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► To cite this version:

Ada Boralevi. On simplicity and stability of tangent bundles of rational homogeneous varieties. 2008.
hal-00441343

HAL Id: hal-00441343

<https://hal.science/hal-00441343>

Preprint submitted on 16 Dec 2009

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ON SIMPLICITY AND STABILITY OF THE TANGENT BUNDLE OF RATIONAL HOMOGENEOUS VARIETIES

ADA BORALEVI

ABSTRACT. Given a rational homogeneous variety G/P where G is complex simple and of type ADE , we prove that the tangent bundle $T_{G/P}$ is simple, meaning that its only endomorphisms are scalar multiples of the identity. This result combined with Hitchin-Kobayashi correspondence implies stability of the tangent bundle with respect to the anticanonical polarization. Our main tool is the equivalence of categories between homogeneous vector bundles on G/P and finite dimensional representations of a given quiver with relations.

1. INTRODUCTION

In [Ram67] Ramanan proved that irreducible homogeneous bundles on rational homogeneous varieties are stable, and hence in particular simple. If the underlying variety is Hermitian symmetric then this result applies to tangent bundles. For the general case, the Hitchin-Kobayashi correspondence gives a weaker result for the tangent bundle, polystability. In this paper we show that in fact the tangent bundle of any G/P is simple, where G is complex, simple and of type ADE . Simplicity and polystability combined give stability.

Our main tool is the equivalence of categories between homogeneous bundles on G/P and finite dimensional representations of a given quiver with relations. Once the machinery of this equivalence of categories is set up, the simplicity of the tangent bundle turns out to be an immediate and surprisingly easy consequence of it. Indeed one only needs to look at endomorphisms of the bundle as endomorphisms of the associated quiver representation.

Homogeneous vector bundles have been classically studied using another equivalence of categories, namely that between homogeneous bundles on G/P and finite dimensional representations of the parabolic subgroup P .

In [BK90] Bondal and Kapranov had the idea of associating to any rational homogeneous variety a quiver with relations. By putting the appropriate relations one gets the aforementioned equivalence of categories between G -homogeneous vector bundles on G/P and finite dimensional representations of the quiver. The relations were later refined by Hille in [Hil94].

2000 *Mathematics Subject Classification.* 14F05, 14M17, 14D20, 16G20.

Key words and phrases. Homogeneous Vector Bundle, Simplicity, Stability, Quiver Representation.

In [ACGP03] Álvarez-Cónsul and García-Prada gave an equivalent construction, while in [OR06] Ottaviani and Rubei used the quiver for computing cohomology, obtaining a generalization of the well-known Borel-Weil-Bott theorem holding on Hermitian symmetric varieties of *ADE* type.

We describe both equivalences of categories and give details on the quiver, its relations and its representations in Sections 2 and 3.

Sections 4 and 5 contain results on simplicity and stability. We use the quiver to prove that homogeneous vector bundles whose associated quiver representation has a particular configuration—we call such bundles multiplicity free—are weakly simple, which means that their only G -invariant endomorphisms are scalar multiples of the identity. Our result holds on any G/P , where G is complex, simple and of type *ADE*:

Proposition A. *Let E be a multiplicity free homogeneous vector bundle on G/P . Let k be the number of connected components of the quiver $\mathcal{Q}|_E$. Then $H^0(\text{End } E)^G = \mathbb{C}^k$. In particular if $\mathcal{Q}|_E$ is connected, then E is weakly simple.*

It turns out that the tangent bundle $T_{G/P}$ of a rational homogeneous variety of *ADE* type is multiplicity free and connected, and that moreover the isotypical component $H^0(\text{End } T_{G/P})^G$ coincides with the whole space $H^0(\text{End } T_{G/P})$, or in other words that the bundle is simple.

Theorem B. *Let $T_{G/P}$ the tangent bundle on a rational homogeneous variety G/P , where G is a complex simple Lie group of type *ADE* and P one of its parabolic subgroups. Then $T_{G/P}$ is simple.*

If algebraic geometry, representation theory and quiver representations give us simplicity, for stability differential geometry also joins the team. A homogeneous variety G/P is in fact also a Kähler manifold, and as such it admits a Hermite-Einstein structure. In virtue of the Hitchin-Kobayashi correspondence this is equivalent to the polystability of its tangent bundle. This together with Theorem B gives:

Theorem C. *Let $T_{G/P}$ be the tangent bundle on a rational homogeneous variety G/P , where G is a complex simple Lie group of type *ADE*, and P one of its parabolic subgroups. Then $T_{G/P}$ is stable with respect to the anti-canonical polarization $-K_{G/P}$ induced by the Hermite-Einstein structure.*

In the case where G/P is a point-hyperplane incidence variety in \mathbb{P}^n , we obtain a complete understanding of the stability of the tangent bundle with respect to different polarizations:

Proposition D. *Let $\mathbb{F} = \mathbb{F}(0, n-1, n)$ be the point-hyperplane incidence variety, and set:*

$$m(n) = \frac{-n + n\sqrt{4n^2 + 4n - 3}}{2(n^2 + n - 1)}.$$

Then the tangent bundle $T_{\mathbb{F}}$ is stable with respect to the polarization $\mathcal{O}_{\mathbb{F}}(a, b)$ if and only if it is semistable if and only if $m(n)a \leq b \leq m(n)^{-1}a$.

We also show similar computations for SL_4/B .

In the last Section 6 we deal with moduli spaces. We quote and generalize the results from [OR06], where the authors showed that King's notion of semistability [Kin94] for a representation $[E]$ of the quiver $\mathcal{Q}_{G/P}$ is in fact equivalent to the Mumford-Takemoto semistability of the associated bundle E on G/P , when the latter is a Hermitian symmetric variety.

We can thus construct moduli spaces of G -homogeneous semistable bundles E with fixed $\mathrm{gr} E$ on any homogenous variety G/P of ADE type.

Acknowledgements. This paper is part of my PhD thesis. I am very grateful to my advisor Giorgio Ottaviani for the patience with which he followed this work very closely and for always transmitting me lots of encouragement and mathematical enthusiasm. I would also like to thank Jean-Pierre Demailly for inviting me to Grenoble and for his warm hospitality and Michel Brion and the referee for many helpful comments.

2. PRELIMINARIES

2.1. Notations and first fundamental equivalence of categories. Let G be a complex semisimple Lie group. We make a choice $\Delta = \{\alpha_1, \dots, \alpha_n\}$ of simple roots of $\mathfrak{g} = \mathrm{Lie} G$ and thus of some Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. We call Φ^+ (respectively Φ^-) the set of positive (negative) roots. Then \mathfrak{g} decomposes as:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_{\alpha} \oplus \bigoplus_{\alpha \in \Phi^-} \mathfrak{g}_{\alpha}.$$

A parabolic subgroup $P \leq G$ is a subgroup P conjugated to one of the standard parabolic subgroups $P(\Sigma)$, where:

$$\mathrm{Lie}(P(\Sigma)) = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_{\alpha} \oplus \bigoplus_{\alpha \in \Phi^-(\Sigma)} \mathfrak{g}_{\alpha},$$

for a subset $\Sigma \subset \Delta$ that induces $\Phi^-(\Sigma) = \{\alpha \in \Phi^- | \alpha = \sum_{\alpha_i \notin \Sigma} p_i \alpha_i\}$.

If $\Sigma = \Delta$, then $P(\Delta) = B$ is the Borel subgroup.

A rational homogeneous variety is a quotient G/P .

A vector bundle E on G/P is called (G) -homogeneous if there is an action of G on E such that the following diagram commutes:

$$\begin{array}{ccc} G \times E & \longrightarrow & E \\ \downarrow & & \downarrow \\ G \times G/P & \longrightarrow & G/P \end{array}$$

where the bottom row is just the natural action of G on the cosets G/P . Note that the tangent bundle $T_{G/P}$ on any rational homogeneous variety G/P is obviously a G -homogeneous bundle.

The category of G -homogeneous vector bundles on G/P is equivalent to the category P -mod of representations of P , and also to the category of integral \mathfrak{p} -modules, where $\mathfrak{p} = \text{Lie}(P)$, see for example [BK90].

More in detail, the group G is a principal bundle over G/P with fiber P . Any G -homogeneous vector bundle E of rank r is induced by this principal bundle via a representation $\rho : P \rightarrow \text{GL}(r)$. We denote $E = E_\rho$. Indeed, E of rank r over G/P is homogeneous if and only if there exists a representation $\rho : P \rightarrow \text{GL}(r)$ such that $E \simeq E_\rho$, and this entails the aforementioned equivalence of categories.

For any weight λ we denote by E_λ the homogeneous bundle corresponding to Σ_λ^* , the dual of the irreducible representation Σ_λ of the parabolic subgroup with maximal weight λ . Here λ belongs to the fundamental Weyl chamber of the reductive part of P . Indeed, P decomposes as $P = R \cdot N$ into a reductive part R and a unipotent part N . At the level of Lie algebras this decomposition entails a splitting $\mathfrak{p} = \mathfrak{r} \oplus \mathfrak{n}$, with the obvious notation $\mathfrak{r} = \text{Lie } R$ and $\mathfrak{n} = \text{Lie } N$. Moreover from a result by Ise [Ise60] we learn that a representation of \mathfrak{p} is completely reducible if and only if it is trivial on \mathfrak{n} , hence it is completely determined by its restriction to \mathfrak{r} .

The well-known Borel-Weil-Bott theorem [Bot57] computes the cohomology of such E_λ 's by using purely Lie algebra tools, namely the orbit of the weight λ under the action of the Weyl group. In particular the theorem states that if λ is G -dominant then $H^0(E_\lambda)$ is the dual of the irreducible representation of G with highest weight λ and all higher cohomology vanishes.

3. THE QUIVER $\mathcal{Q}_{G/P}$

3.1. Definition of the quiver $\mathcal{Q}_{G/P}$ and its representations. Other than looking at homogeneous bundles as P -modules, it is useful to try a different point of view and look at these same bundles as representations of a given quiver with relations. For basics on quiver theory we refer the reader to [DJ05] or [Kin94].

To any rational homogeneous variety G/P we can associate a quiver with relations, that we denote by $\mathcal{Q}_{G/P}$. The idea is to exploit all the information given by the choice of the parabolic subgroup P , with its Levi decomposition $P = R \cdot N$.

Let Λ be the fundamental Weyl chamber of G , and let Λ^+ be the Weyl chamber of the reductive part R . Then we can give the following:

Definition 3.1. Let G/P be any rational homogeneous variety. The quiver $\mathcal{Q}_{G/P}$ is constructed as follows. The set of vertices is:

$$\mathcal{Q}_0 = \Lambda^+ = \{\lambda \mid \lambda \text{ dominant weight for } R\}.$$

There is an arrow connecting the vertices λ and μ if and only if the vector space $\text{Hom}(\Omega_{G/P}^1 \otimes E_\lambda, E_\mu)^G$ is non-zero.

Remark 3.1. Definition 3.1 is precisely the original one of Bondal and Kapranov [BK90], later used also by Alvarez-Cónsul and García-Prada [ACGP03]. Arrows correspond to weights of the nilpotent algebra \mathfrak{n} , considered as an \mathfrak{r} -module with the adjoint action.

In fact one could obtain an equivalent theory by considering the same vertices with a smaller number of arrows, i.e. by taking only weights of the quotient $\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$. This is for example the choice made by Hille (see [Hil98] and [Hil96]).

Note that vertices λ of $\mathcal{Q}_{G/P}$ correspond to irreducible homogeneous bundles E_λ on G/P .

The relations on the quiver $\mathcal{Q}_{G/P}$ will be defined in Section 3.2.

Let now E be an homogeneous vector bundle over G/P . Let us use the same notation E to denote the associated P -module. We want to associate to E a representation $[E]$ of the quiver $\mathcal{Q}_{G/P}$. The bundle E comes with a filtration:

$$(3.1) \quad 0 \subset E_1 \subset E_2 \subset \dots \subset E_k = E,$$

where each E_i/E_{i-1} is completely reducible. Of course the filtration does not split in general. We define $\text{gr } E = \bigoplus_i E_i/E_{i-1}$ for any filtration (3.1). The graded bundle $\text{gr } E$ does not depend on the filtration: in fact it is given by looking at the \mathfrak{p} -module associated to E as a module over \mathfrak{r} , so that it decomposes as a direct sum of irreducibles:

$$(3.2) \quad \text{gr } E = \bigoplus_{\lambda} E_{\lambda} \otimes V_{\lambda},$$

with multiplicity spaces $V_{\lambda} \simeq \mathbb{C}^k$, where $k \in \mathbb{Z}_{\geq 0}$ is the number of times E_{λ} occurs.

The representation $[E]$ associates to the vertex λ of the quiver $\mathcal{Q}_{G/P}$ precisely the multiplicity space V_{λ} in the decomposition (3.2).

Going on with the definition of the representation $[E]$, given any $\lambda, \mu \in \mathcal{Q}_0$ such that there is an arrow $\lambda \rightarrow \mu \in \mathcal{Q}_1$, we now need to define a linear map $V_{\lambda} \rightarrow V_{\mu}$. This information is given by the nilpotent part \mathfrak{n} . More precisely, it is given by the natural action of \mathfrak{n} on the P -module associated with E , both viewed as \mathfrak{r} -modules:

$$\theta : \mathfrak{n} \otimes \text{gr } E \rightarrow \text{gr } E.$$

The morphism θ encodes all the information we need, including that on the relations of the quiver.

If we have a vector bundle E we then have the graded $\text{gr } E$ and the morphism θ . Viceversa, if we have an \mathfrak{r} -module and a morphism that behaves “just like θ ”, we can reconstruct a \mathfrak{p} -module and hence a vector bundle. More in detail, let us state and prove the following generalization of [OR06, Theorem 3.1]:

Theorem 3.1. *Consider \mathfrak{n} as an \mathfrak{r} -module with the adjoint action.*

- (1) *Given a \mathfrak{p} -module E on X , the action of \mathfrak{n} over E induces a morphism of \mathfrak{r} -modules:*

$$\theta : \mathfrak{n} \otimes \text{gr } E \rightarrow \text{gr } E.$$

The morphism

$$\theta \wedge \theta : \wedge^2 \mathfrak{n} \otimes \text{gr } E \rightarrow \text{gr } E$$

defined by $\theta \wedge \theta((n_1 \wedge n_2) \otimes f) := n_1 \cdot (n_2 \cdot f) - n_2 \cdot (n_1 \cdot f)$ satisfies the equality $\theta \wedge \theta = \theta\varphi$ in $\text{Hom}(\wedge^2 \mathfrak{n} \otimes \text{gr } E, \text{gr } E)$, where φ is given by:

$$\begin{aligned} \varphi : \wedge^2 \mathfrak{n} \otimes \text{gr } E &\rightarrow \mathfrak{n} \otimes \text{gr } E \\ (n_1 \wedge n_2) \otimes e &\mapsto [n_1, n_2] \otimes e. \end{aligned}$$

- (2) *Conversely, given an \mathfrak{r} -module F on X and a morphism of \mathfrak{r} -modules*

$$\theta : \mathfrak{n} \otimes F \rightarrow F$$

such that $\theta \wedge \theta = \theta\varphi$, we have that θ extends uniquely to an action of \mathfrak{p} over F , giving a bundle E such that $\text{gr } E = F$.

Proof. (1) Obviously θ is \mathfrak{r} -equivariant, almost by definition.

Take n_1 and n_2 in \mathfrak{n} . We have that:

$$\theta \wedge \theta((n_1 \wedge n_2) \otimes f) = n_1 \cdot (n_2 \cdot f) - n_2 \cdot (n_1 \cdot f) = [n_1, n_2] \cdot f,$$

which means exactly that $\theta \wedge \theta = \theta\varphi$.

- (2) For any $f \in F$ and any $r + n \in \mathfrak{p} = \mathfrak{r} \oplus \mathfrak{n}$ we set:

$$(3.3) \quad (r + n) \cdot f := r \cdot f + \theta(n \otimes f).$$

We need to show that given any $p_1, p_2 \in \mathfrak{p}$, the action (3.3) respects the bracket, i.e. that for every $f \in F$:

$$[p_1, p_2] \cdot f = p_1 \cdot (p_2 \cdot f) - p_2 \cdot (p_1 \cdot f).$$

Now if both $p_1, p_2 \in \mathfrak{r}$, then there is nothing to prove.

If both $p_1, p_2 \in \mathfrak{n}$, then from the equality $\theta \wedge \theta = \theta\varphi$, we get:

$$\begin{aligned} [p_1, p_2] \cdot f &= \theta([p_1, p_2] \otimes f) = \theta\varphi((p_1 \wedge p_2) \otimes f) = \\ &= \theta \wedge \theta((p_1 \wedge p_2) \otimes f) = p_1 \cdot (p_2 \cdot f) - p_2 \cdot (p_1 \cdot f). \end{aligned}$$

Finally, in case $p_1 \in \mathfrak{r}$ and $p_2 \in \mathfrak{n}$, then $[p_1, p_2] \in \mathfrak{n}$ and we have:

$$\begin{aligned} p_1 \cdot (p_2 \cdot f) &= \theta(p_1 \cdot (p_2 \otimes f)) = \theta([p_1, p_2] \otimes f + p_2 \otimes (p_1 \cdot f)) = \\ &= [p_1, p_2] \cdot f + \theta(p_2 \otimes (p_1 \cdot f)) = [p_1, p_2] \cdot f + p_2 \cdot (p_1 \cdot f). \end{aligned}$$

□

Remark that (3.2) entails that we have a decomposition:

$$(3.4) \quad \theta \in \text{Hom}(\mathfrak{n} \otimes \text{gr } E, \text{gr } E) = \bigoplus_{\lambda, \mu \in \mathcal{Q}_0} \text{Hom}(V_\lambda, V_\mu) \otimes \text{Hom}(\mathfrak{n} \otimes \Sigma_\lambda^*, \Sigma_\mu^*).$$

Lemma 3.2. [BK90, Proposition 2] *In the ADE case $\dim \text{Hom}(\mathfrak{n} \otimes \Sigma_\lambda^*, \Sigma_\mu^*)^P$ is either 0 or 1 for every $\lambda, \mu \in \Lambda^+$.*

Remark 3.2. From now on G will thus always denote a complex Lie group of ADE type. Nevertheless, the construction of the quiver with its relation can be done for any type of Lie group, like in [ACGP03].

We can now conclude the construction of the representation of the quiver $[E]$ associated to the bundle E .

For any \mathfrak{r} -dominant weight λ fix a maximal vector v^λ of Σ_λ^* (it is unique up to constants). For any weight α of \mathfrak{n} , fix an eigenvector $e_\alpha \in \mathfrak{n}$.

Now suppose that there is an arrow $\lambda \rightarrow \mu$ in the quiver. Then the vector space $\text{Hom}(\mathfrak{n} \otimes \Sigma_\lambda^*, \Sigma_\mu^*)^P$ is non-zero, and in particular is 1-dimensional. Notice that by definition, being given by the action of \mathfrak{n} , the arrow will send the weight λ into a weight $\mu = \lambda + \alpha$, for some negative root $\alpha \in \Phi^-(\Sigma)$ (for we have $\mathfrak{g}_\alpha \cdot W_\lambda \subseteq W_{\lambda+\alpha}$).

Then fix the generator $f_{\lambda\mu}$ of $\text{Hom}(\mathfrak{n} \otimes \Sigma_\lambda^*, \Sigma_\mu^*)^P$ that takes $e_\alpha \otimes v^\lambda \mapsto v^\mu$. Once all the generators are fixed, from (3.4) we write the map θ uniquely as:

$$(3.5) \quad \theta = \sum_{\lambda, \mu} g_{\lambda\mu} f_{\lambda\mu},$$

and thus we can associate to the arrow $\lambda \xrightarrow{f_{\lambda\mu}} \mu$ exactly the element $g_{\lambda\mu}$ in $\text{Hom}(V_\lambda, V_\mu)$. All in all:

Definition 3.2. To any homogeneous vector bundle E on G/P we associate a representation $[E]$ of the quiver $\mathcal{Q}_{G/P}$ as follows.

To any vertex $\lambda \in \mathcal{Q}_0$ we associate the vector space V_λ from the decomposition (3.2).

To any arrow $\lambda \rightarrow \mu$ we associate the element $g_{\lambda\mu} \in \text{Hom}(V_\lambda, V_\mu)$ from the decomposition (3.4).

Notice that a different choice of generators would have led to an equivalent construction.

3.2. Second fundamental equivalence of categories. We introduce here the equivalence of categories between homogeneous vector bundles on G/P and finite dimensional representations of the quiver $\mathcal{Q}_{G/P}$.

From Proposition 3.1 it is clear that by putting the appropriate relations on the quiver, namely the equality $\theta \wedge \theta = \theta\varphi$, we can get the following:

Theorem 3.3. [BK90, Hil94, ACGP03] *Let G/P a rational homogeneous variety of ADE type. The category of integral \mathfrak{p} -modules is equivalent to the category of finite dimensional representations of the quiver $\mathcal{Q}_{G/P}$ with certain relations \mathcal{R} , and it is equivalent to the category of G -homogeneous bundles on G/P .*

We show here how one can derive the relations. For details, see [ACGP03].

Let $\lambda, \mu, \nu \in \mathcal{Q}_0$. We start by defining the morphism $\phi_{\lambda\mu\nu}$:

$$\mathrm{Hom}(\mathfrak{n} \otimes \Sigma_\lambda^*, \Sigma_\mu^*) \otimes \mathrm{Hom}(\mathfrak{n} \otimes \Sigma_\mu^*, \Sigma_\nu^*) \xrightarrow{\phi_{\lambda\mu\nu}} \mathrm{Hom}(\wedge^2 \mathfrak{n} \otimes \Sigma_\lambda^*, \Sigma_\nu^*)$$

by setting $\phi_{\lambda\mu\nu}(a \otimes a') : (n \wedge n') \otimes x \mapsto z$, where $a : n \otimes x \mapsto y$ and $a' : n' \otimes y \mapsto z$.

The image of the P -invariant part:

$$\phi_{\lambda\mu\nu}(\mathrm{Hom}(\mathfrak{n} \otimes \Sigma_\lambda^*, \Sigma_\mu^*)^P \otimes \mathrm{Hom}(\mathfrak{n} \otimes \Sigma_\mu^*, \Sigma_\nu^*)^P) \subseteq \mathrm{Hom}(\wedge^2 \mathfrak{n} \otimes \Sigma_\lambda^*, \Sigma_\nu^*)^P.$$

In particular recall once the choice of constants has been made, there are fixed generators $f_{\lambda\mu}$, where $f_{\lambda\mu} : e_\alpha \otimes v_\lambda \mapsto v_\mu$, and $\alpha = \lambda - \mu$. Then if we set $\beta = \mu - \nu$, all in all:

$$\phi_{\lambda\mu\nu}(f_{\lambda\mu} \otimes f_{\mu\nu}) : (e_\alpha \wedge e_\beta) \otimes v^\lambda \mapsto v^\nu.$$

Now consider the natural morphism $\wedge^2 \mathfrak{n} \rightarrow \mathfrak{n}$ sending $n \wedge n' \mapsto [n, n']$.

It induces a morphism $\phi_{\lambda\nu}$:

$$\mathrm{Hom}(\mathfrak{n} \otimes \Sigma_\lambda^*, \Sigma_\nu^*) \xrightarrow{\phi_{\lambda\nu}} \mathrm{Hom}(\wedge^2 \mathfrak{n} \otimes \Sigma_\lambda^*, \Sigma_\nu^*).$$

Once again it is clear that the invariant part $\phi_{\lambda\nu}(\mathrm{Hom}(\mathfrak{n} \otimes \Sigma_\lambda^*, \Sigma_\nu^*)^P)$ is contained in $\mathrm{Hom}(\wedge^2 \mathfrak{n} \otimes \Sigma_\lambda^*, \Sigma_\nu^*)^P$.

Theorem 3.1 together with the splitting (3.5) entail that we have an equality in $\mathrm{Hom}(\wedge^2 \mathfrak{n} \otimes \Sigma_\lambda^*, \Sigma_\nu^*)^P$:

$$(3.6) \quad \sum_{\lambda, \nu} \sum_{\mu} \left(\phi_{\lambda\mu\nu}(f_{\lambda\mu} \otimes f_{\mu\nu})(g_{\lambda\mu} g_{\mu\nu}) + \phi_{\lambda\nu}([f_{\lambda\mu}, f_{\mu\nu}])g_{\lambda\nu} \right) = 0.$$

Finally, let $\{c_{\lambda\nu}^k\}$ be a basis of the vector space $\mathrm{Hom}(\wedge^2 \mathfrak{n} \otimes \Sigma_\lambda^*, \Sigma_\nu^*)^P$, for $k = 1, \dots, m_{\lambda\nu}$, with $m_{\lambda\nu} = \dim(\mathrm{Hom}(\wedge^2 \mathfrak{n} \otimes \Sigma_\lambda^*, \Sigma_\nu^*)^P)$. Expand:

$$\begin{aligned} \phi_{\lambda\mu\nu}(f_{\lambda\mu} \otimes f_{\mu\nu}) &= \sum_{k=1}^{m_{\lambda\nu}} x_{\lambda\mu\nu}^k c_{\lambda\nu}^k \\ \phi_{\lambda\nu}([f_{\lambda\mu}, f_{\mu\nu}]) &= \sum_{k=1}^{m_{\lambda\nu}} y_{\lambda\nu}^k c_{\lambda\nu}^k, \end{aligned}$$

Then for every couple of vertices λ, ν , equality (3.6) gives us a system of $m_{\lambda\nu}$ equations that the maps $g_{\gamma\delta}$ satisfy:

$$(3.7) \quad \mathcal{R}_k^{\lambda\nu} : \sum_{\mu \in \mathcal{Q}_0} x_{\lambda\mu\nu}^k g_{\lambda\mu} g_{\mu\nu} + y_{\lambda\nu}^k g_{\lambda\nu} = 0$$

Definition 3.3. We define the relations \mathcal{R} on the quiver $\mathcal{Q}_{G/P}$ as the ideal generated by all the equations (that with a slight abuse of notation we keep calling \mathcal{R}_k):

$$\mathcal{R}_k^{\lambda\nu} : \sum_{\mu \in \mathcal{Q}_0} x_{\lambda\mu\nu}^k f_{\lambda\mu} f_{\mu\nu} + y_{\lambda\nu}^k f_{\lambda\nu} = 0,$$

for $k = 1, \dots, m_{\lambda\nu}$ and for any couple of weights $\lambda, \nu \in \mathcal{Q}_0$.

If $P = B$ is the Borel subgroup, Definition 3.3 can be made more precise. We show this with some details in Section 5 (see Definition 5.7).

4. SIMPLICITY

4.1. Simplicity of multiplicity free bundles. In this section we work on rational homogeneous varieties G/P , where G is complex, simple and of type ADE .

Let E be a rank r homogeneous vector bundle on G/P , and let $[E]$ be the associated representation of the quiver $\mathcal{Q}_{G/P}$. Denote by $\mathcal{Q}|_E$ the subquiver of $\mathcal{Q}_{G/P}$ given by all vertices where $[E]$ is non-zero and all arrows connecting any two such vertices. Clearly the support of $\mathcal{Q}|_E$ has at most r vertices, call them $\lambda_1, \dots, \lambda_n$, $n \leq r$. Notice also that the representation $[E]$ of $\mathcal{Q}_{G/P}$ induces a representation of the subquiver $\mathcal{Q}|_E$.

Definition 4.1. A homogeneous vector bundle E of rank r is *multiplicity free* if and only if $\text{gr } E$ is multiplicity free.

Let now E be multiplicity free. Just by looking at the associated quiver representation $[E]$, we will show that the only G -invariant endomorphisms that such a bundle can have are scalar multiple of the identity, i.e. that the isotypical component $H^0(\text{End } E)^G = \mathbb{C}$. If this holds we call the bundle weakly simple.

Proposition 4.1. *Let E be a multiplicity free homogeneous vector bundle on G/P . Let k be the number of connected components of the quiver $\mathcal{Q}|_E$. Then $H^0(\text{End } E)^G = \mathbb{C}^k$. In particular if $\mathcal{Q}|_E$ is connected, then E is weakly simple.*

Proof. Suppose first that $k = 1$, so that the subquiver $\mathcal{Q}|_E$ is connected. Any element $\varphi \in H^0(\text{End } E)^G$ is a G -invariant endomorphism $\varphi : E \rightarrow E$. In particular we can look at φ as a morphism $[E] \rightarrow [E]$ between representations of the same quiver. This means that we can look at φ as a family of morphisms $\{\varphi_i : V_i \rightarrow V_i \mid i = 1, \dots, n\}$. The hypothesis that E is multiplicity free entails that in particular each $\varphi_i = k_i \text{Id}$ and hence $\varphi = (k_1, \dots, k_n)$ is in fact just an element of \mathbb{C}^n .

By definition of morphisms of quiver representations, every time that there is an arrow $\lambda_i \rightarrow \lambda_j$ in the quiver $\mathcal{Q}|_E$, there is a commutative diagram:

$$\begin{array}{ccc} V_i & \longrightarrow & V_j \\ k_i \downarrow & & \downarrow k_j \\ V_i & \longrightarrow & V_j \end{array}$$

Remark that the two horizontal arrows are the same. This means that if we fix the first constant k_1 , then φ is completely determined thanks to connectedness. So this proves that $H^0(\text{End } E)^G \subseteq \mathbb{C}$.

On the other hand, notice we have homotheties, hence $\mathbb{C} \subseteq H^0(\text{End } E)^G$, and the thesis follows for the case $k = 1$.

The same argument applies for each connected component, and this completes the proof. \square

4.2. Simplicity of tangent bundles. The results of the previous section can be applied to a “special” multiplicity free homogeneous bundle: the tangent bundle $T_{G/P}$, with G simple of type ADE .

Let $T_{G/P}$ be the tangent bundle on G/P . Recall that any standard parabolic subgroup $P = P(\Sigma)$ is given by a subset $\Sigma \subseteq \Delta$ of simple roots. Define also the subset of positive roots $\Phi_P^+ = \{\alpha \in \Phi^+ \mid -\alpha \notin \Phi^-(\Sigma)\}$. Notice that when $P = B$ is the Borel, $\Sigma = \Delta$ and $\Phi_B^+ = \Phi^+$.

The bundle $T_{G/P}$ is a homogeneous bundle of rank $r = \dim G/P = |\Phi_P^+|$, whose weights are exactly the elements of Φ_P^+ .

Remark 4.1. It is convenient for us to take into account all the weights of the tangent bundle, and not only the highest weights. Obviously in the case of the Borel it doesn't make any difference. If P is any other parabolic subgroup of G , this means that instead of the tangent bundle $T_{G/P}$ we are considering its pull-back $\pi^* T_{G/P}$ via the (flat!) projection

$$\pi : G/B \rightarrow G/P.$$

Projection formula (see [Har77], II.5) together with the fact that $\pi_* \mathcal{O}_{G/B} = \mathcal{O}_{G/P}$ guarantee that:

$$H^0(G/B, \text{End}(\pi^* T_{G/P})) = H^0(G/B, \pi^*(\text{End } T_{G/P})) = H^0(G/P, \text{End } T_{G/P}).$$

Hence we are allowed to work on $\pi^* T_{G/P}$ instead of $T_{G/P}$. To simplify the notation we write $\mathcal{T}_{G/P} := \pi^* T_{G/P}$.

Now we make the easy but fundamental remark that for every homogeneous variety G/P , the rank of $\mathcal{T}_{G/P}$ (that is, the dimension of G/P) coincides with the number of weights of the associated representation and also with the number of nonzero vertices in the quiver representation $[\mathcal{T}_{G/P}]$. Hence these vertices must all have multiplicity one. Moreover, notice that whenever $\alpha \neq -\beta$ we have $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$ (recall that we are assuming \mathfrak{g} simple here). All in all:

Theorem 4.2. *For every homogeneous variety G/P the bundle $\mathcal{T}_{G/P}$ is multiplicity free and connected.*

We will now show that in the case of the tangent bundle $\mathcal{T}_{G/P}$ all endomorphisms are G -invariant endomorphisms, or in other words that

$$H^0(\text{End}(\mathcal{T}_{G/P}))^G = H^0(\text{End}(\mathcal{T}_{G/P})).$$

Lemma 4.3. *Let G/P any homogeneous variety of type ADE. For any simple G -module $W \neq \mathbb{C}$, $H^0(\text{gr End}(\mathcal{T}_{G/P}))^W = 0$.*

Proof. This is a direct computation.

We use here the notation for positive and negative roots Φ^+ and Φ^- used for example in [FH91], and we call $\{L_i\}$ the standard basis of \mathfrak{h}^* . In the A_{N-1} case ($G = \text{SL}_N$):

$$\Phi^+ = \{L_i - L_j\}_{1 \leq i < j \leq N} \quad \text{and} \quad \Phi^- = \{L_i - L_j\}_{1 \leq j < i \leq N},$$

and the fundamental Weyl chamber associated is the set:

$$(4.1) \quad \Lambda = \left\{ \sum a_i L_i \mid a_1 \geq a_2 \geq \dots \geq a_N \right\}.$$

For the D_N case ($G = \text{SO}_{2N}$):

$$\Phi^+ = (\{L_i - L_j\} \cup \{L_i + L_j\})_{i < j} \quad \text{and} \quad \Phi^- = (\{L_i - L_j\} \cup \{-L_i - L_j\})_{i > j},$$

and the fundamental Weyl chamber associated:

$$(4.2) \quad \Lambda = \left\{ \sum a_i L_i \mid a_1 \geq \dots \geq a_{N-1} \geq |a_N| \right\}.$$

By the Borel-Weil-Bott theorem [Bot57], an irreducible bundle E_ν has non-zero $H^0(E_\nu)$ exactly if and only if ν lies in the fundamental Weyl chamber.

For the case of A_{N-1} : let ν be an irreducible summand of $\text{End } \mathcal{T}_{\text{SL}_N/P}$. Hence ν is of the form $\alpha + \beta$ with $\alpha \in \Phi^+$ and $\beta \in \Phi^-$.

Practically speaking, α is a vector in \mathbb{Z}^N having 1 at the i -th place, -1 at the j -th place and 0 everywhere else, with $1 \leq i < j \leq N$; similarly β has a -1 at the h -th place, a 1 at the k -th place and 0 everywhere else, with again $1 \leq h < k \leq N$. What does a sum $\nu = \alpha + \beta$ look like? Fix α . Then only six possibilities can occur for β , namely

$$\begin{aligned} \text{either: } & h < k \leq i < j, \\ \text{or: } & h \leq i \leq k \leq j, \\ \text{or: } & h \leq i < j \leq k, \\ \text{or: } & i \leq h < k \leq j, \\ \text{or: } & i \leq h \leq j \leq k, \\ \text{or else: } & i < j \leq h < k. \end{aligned}$$

One can check directly that the condition (4.1) is satisfied exactly when $\beta = -\alpha$. In this case $\nu = 0$ and $H^0(E_0) = \mathbb{C}$, which is what we wanted.

Let us now move to the case D_N . Here the situation is complicated by the fact that we deal with more roots, and thus with more possible combinations for the sum $\nu = \alpha + \beta$. Nevertheless the condition (4.2) for ν to belong to the fundamental Weyl chamber is stronger than that for A_{N-1} (4.1), thus

making our life easier.

A direct check shows that the thesis holds true for $N = 4$.

So let us suppose we are in the case D_N , with $N \geq 5$. The weight ν is a vector of \mathbb{Z}^N having at most 4 non-zero elements $a_i \in \{\pm 1, \pm 2\}$ (thus since $N \geq 5$ there is at least one coordinate equal to 0).

Look at the last coordinate a_N : if $a_N \neq 0$, then there is no way for condition (4.2) to be satisfied, and we are done. If instead $a_N = 0$, we look at the other coordinates: if all the other elements are non-zero, then it means that we are necessarily in the case D_5 , and such elements are alternating 1's and -1's, hence we are done again. If there is another $a_i = 0$ with $i \neq N - 1$, we are also done.

Finally, if we are in the case $a_{N-1} = a_N = 0$, we repeat the argument above. We can go on until we either get to $a_1 = a_2 = \dots = a_N = 0$, or we encounter an element that cannot satisfy (4.2), and this concludes the proof for the D -case.

The proof for the three exceptional cases E_6 , E_7 and E_8 is nothing but a brute force check. First write down all the positive roots λ_i , $i = 1, \dots, m$ for E_n (in the three cases respectively $m = 36, 73$ and 120). The most convenient thing is writing them down in the basis formed by simple roots. By computing as usual all differences of type $\lambda_i - \lambda_j$, we get all the irreducible summands of $\text{gr End } T$. Using the Cartan matrix we can find their coefficients in the basis of fundamental weights, where the condition of belonging to the fundamental Weyl chamber just means having all non-negative coefficients. Then one simply checks that the only irreducible summands of $\text{gr End } T$ satisfying this condition are those where $\lambda_i = \lambda_j$. \square

Theorem 4.4. *Let $T_{G/P}$ the tangent bundle on a flag manifold G/P , where G is a complex simple Lie group of type ADE , and P one of its parabolic subgroups. Then $T_{G/P}$ is simple.*

Proof. Proposition 4.1, Theorem 4.2 and Remark 4.1 imply that the isotypical component $H^0(\text{End}(T_{G/P}))^G = H^0(\text{End}(\mathcal{T}_{G/P}))^G = \mathbb{C}$. But from Lemma 4.3 we get that $H^0(\text{End}(T_{G/P})) = H^0(\text{End}(T_{G/P}))^G$, and we are done. \square

5. STABILITY

5.1. Simplicity and stability. Let us start this section with some basic definitions.

Definition 5.1. Let H be an ample line bundle on a projective variety X of dimension d . For any coherent sheaf E on X define the *slope* $\mu_H(E)$ as:

$$\mu_H(E) = \frac{c_1(E) \cdot H^{d-1}}{\text{rk } E}.$$

E is called *H-stable* (respectively *H-semistable*) if for every coherent subsheaf $F \subset E$ such that E/F is torsion free and $0 < \text{rk } F < \text{rk } E$ we have:

$$\mu_H(F) < \mu_H(E) \quad (\text{respectively } \leq).$$

This notion of stability is known as Mumford-Takemoto stability.

Definition 5.2. In the same setting as above, E is called *H-polystable* if it decomposes as a direct sum of H -stable vector bundles with the same slope.

It is a well-known fact that for vector bundles stability implies polystability and the latter implies semistability, see for example [Kob87]. Also stability implies simplicity, and the converse is not true in general (see [OSS80], or [Fai06] for a homogeneous counterexample).

We now want to look at our homogeneous vector bundles from the point of view of differential geometry. A homogeneous variety G/P is in particular a homogeneous Kähler manifold. For an exhaustive introduction on Kähler-Einstein manifolds we refer the reader to [Bes87]. Here we content ourselves with quoting the results on Kähler-Einstein and Hermite-Einstein structures that we need. The following holds:

Theorem 5.1. [Bes87, Theorem 8.95] *Every compact, simply connected homogeneous Kähler manifold admits a unique (up to homothety) invariant Kähler-Einstein structure.*

Theorem 5.1 above implies that the tangent bundle $T_{G/P}$ admits a Kähler-Einstein structure, and hence in particular an Hermite-Einstein structure.

If X is a compact Kähler manifold and E a holomorphic bundle over X , a Hermitian metric on E determines a canonical unitary connection whose curvature is a $(1, 1)$ -form F with values in $\text{End } E$.

The inner product of F with the Kähler form is then an endomorphism of E . Metrics which give rise to connections such that the endomorphism is a multiple of the identity are called Hermite-Einstein metrics.

Indeed, the notion of an Hermite-Einstein connection originated in physics. Hitchin and Kobayashi made a very precise conjecture connecting this notion to that of Mumford-Takemoto stability, which is known as the Hitchin-Kobayashi correspondence. Uhlenbeck and Yau showed in [UY86] that the conjecture holds true for compact Kähler manifolds.

Theorem 5.2. [UY86] *A holomorphic vector bundle over a compact Kähler manifold admits an Hermite-Einstein structure if and only if it is polystable.*

As an immediate consequence we get that:

Corollary 5.3. *Let G/P a rational homogeneous variety of type ADE. Then the tangent bundle $T_{G/P}$ is polystable with respect to the anticanonical polarization $-K_{G/P}$ induced by the Hermite-Einstein structure.*

Recall now that a simple bundle is in particular indecomposable. Thus the polystability of the tangent bundle $T_{G/P}$ combined with its simplicity implies that the direct sum of stable bundles in which it decomposes is reduced in reality to only one summand, or in other words that:

Theorem 5.4. *Let G/P a rational homogeneous variety of type ADE. Then the tangent bundle $T_{G/P}$ is stable with respect to the anticanonical polarization $-K_{G/P}$ induced by the Hermite-Einstein structure.*

5.2. Some bounds on stability and polarizations in the A_n case. A natural question arising from Theorem 5.4 is whether or not there are other polarizations having the same property of the anticanonical one, and in case we get a positive answer, can we describe them? This section contains an answer to these questions in some specific cases: in particular here we assume $G = \mathrm{SL}_{n+1}$.

We start with point-hyperplane incidence variety $\mathbb{F}(0, n-1, n)$ in \mathbb{P}^n : these are homogeneous varieties of dimension $2n-1$ and of the form SL_{n+1}/P , where $P = P(\Sigma)$ is the parabolic obtained removing only the first and the last simple roots of the Lie algebra, i.e. $\Sigma = \{\alpha_1, \alpha_n\}$.

Proposition 5.5. *Let $\mathbb{F} = \mathbb{F}(0, n-1, n)$ be the point-hyperplane incidence variety, and set:*

$$m(n) = \frac{-n + n\sqrt{4n^2 + 4n - 3}}{2(n^2 + n - 1)}.$$

Then the tangent bundle $T_{\mathbb{F}}$ is stable with respect to the polarization $\mathcal{O}_{\mathbb{F}}(a, b)$ if and only if it is semistable if and only if:

$$m(n)a \leq b \leq m(n)^{-1}a.$$

Proof. Start by noticing that:

$$\mathbb{F}(0, n-1, n) = \mathbb{P}(Q_{\mathbb{P}^n}),$$

meaning that we can look at our varieties as the projectivization $\mathbb{F} = \mathbb{P}(Q_{\mathbb{P}^n})$ of the quotient bundle $Q_{\mathbb{P}^n} \simeq T_{\mathbb{P}^n}(-1)$ on \mathbb{P}^n . We get projections:

$$\begin{array}{ccc} & \mathbb{F} & \\ \alpha \swarrow & & \searrow \beta \\ \mathbb{P}^n & & \mathbb{P}^{n\vee} \end{array}$$

Hence $\mathrm{Pic}(\mathbb{F}(0, n-1, n)) = \mathbb{Z}^2$ is spanned by $F = \beta^* \mathcal{O}_{\mathbb{P}^{n\vee}}(1) = \mathcal{O}_{\mathbb{F}}(1, 0)$ and $G = \alpha^* \mathcal{O}_{\mathbb{P}^n}(1) = \mathcal{O}_{\mathbb{F}}(0, 1)$.

Recalling that the elements of \mathbb{F} are couples $(p, H) = (\text{point}, \text{hyperplane})$ such that $p \in H \subset \mathbb{P}^n$ we also get the identification:

$$F = \{(p, H) \mid p \in H, p = p_0\} \quad \text{and} \quad G = \{(p, H) \mid p \in H, H = H_0\}.$$

Moreover, we have two short exact sequences:

$$(5.1) \quad 0 \rightarrow \mathcal{O}_{\mathbb{F}} \rightarrow \pi^* Q_{\mathbb{P}^n}^* \otimes \mathcal{O}(1)_{\mathrm{rel}} \rightarrow T_{\mathrm{rel}} \rightarrow 0$$

$$(5.2) \quad 0 \rightarrow T_{\mathrm{rel}} \rightarrow T_{\mathbb{F}} \rightarrow \pi^* T_{\mathbb{P}^n} \rightarrow 0.$$

All in all, the (quiver associated to the) tangent bundle $T_{\mathbb{F}}$ to these varieties has the simple look:

$$(5.3) \quad \begin{array}{ccc} & \bullet^{n-1} & \\ \uparrow & & \\ \bullet^1 & \xrightarrow{\quad} & \bullet^{n-1} \end{array}$$

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So $\text{gr } T_{\mathbb{F}}$ has 3 irreducible summands, all with multiplicity 1 and whose rank is described in the picture (5.3) above.

Now that we have understood the tangent bundle, we can look at its subbundles. Of course there are more than two subbundles. Yet it is enough to check the stability condition only on the two homogeneous subbundles, in virtue of a criterion given by Rohmfeld in [Roh91], and later refined by Faini in [Fai06]:

Theorem 5.6 (Rohmfeld-Faini). *Let H be an ample line bundle. If a homogeneous bundle $E = E_\rho$ is not H -semistable then there exists a homogeneous subbundle F such that $\mu_H(F) > \mu_H(E)$.*

Since our tangent bundles have the particular configuration shown in (5.3), all we need to do is just look at the polarizations $H = aF + bG$ such that:

$$\begin{cases} \mu_H(E') < \mu_H(T_{\mathbb{F}}) \\ \mu_H(E'') < \mu_H(T_{\mathbb{F}}) \end{cases}$$

where the subbundles E' and E'' are the irreducible rank $n - 1$ bundles in the following picture:

$$\begin{array}{ccccccc} E' = & \bullet & & & & & E'' = \circ \\ & & & & & & \\ & & & & & & \\ & \circ & & \circ & & \circ & \bullet \end{array}$$

Knowing all the weights of the representation associated to our bundles, we easily compute their first Chern class, so that the two inequalities above read:

$$(5.4) \quad \begin{cases} \frac{(-\lambda_1 + n\lambda_2) \cdot (a\lambda_1 + b\lambda_2)^{2n-2}}{\frac{n-1}{n-1}} < \frac{(n\lambda_1 + n\lambda_2) \cdot (a\lambda_1 + b\lambda_2)^{2n-2}}{\frac{2n-1}{2n-1}} \\ \frac{(n\lambda_1 - \lambda_2) \cdot (a\lambda_1 + b\lambda_2)^{2n-2}}{\frac{n-1}{n-1}} < \frac{(n\lambda_1 + n\lambda_2) \cdot (a\lambda_1 + b\lambda_2)^{2n-2}}{\frac{2n-1}{2n-1}} \end{cases}$$

Intersection theory is easy to understand in this particular case; out of all products

$$F^i G^j, \quad i + j = 2n - 1,$$

the only non-vanishing ones will be (recall that we are pulling back from a \mathbb{P}^n !):

$$\begin{cases} F^{n-1} G^n = F^n G^{n-1} = 1 \\ F^i G^j = 0 \quad \text{for } i, j \neq n, n-1, i + j = 2n - 1 \end{cases}$$

We stress the fact that there is a complete symmetry $\mathbb{P}^n \leftrightarrow \mathbb{P}^{n\vee}$, and thus $F \leftrightarrow G$. Simplifying (5.4), we get that $T_{\mathbb{F}}$ is stable if and only if:

$$\begin{cases} (n^2 + n - 1)b^2 + nab - n^2 a^2 > 0 \\ -n^2 b^2 + nab + (n^2 + n - 1)a^2 > 0 \end{cases}$$

And from these two inequalities one gets that $T_{\mathbb{F}}$ is stable with respect to $H = aF + bG$ if and only $m(n)a \leq b \leq m(n)^{-1}a$, where $m(n)$ is defined as in the statement of Proposition 5.5.

The only thing left to check is the equivalence “stable \Leftrightarrow semistable”, but this simply follows from the fact that the conditions for semistability are just the conditions (5.4) where we substitute the sign $<$ with a \leq .

But in reality equality never holds, for $m(n)$ is an irrational coefficient (because $4n^2 + 4n - 3 = (2n + 1)^2 - 4$), while we need $(a, b) \in \mathbb{Z}^2$. \square

Remark 5.1. An interesting observation is that as n grows bigger $m(n)$ approaches 1. Hence the cone of polarizations with respect to which $T_{\mathbb{F}}$ is stable collapses to the line $a = b$, which is the one corresponding to the anticanonical. The collapsing process is illustrated in Figure 1 below, where we have drawn the cone for—respectively— $n = 2$ and $n = 20$ in the space of polarizations (a, b) . The dotted line is the anticanonical polarization $\{a = b\}$.

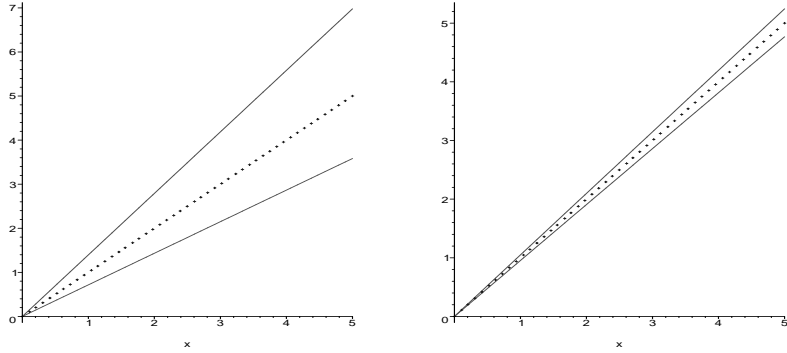


FIGURE 1. The cone for $n = 2$ (left) and $n = 20$ (right).

We wish to obtain the same type of characterization as Proposition 5.5 for other homogeneous varieties.

The next simplest case after the incidence varieties with Picard group \mathbb{Z}^2 is the full flag manifold $\mathbb{F} = \mathrm{SL}_4/B$, with $\mathrm{Pic}(\mathbb{F}) = \mathbb{Z}^3$ and dimension 6. The weights of the tangent bundle are the 6 positive roots of SL_4 . Before we can show how the quiver representation $[T_{\mathbb{F}}]$ looks like, we need to explain here how the relations on the quiver work. In the Borel case, Definition 3.3 can be made more explicit.

For each root α , let $e_\alpha \in \mathfrak{g}_\alpha$ be the corresponding Chevalley generator, and define the Chevalley coefficients $N_{\alpha\beta}$ by $[e_\alpha, e_\beta] = N_{\alpha\beta}e_{\alpha+\beta}$, if $\alpha + \beta \in \Phi^+$, and $N_{\alpha\beta} = 0$ otherwise.

Proposition 5.7. [ACGP03, Proposition 1.21] *The relations \mathcal{R} on the quiver $\mathcal{Q}_{G/P}$ are the ideal generated by all the equations:*

$$\mathcal{R}_{(\alpha,\beta)} = f_{\lambda\mu}f_{\mu\nu} - f_{\lambda\mu'}f_{\mu'\nu} - N_{\alpha\beta}g_{\lambda\nu}$$

for $\alpha < \beta \in \Phi^+$ and for any couple of weights $\lambda, \nu \in \mathcal{Q}_0$, where $\alpha + \beta = \lambda - \nu$, $\mu = \lambda + \alpha$ and $\mu' = \lambda + \beta$.

Let now $G = \mathrm{SL}_N$. For the sake of simplicity we indicate with e_{ij} the elementary matrix with weight $L_i - L_j$.

Take a couple of roots $\alpha < \beta \in \Phi^+$, $\alpha = L_i - L_j$ and $\beta = L_h - L_k$ (with $i < j$). Note that in this case the non-zero coefficients $N_{\alpha\beta}$ are all ± 1 . The only possibility for the coefficient $N_{\alpha\beta}$ to be non-zero is if either $j = h$ ($\Rightarrow N_{\alpha\beta} = 1$), or $i = k$ ($\Rightarrow N_{\alpha\beta} = -1$). But then from Proposition 5.7:

$$\begin{aligned}\mathcal{R}_{(L_i-L_h, L_h-L_k)} &= e_{ih}e_{hk} - e_{hk}e_{ih} - e_{ik} = 0, \\ \mathcal{R}_{(L_i-L_j, L_h-L_i)} &= e_{ij}e_{hi} - e_{hi}e_{ij} + e_{hj} = 0.\end{aligned}$$

In other words the relations tell us nothing other than $[e_{ih}, e_{hk}] = e_{ik}$. Now suppose $j \neq h$ and $i \neq k$, so that $N_{\alpha\beta} = 0$. The relations are thus:

$$\mathcal{R}_{(L_i-L_j, L_h-L_k)} = e_{ij}e_{hk} - e_{hk}e_{ij} = 0,$$

for any $i, j, h, k+1, \dots, N$.

All in all the relations that we need to put on the quiver $\mathcal{Q}_{\mathbb{F}}$ for a full flag manifold $\mathbb{F} = \mathrm{SL}_N/B$ are nothing but the Serre relations:

$$(5.5) \quad \mathcal{R}_{(L_i-L_h, L_h-L_k)} = [e_{ih}, e_{hk}] = e_{ik},$$

$$(5.6) \quad \mathcal{R}_{(L_i-L_j, L_h-L_i)} = [e_{ij}, e_{hi}] = -e_{hj},$$

$$(5.7) \quad \mathcal{R}_{(L_i-L_j, L_h-L_k)} = e_{ij}e_{hk} - e_{hk}e_{ij} = 0, \text{ for } i \neq k, j \neq h.$$

For $N = 4$, we will have in particular that $[e_{12}, e_{34}] = 0$, so the corresponding arrows commute.

For SL_4/B the quiver representation $[\mathrm{T}_{\mathbb{F}}]$ looks like in (5.8). We have indicated to which element of \mathfrak{n} correspond the arrows.

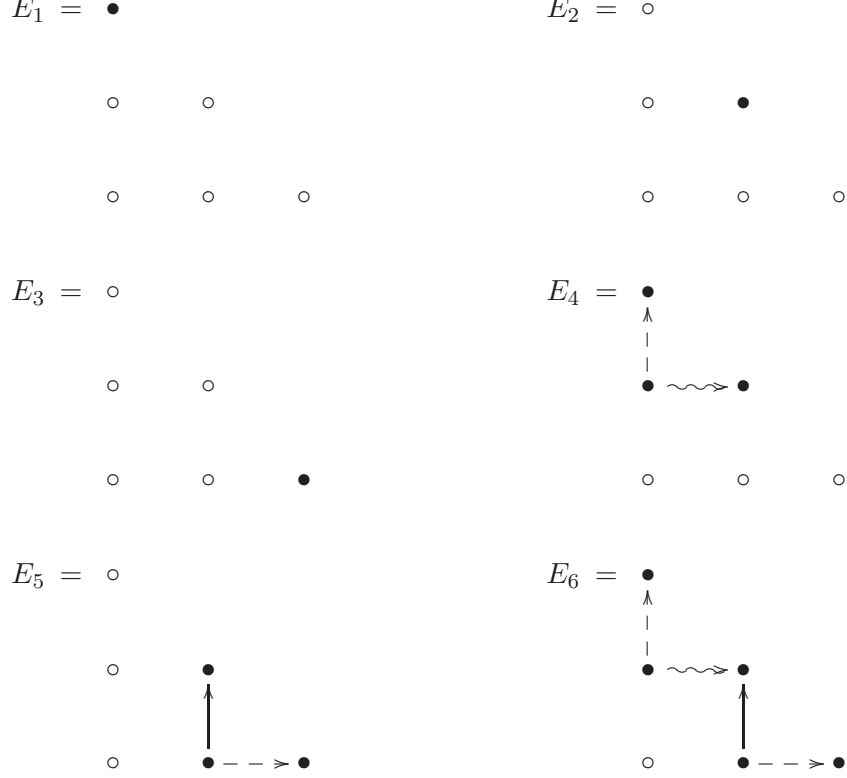
$$(5.8) \quad \mathrm{T}_{\mathbb{F}} = \begin{array}{ccccc} & & \bullet & & \\ & & \uparrow & & \\ & e_{23} & | & & \\ & \bullet & \xrightarrow{e_{12}} & \bullet & \\ e_{34} \uparrow & & & & \uparrow e_{34} \\ \bullet & \xrightarrow{e_{12}} & \bullet & \xrightarrow{e_{23}} & \bullet \end{array}$$

The relations tell us that the square below is commutative:

$$\begin{array}{ccc} \bullet & \xrightarrow{e_{12}} & \bullet \\ e_{34} \uparrow & & \uparrow e_{34} \\ \bullet & \xrightarrow{e_{12}} & \bullet \end{array}$$

Now let's go back to stability computations. Again by Theorem 5.6 all we need to do is identify all the homogenous subbundles F of the tangent bundle $\mathrm{T}_{\mathbb{F}}$, and then impose the stability condition $\mu_H(F) < \mu_H(\mathrm{T}_{\mathbb{F}})$; this will give us a necessary and sufficient condition for the polarization H to be such that $\mathrm{T}_{\mathbb{F}}$ is H -stable.

All the homogenous subbundles we need to analyze are thus the following six:



Now we need to compute the first Chern class of all these bundles. The elements of \mathbb{F} are triples $(p, \ell, \pi) = (\text{point}, \text{line}, \text{plane})$ such that $p \in \ell \subset \pi \subset \mathbb{P}^3$. A basis for the Picard group is given by:

$$\begin{cases} F = \{(p, \ell, \pi) \mid p \in \pi_0, \pi_0 \text{ fixed} \} \\ G = \{(p, \ell, \pi) \mid \ell \cap \ell_0 \neq \emptyset, \ell_0 \text{ fixed} \} \\ H = \{(p, \ell, \pi) \mid p_0 \in \pi, p_0 \text{ fixed} \} \end{cases}$$

They correspond to the pull-back (via the standard projection) of the tautological bundle $\mathcal{O}(1)$ from respectively \mathbb{P}^3 ($\leftrightarrow F$), the Grassmannian of lines in \mathbb{P}^3 $\mathbb{G}(1, 3)$ ($\leftrightarrow G$) and \mathbb{P}^{3^\vee} ($\leftrightarrow H$). We underline the symmetry between F and H .

Since we are looking for all polarizations $\mathcal{O}_{\mathbb{F}}(a, b, c) = aF + bG + cH$ such that for all $i = 1, \dots, 6$:

$$(5.9) \quad \frac{c_1(E_i)(aF + bG + cH)^5}{\text{rk } E_i} < \frac{c_1(\mathbb{T}_{\mathbb{F}})(aF + bG + cH)^5}{6},$$

we are interested in intersections $F^i G^j H^k$, where $i + j + k = 6$. Intersection theory brings us to:

$$\begin{cases} FG^4H = HG^4F = F^2G^2H^2 = FG^3H^2 = F^2G^3H = 2 \\ F^3G^2H = FG^2H^3 = F^3GH^2 = F^2GH^3 = 1 \\ F^iG^jH^k = 0 \text{ for all other } i, j, k \text{ s.t. } i + j + k = 6 \end{cases}$$

With the help of a computer algebra system and the intersections above we see that the six inequalities (5.9) define a cone around the line $\{a = b = c\}$ that corresponds to the anticanonical polarization $-K_{\mathbb{F}} = \mathcal{O}_{\mathbb{F}}(2, 2, 2)$.

Figure 2 below shows a section of this cone cut by the plane $\{a + b + c = 3\}$ orthogonal to the “anticanonical line”. It is somewhat unexpected that the region that we obtain is not convex. In fact from general theory we learn that the area would be convex if we were considering all possible characters in the definition of stability, and not just the ones arising from geometric polarizations given by ample line bundles like in our case. In the next section we will explain with some more detail the question of characters, stability and moduli spaces.

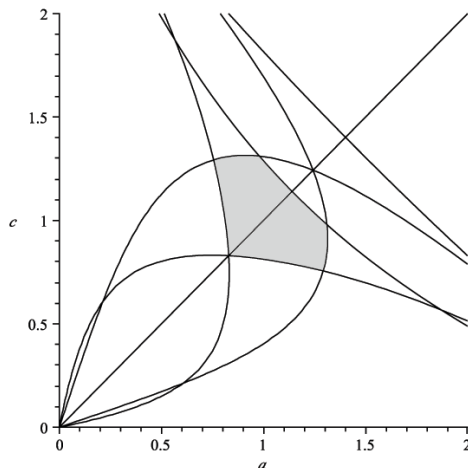


FIGURE 2. Polarizations for SL_4/B

6. MODULI AND STABILITY

There is a notion of semistability of representations of quivers introduced by King in [Kin94], which is suitable to construct moduli spaces according to the Geometric Invariant Theory (GIT from now on).

In their paper [OR06] Ottaviani and Rubei showed that King’s notion of semistability for a representation $[E]$ of the quiver $\mathcal{Q}_{G/P}$ is in fact equivalent to the Mumford-Takemoto semistability of the associated bundle E on $X = G/P$, when the latter is a Hermitian symmetric variety. They thus obtained moduli spaces of G -homogeneous semistable bundles with fixed $\mathrm{gr} E$.

In this section we recall some of these results and show how they can be extended to our more general setting where X is any—not necessarily

Hermitian symmetric—homogeneous variety.

Consider the moduli problem of homogeneous vector bundles E on X with the same $\text{gr } E$ and thus with the same dimension vector $\alpha = (\alpha_\lambda) \in \mathbb{Z}^{|\mathcal{Q}_0|}$.

Once we have made the choice of vector spaces V_λ with dimension α_λ , the isomorphism classes of representations of the quiver \mathcal{Q}_X with the same dimension vector α are associated with certain orbits of the group:

$$\text{GL}(\alpha) := \prod_{\lambda \in \mathcal{Q}_0} \text{GL}(V_\lambda)$$

acting over

$$\mathcal{R}(\mathcal{Q}_X, \alpha) := \oplus_{a \in \mathcal{Q}_1} \text{Hom}(V_{ta}, V_{ha})$$

by $(g \cdot \phi)_a = g_{ha} \phi_a g_{ta}^{-1}$, and specifically over the closed subvariety $V_X(\alpha) \subseteq \mathcal{R}(\mathcal{Q}_X, \alpha)$ defined by the relations in our quiver.

The affine quotient $\text{Spec}(\mathbb{C}[V_X(\alpha)]^{\text{GL}(\alpha)})$ is a single point, represented by $\text{gr } E$ itself, and it thus has no interest for our purposes.

Following [Kin94], we call a *character* of the category $\mathbb{C}\mathcal{Q}_X - \text{mod}$ an additive function $\sigma : K_0(\mathbb{C}\mathcal{Q}_X - \text{mod}) \rightarrow \mathbb{R}$ on the Grothendieck group. (For the sake of simplicity we denote by $\mathbb{C}\mathcal{Q}_X - \text{mod}$ the category of left modules on the path algebra $(\mathbb{C}\mathcal{Q}_X, \mathcal{R})$ of the quiver \mathcal{Q}_X with relations \mathcal{R} : by writing only $\mathbb{C}\mathcal{Q}_X$ it is understood that we are modding out the path algebra by the ideal of relations.)

A representation $[E]$ of the quiver is called σ -*semistable* if $\sigma([E]) = 0$ and every subrepresentation $[E'] \subseteq [E]$ satisfies $\sigma([E']) \leq 0$. Moreover $[E]$ is called σ -*stable* if the only subrepresentations $[E']$ satisfying $\sigma([E']) = 0$ are $[E]$ itself and 0.

When σ takes integer values, there is an associated character χ_σ for $\text{GL}(\alpha)$ acting on $\mathcal{R}(\mathcal{Q}_X, \alpha)$. More precisely, King shows that the characters of $\text{GL}(\alpha)$ $\chi_\sigma, \chi_\sigma : \text{GL}(\alpha) \rightarrow \mathbb{C}^*$ are given by:

$$\chi_\sigma(g) = \prod_{\lambda \in (\mathcal{Q}_X)_0} \det(g_\lambda)^{\sigma_\lambda}$$

for $\sigma \in \mathbb{Z}^{|\mathcal{Q}_0|}$ such that $\sum_\lambda \sigma_\lambda \alpha_\lambda = 0$.

A point in $\mathcal{R}(\mathcal{Q}_X, \alpha)$ corresponding to a representation $[E] \in \mathbb{C}\mathcal{Q}_X - \text{mod}$ is χ_σ -semistable (respectively χ_σ -stable) if and only if $[E]$ is σ -semistable (respectively σ -stable).

We stress the fact that $\sigma \in \text{Hom}(\mathbb{C}\mathcal{Q}_X - \text{mod}, \mathbb{Z})$ can be simply seen as an homomorphism that applied to E_λ gives σ_λ .

A function $f \in \mathbb{C}[V_X(\alpha)]$ is called a *relative invariant of weight σ* if

$$f(g \cdot x) = \chi_\sigma(g)f(x),$$

and the space of such relatively invariant functions is denoted by:

$$\mathbb{C}[V_X(\alpha)]^{\text{GL}(\alpha), \sigma}.$$

So once we have fixed the dimension vector α and a character σ , we can define the moduli space $M_X(\alpha, \sigma)$ by:

$$M_X(\alpha, \sigma) := \text{Proj} \left(\oplus_{n \geq 0} \mathbb{C}[V_X(\alpha)]^{\text{GL}(\alpha), n\sigma} \right),$$

which is projective over $\text{Spec}(\mathbb{C}[V_X(\alpha)]^{\text{GL}(\alpha)})$, hence it is a projective variety.

In fact $M_X(\alpha, \sigma)$ has a more geometrical description as the GIT quotient of the open set $V_X(\alpha)^{ss}$ of χ_σ -semistable points.

Fix an ample line bundle H (a polarization).

Every ample line bundle H defines a character σ_H by:

$$\sigma_H(\alpha)_\lambda = \text{rk}(E)(c_1(E_\lambda)H^{n-1}) - (c_1(\text{gr } E)H^{n-1}) \text{rk } E_\lambda,$$

where n is the dimension of the underlying variety.

Notice that given an $F \in \mathbb{C}\mathcal{Q}_X - \text{mod}$ with dimension vector α and given a fixed character σ , we can define the *slope of F with respect to σ* (or *slope of α w.r.t. σ*):

$$\mu_\sigma(F) = \mu_\sigma(\alpha) = \sum_{\lambda \in \mathcal{Q}_0} \sigma_\lambda \alpha_\lambda.$$

An object is then called μ_σ -(semi)stable if and only if it is σ -(semi)stable.

Recall now from Theorem 3.1 that a homogeneous bundle E is determined by $\theta \in \text{Hom}(\text{gr } E, \text{gr } E \otimes T_X)$ such that $\theta \wedge \theta = \varphi\theta$.

Theorem 6.1. [OR06, Generalization of Theorem 7.1] *Let E be a homogeneous bundle on a rational homogeneous variety X , and let α be the dimension vector corresponding to $\text{gr } E$. Fix an ample line bundle H giving the character $\sigma \in \text{Hom}(\mathbb{C}\mathcal{Q}_X - \text{mod}, \mathbb{Z})$. Then the following facts are equivalent:*

- (i) *for every G -invariant subbundle K , we have $\mu_\sigma(K) \leq \mu_\sigma(E)$ (equivariant semistability);*
- (ii) *for every subbundle K such that $\theta_E(\text{gr } K) \subset \text{gr } K \otimes T_X$, we have $\mu_\sigma(K) \leq \mu_\sigma(E)$ (Higgs semistability);*
- (iii) *the representation $[E]$ if \mathcal{Q}_X is σ -semistable, according to [Kin94] (quiver semistability);*
- (iv) *E is a χ_σ -semistable point in $V_X(\alpha)$ for the action of $\text{GL}(\alpha)$ [Kin94] (GIT semistability);*
- (v) *for every subsheaf K , we have $\mu_H(K) \leq \mu_H(E)$ (Mumford-Takemoto semistability).*

Proof. The equivalence (i) \Leftrightarrow (ii) follows from the fact that a subbundle $K \subset E$ is G -invariant if and only if $\theta_E(\text{gr } K) \subset \text{gr } K \otimes T_X$. The equivalence (ii) \Leftrightarrow (iii) is just a rephrasing of the second fundamental equivalence of categories. The equivalence (iii) \Leftrightarrow (iv) is proved in [Kin94, Proposition 3.1]. In fact this equivalence holds true even for those characters σ that do not have a geometric interpretation as the one induced by a polarization that we chose. Finally, the equivalence (i) \Leftrightarrow (v) is proved for example in [Mig96]. \square

With the same reasoning one can prove that:

Theorem 6.2. [OR06, Generalization of Theorem 7.2] *Let E be a homogeneous bundle on a rational homogeneous variety X , and let α be the dimension vector corresponding to $\operatorname{gr} E$. Fix an ample line bundle H giving the character $\sigma \in \operatorname{Hom}(\mathbb{C}\mathcal{Q}_X - \operatorname{mod}, \mathbb{Z})$. Then the following facts are equivalent:*

- (i) *for every proper G -invariant subbundle K , we have $\mu_\sigma(K) < \mu_\sigma(E)$ (equivariant stability);*
- (ii) *for every proper subbundle K such that $\theta_E(\operatorname{gr} K) \subset \operatorname{gr} K \otimes T_X$, we have $\mu_\sigma(K) < \mu_\sigma(E)$ (Higgs stability);*
- (iii) *the representation $[E]$ if \mathcal{Q}_X is σ -stable, according to [Kin94] (quiver stability);*
- (iv) *E is a χ_σ -stable point in $V_X(\alpha)$ for the action of $\operatorname{GL}(\alpha)$ [Kin94] (GIT stability);*
- (v) *$E \simeq W \otimes E'$ where W is an irreducible G -module, and for every proper subsheaf $K \subset E'$, we have $\mu_H(K) < \mu_H(E')$ (Mumford-Takemoto stability).*

Proof. The proof is the same of Theorem 6.1. The equivalence (i) \Leftrightarrow (v) is proved in [Fai06]. \square

We remark that case (v) in Theorem 6.2 is more involved with respect to the naive expectation coming from (v) in Theorem 6.1.

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