right)}{\frac{1}{n}P(A)} \rightarrow P(A)f'(0)(n \rightarrow \infty) \end{aligned}$$

For disjoint sets $A_1, A_2 \in \mathcal{A}$, we have $f(P(A_1 + A_2)) - f(P(A_2)) \leq f'(0)P(A_1) = \bar{w}(A_1)$, i.e. w is a.-subadditive.

Let us prove that w is semicontinuous from below. By assumption, $f(p) \leq f(q) + f'(0)(p - q)$ for $0 \leq q \leq p \leq 1$. Hence for $q \uparrow p$, it comes $f(p) \leq f(p-)$. Let $A_n \uparrow A$. Since P is continuous from below, $P(A_n) \uparrow P(A)$, thus $\liminf_n w(A_n) \geq w(A)$. \square

The assumption that $f'(0) < \infty$ can not be dropped if we wish that $w = f \circ P$ to be of bounded sum. Otherwise \bar{w} takes its value in $\{0, \infty\}$ whenever P is non-atomic. For instance consider the entropy function

$$f : [0, 1] \rightarrow \mathbb{R}^+ : p \mapsto -p \ln(p)$$

with $f(0) = 0$. Then, $f'(0) = \lim_0 \ln(1/p) = \infty$. Put $w = f \circ P$. w is a subadditive continuous from below set function. However, w is not 0-H⁶ and w is not of bounded sum.

Example 3 provides a way to construct a.-subadditive semicontinuous from below set function of finite sum that are neither subadditive nor continuous from below. Consider the following piecewise linear probability distortion,

$$f(p) = \begin{cases} 2p, & \text{if } 0 \leq p \leq 1/8 \\ 1/4, & \text{if } 1/8 \leq p \leq 1/2 \\ 2p - 3/4, & \text{if } 1/2 \leq p < 1 \\ 1, & \text{if } p = 1 \end{cases}$$

⁵For instance if f is concave or when f is derivable with $f' \leq f'(0)$.

⁶Since $w(\Omega) = 0$ and $w \geq 0, w \neq 0$.

Then, $f \circ P$ is not subadditive since $f(1/2) = 1/4 < 1/2$ and $f \circ P$ is not continuous from below since $f(1-) = 5/4 > 1$.

3 Super Radon-Nikodym derivatives

Let μ, w be set functions. μ dominates w i.e., $\mu \gg w$, if for all $A \in \mathcal{A}$, $\mu(A) = 0 \Rightarrow w(A) = 0$.

μ strongly dominates w i.e., $\mu \gg_s w$, if for all $\epsilon > 0$ there exists $\eta > 0$ such that for all $A \in \mathcal{A}$, $w(A) < \epsilon$ whenever $\mu(A) < \eta$.

These notions of absolute continuity coincide whenever μ, w are measures, and more generally if w is an a.-subadditive set function,

Property 1 *Let w be an a.-subadditive semicontinuous from below set function with $w \in BS$ and μ a measure. Then, $\mu \gg w$ if and only if $\mu \gg_s w$.*

Proof: (If). It is immediate.

(Only if). Assume $\mu \gg w$, then $\mu \gg \bar{w}$. Since w is an a.-subadditive semicontinuous from below, by Proposition 1, \bar{w} is σ -additive thus $\mu \gg_s \bar{w}$ and $\mu \gg_s w$ follows. \square

The classical Radon-Nikodym theorem for measures is,

Theorem :(RADON-NIKODYM) *Let μ, ν be measures. Then, $\mu \gg \nu$ if and only if there exists a function $f \geq 0$, \mathcal{A} -measurable and integrable such that $\nu = f\mu$, i.e., for all $A \in \mathcal{A}$,*

$$\nu(A) = \int_A f d\mu = f\mu(A),$$

where $f = \frac{d\nu}{d\mu}$ is the Radon-Nikodym derivative of ν w.r.t. μ .

Hence, for any bounded non-negative measurable function X , $\int X d\nu = \int X f d\mu$.

Let w be a set function. For a non-negative measurable function we may define the Choquet integral ([3, 14]) on $A \in \mathcal{A}$ w.r.t. w by the following quantity,

$$\int_A X dw = \int_0^\infty w(\{X > t\} \cap A) dt = \int_0^\infty w(\{X 1_A > t\}) dt \in [0, \infty]$$

where the integral under consideration is the Lebesgue integral on $[0, \infty)$. This integral is well defined as soon as w is semicontinuous from below. And is finite for all bounded X whenever w is bounded (see Appendix). Denote the space of w -integrable functions,

$$\mathcal{L}^+(w) = \{X : X \geq 0, \mathcal{A} - \text{measurable with } \int X dw < \infty\}.$$

This integral coincide with Lebesgue's integral whenever w is σ -additive.

Definition 1 Let w be a set function and μ a measure. w is majorized by μ if there exists a function $g \geq 0$, \mathcal{A} -measurable and μ -integrable such that

$$w \leq g\mu$$

That is to say, g majorizes w .

So whenever w is majorized, w is necessarily of bounded sum, order-continuous and $\mu \gg g\mu \gg w$.

Definition 2 Let w be a set function and μ a measure. w admits a superior Radon-Nikodym derivative w.r.t. μ if there exists f^s that majorizes w and f^s is minimal i.e.

$$\forall g, w \leq g\mu \Rightarrow f^s \leq g, \mu - a.e.$$

We shall say that $\frac{dw^s}{d\mu}$ exists and write $\frac{dw^s}{d\mu} = f^s$ where no confusion is possible.

In particular, for any bounded non-negative measurable function X ,

$$\int X dw \leq \int X f^s d\mu$$

Indeed, $\int X dw \leq \int X df^s \mu$ and $f^s \mu$ is σ -additive so $\int X df^s \mu = \int X f^s d\mu$.

Property 2 Let w, W be set functions.

If w is 0-H and $w \equiv W$ then W is 0-H.

Moreover, w is 0-H if and only if $w \equiv \bar{w}$.

Proof: Let $A, B \in \mathcal{A}$, $A \supset B$ where $W(A) = 0$. Since $W \gg w$, we have $w(A) = 0$. w is 0-H thus $w(B) = 0$. So $w \gg W$ entails $W(B) = 0$.

(Moreover). (If). Let $A, B \in \mathcal{A}$, $A \supset B$ where $w(A) = 0$. So $\bar{w}(A) = 0$. By construction $\bar{w}(A) \geq \bar{w}(B) \geq w(B) \geq 0$ thus $w(B) = 0$.

(Only if). It suffices to prove that $w \gg \bar{w}$. Let $A, A_1, \dots, A_n \in \mathcal{A}$ with $A = \sum_{i=1}^n A_i$ and $w(A) = 0$. Since w is 0-H, for all i , $w(A_i) = 0$. Thus $\sum_{i=1}^n w(A_i) = 0$, so $\bar{w}(A) = 0$. \square

Theorem 1 Let w be an a -subadditive set function and μ a measure. Then, w admits a superior Radon-Nikodym derivative w.r.t. μ if and only if

$$w \in BS, w \uparrow_s, \mu \gg w$$

Moreover, $f^s \mu \equiv w$ if and only if w is 0-H.

Hence, for any bounded non-negative measurable function X , $\int X dw = 0$ if and only if $\int X f^s d\mu = 0$.

So all the pertinent information contained in w can be summarized through a factorization and this is made precisely with the superior Radon-Nikodym derivative.

Proof: (If). Since $w \in BS$ and $w \uparrow_s$, according to Proposition 1, \bar{w} is σ -additive. Moreover $\mu \gg w$ thus $\mu \gg \bar{w}$ hence by the classical Radon-Nikodym theorem there exists f^s such that $\bar{w} = f^s \mu$. But $w \leq \bar{w}$, so $w \leq f^s \mu$ i.e. f^s majorizes w .

It remains to show that f^s is the superior R-N derivative. Let g majorizes w . Since $g\mu$ is a measure and $w \leq g\mu$, Lemma 2 gives $\bar{w} \leq g\mu$, thus $f^s \mu \leq g\mu$ hence $f^s \leq g$ μ -a.e.

(Only if). Let w admit a superior R-N derivative f^s thus $w \leq f^s \mu$.

But $f^s \mu$ is σ -additive, thus $w \in BS$ and $\bar{w} \leq f^s \mu$ hence $\bar{w} \downarrow \emptyset$. And \bar{w} is additive according to Lemma 3, thus $w \uparrow_s$ by Proposition 1. Finally, $\mu \gg f^s \mu$ and $f^s \mu \geq w$ thus $\mu \gg w$.

(Moreover). (If). Assume w is 0-H, then by Property 2, $\bar{w} \equiv w$. And since $\bar{w} = f^s \mu$, we have $w \equiv f^s \mu$.

(Only if). Since $f^s \mu$ is 0-H and $f^s \mu \equiv w$, by Property 2, w is 0-H.

(Hence). We have, $\int X dw = 0 \iff w(\{X > t\}) = 0, \lambda - \text{a.e.} \iff f^s \mu(\{X > t\}) = 0, \lambda - \text{a.e.} \iff \int X df^s \mu = 0 \iff \int f^s X d\mu = 0. \quad \square$

Thanks to Theorem 1 we can extend the informative part of the Radon-Nikodym derivative to couples of a.-subadditive set functions.

Theorem 2 *Let $w, W \in BS$ be a.-subadditive, 0-H, semicontinuous from below set functions. The following statements are equivalent.*

(i) $w \gg W$,

(ii) *There exists $f \in \mathcal{L}^+(w)$ s.t. $fw \equiv W$,*

(iii) $f^s w \equiv W$ with $f^s = \frac{dW}{d\bar{w}}$.

Hence, for any bounded non-negative measurable function X , $\int X dW = 0$ if and only if $\int X f^s dw = 0$.

Proof: (ii) \Rightarrow (i). Since w is 0-H, $w \gg fw \equiv W$.

(iii) \Rightarrow (ii). Since $f^s \in \mathcal{L}^+(\bar{w}) \subset \mathcal{L}^+(w)$ we can pick $f = f^s$.

(i) \Rightarrow (iii). Since $w \in BS$ is a.-subadditive semicontinuous from below and $w \gg W$, \bar{w} is a measure and $\bar{w} \gg W$. So according to the ‘‘Moreover part’’ of Theorem 1 with W 0-H,

$$f^s \bar{w} \equiv W.$$

It remains to check that $f^s \bar{w} \equiv f^s w$. By construction, $f^s \bar{w} \gg f^s w$ since $\bar{w} \geq w$. As for the other inequality. Let $A \in \mathcal{A}$ such that, $f^s w(A) = \int_A f^s dw = \int_0^\infty w(\{f^s \geq t\} \cap A) = 0$. Since $w \geq 0$, we have $w(\{f^s \geq .\} \cap A) = 0, \lambda$ a.e. But w is 0-H so by Property 2, $\bar{w} \equiv w$. Hence $\bar{w}(\{f^s \geq .\} \cap A) = 0, \lambda$ a.e. And \bar{w} is monotone so $\bar{w}(\{f^s \geq .\} \cap A)$ is non-increasing on \mathbb{R}^+ thus measurable, so $f^s \bar{w}(A) = 0$.

(Hence). Since $f^s \bar{w} \equiv W$ we have, $\int X dW = 0 \iff W(\{X > t\}) = 0, \lambda - \text{a.e.} \iff f^s \bar{w}(\{X > t\}) = 0, \lambda - \text{a.e.} \iff \int X df^s \bar{w} = 0 \iff \int f^s X d\bar{w} = 0 \iff \int f^s X dw = 0. \quad \square$

Corollary 1 *Let w be an a.-subadditive set function and μ a measure such that $\frac{dw}{d\mu}$ exists then w is σ -additive if and only if $w = f^s \mu$*

Proof: (If) immediate.

(Only if). We have $\frac{dw}{d\mu}\mu = w \leq \frac{dw^s}{d\mu}\mu \leq \frac{dw}{d\mu}\mu$ by minimality of the superior Radon-Nikodym derivative, thus $\frac{dw}{d\mu} = \frac{dw^s}{d\mu}$ μ -a.e. \square

Corollary 2 *Let w be an a.-subadditive set function and μ such that $\frac{dw^s}{d\mu}$ exists. For X a bounded non-negative measurable function define, $W(A) = \int_A X dw$ for all $A \in \mathcal{A}$. Then W is majorized by $X \frac{dw^s}{d\mu}$.*

Moreover, if w is subadditive then $\frac{dW^s}{d\mu}$ exists and

$$\frac{dW^s}{d\mu} \leq X \frac{dw^s}{d\mu}$$

and if w is 0-H then W is 0-H.

Proof: Since $\int Y dw \leq \int Y \frac{dw^s}{d\mu} d\mu$ for any bounded non-negative measurable function Y , it holds for $Y = X1_A$, thus $W(A) = \int_A X dw \leq \int_A X \frac{dw^s}{d\mu} d\mu = M(A)$ i.e., W is majorized by $X \frac{dw^s}{d\mu}$ and $W \ll \mu$.

Since M is σ -additive, $W \in BS$ and W is order-continuous.

Assume moreover that w is subadditive. For disjoint sets $A, B \in \mathcal{A}$, and $t > 0$

$$w(\{X > t\} \cap (A \cup B)) \leq w(\{X > t\} \cap A) + w(\{X > t\} \cap B)$$

by integration it comes $W(A \cup B) \leq W(A) + W(B)$. And by Proposition 2, W is semicontinuous from below, so according to Theorem 1, W admits a superior Radon-Nikodym derivative and

$$\frac{dW^s}{d\mu} \leq X \frac{dw^s}{d\mu}.$$

The last part is immediate. \square

The following example is a particular case of Corollary 3 where $w_i = P_i$,

EXAMPLE 2 (CONTINUATION): *Let P_1, P_2, μ be measures with $P_1, P_2 \ll \mu$ and define $w = \max\{P_1, P_2\}$. Then*

$$\frac{dw^s}{d\mu} = \frac{d(P_1 \vee P_2)}{d\mu} = \max\left\{\frac{dP_1}{d\mu}, \frac{dP_2}{d\mu}\right\},$$

moreover there exists a set $H \in \mathcal{A}$ such that

$$\frac{dw^s}{d\mu} = \frac{dP_1}{d\mu} 1_H + \frac{dP_2}{d\mu} 1_{H^c}.$$

Proof: On one hand $P_1, P_2 \leq w \leq \bar{w}$ and \bar{w} is additive, thus $P_1 \vee P_2 \leq \bar{w}$ holds. On the other hand $P_1, P_2 \leq P_1 \vee P_2$ thus $w \leq P_1 \vee P_2$ and $\bar{w} \leq P_1 \vee P_2$ follows. The superior Radon-Nikodym derivative can be precised.

For $P_i \leq w \leq \bar{w}$, we have $\frac{dP_i}{d\mu} \leq \frac{d\bar{w}}{d\mu}$, thus $\max\{\frac{dP_1}{d\mu}, \frac{dP_2}{d\mu}\} \leq \frac{dw^s}{d\mu}$. And since, $P_i = \frac{dP_i}{d\mu}\mu \leq (\max\{\frac{dP_1}{d\mu}, \frac{dP_2}{d\mu}\})\mu$, we have $w \leq (\max\{\frac{dP_1}{d\mu}, \frac{dP_2}{d\mu}\})\mu$, thus $\frac{dw^s}{d\mu} \leq \max\{\frac{dP_1}{d\mu}, \frac{dP_2}{d\mu}\}$. We have proved that,

$$\frac{dw^s}{d\mu} = \max\left\{\frac{dP_1}{d\mu}, \frac{dP_2}{d\mu}\right\}$$

Take $H = \left\{\frac{dP_1}{d\mu} \geq \frac{dP_2}{d\mu}\right\}$.

□

EXAMPLE 3 (CONTINUATION): Since $f'(0)P = \bar{w}$, we have $\frac{dw^s}{dP} = f'(0)$.

A Choquet integral

In order to show that the Choquet integral is well-defined we shall need some technical material related to measurability issues.

A function $X : \Omega \rightarrow \mathbb{R}$ is \mathcal{A} -measurable if $\forall y \in \mathbb{R}, \{X > y\} \in \mathcal{A}$.

Definition 3 Let $O \subset \mathbb{R}^+$. O is right open if

$$\forall x \in O, \exists \eta > 0 / [x, x + \eta) \subset O.$$

Let $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ be a function. f is right lower semicontinuous if

$$\forall y \in \mathbb{R}, \{f > y\} \text{ is right open.}$$

Right lower semicontinuity can be characterized in term of sequences.

Lemma A.1 Let $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ be a function. f is right lower semicontinuous if and only if

$$\forall t_0 \in \mathbb{R}^+, \forall t_n \downarrow t_0, f(t_0) \leq \liminf_n f(t_n).$$

Proof: (Only if). Let $t_0 \in \mathbb{R}^+, t_n \downarrow t_0$ and $\epsilon > 0$. Since f is right lower semicontinuous, $\{f > f(t_0) - \epsilon\}$ is right open and $t_0 \in \{f > f(t_0) - \epsilon\}$. Thus there exists $\eta > 0$ s.t. $[t_0, t_0 + \eta) \subset \{f > f(t_0) - \epsilon\}$. So for n large enough, $f(t_n) > f(t_0) - \epsilon$. Hence, $\liminf_n f(t_n) \geq f(t_0) - \epsilon$. And since ϵ can be arbitrary chosen close to 0, it gives $\liminf_n f(t_n) \geq f(t_0)$.

(If). Assume on the contrary that there exists y s.t. $\{f > y\}$ is not right open. Thus there exists $t_0 \in \{f > y\}$ s.t. for all $\eta > 0, \exists t_\eta \in (t_0, t_0 + \eta)$ satisfying $t_\eta \notin \{f > y\}$. So we may pick a decreasing sequence t_n for $t_n \downarrow t_0$ such that $f(t_n) \leq y$ for all n . So a fortiori, $\liminf_n f(t_n) \leq y < f(t_0)$. □

Lemma A.2 Let $O \subset \mathbb{R}^+$. If O is right open then O is a Borel set.

Let $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ be a function. If f is right lower semicontinuous then f is a Borel function.

Proof: The second statement is a consequence of the first statement. Let $y \in \mathbb{R}$. $\{f > y\}$ is right open by definition of right lower semicontinuity of f . Thus, $\{f > y\}$ is a Borel set. Since it holds for all $y \in \mathbb{R}$, f is a Borel function.

Let us prove the first statement. Let O be right open. We introduce the following “excess” function,

$$\kappa : \mathbb{R}^+ \longrightarrow \mathbb{R}^+ : x \mapsto \kappa(x) = \sup\{\eta : \eta \in [0, 1], [x, x + \eta) \subset O\}$$

By construction, $O = \{\kappa > 0\}$.

(\subset). Let $x \in O$. Since O is right open there exists $\eta > 0$ s.t. $[x, x + \eta) \subset O$ so a fortiori $[x, x + \min\{\eta, 1\}) \subset O$. Hence, $0 < \min\{\eta, 1\} \leq \kappa(x)$.

(\supset). Let $x \in \{\kappa > 0\}$. There exists $0 < \eta \leq \kappa(x)$ s.t. $[x, x + \eta) \subset O$ thus $x \in O$. We shall prove now that $(x \mapsto x + \kappa(x))$ is non-decreasing.

Let $0 \leq x < y$.

1st CASE. If $x + \kappa(x) \leq y$ then $x + \kappa(x) \leq y + \kappa(y)$ since $\kappa(y) \geq 0$.

2nd CASE. If $x + \kappa(x) > y$. There exists $0 < \eta \leq 1$ s.t. $x + \eta > y$ and $[x, x + \eta) \subset O$, thus $y \in O$. And a fortiori, $[y, x + \eta) = [y, y + (x - y) + \eta) \subset O$, so $0 \leq (x - y) + \eta \leq \eta \leq 1$. Thus, $(x - y) + \eta \leq \kappa(y)$. Since η can be arbitrary chosen close to $\kappa(x)$, it comes $(x - y) + \kappa(x) \leq \kappa(y)$.

Since $(x \mapsto x + \kappa(x))$ and $(x \mapsto x)$ are non-decreasing, κ as a difference is (of bounded variation) a Borel function. \square

Proposition 3 *Let $w : \mathcal{A} \longrightarrow \mathbb{R}^+$ be semicontinuous from below. Then the Choquet integral $\int X dw$ is well-defined for all non-negative \mathcal{A} -measurable function X . Moreover, if $w \in BS$ and X is bounded then the Choquet integral is finite.*

Proof: Define the function,

$$w_X : \mathbb{R}^+ \longrightarrow \mathbb{R} : t \mapsto w_X(\{X > t\})$$

Let us check that w_X is right lower semicontinuous. Take $t_n \downarrow t_0$. It holds $\{X > t_n\} \uparrow \{X > t_0\}$ and since w is semicontinuous from below

$$w_X(t_0) = w(\{X > t_0\}) \leq \liminf_n w(\{X > t_n\}) = w_X(t_n)$$

Hence, according to Lemma A.1 w_X is right lower semicontinuous. And by Lemma A.2 w_X is measurable, thus $\int X dw$ is well-defined.

Assume now that $w \in BS$ and X is bounded. Then w_X is bounded and has a bounded support, i.e. $\{w_X \neq 0\} \subset [0, \sup X)$. So, $|w_X| \leq \|w\|_\infty \cdot 1_{[0, \sup X)}$ thus w_X is integrable and $\int X dw$ is finite. \square

To be more precise it suffices that $\sup\{w(A) : A \in \mathcal{A}\} < \infty$.

A more general situation where the Choquet is finite when X is bounded is where w is *signed* and *bounded* i.e.

$$w : \mathcal{A} \longrightarrow \mathbb{R}, \quad \|w\|_\infty = \sup\{|w(A)| : A \in \mathcal{A}\} < \infty$$

and \mathcal{A} might not be a σ -algebra but simply a *paving* i.e. $\emptyset, \Omega \in \mathcal{A}$. This situation is studied extensively in [12] for signed, bounded and continuous from below set functions.

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