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# On the Laplace transform of perpetuities with thin tails

Jean-Baptiste BARDET, H el ene GU ERIN, Florent MALRIEU

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## Abstract

We consider the random variables  $R$  which are solutions of the distributional equation  $R \stackrel{\mathcal{L}}{=} MR + Q$ , where  $(Q, M)$  is independent of  $R$  and  $|M| \leq 1$ . Goldie and Gr ubel showed that the tails of  $R$  are no heavier than exponential. Alsmeyer and *al* provide a complete description of the domain of the Laplace transform of  $R$ . We present here a simple proof in a particular case and an extension to the Markovian case.

**AMS Classification 2000:** Primary 60H25; secondary 60E99

## 1 Introduction

We define on some probability space  $(\Omega, \mathcal{A}, \mathbb{P})$  a couple of random variables  $(M, Q)$ , a sequence  $(M_n, Q_n)_{n \geq 0}$  of independent and identically distributed random vectors with the same law as  $(M, Q)$ , and  $R_0$  a random variable independent of the sequence  $(M_n, Q_n)_{n \geq 0}$ . Define the sequence  $(R_n)_{n \geq 0}$  by

$$R_{n+1} = M_n R_n + Q_n, \quad (1)$$

for any  $n \geq 0$ . This sequence has been extensively studied in the last decades. Under weak assumptions (see [8]) which are obviously fulfilled in our setting, it can be shown that the sequence  $(R_n)_{n \geq 0}$  converges almost surely to a random variable  $R$  such that

$$R \stackrel{\mathcal{L}}{=} MR + Q, \quad (2)$$

where  $R$  is independent of  $(M, Q)$ .

In [7], Kesten established that  $R$  is in general heavy-tailed (*i.e.* not all the moments of  $R$  are finite) even if  $Q$  is light-tailed as soon as  $|M|$  can be greater than 1. Nevertheless, Goldie and Gr ubel [4] have shown that  $R$  can have some exponential moments if  $|M| \leq 1$ . In particular, if  $Q$  and  $M$  are nonnegative the following result holds.

**Theorem 1.1** (Goldie, Gr ubel [4]). *Assume that*

$$\mathbb{P}(Q \geq 0, 0 \leq M \leq 1) = 1, \quad \mathbb{P}(M < 1) > 0$$

*and that there is  $v_Q > 0$  (possibly infinite) such that*

$$\mathbb{E}(e^{vQ}) \begin{cases} < +\infty & \text{if } v < v_Q, \\ = +\infty & \text{if } v > v_Q. \end{cases} \quad (3)$$

*Then, the Laplace transform  $v \mapsto \mathbb{E}(e^{vR})$  of the solution  $R$  of (2) is finite on the set  $(-\infty, v_{GG})$  with  $v_{GG} = v_Q \wedge \sup \{v \geq 0, \mathbb{E}(e^{vQ}M) < 1\}$ .*

In fact, the domain of the Laplace transform of  $R$  is larger than  $(-\infty, v_{GG})$ . In [1], a full description of this domain is established. Let us provide a simple proof under the assumptions of Theorem 1.1.

## 2 The main result

**Theorem 2.1.** *Under the assumptions of Theorem 1.1, assuming furthermore that  $R_0$  is non-negative and has all its exponential moments finite, then*

$$\sup_{n \geq 0} \mathbb{E}(e^{vR_n}) < +\infty \quad \text{and} \quad \mathbb{E}(e^{vR}) < +\infty$$

for any  $v < v_c$ , where

$$v_c = v_Q \wedge \sup\{v \geq 0, \mathbb{E}(e^{vQ} 1_{\{M=1\}}) < 1\}.$$

Moreover, for any  $v > v_c$ ,  $\sup_{n \geq 0} \mathbb{E}(e^{vR_n}) = +\infty$  and  $\mathbb{E}(e^{vR}) = +\infty$ .

For other recent generalizations of [4], the interested reader is referred to [6], where the authors give sharper results than ours on the tails for some specific examples.

*Proof of Theorem 2.1.* Let us start this section with the main lines of the proof of Theorem 1.1 of Goldie and Grübel [4]. For  $\rho > 0$ , let  $\mathcal{M}_\rho$  be the set of probability measures on  $\mathbb{R}_+$  with finite exponential moment of order  $\rho$ , and  $d_\rho$  a distance defined on  $\mathcal{M}_\rho$  by: for  $\mu, \nu \in \mathcal{M}_\rho$ ,

$$d_\rho(\mu, \nu) = \int_0^\infty e^{\rho u} |\mu[u, \infty) - \nu[u, \infty)| du.$$

Define the application  $T$  on  $\mathcal{M}_\rho$  as follows: for  $X$  with law  $\mu \in \mathcal{M}_\rho$ ,  $T\mu$  is the law of  $Q + MX$  with  $(M, Q)$  independent of  $X$ . It is shown in [4] that,

$$d_\rho(T\mu, T\nu) \leq \mathbb{E}(e^{\rho Q} M) d_\rho(\mu, \nu).$$

Since

$$\mathbb{E}(e^{vX}) = v \int_0^\infty e^{vu} \mathbb{P}(X \geq u) du,$$

one can show that, for any  $n \geq 0$  and  $v < \min(v_0, v_Q)$  with  $v_0 = \sup\{v \geq 0, \mathbb{E}(e^{vQ} M) < 1\}$ ,

$$\mathbb{E}(e^{vR_n}) \leq v \frac{1 - \mathbb{E}(e^{vQ} M)^n}{1 - \mathbb{E}(e^{vQ} M)} d_v(T\mu_0, \mu_0) + \mathbb{E}(e^{vR_0}).$$

In others words, Goldie and Grübel [4] established that for any  $v < \min(v_0, v_Q)$ ,  $(\mathbb{E}(e^{vR_n}))_n$  is uniformly bounded. This estimate can be extended to a larger domain.

Let us define  $v_1 = \sup\{v \geq 0, \mathbb{E}(e^{vQ} 1_{\{M=1\}}) < 1\}$  and  $v_c = \min(v_1, v_Q)$ . Let us fix  $v < v_c$  and choose  $\varepsilon > 0$  such that

$$\rho := \mathbb{E}(e^{vQ} 1_{\{1-\varepsilon < M \leq 1\}}) < 1.$$

Then we get, for any  $n \geq 0$ ,

$$\begin{aligned} L_{n+1}(v) &:= \mathbb{E}(e^{vR_{n+1}}) = \mathbb{E}(e^{v(M_n R_n + Q_n)}) \\ &\leq \mathbb{E}(e^{v((1-\varepsilon)R_n + Q_n)} 1_{\{M_n \leq 1-\varepsilon\}}) + \mathbb{E}(e^{v(R_n + Q_n)} 1_{\{1-\varepsilon < M_n \leq 1\}}) \\ &\leq L_n((1-\varepsilon)v) L_Q(v) + \rho L_n(v) \end{aligned}$$

where  $L_Q(v) = \mathbb{E}(e^{vQ})$ . By iteration of this estimate, one gets for any  $n \geq 0$

$$L_n(v) \leq \left( \sum_{k=0}^{n-1} \rho^k L_{n-k}((1-\varepsilon)v) \right) L_Q(v) + \rho^n L_0(v).$$

Let us notice that we have in fact more: for the same  $\varepsilon$ , and for any  $\tilde{v} \leq v$ ,  $\tilde{\rho} := \mathbb{E}(e^{\tilde{v}Q} 1_{\{1-\varepsilon < M \leq 1\}}) < \rho$ , hence, by the same method as before,

$$L_n(\tilde{v}) \leq \left( \sum_{k=0}^{n-1} \tilde{\rho}^k L_{n-k}((1-\varepsilon)\tilde{v}) \right) L_Q(\tilde{v}) + \tilde{\rho}^n L_0(\tilde{v}). \quad (4)$$

Let us define  $\bar{L} = \sup_{n \geq 0} L_n$ . Taking the supremum over  $n$  in (4), one gets for any  $\tilde{v} \leq v$

$$\bar{L}(\tilde{v}) \leq \frac{1}{1-\rho} \bar{L}((1-\varepsilon)\tilde{v}) L_Q(\tilde{v}) + L_0(\tilde{v}). \quad (5)$$

There is  $k \in \mathbb{N}$  such that  $(1-\varepsilon)^k v < v_Q$ , hence  $\bar{L}((1-\varepsilon)^k v) < +\infty$ . Applying  $k$  times estimate (5), one then obtains immediatly that  $\bar{L}(v) < +\infty$ , which achieves the first part of the proof.

On the other hand, if  $v > v_Q$ ,  $R_1 \geq Q_0$  immediatly implies  $L_1(v) = +\infty$ ; if  $v > v_1$ ,  $\rho_0 := \mathbb{E}(e^{vQ} 1_{\{M=1\}}) > 1$  (except in a trivial case, left to the reader) and, for all  $n \geq 0$ ,

$$L_{n+1}(v) \geq \rho_0 L_n(v),$$

implying that  $\bar{L}(v) = +\infty$ . □

### 3 Some extensions and perspectives

What happens if the random variables  $(M_n, Q_n)_{n \geq 0}$  are no longer independent? We provide here a partial result under a Markovian assumption when the contractive term  $M$  is less than 1.

Let us introduce  $X = (X_n)_{n \geq 0}$  an irreducible recurrent Markov process with finite space  $E$  and  $((M_n(x), Q_n(x))_{x \in E})_{n \geq 0}$  a sequence of i.i.d. random vectors supposed to be independent of  $X$ . We assume that, for all  $x \in E$ ,

$$\mathbb{P}(0 \leq M(x) < 1) = 1,$$

but we do not assume in the sequel that  $Q$  is non negative. The sequence  $(R_n)_{n \geq 0}$  is defined by

$$R_{n+1} = M_n(X_n)R_n + Q_n(X_n),$$

$R_0$  being arbitrary (with all exponential moments). Notice that the process  $(X_n, R_n)_{n \geq 0}$  is a Markov process whereas  $(R_n)_{n \geq 0}$  is not (in general).

**Proposition 3.1.** Introduce  $\underline{v} = \inf_{x \in E} v_{|Q(x)|}$ , with  $v_{|Q(x)|}$  defined as in (3). For any  $v < \underline{v}$ ,

$$\sup_{n \geq 0} \mathbb{E} \left( e^{v|R_n|} \right) < +\infty.$$

Moreover, if  $v > \underline{v}$ , then this supremum is infinite.

*Proof.* Let us introduce  $\overline{M}_n = \max_{x \in E} M_n(x)$  and  $\overline{Q}_n = \max_{x \in E} |Q_n(x)|$ . The random variables  $((\overline{M}_n, \overline{Q}_n))_{n \geq 0}$  are i.i.d. Define the sequence  $(\overline{R}_n)_{n \geq 0}$  by

$$\overline{R}_0 = |R_0| \quad \text{and} \quad \overline{R}_{n+1} = \overline{M}_n \overline{R}_n + \overline{Q}_n \quad \text{for } n \geq 1.$$

Obviously,  $|R_n| \leq \overline{R}_n$  for all  $n \geq 0$ . Thus it is sufficient to study the Laplace transforms of  $(\overline{R}_n)_{n \geq 0}$ . On the other hand, Theorem 2.1 ensures that  $(\mathbb{E}(e^{v\overline{R}_n}))_n$  is uniformly bounded as soon as  $v < \overline{v}_c = \min(\overline{v}_1, v_{\overline{Q}})$  with  $\overline{v}_1 = \sup\{v \geq 0 : \mathbb{E}(e^{v\overline{Q}} \mathbf{1}_{\{\overline{M}=1\}}) < 1\} > 0$ . In our case,  $\overline{v}_1$  is infinite since  $\mathbb{P}(\overline{M} < 1) = 1$ . At last, for  $v \geq 0$ ,

$$\sup_{x \in E} \mathbb{E} \left( e^{v|Q(x)|} \right) \leq \mathbb{E} \left( e^{v\overline{Q}} \right) = \mathbb{E} \left( \sup_{x \in E} e^{v|Q(x)|} \right) \leq \sum_{x \in E} \mathbb{E} \left( e^{v|Q(x)|} \right).$$

Thus  $v_{\overline{Q}} = \inf_{x \in E} v_{|Q(x)|}$ .

On the other hand, choose  $v > \underline{v}$ . There exists  $x_0 \in E$  such that  $\mathbb{E}(e^{v|Q(x_0)|})$  is infinite. Then, for any  $n \geq 0$ ,

$$\begin{aligned} \mathbb{E} \left( e^{v|R_{n+1}|} \right) &\geq \mathbb{E} \left( e^{v|R_{n+1}|} \mathbf{1}_{\{X_n=x_0\}} \right) \\ &\geq \mathbb{E} \left( e^{-v|R_n|} e^{v|Q_n(x_0)|} \mathbf{1}_{\{X_n=x_0\}} \right) \\ &\geq \mathbb{E} \left( \mathbf{1}_{\{X_n=x_0\}} e^{-v|R_n|} \right) \mathbb{E} \left( e^{v|Q_n(x_0)|} \right). \end{aligned}$$

The recurrence of  $X$  ensures that  $\{n \geq 0, \mathbb{E}(e^{v|R_n|}) = +\infty\}$  is infinite.  $\square$

**Remark 3.2.** In [2], we use the previous estimates to improve the results of [5, 3] on the tails of the invariant measure of a diffusion process with Markov switching.

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Jean-Baptiste BARDET e-mail: `jean-baptiste.bardet(AT)univ-rouen.fr`

UMR 6085 CNRS LABORATOIRE DE MATHÉMATIQUES RAPHAËL SALEM (LMRS)  
UNIVERSITÉ DE ROUEN, AVENUE DE L'UNIVERSITÉ, BP 12, F-76801 SAINT ETIENNE DU ROUVRAY

Hélène GUÉRIN, e-mail: `helene.guerin(AT)univ-rennes1.fr`

UMR 6625 CNRS INSTITUT DE RECHERCHE MATHÉMATIQUE DE RENNES (IRMAR)  
UNIVERSITÉ DE RENNES I, CAMPUS DE BEAULIEU, F-35042 RENNES CEDEX, FRANCE.

Florent MALRIEU, corresponding author, e-mail: `florent.malrieu(AT)univ-rennes1.fr`

UMR 6625 CNRS INSTITUT DE RECHERCHE MATHÉMATIQUE DE RENNES (IRMAR)  
UNIVERSITÉ DE RENNES I, CAMPUS DE BEAULIEU, F-35042 RENNES CEDEX, FRANCE.