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Moduli of linear representations, symmetric products and the non commutative Hilbert scheme

Francesco Vaccarino

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1 Introduction

Let k be a commutative ring and let R be a commutative k -algebra. Given a positive integer n and a R -algebra A one can consider three functors of points from the category \mathcal{C}_R of commutative R -algebras to the small category of sets. All these functors are representable, namely

- $\mathcal{R}ep_A^n$ represents the functor induced by $B \rightarrow \text{hom}_R(A, M_n(B))$, where $M_n(B)$ are the $n \times n$ matrices over B , for all $B \in \mathcal{C}_R$.
- the non-commutative Hilbert scheme Hilb_A^n represents the functor induced by

$$B \rightarrow \{ \text{left ideals } I \subset A \otimes_k B : A \otimes_k B / I \text{ is a projective } R\text{-module of rank } n \}$$

for all $B \in \mathcal{C}_R$.

- $\text{Spec } \Gamma_R^n(A)^{ab}$ represents the functor induced by

$$B \rightarrow \{ \text{multiplicative polynomial laws homogeneous of degree } n \text{ from } A \text{ to } B \}$$

for all $B \in \mathcal{C}_R$.

When A is commutative $Hilb_n^A$ is the usual Hilbert scheme of n -points of $X = \text{Spec } A$. A polynomial law is a kind of map generalizing polynomial mapping and coinciding with it over flat R -modules. The typical example of multiplicative polynomial law homogeneous of degree n is the determinant of $n \times n$ matrices. The R -algebra $\Gamma_R^n(A)^{ab}$ is the quotient of the R -algebra $\Gamma_R^n(A)$ of the divided powers of degree n on A by the ideal generated by commutators. When A is flat as R -module this coincides with the symmetric tensors of order n that is $\Gamma_R^n(A) \cong (A^{\otimes n})^{S_n}$, where S_n is the symmetric group. Therefore when A is commutative and flat we have $\text{Spec } \Gamma_R^n(A)^{ab} \cong X^{(n)}$, the n -th symmetric product of $X = \text{Spec } A$.

We discuss the connections between the coarse moduli space $\mathcal{R}ep_A^n // GL_n$ of the n -dimensional representations of A , the scheme $Hilb_A^n$ and the affine scheme $\text{Spec } \Gamma_R^n(A)^{ab}$. We build a norm map from $Hilb_A^n$ to $\Gamma_R^n(A)^{ab}$ which specializes to the Hilbert-Chow morphism on the geometric points when A is commutative and k is an algebraically closed field. This generalizes the construction done by Grothendieck, Deligne and others. When k is an infinite field and $A = k\{x_1, \dots, x_m\}$ is the free k -associative algebra on m letters, we use the isomorphism $\text{Spec } \Gamma_k^n(A)^{ab} \cong \mathcal{R}ep_A^n // GL_n$ following from Theorem 1 in [35] to give a simple description of this norm map.

A field of application of our construction can be the extension to the positive characteristic case of some of the results due to L. Le Bruyn on noncommutative desingularization which can be found in [16]. Another intriguing possibility is to use our construction to extend the work done by C.H.Liu and S.T.Yau on D-Branes in [17] to the more recent non commutative case in [18].

This survey paper is based on [9, 33, 34, 35, 36].

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Notations

Unless otherwise stated we adopt the following notations:

- k is a fixed commutative ground ring.
- R is a commutative k -algebra.

- B is a commutative R -algebra.
- A is an arbitrary R -algebra.
- $F = k\{x_1, \dots, x_m\}$ denotes the associative free k -algebra on m letters.
- $\mathcal{N}_-, \mathcal{C}_-, \text{Mod}_-$ and Sets denote the categories of $-$ algebras, commutative $-$ algebras, $-$ modules and sets, respectively.
- we write $\mathcal{A}(B, C)$ for the $\text{hom}_{\mathcal{A}}(B, C)$ set in a category \mathcal{A} with B, C objects in \mathcal{A} .

2 Moduli of representations

2.1 The universal representation

We denote by $M_n(B)$ the full ring of $n \times n$ matrices over B . If $f : B \rightarrow C$ is a ring homomorphism we denote with $M_n(f) : M_n(B) \rightarrow M_n(C)$ the homomorphism induced on matrices.

Definition 2.1. By an n -dimensional representation of A over B we mean a homomorphism of R -algebras $\rho : A \rightarrow M_n(B)$.

The assignment $B \rightarrow \mathcal{N}_R(A, M_n(B))$ defines a covariant functor $\mathcal{N}_R \rightarrow \text{Sets}$. This functor is represented by a commutative R -algebra. We report here the proof of this fact to show how this algebra comes up using generic matrices. These objects will be also crucial in the construction of the norm map in Section 4.

Lemma 2.1. [4, Lemma 1.2.] For all $A \in \mathcal{N}_R$ there exist a commutative R -algebra $V_n(A)$ and a representation $\pi_A : A \rightarrow M_n(V_n(A))$ such that $\rho \mapsto M_n(\rho) \cdot \pi_A$ gives an isomorphism

$$\mathcal{C}_R(V_n(A), B) \xrightarrow{\cong} \mathcal{N}_R(A, M_n(B)) \quad (1)$$

for all $B \in \mathcal{C}_R$.

Proof. Let A be an arbitrary R -algebra and suppose that there exist π_A and $V_n(A)$ as in the statement. Let $I \subset A$ be a bilateral ideal of A . Let J be the ideal generated by $\pi_A(I)$ in $M_n(V_n(A))$. There exists then an ideal $K \subset$

$V_n(A)$ such that $J = M_n(K)$. It should be clear that $V_n(A/I) = V_n(A)/K$ and that $\pi_{A/I} : A/I \rightarrow M_n(V_n(A/I))$ is given by

$$\begin{array}{ccc} A & \xrightarrow{\pi_A} & M_n(V_n(A)) \\ \downarrow & & \downarrow \\ A/I & \xrightarrow{\pi_{A/I}} & M_n(V_n(A/I)) = M_n(V_n(A))/J \end{array}$$

It is enough now to prove the statement for $A = R\{x_1, \dots, x_m\} = R \otimes_k F$ the free associative R -algebra on m letters. Let $V_n(A) = R[\xi_{kij}]$ be the polynomial ring in variables $\{\xi_{kij} : i, j = 1, \dots, n, k = 1, \dots, m\}$ over the base ring R . To every n -dimensional representation of A over B it corresponds a unique m -tuple of $n \times n$ matrices, namely the images of x_1, \dots, x_m , hence a unique $\bar{\rho} \in \mathcal{C}_R(R[\xi_{kij}], B)$ such that $\bar{\rho}(\xi_{kij}) = (\rho(x_k))_{ij}$. Following C.Procesi [23, 4] we introduce the generic matrices. Let $\xi_k = (\xi_{kij})$ be the $n \times n$ matrix whose (i, j) entry is ξ_{kij} for $i, j = 1, \dots, n$ and $k = 1, \dots, m$. We call ξ_1, \dots, ξ_m the generic $n \times n$ matrices. Consider the map

$$\pi_A : A \rightarrow M_n(V_n(R)), \quad x_k \mapsto \xi_k, \quad k = 1, \dots, m.$$

It is then clear that the map $\mathcal{C}_R(V_n(A), B) \ni \sigma \mapsto M_n(\sigma) \cdot \pi_A \in \mathcal{N}_R(A, M_n(B))$ gives the isomorphism (1) in this case. \square

Remark 2.1. It should be clear that the number of generators m of A is immaterial and we can extend the above isomorphism to an arbitrary R -algebra.

Definition 2.2. We write $\mathcal{R}ep_A^n$ to denote $\text{Spec } V_n(A)$. It is considered as an R -scheme.

The map

$$\pi_A : A \rightarrow M_n(V_n(A)), \quad x_k \mapsto \xi_k. \quad (2)$$

is called *the universal n -dimensional representation*.

Given a representation $\rho : A \rightarrow M_n(B)$ we denote by $\bar{\rho}$ its classifying map $\bar{\rho} : V_n(A) \rightarrow B$.

Example 2.1. For the free algebra one has $\mathcal{R}ep_A^n \cong M_n^m$ the scheme whose B -points are the m -tuples of $n \times n$ matrices with entries in B .

Example 2.2. Note that $\mathcal{R}ep_A^n$ could be quite complicated, as an example, when $A = \mathbb{C}[x, y]$ we obtain that $\mathcal{R}ep_A^n$ is the *commuting scheme* i.e. the couples of commuting matrices and it is not even known (but conjecturally true) if it is reduced or not, see [34].

2.2 GL_n -action

Definition 2.3. We denote by GL_n the affine group scheme whose group of B -points form the group $\mathrm{GL}_n(B)$ of $n \times n$ invertible matrices with entries in B , for all $B \in \mathcal{C}_R$.

Define a GL_n -action on $\mathcal{R}ep_A^n$ as follows. For any $\varphi \in \mathcal{R}ep_A^n(B)$, $g \in \mathrm{GL}_n(B)$, let $\varphi^g : V_n(A) \rightarrow B$ be the R -algebra homomorphism corresponding to the representation given by

$$\begin{aligned} A &\rightarrow M_n(B) \\ a &\mapsto g(M_n(\varphi) \cdot \pi_A(a))g^{-1}. \end{aligned} \quad (3)$$

Note that if φ, φ' are B -points of $\mathcal{R}ep_A^n$, then the A -module structures induced on B^n by φ and φ' are isomorphic if and only if there exists $g \in \mathrm{GL}_n(B)$ such that $\varphi' = \varphi^g$.

Definition 2.4. We denote by $\mathcal{R}ep_A^n // \mathrm{GL}_n = \mathrm{Spec} V_n(A)^{\mathrm{GL}_n(R)}$ the categorical quotient (in the category of R -schemes) of $\mathcal{R}ep_A^n$ by GL_n . It is the (coarse) moduli space of n -dimensional linear representations of A .

2.2.1 Determinant

Definition 2.5. We denote by $C_n(A)$ the k -subalgebra of $V_n(A)$ generated by the coefficients $e_i(\pi_A(f))$ of characteristic polynomials

$$\det(t - \pi_A(f)) = t^n + \sum_{i=1}^n (-1)^i e_i(\pi_A(f)) t^{n-i}$$

as f varies in A .

Remark 2.2. Let $B \in \mathcal{C}_R$ and let $b \in M_n(B)$ and $1 \leq i \leq n$. Note that $e_i(b)$ is the trace of $\wedge^i(b)$. Obviously $e_1(b)$ is the trace of b and $e_n(b) = \det(b)$ is the determinant of b .

Remark 2.3. We have just seen that the general linear group $\mathrm{GL}_n(R)$ acts on $V_n(A)$. It is clear that $C_n(A) \subset V_n(A)^{\mathrm{GL}_n(R)}$ for all $k, R \in \mathcal{C}_k$ and $A \in \mathcal{N}_k$. It has been shown in [6, 25, 39] that $C_n(F) = V_n(F)^{\mathrm{GL}_n(k)}$ when k is \mathbb{Z} or an infinite field. When k is a characteristic zero field it follows that $C_n(A) = V_n(A)^{\mathrm{GL}_n(k)}$ for all $A \in \mathcal{N}_k$ because $\mathrm{GL}_n(k)$ is linear reductive in this case.

3 Symmetric products and divided powers

In this part of the paper we introduce a representable functor whose representing scheme will play the same role played by the symmetric product in the classical Hilbert-Chow morphism.

3.1 Polynomial laws

We first recall the definition of polynomial laws between k -modules. These are mappings that generalize the usual polynomial mappings between free k -modules. We mostly follow [27, 28, 36, 38] and we refer the interested reader to these papers for detailed descriptions and proof.

Definition 3.1. Let M and N be two k -modules. A *polynomial law* φ from M to N is a family of mappings $\varphi_A : A \otimes_k M \rightarrow A \otimes_k N$, with $A \in \mathcal{C}_k$ such that the following diagram commutes

$$\begin{array}{ccc} A \otimes_k M & \xrightarrow{\varphi_A} & A \otimes_k N \\ f \otimes id_M \downarrow & & \downarrow f \otimes id_N \\ B \otimes_k M & \xrightarrow{\varphi_B} & B \otimes_k N \end{array} \quad (4)$$

for all $A, B \in \mathcal{C}_k$ and all $f \in \mathcal{C}_k(A, B)$.

Definition 3.2. Let $n \in \mathbb{N}$. If $\varphi_A(au) = a^n \varphi_A(u)$, for all $a \in A$, $u \in A \otimes_k M$ and all $A \in \mathcal{C}_k$, then φ is called *homogeneous of degree n* .

Definition 3.3. If M and N are two k -algebras and

$$\begin{cases} \varphi_A(xy) & = \varphi_A(x)\varphi_A(y) \\ \varphi_A(1_{A \otimes M}) & = 1_{A \otimes N} \end{cases}$$

for $A \in \mathcal{C}_k$ and for all $x, y \in A \otimes_k M$, then φ is called *multiplicative*.

We need the following

Definition 3.4. For S a set and any additive monoid M , we denote by $M^{(S)}$ the set of functions $f : S \rightarrow M$ with finite support.

Let $\alpha \in M^{(S)}$, we denote by $|\alpha|$ the (finite) sum $\sum_{s \in S} \alpha(s)$,

Let A and B be two k -modules and $\varphi : A \rightarrow B$ be a polynomial law. The following result on polynomial laws is a restatement of Théorème I.1 of [27].

Theorem 3.1. *Let S be a set.*

1. *Let $L = k[x_s]_{s \in S}$ and let a_s be elements of A such that a_s is 0 except for a finite number of $s \in S$, then there exist $\varphi_\xi((a_s)) \in B$, with $\xi \in \mathbb{N}^{(S)}$, such that:*

$$\varphi_L\left(\sum_{s \in S} x_s \otimes a_s\right) = \sum_{\xi \in \mathbb{N}^{(S)}} x^\xi \otimes \varphi_\xi((a_s)) \quad (5)$$

where $x^\xi = \prod_{s \in S} x_s^{\xi_s}$.

2. *Let R be any commutative k -algebra and let $r_s \in R$ for $s \in S$, then:*

$$\varphi_R\left(\sum_{s \in S} r_s \otimes a_s\right) = \sum_{\xi \in \mathbb{N}^{(S)}} r^\xi \otimes \varphi_\xi((a_s)) \quad (6)$$

where $r^\xi = \prod_{s \in S} r_s^{\xi_s}$.

3. *If φ is homogeneous of degree n , then one has $\varphi_\xi((a_s)) = 0$ if $|\xi|$ is different from n . That is:*

$$\varphi_R\left(\sum_{s \in S} r_s \otimes a_s\right) = \sum_{\xi \in \mathbb{N}^{(S)}, |\xi|=n} r^\xi \otimes \varphi_\xi((a_s)) \quad (7)$$

In particular, if φ is homogeneous of degree 0 or 1, then it is constant or linear, respectively.

Remark 3.1. The above theorem means that a polynomial law $\varphi : A \rightarrow B$ is completely determined by its coefficients $\varphi_\xi((a_s))$, where $\{a_s : s \in S\}$ is a set of linear generator of A and $\xi \in \mathbb{N}^{(S)}$.

Remark 3.2. If A is a free k -module and $\{a_t : t \in T\}$ is a basis of A , then φ is completely determined by its coefficients $\varphi_\xi((a_t))$, with $\xi \in \mathbb{N}^{(T)}$. If also B is a free k -module with basis $\{b_u : u \in U\}$, then $\varphi_\xi((a_t)) = \sum_{u \in U} \lambda_u(\xi) b_u$. Let $a = \sum_{t \in T} \mu_t a_t \in A$. Since only a finite number of μ_t and $\lambda_u(\xi)$ are different from zero, the following makes sense:

$$\begin{aligned} \varphi(a) &= \varphi\left(\sum_{t \in T} \mu_t a_t\right) = \sum_{\xi \in \mathbb{N}^{(T)}} \mu^\xi \varphi_\xi((a_t)) = \sum_{\xi \in \mathbb{N}^{(T)}} \mu^\xi \left(\sum_{u \in U} \lambda_u(\xi) b_u\right) \\ &= \sum_{u \in U} \left(\sum_{\xi \in \mathbb{N}^{(T)}} \lambda_u(\xi) \mu^\xi\right) b_u. \end{aligned}$$

Hence, if both A and B are free k -modules, a polynomial law $\varphi : A \rightarrow B$ is simply a polynomial map.

Definition 3.5. Let k be a commutative ring.

- (1) For M, N two k -modules we set $P_k^n(M, N)$ for the set of homogeneous polynomial laws $M \rightarrow N$ of degree n .
- (2) If M, N are two k -algebras we set $MP_k^n(M, N)$ for the multiplicative homogeneous polynomial laws $M \rightarrow N$ of degree n .

The assignment $N \rightarrow P_k^n(M, N)$ (resp. $N \rightarrow MP_k^n(M, N)$) determines a functor from Mod_k (resp. \mathcal{N}_k) to $Sets$.

Example 3.1. 1. For all $A, B \in \mathcal{N}_k$ it holds that $MP_k^1(A, B) = \mathcal{N}_k(A, B)$. A linear multiplicative polynomial law is a k -algebra homomorphism.

2. For $B \in \mathcal{C}_k$ the usual determinant $det : M_n(B) \rightarrow B$ belongs to $MP_k^n(M_n(B), B)$
3. For $B \in \mathcal{C}_k$ the mapping $b \mapsto b^n$ belongs to $MP_k^n(B, B)$
4. For $B, C \in \mathcal{C}_B$ consider $p \in MP_B^n(B, C)$, since $p(b) = p(b1) = b^n p(1) = b^n 1_C$ it is clear that in this case raising to the power n is the unique multiplicative polynomial law homogeneous of degree n .
5. When $A \in \mathcal{N}_k$ is an Azumaya algebra of rank n^2 over its center k , then its reduced norm N belongs to $MP_k^n(A, k)$.

3.2 Divided powers

The functors just introduced in Def. 3.5 are represented by the divided powers which we introduce right now.

Definition 3.6. For a k -module M the *divided powers algebra* $\Gamma_k(M)$ (see [27, 28]) is an associative and commutative k -algebra with identity 1_k and product \times , with generators $m^{(k)}$, with $m \in M$, $k \in \mathbb{Z}$ and relations, for all $m, n \in M$:

1. $m^{(i)} = 0, \forall i < 0$;
2. $m^{(0)} = 1_k, \forall m \in M$;

3. $(\alpha m)^{(i)} = \alpha^i m^{(i)}, \forall \alpha \in k, \forall i \in \mathbb{N}$;
4. $(m + n)^{(k)} = \sum_{i+j=k} m^{(i)} \times n^{(j)}, \forall k \in \mathbb{N}$;
5. $m^{(i)} \times m^{(j)} = \binom{i+j}{i} m^{(i+j)}, \forall i, j \in \mathbb{N}$.

The k -module $\Gamma_k(M)$ is generated by finite products $\times_{i \in I} x_i^{(\alpha_i)}$ of the above generators. The divided powers algebra $\Gamma_k(M)$ is an \mathbb{N} -graded algebra with homogeneous components $\Gamma_k^n(M)$, ($n \in \mathbb{N}$), the submodule generated by $\{\times_{i \in I} x_i^{(\alpha_i)} : |\alpha| = \sum_i \alpha_i = n\}$. One easily checks that Γ_k is a functor from Mod_k to \mathcal{C}_k .

Remark 3.3. Suppose $n!$ is invertible in k e.g. $k \supset \mathbb{Q}$. The map given by $x^{(n)} \mapsto x^n/n!$ induces an isomorphism of k -algebras between $\Gamma_k(M)$ and the symmetric algebra over M . Whence the name *divided powers*.

3.3 Universal properties

3.3.1 Functoriality and adjointness

Γ_k^n is a covariant functor from Mod_k to Mod_k and one can easily check that it preserves surjections. The map $\gamma^n : r \mapsto r^{(n)}$ is a polynomial law $M \rightarrow \Gamma_k^n(M)$ homogeneous of degree n . We call it *the universal map* for the following reason: consider another k -module N and the set $Mod_k(\Gamma_k^n(M), N)$ of homomorphisms of k -modules between $\Gamma_k^n(M)$ and N , we have an isomorphism

$$Mod_k(\Gamma_k^n(M), N) \xrightarrow{\cong} P_k^n(M, N) \quad (8)$$

given by $\varphi \mapsto \varphi \circ \gamma^n$.

3.3.2 The algebra $\Gamma_k^n(A)$

If A is k -algebra then $\Gamma_k^n(A)$ inherits a structure of k -algebra by

$$a^{(n)} \star b^{(n)} = (ab)^{(n)} \quad (9)$$

The unit in $\Gamma_k^n(A)$ is $1^{(n)}$. It was proved by N.Roby [28] that in this way $A \mapsto \Gamma_k^n(A)$ gives a functor from k -algebras to k -algebras such that $\gamma^n(a)\gamma^n(b) = \gamma^n(ab)$, $\forall a, b \in A$. Hence

$$\mathcal{N}_k(\Gamma_k^n(A), B) \xrightarrow{\cong} MP_k^n(A, B) \quad (10)$$

and the map given by $\varphi \mapsto \varphi \circ \gamma^n$ is an isomorphism for all $A, B \in \mathcal{N}_k$.

3.3.3 Base change

[27, Thm. III.3, p. 262] For any $R \in \mathcal{C}_k$ and $A \in \text{Mod}_k$ it holds that

$$R \otimes_k \Gamma_k(A) \xrightarrow{\cong} \Gamma_R(R \otimes_k A). \quad (11)$$

When A is a k -algebra this gives an isomorphism of R -algebras

$$R \otimes_k \Gamma_k^n(A) \xrightarrow{\cong} \Gamma_R^n(R \otimes_k A) \quad (12)$$

for all $n \geq 0$.

3.4 Symmetric tensors

Definition 3.7. Let M be a k -module and consider the n -fold tensor power $M^{\otimes n}$. The symmetric group S_n acts on $M^{\otimes n}$ by permuting the factors and we denote by $\text{TS}_k^n(M)$ or simply by $\text{TS}^n(M)$ the k -submodule of $M^{\otimes n}$ of the invariants for this action. The elements of $\text{TS}^n(M)$ are called symmetric tensors of degree n over M .

Remark 3.4. If M is a k -algebra then S_n acts on $M^{\otimes n}$ as a group of k -algebra automorphisms. Hence $\text{TS}^n(M)$ is a k -subalgebra of $M^{\otimes n}$.

3.4.1 Flatness and Symmetric Tensors

Suppose $M \in \mathcal{N}_k$ (resp. $M \in \mathcal{C}_k$). The homogeneous polynomial law $M \rightarrow \text{TS}_k^n(M)$ given by $x \mapsto x^{\otimes n}$ gives a morphism $\tau_n : \Gamma_k^n(M) \rightarrow \text{TS}_k^n(M)$ which is an isomorphism when M is flat over k . Indeed one can easily prove that τ_n is an isomorphism in case M is free. The flat case then follows because any flat k -module is a filtered direct limit of free modules and both Γ^n and TS^n commute with filtered direct limits.

Remark 3.5. Divided powers and symmetric tensors are not always isomorphic. See [19] for counterexamples.

3.5 Generators

Following [35] we give here a generating set for $\Gamma_k^n(A)$.

We need a Lemma.

Lemma 3.2. *Suppose $M \in \text{Mod}_k$ is generated by $\{m_i\}_{i \in I}$ then $\Gamma_k^n(M)$ is generated by $\{\times_{i \in I} m_i^{(\alpha_i)} : |\alpha| = \sum_i \alpha_i = n\}$ for all $n \in \mathbb{N}$. In particular if $\{m_i\}_{i \in I}$ is a basis of M then $\{\times_{i \in I} m_i^{(\alpha_i)} : |\alpha| = \sum_i \alpha_i = n\}$ is a basis of $\Gamma_k^n(M)$ for all $n \in \mathbb{N}$.*

Proof. The first assertion follows directly from the definition.

Suppose M is free of finite rank p . By Remark 3.2 it follows that any polynomial law $M \rightarrow k$ homogeneous of degree n can be identified with an homogeneous polynomial in p variables $\sum_{h_1+\dots+h_p=n} a_{h_1,\dots,h_p} x_1^{h_1} \cdots x_p^{h_p} \in k[x_1, \dots, x_p]$. This means that one can find a linear form taking arbitrary values on the $m_1^{(\alpha_1)} \times \cdots \times m_p^{(\alpha_p)}$. This argument easily extends to not finitely generated free k -modules by applying it to arbitrary finite subsets of a basis. \square

Definition 3.8. Let \mathfrak{M} denote the set of monomials in F . There is a natural degree d on F given by $d(x_i) = 1$ for all $i = 1, \dots, m$ and $d(1) = 0$. We denote by \mathfrak{M}^+ the set of monomials of positive degree. Thus $\mathfrak{M} = \mathfrak{M}^+ \cup \{1\}$. We set F^+ for the ideal of F generated by \mathfrak{M}^+ .

It is clear that \mathfrak{M} is a basis of F so that $\mathfrak{M}_n = \{\times_{\mu \in \mathfrak{M}} \mu^{(\alpha_\mu)} : |\alpha| = n\}$ is a basis of $\Gamma_k^n(M)$ for all $n \in \mathbb{N}$ by Lemma 3.2.

Proposition 3.3 (Product Formula). *Let $h, k \in \mathbb{N}$, $\alpha \in \mathbb{N}^h$, $\beta \in \mathbb{N}^k$ be such that $|\alpha|, |\beta| \leq n$. Let $r_1, \dots, r_h, s_1, \dots, s_k \in F^+$. Then*

$$\begin{aligned} 1^{(n-|\alpha|)} \times_{i=1}^h r_i^{(\alpha_i)} \star 1^{(n-|\beta|)} \times_{j=1}^k s_j^{(\beta_j)} &= \\ &= \sum_{\gamma} 1^{(n-|\gamma|)} \times_{i=1}^h r_i^{(\gamma_{i0})} \times_{j=1}^k s_j^{(\gamma_{0j})} \times_{i,j=1}^{h,k} (r_i s_j)^{(\gamma_{ij})} \end{aligned} \quad (13)$$

where

$$\gamma = (\gamma_{10}, \gamma_{20}, \dots, \gamma_{h0}, \gamma_{01}, \dots, \gamma_{0k}, \gamma_{11}, \dots, \gamma_{1k}, \dots, \gamma_{h1}, \dots, \gamma_{hk})$$

is such that

$$\begin{cases} \gamma_{ij} \in \mathbb{N} \\ \sum_{i,j} \gamma_{ij} \leq n \\ \sum_{j=0}^k \gamma_{ij} = \alpha_i \text{ for } i = 1, \dots, h \\ \sum_{i=0}^h \gamma_{ij} = \beta_j \text{ for } j = 1, \dots, k. \end{cases} \quad (14)$$

Proof. Let t_1, t_2 be two commuting independent variables and let $a, b \in F^+$. We have

$$(1 + t_1 \otimes a)^{(n)} \star (1 + t_2 \otimes b)^{(n)} = (1 + t_1 \otimes a + t_2 \otimes b + t_1 t_2 \otimes ab)^{(n)} \quad (15)$$

hence

$$\begin{aligned} & (1^{(n)} + \sum_{i=1}^n t_1^i \otimes 1^{(n-i)} \times a^{(i)}) \star (1^{(n)} + \sum_{j=1}^n t_2^j \otimes 1^{(n-j)} \times b^{(j)}) = \\ & = 1^{(n)} + \sum_{i,j} t_1^i t_2^j \otimes 1^{(n-i)} \times a^{(i)} \star 1^{(n-j)} \times b^{(j)} = \\ & = 1^{(n)} + \sum_{l_1, l_2, l_{12}} t_1^{l_1+l_{12}} t_2^{l_2+l_{12}} \otimes 1^{(n-|\gamma|)} \times_{i=1}^h r_i^{(\gamma_{i0})} \times_{j=1}^k s_j^{(\gamma_{0j})} \times_{i,j=1}^{h,k} (r_i s_j)^{(\gamma_{ij})} \end{aligned}$$

The desired equation then easily follows. \square

Remark 3.6. The product formula could be more easily visualized by observing that we are summing over those matrices of positive integers

$$\bar{\gamma} = \begin{pmatrix} 0 & \gamma_{01} & \cdots & \gamma_{0k} \\ \gamma_{10} & \gamma_{11} & \cdots & \gamma_{1k} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{h0} & \gamma_{h1} & \cdots & \gamma_{hk} \end{pmatrix}$$

having the last h rows and the last k columns with sums $\alpha_1, \dots, \alpha_h$ and β_1, \dots, β_k respectively.

Remark 3.7. The above Product Formula can be derived from the one found by N.Roby in the context of divided powers (see [27]). It has also been derived by D.Ziplies in his paper on the divided powers algebra $\widehat{\Gamma}$ (see[38]).

Corollary 3.4. *Let $k \in \mathbb{N}$, $a_1, \dots, a_k \in F^+$, $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{N}^k$ with $|\alpha| \leq n$. Then $1^{(n-|\alpha|)} \times_{i=1}^k a_i^{(\alpha_i)}$ belongs to the subalgebra of $\Gamma_k^n(F)$ generated by the $1^{(n-i)} \times \mu^{(i)}$, where $i = 1, \dots, n$ and μ is a monomial in the a_1, \dots, a_k .*

Proof. We prove the claim by induction on $|\alpha|$ assuming that $\alpha_i > 0$ for all i (note that $1 \leq k \leq \sum_j \alpha_j$). Since n is fixed we suppress the superscript n for all the proof.

If $\sum_j \alpha_j = 1$ then $k = 1$ and $1^{(n-|\alpha|)} \times_{i=1}^k a_i^{(\alpha_i)} = 1^{(n-1)} \times a_1^{(1)}$. Suppose the claim true for all $1^{(n-|\beta|)} \times_{i=1}^h b_i^{(\beta_i)}$ with $b_1, \dots, b_h \in F^+$ and $|\beta| < |\alpha|$.

Let $k, a_1, \dots, a_k, \alpha$ be as in the statement, then we have by the Product Formula

$$1^{(n-1)} \times a_1^{(\alpha_1)} \star 1^{(n-(\alpha_2+\dots+\alpha_k))} \times_{i=2}^k a_i^{(\alpha_i)} = 1^{(n-|\alpha|)} \times_{i=1}^k a_i^{(\alpha_i)} + \sum_{\gamma} 1^{(n-|\gamma|)} a_1^{(\gamma_{10})} \times_{j=2}^k a_j^{(\gamma_{0j})} \times_{l=2}^k (a_1 a_l)^{(\gamma_{1l})} \quad (16)$$

where

$$\gamma = (\gamma_{10}, \gamma_{02}, \dots, \gamma_{0k}, \gamma_{12}, \dots, \gamma_{1k})$$

with $\gamma_{10} + \gamma_{12} + \dots + \gamma_{1k} = \alpha_1$ with $\sum_{l=2}^k \gamma_{1l} > 0$, and $\gamma_{0j} + \gamma_{1j} = \alpha_j$ for $j = 2, \dots, k$. Thus

$$\gamma_{10} + \gamma_{02} + \dots + \gamma_{0k} + \gamma_{12} + \dots + \gamma_{1k} = \sum_j \alpha_j - \sum_{l=2}^k \gamma_{1j} < \sum_j \alpha_j.$$

Hence

$$1^{(n-|\alpha|)} \times_{i=1}^k a_i^{(\alpha_i)} = 1^{(n-1)} \times a_1^{(\alpha_1)} \star 1^{(n-(\alpha_2+\dots+\alpha_k))} \times_{i=2}^k a_i^{(\alpha_i)} - \sum_{\gamma} 1^{(n-|\gamma|)} a_1^{(\gamma_{10})} \times_{j=2}^k a_j^{(\gamma_{0j})} \times_{l=2}^k (a_1 a_l)^{(\gamma_{1l})} \quad (17)$$

where $|\gamma| = \sum_{r,s} \gamma_{rs} < |\alpha|$. So the claim follows by the induction hypothesis. \square

Corollary 3.5. *The algebra $\Gamma_k^n(F)$ is generated by the $1^{(n-i)} \times \mu^{(i)}$ where $1 \leq i \leq n$ and $\mu \in \mathfrak{M}^+$.*

Proof. It follows from Corollary 3.4 applied to the elements of the basis \mathfrak{M}_n . \square

Remark 3.8. The above corollaries can also be proved using Corollary (4.1) and (4.5) in [38].

Lemma 3.6. *For all $f \in F^+$, and $k, h \in \mathbb{N}$, $1^{(n-h)} \times (f^k)^{(h)}$ belongs to the subalgebra of $\Gamma_k^n(F)$ generated by the $1^{(n-j)} \times f^{(j)}$.*

Proof. Let $f \in F^+$. The evaluation $k[x] \rightarrow F$ induces an homomorphism $\rho_f : \Gamma_k^n(k[x]) \rightarrow \Gamma_k^n(F)$. Now it is clear that

$$\Gamma_k^n(k[x]) \cong \text{TS}_k^n(k[x]) \cong k[x_1, \dots, x_n]^{S_n}$$

so that

$$(1 + t \otimes x)^{(n)} \mapsto (1 + t \otimes x)^{\otimes n} \mapsto \prod_{i=1}^n (1 + t \otimes x_i) = 1 + \sum_{i=1}^n t^i \otimes e_i(x_1, \dots, x_n)$$

where the $e_i(x_1, \dots, x_n)$ are the elementary symmetric functions and t is an independent variable. We have another distinguished kind of functions in the ring of symmetric polynomials $k[x_1, \dots, x_n]^{S_n}$ beside the elementary symmetric ones: the *power sums*. For $r \geq 1$ the r -th power sum is

$$p_r = \sum_{i \geq 1} x_i^r \quad (18)$$

Let $g \in k[x_1, \dots, x_n]^{S_n}$, set $g \cdot p_r = g(x_1^r, x_2^r, \dots, x_n^r)$ for the plethysm of g and p_r (see Section I.8 of [20]). The function $g \cdot p_r$ is again symmetric. Since the e_i freely generate $k[x_1, \dots, x_n]^{S_n}$ we have that $g \cdot p_r$ can be expressed as a polynomial in the e_i and we denote it by

$$P_{h,k} = e_h \cdot p_k \quad (19)$$

It follows that

$$\begin{aligned} 1^{(n-h)} \times (f^k)^{(h)} &= \rho_f(e_h \cdot p_k) = \rho_f(P_{h,k}(e_1, e_2, \dots, e_n)) = \\ &= P_{h,k}(1^{(n-1)} \times f^{(1)}, 1^{(n-2)} \times f^{(2)}, \dots, f^{(n)}) \end{aligned} \quad (20)$$

and the result is proved. \square

Definition 3.9. A monomial $\mu \in \mathfrak{M}^+$ is called *primitive* if it is not the proper power of another one.

Example 3.2. $x_1 x_2 x_1 x_2$ is not primitive while $x_1 x_2 x_1 x_1$ is primitive.

We have then the following refinement of Corollary 3.5.

Theorem 3.7 (Generators). *The algebra $\Gamma_k^n(F)$ is generated by $1^{(n-i)} \times \mu^{(i)}$ with $1 \leq i \leq n$ and μ primitive.*

Proof. It follows from Corollary 3.5 and Lemma 3.6. \square

3.6 The abelianization

In this paragraph we introduce the abelianization functor and we prove some of its properties. Then we deduce the fact that the abelianization of divided powers commutes with base change.

Definition 3.10. Given a k -algebra A we denote by $[A]$ the two-sided ideal of A generated by the commutators $[a, b] = ab - ba$ with $a, b \in A$. We write

$$A^{ab} = A/[A]$$

and call it the abelianization of A .

We denote by \mathbf{ab}_A the surjective homomorphism $\mathbf{ab}_A : A \rightarrow A^{ab}$.

For $A, B \in \mathcal{N}_k$ and $f \in \mathcal{N}_k(A, B)$ we denote by $f^{ab} \in \mathcal{C}_k(A^{ab}, B^{ab})$ the abelianization of f .

We collect some facts regarding this construction.

Proposition 3.8. 1. For all $B \in \mathcal{C}_k$ there is an isomorphism $\mathcal{C}_k(A^{ab}, B) \rightarrow \mathcal{N}_k(A, B)$ by means of $\rho \mapsto \rho \cdot \mathbf{ab}_A$. Equivalently for all $\varphi \in \mathcal{N}_k(A, B)$ there is a unique $\bar{\varphi} : A^{ab} \rightarrow B$ such that the following diagram commutes

$$\begin{array}{ccc} A & \xrightarrow{\mathbf{ab}_A} & A^{ab} \\ & \searrow \varphi & \downarrow \bar{\varphi} \\ & & B \end{array}$$

2. The assignment $A \rightarrow A^{ab}$ induces a covariant functor $\mathcal{N}_k \rightarrow \mathcal{C}_k$ that preserves surjections.

Proof. The first point is obvious. For $A, B \in \mathcal{N}_k$ and $f \in \mathcal{N}_k(A, B)$ we have that $f([A]) \subset [B]$. This proves the functoriality. Let $A, B \in \mathcal{N}_k$ and suppose $f \in \mathcal{N}_k(A, B)$ to be surjective. We have that $[B] = f([A])$. \square

Definition 3.11. We call the just introduced functor the *abelianization functor*.

Proposition 3.9. For all $A \in \mathcal{N}_k$ and all $R \in \mathcal{C}_k$ it holds that

$$(R \otimes_k A)^{ab} \cong R \otimes_k A^{ab}$$

Proof. The homomorphism $id_R \otimes \mathbf{ab}_A : R \otimes_k A \rightarrow R \otimes_k A^{ab}$ of R -algebras induces a unique one $\alpha : (R \otimes_k A)^{ab} \rightarrow R \otimes_k A^{ab}$ making the following diagram commutative

$$\begin{array}{ccc} R \otimes_k A & \xrightarrow{id_R \otimes \mathbf{ab}_A} & R \otimes_k A^{ab} \\ & \searrow \mathbf{ab}_{R \otimes A} & \uparrow \alpha \\ & & (R \otimes_k A)^{ab} \end{array}$$

On the other hand the homomorphism of k -algebras given by the composition

$$A \rightarrow R \otimes_k A \xrightarrow{\mathbf{ab}_{R \otimes A}} (R \otimes_k A)^{ab}$$

induces a unique one $\beta : A^{ab} \rightarrow (R \otimes_k A)^{ab}$. This β extends to a homomorphism of R -algebras $\tilde{\beta} : R \otimes_k A^{ab} \rightarrow (R \otimes_k A)^{ab}$ that is the inverse of α as can be easily checked. \square

Corollary 3.10. *With the notation of Proposition 3.9 we have*

$$\Gamma_R^n(R \otimes_k A)^{ab} \cong R \otimes_k \Gamma_k^n(A)^{ab}$$

Proof. It follows from (12) and Proposition 3.9. \square

Remark 3.9. Corollary 3.10 remain true by replacing Γ^n with TS^n when A is flat. In particular when R is a field.

Definition 3.12. We denote by \mathcal{S}_A^n the affine R -scheme $\mathrm{Spec} \Gamma_R^n(A)^{ab}$.

Proposition 3.11. *Suppose $A = R \otimes F/J \in \mathcal{N}_r$ then the induced morphism $\mathcal{S}_A^n \rightarrow \mathcal{S}_{R \otimes F}^n \cong \mathcal{S}_F^n \times_k \mathrm{Spec} R$ is a closed immersion.*

Proof. It follows from Proposition 3.8 and 3.3.3. \square

Remark 3.10. In view of Definition 3.12 all the results of this section can be rephrased in terms of R -schemes. In particular Corollary 3.10 means that \mathcal{S}_A^n base-changes well.

Remark 3.11. When A is flat (see paragraph 3.4.1) and commutative one sees that

$$\mathcal{S}_A^n \cong X^n/S_n = \mathrm{Spec} \mathrm{TS}_R^n(A)$$

the n -fold symmetric product of $X = \mathrm{Spec} A$.

In particular it follows that $\mathcal{S}_F^n \cong \mathrm{Spec} \mathrm{TS}_k^n(F)^{ab}$.

3.7 Generators II

We extend here the results of Section 3.5 to the abelianization $\Gamma(A)^{ab}$.

Definition 3.13. Consider \mathfrak{M}^+ / \sim the set of the equivalence classes of monomials $\mu \in \mathfrak{M}^+$ where $\mu \sim \mu'$ if and only if there is a cyclic permutation σ such that $\sigma(\mu) = \mu'$. We set Ψ to denote the set of equivalence classes in \mathfrak{M}^+ / \sim made of primitive monomials

Theorem 3.12. *The ring $\Gamma_k^n(F)^{ab}$ is generated as a ring by $\mathbf{ab}(1^{(n-i)} \times \mu^{(i)})$ where $1 \leq i \leq n$ and μ varying in a complete set of representatives of Ψ .*

Proof. Using (15) it easy to see that

$$\mathbf{ab}(1^{(n-i)} \times (rs)^{(i)}) = \mathbf{ab}(1^{(n-i)} \times (sr)^{(i)}) \quad (21)$$

for all $1 \leq i \leq n$ and $r, s \in F^+$. The result then follows from Theorem 3.7 and the surjectivity of \mathbf{ab} . \square

Corollary 3.13. *Let A be a k -algebra generated by $\{a_i\}_{i \in I}$ then*

1. $\Gamma_k^n(A)$ is generated by $1^{(n-j)}\mu^{(j)}$ where $1 \leq j \leq n$ and μ varies in the set of primitive monomials in $\{a_i\}_{i \in I}$
2. $(\Gamma_k^n(A))^{ab}$ is generated by $\mathbf{ab}(1^{(n-j)}\nu^{(j)})$ where ν varies in a complete set of representative of equivalence classes (under cyclic permutations) of primitive monomials in the $\{a_i\}_{i \in I}$.

Proof. Given the k -algebra A and a set $\{a_i\}_{i \in I}$ of its generators we have a surjective homomorphism $F_I = k\{x_i\}_{i \in I} \rightarrow A$ hence another surjective one $\Gamma_k^n(F_I) \rightarrow \Gamma_k^n(A)$ and the result follows from Theorem 3.7. The result on the abelianization follows from Theorem 3.12 since the abelianization functor preserves surjections. \square

Remark 3.12. Corollary 3.13(1) is a refinement of Cor.(4.5) in [38].

Corollary 3.14. *The following hold*

1. *The ring $\text{TS}_k^n(k[x_1, \dots, x_m]) \cong k[x_{11}, x_{12}, \dots, x_{m1}, \dots, x_{mn}]^{S_n}$ of the multisymmetric functions also known as the ring of the vector invariants of the symmetric group is generated by the $1^{(n-i)} \times \nu^{(i)}$ with $\nu = x_1^{\alpha_1} \cdots x_m^{\alpha_m}$ such that $\alpha_1, \dots, \alpha_m$ are coprime.*

2. Let A be a commutative k -algebra generated by $\{a_i\}_{i \in I}$ then $\Gamma_k^n(A)$ is generated by $1^{(n-j)}v^{(j)}$ where $v = \prod_i a_i^{\alpha_i}$ is such that $\sum_i \alpha_i$ is finite and the α_i are coprime.

Proof. It follows from Theorem 3.12 since Γ^n preserves surjections. Write $P = K[x_1, \dots, x_m]$, then P is a free k -module hence the surjection $F \rightarrow P$ induces another surjection $TS_k^n(F)^{ab} \rightarrow TS_k^n(P)$. Use Theorem 3.12. \square

Remark 3.13. This gives another proof of Th.1 in [32]

Remark 3.14. A *minimal* generating set for $\Gamma_k^n(k[x_1, \dots, x_m])$ has been found by David Rydh [29] improving Corollary 3.14.

4 The norm map

In this section we introduce a morphism which connect $\mathcal{R}ep_A^n // \text{GL}_n$ and \mathcal{S}_A^n .

Let $\rho \in \mathcal{N}_R(A, B)$ be a representation of A over $B \in \mathcal{C}_R$. The composition $\det \cdot \rho$ is a multiplicative polynomial law homogeneous of degree n belonging to $MP_R^n(A, B)$. We denote by \det_ρ the unique homomorphism in $\mathcal{C}_R(\Gamma_R^n(A)^{ab}, B)$ such that $\det \cdot \rho = \det_\rho \cdot \gamma^n$, see (10). The correspondence $\rho \mapsto \det_\rho$ is clearly functorial in B giving then a morphism $\Gamma_R^n(A)^{ab} \rightarrow V_n(A)$ by universality and henceforth a morphism of R -schemes

$$\underline{\det} : \mathcal{R}ep_A^n \rightarrow \mathcal{S}_A^n \quad (22)$$

Let $\bar{\rho} : V_n(A) \rightarrow B$ be the unique B -point of $\mathcal{R}ep_A^n$ such that $\rho = M_n(\bar{\rho}) \cdot \pi_A$, see Definition 2.2. It is easy to check that

$$\bar{\rho} \cdot \det \cdot \pi_A = \det \cdot M_n(\bar{\rho}) \cdot \pi_A = \det \cdot \rho$$

so that the following commutes

$$\begin{array}{ccccc} A & \xrightarrow{\pi_A} & M_n(V_n(A)) & \xrightarrow{\det} & V_n(A) \\ \parallel & & \downarrow M_n(\bar{\rho}) & & \downarrow \bar{\rho} \\ A & \xrightarrow{\rho} & M_n(B) & \xrightarrow{\det} & B \end{array} \quad (23)$$

It follows that $\underline{\det}$ is the affine morphism corresponding to the composition of the universal map π_A introduced in (2.2) with the determinant i.e. to the top horizontal arrows of diagram (23).

The determinant is invariant under basis changes and $\mathcal{R}ep_A^n // \mathrm{GL}_n$ is a categorical quotient. Hence there exists a unique morphism $\underline{\underline{\det}} : \mathcal{R}ep_A^n // \mathrm{GL}_n \rightarrow \mathcal{S}_A^n$ such that the following commutes

$$\begin{array}{ccc} \mathcal{R}ep_A^n & & \\ \downarrow & \searrow^{\underline{\underline{\det}}} & \\ \mathcal{R}ep_A^n // \mathrm{GL}_n & \xrightarrow{\underline{\underline{\det}}} & \mathcal{S}_A^n \end{array} \quad (24)$$

We have the following result.

Theorem 4.1. *The morphism $\underline{\underline{\det}} : \mathcal{R}ep_A^n // \mathrm{GL}_n \rightarrow \mathcal{S}_A^n$ has the following properties.*

1. *If $A = k\{x_1, \dots, x_m\}$ then $\underline{\underline{\det}}$ is an isomorphism when $k = R$ is an infinite field or $k = R = \mathbb{Z}$.*
2. *Suppose $k = R$ is an infinite field. If $A \in \mathcal{C}_k$ then $\underline{\underline{\det}}$ induces an isomorphism between the associated reduced schemes.*
3. *Suppose $k = R$ is a characteristic zero field then*
when $A \in \mathcal{N}_k$ then $\underline{\underline{\det}}$ is a closed embedding,
when $A \in \mathcal{C}_k$ then $\underline{\underline{\det}}$ is an isomorphism.
4. *When A is commutative and flat as R -module then $\Gamma_R(A) \cong \mathrm{TS}_R^n(A) \rightarrow V_n(A)$ is an injective homomorphism.*

Proof. First of all observe that $\underline{\underline{\det}}$ gives $1^{(n-i)} \times a^{(i)} \mapsto e_i(\pi_A(a))$ for all $i = 1, \dots, n$ and $a \in A$. Therefore by Corollary 3.13 we have that $\underline{\underline{\det}}$ is a surjection $\Gamma_k(A) \rightarrow C_n(A) \subset V_n(A)^{\mathrm{GL}_n(R)}$ by Remark 2.3.

1. When A is the free k -algebra $\Gamma_k(A)^{ab} \cong C_n(A) = V_n(A)^{\mathrm{GL}_n(k)}$ by Theorem 1.1 in [35]. This is obtained by showing that $\Gamma_k(A)^{ab}$ and $C_n(A) = V_n(A)^{\mathrm{GL}_n(k)}$ shares a common presentation by generators and relations.

2. and 4. When A is commutative and flat as k -module the above mapping has an inverse $C_n(A) \rightarrow \mathrm{TS}_k^n(A) \cong \Gamma_k^n(A)$ given by the restriction to $C_n(A)$ of the unique homomorphism $V_n(A) \rightarrow A^{\otimes n}$ that corresponds to the linear representation $A \rightarrow M_n(A^{\otimes n})$ defined by mapping $a \in A$ to the diagonal matrix whose diagonal is $(a \otimes 1^{\otimes n-1}, 1 \otimes a \otimes 1^{\otimes n-2}, \dots, 1^{\otimes n-1} \otimes a)$. This gives $\Gamma_k^n(A) \cong C_n(A)$. In this way it is obvious that $\underline{\underline{\det}}$ corresponds

to the restriction to diagonal matrices. It is also clear that this gives a surjection $V_n(A)^{\mathrm{GL}_n(k)} \rightarrow \Gamma_k^n(A)$. When $k = R$ is an infinite field we have an isomorphism of the associated reduce schemes. Indeed the surjection $V_n(A)^{\mathrm{GL}_n(k)} \rightarrow \Gamma_k^n(A)$ induced by the restriction to diagonal matrices is also injective because any orbit has in its closure a tuple of diagonal matrices, i.e. a semisimple representation.

3. The fact that $\underline{\det}$ is a closed embedding in characteristic zero follows considering the following diagram

$$\begin{array}{ccc} \mathrm{Rep}_F^n // \mathrm{GL}_n & \xrightarrow[\cong]{\underline{\det}} & \mathcal{S}_F^n \\ \uparrow & & \uparrow \\ \mathcal{R}ep_A^n // \mathrm{GL}_n & \xrightarrow{\underline{\det}} & \mathcal{S}_A^n \end{array}$$

recalling Proposition 3.11 and the fact that GL_n in this case is linear reductive. This fact together with the isomorphism of reduced structures give the isomorphism for $A \in \mathcal{C}_k$ when k is a characteristic zero field. \square

5 The non commutative Hilbert scheme

For $A \in \mathcal{N}_R$ we recall the definition and the main properties of a representable functor of points whose representing scheme is the usual Hilbert scheme of points when A is commutative. For references see for example [2, 9, 22, 26, 30, 31, 37].

Definition 5.1. For any algebra $B \in \mathcal{C}_R$ we write

$$\mathrm{Hilb}_A^n(B) := \{ \text{left ideals } I \text{ in } A \otimes_R B \text{ such that } M = A \otimes_R B / I \text{ is projective of rank } n \text{ as a } B\text{-module} \}.$$

Proposition 5.1. *The correspondence $\mathcal{C}_R \rightarrow \mathrm{Sets}$ induced by $B \mapsto \mathrm{Hilb}_A^n(B)$ gives a covariant functor denoted by Hilb_A^n .*

Proof. Straightforward verification. \square

Proposition 5.2. [37, Proposition 2] *The contravariant functor $R\text{-Schemes} \rightarrow \mathrm{Sets}$ induced by Hilb_A^n is representable by an R -scheme denoted by Hilb_A^n .*

Proof. The functor Hilb_A^n is a closed subfunctor of the grassmanian functor. \square

Let now B be a commutative k -algebra. Consider triples (ρ, m, M) where M is a projective B -module of rank n , $\rho : A \rightarrow \text{End}_B(M)$ is a k -algebra homomorphism such that $\rho(R) \subset B$ and $\rho(A)(Bm) = M$.

Definition 5.2. The triples (ρ, m, M) and (ρ', m', M') are *equivalent* if there exists a B -module isomorphism $\alpha : M \rightarrow M'$ such that $\alpha(m) = m'$ and $\alpha\rho(a)\alpha^{-1} = \rho'(a)$, for all $a \in A$.

These equivalence classes represent B -points of Hilb_A^n as stated in the following

Lemma 5.3. *If B is a k -algebra, the B -points of Hilb_A^n are in one-one correspondence with equivalence classes of triples (ρ, m, M) .*

Proof. Let $I \in \text{Hilb}_A^n(B)$. Choose $M = A \otimes_R B/I$ and $\rho : A \rightarrow \text{End}_B(M)$ given by the composition of the left regular action of A on itself and the B -module homomorphism $A \otimes_R B \rightarrow M$. Finally let $m = 1_M$.

On the other hand, let (ρ, m, M) be as in the statement, we consider the map

$$A \otimes_R B \rightarrow M, \quad a \otimes b \longmapsto \rho(a)(bm). \quad (25)$$

This map is surjective and its kernel I is a B -point in Hilb_A^n . If we consider a triple (ρ', m', M') in the same equivalence class of (ρ, m, M) , we have that

$$\sum \rho'(a)(b\alpha(m)) = \sum \alpha\rho(a)(b\alpha^{-1}\alpha(m)) = \alpha \sum \rho(a)(bm)$$

so that $I' = I$ and the two triples define the same B -point in Hilb_A^n . (See also [30, Lemma 0.1], and [37, Lemma 3]). \square

Remark 5.1. If the triple (ρ, m, M) represents a B -point of Hilb_A^n , then the homomorphism ρ induces an R -algebra structure on B . Moreover a homomorphism ρ' where (ρ', m', M') is in the same equivalence class of (ρ, m, M) , induces the same R -algebra structure on B .

5.1 Examples

5.1.1 Free algebras

Any $A \in \mathcal{N}_R$ is a quotient of an opportune free R -algebra $R\{x_\alpha\}$ and in this case it is easy to see that the scheme Hilb_A^n is a closed subscheme of $\text{Hilb}_{R\{x_\alpha\}}^n$.

5.1.2 Van den Bergh's results

In [37] $Hilb_A^n$ was also defined by M. Van den Bergh in the framework of Brauer-Severi schemes. He proved that the scheme $Hilb_F^n$ is irreducible and smooth of dimension $n + (m - 1)n^2$ if k is an algebraically closed field (see [37, Theorem 6]).

5.1.3 Commutative case: Hilbert schemes of n -points.

Let now $R = k$ be an algebraically closed field and let A be commutative. Let $X = \text{Spec } A$, the k -points of $Hilb_A^n$ parameterize zero-dimensional subschemes $Y \subset X$ of length n . It is the simplest case of Hilbert scheme parameterizing closed subschemes of X with fixed Hilbert polynomial P , in this case P is the constant polynomial n . The scheme $Hilb_A^n$ is usually called the *Hilbert scheme of n -points on X* (see for example [3, 14]).

5.2 A principal bundle over the Hilbert scheme

For any $B \in \mathcal{C}_R$ identify B^n with $\mathbb{A}_R^n(B)$, the B -points of the n -dimensional affine scheme over R . We introduce another functor that is one of the cornerstones of our construction.

Definition 5.3. For each $B \in \mathcal{C}_R$, let $U_A^n(B)$ denote the set of B -points $(\bar{\rho}, v)$ of $\mathcal{R}ep_A^n \times_R \mathbb{A}_R^n$ such that $\rho(A)(Bv) = B^n$, i.e. such that v generates B^n as A -module via $\rho : A \rightarrow M_n(B)$.

Remark 5.2. It is easy to check that the assignment $B \mapsto U_A^n(B)$ is functorial in B . Therefore we get a subfunctor U_A^n of $\mathcal{R}ep_A^n \times_R \mathbb{A}_R^n$ that is clearly open and we denote by U_A^n the open subscheme of $\mathcal{R}ep_A^n \times_R \mathbb{A}_R^n$ which represents it.

We can define an action of GL_n on $\mathcal{R}ep_A^n \times_R \mathbb{A}_R^n$ similar to the one on $\mathcal{R}ep_A^n$, namely for any $B \in \mathcal{C}_R$ let

$$g(\alpha, v) = (\alpha^g, gv), \quad g \in GL_n(B), \quad \alpha \in \mathcal{R}ep_A^n(B), \quad v \in \mathbb{A}_R^n(B). \quad (26)$$

It is clear that U_A^n is stable under the above action. Therefore we have that U_A^n is an open GL_n -subscheme of $\mathcal{R}ep_A^n \times_R \mathbb{A}_R^n$.

Proposition 5.4. *For the action described above, $U_A^n \rightarrow U_A^n/GL_n$ is a locally-trivial principal GL_n -bundle which is a universal categorical quotient.*

Proof. In [22, Proposition 1] the proposition is proved for the case $A = F$ and $k = R = \mathbb{Z}$. It has been extended in [2, Theorem 7.16] for arbitrary R . The statement then follows by observing that $\mathcal{R}ep_A^n \times_R \mathbb{A}_R^n$ is a closed GL_n -subscheme of $\mathrm{Rep}_F^n \times_R \mathbb{A}_R^n$. \square

We have the following result.

Theorem 5.5. *The scheme U_A^n/GL_n represents Hilb_A^n and $U_A^n \rightarrow \mathrm{Hilb}_A^n$ is a universal categorical quotient and a GL_n -principal bundle.*

Proof. In [22, Proposition 1] the proposition is proved for the case $A = F$ and $k = R = \mathbb{Z}$. It has been extended in [2, Theorem 7.16] for arbitrary R . The statement then follows by observing that $\mathcal{R}ep_A^n \times_R \mathbb{A}_R^n$ is a closed GL_n -subscheme of $\mathrm{Rep}_F^n \times_R \mathbb{A}_R^n$. \square

Remark 5.3. Madhav Nori gave a direct proof of the representability of Hilb_A^n for $k = R = \mathbb{Z}$ and $A = F$ in [22]. In this case Lemma 5.3 says that B -points in Hilb_F^n are represented by equivalence classes of triples (ρ, v, k^n) where

$$\rho : F \rightarrow M_n(k)$$

is an n -dimensional representation of the algebra F over k and $v \in k^n$ is such that $\rho(F)v = k^n$. It is then easy to see that $U_A^n \rightarrow U_A^n/\mathrm{GL}_n$ is a principal GL_n -bundle and that the above equivalence classes are in bijection with the GL_n -orbits of U_A^n (see the proof of Lemma 5.3). The isomorphism $U_A^n/\mathrm{GL}_n \cong \mathrm{Hilb}_A^n$ is then obtained by proving that Hilb_A^n is $\mathrm{Proj} P$ where $P = \bigoplus_d P_d$ with

$$P_d = \{f : f(g(a_1, \dots, a_m, v)) = (\det g)^d f((a_1, \dots, a_m, v)), \text{ for all } g \in \mathrm{GL}_n(k)\}.$$

Example 5.1. Let $A = \mathbb{C}[x, y]$ then $\mathrm{Hilb}_A^n(\mathbb{C})$ is described by the above Theorem as

$$\{(X, Y, v) : X, Y \in M_n(\mathbb{C}), XY = YX, \mathbb{C}[X, Y]v = \mathbb{C}^n\}.$$

This description of the Hilbert scheme of n -points of \mathbb{C}^2 is one of the key ingredients of the celebrated Haiman's proof of the $n!$ Theorem [11]. It has also been widely used by H.Nakajima [21].

6 Morphisms

In this section we introduce morphisms which connect $\mathcal{R}ep_A^n // \mathrm{GL}_n$, $Hilb_A^n$ and \mathcal{S}_A^n .

6.1 The forgetful map

Recall from Section 5.2 that we have $U_A^n \rightarrow U_A^n / \mathrm{GL}_n \cong Hilb_A^n$. We have then a commutative diagram

$$\begin{array}{ccc}
 & \mathcal{R}ep_A^n \times_R \mathbb{A}_R^n & \\
 U_A^n \hookrightarrow & \nearrow & \searrow \\
 U_A^n & \xrightarrow{\quad} & \mathcal{R}ep_A^n \\
 \downarrow & & \downarrow \\
 Hilb_A^n & \xrightarrow{\quad p \quad} & \mathcal{R}ep_A^n // \mathrm{GL}_n \\
 & \searrow & \nearrow \\
 & \mathcal{R}ep_A^n \times_R \mathbb{A}_R^n // \mathrm{GL}_n &
 \end{array} \tag{27}$$

Theorem 6.1. *The morphism*

$$p : Hilb_A^n \rightarrow \mathcal{R}ep_A^n // \mathrm{GL}_n$$

in (27) is projective.

Proof. By Theorem 7.16 and Remark 7.17 in [2] this is true for $A = F$. The result follows since $Hilb_A^n$ is a closed subscheme of $Hilb_F^n$ and $\mathcal{R}ep_A^n // \mathrm{GL}_n$ is affine. \square

The fibers of the map p are very difficult to study. Some results is known for the case $A = F$ and k algebraically closed field [15].

6.2 Non commutative Hilbert-Chow

Definition 6.1. We denote by hc the morphism given by

$$\begin{array}{ccc}
 Hilb_A^n & \xrightarrow{p} & \mathcal{R}ep_A^n // \mathrm{GL}_n \xrightarrow{\det} \mathcal{S}_A^n \\
 & \searrow hc & \nearrow \\
 & &
 \end{array} \tag{28}$$

There is the following

Theorem 6.2. *The morphism hc is projective.*

Proof. We know p to be projective by Theorem 6.1. Since $\underline{\det}$ is affine and hence separated the result follows. \square

Theorem 6.3. *The image of the non commutative Hilbert-Chow morphism is isomorphic to the one of the forgetful map in the following cases*

1. *when A is the free algebra and $k = R = \mathbb{Z}$ or it is an infinite field.*
2. *when A is commutative and $k = R$ is a characteristic zero field.*

Proof. In the listed cases we have that $\underline{\det}$ is an isomorphism hence we have an isomorphism between the image of hc with the one of the forgetful map. \square

Example 6.1. Suppose $A = F$ the free algebra. In this case $\mathcal{R}ep_A^n // \text{GL}_n = M_n^m(k) // \text{GL}_n$ the coarse moduli space of m -tuples of $n \times n$ matrices modulo the action of GL_n by simultaneous conjugation. The scheme Hilb_A^n is in this case identified with the quotient by the above action of the open of $\mathcal{R}ep_A^n \times_R \mathbb{A}_R^n$ of the tuples (a_1, \dots, a_m, v) such that v is cyclic i.e. $\{f(a_1, \dots, a_m)v : f \in A\}$ linearly generates k^n . In this case the Hilbert-Chow morphism can be identified with the map induced by $(a_1, \dots, a_m, v) \mapsto (a_1, \dots, a_m)$.

This picture works for $A = k[x_1, \dots, x_m]$ adding the extra condition that $a_i a_j = a_j a_i$ for all i, j .

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