Approximation of stationary solutions of Gaussian driven Stochastic Differential Equations

Serge Cohen, Fabien Panloup

Existence of the stationary regime of a Non-Markovian Stochastic Differential Equation

Serge Cohen*, Fabien Panloup†

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Abstract

In this paper, we obtain some existence results of stationary solutions to a class of SDEs driven by continuous Gaussian processes with stationary increments. We propose a constructive approach based on the study of some sequences of empirical measures of Euler schemes of these SDEs. In our main result, we obtain the functional convergence of this sequence to a stationary solution to the SDE. We also obtain some specific properties of the stationary solution. In particular, we show that, in contrast to Markovian SDEs, the initial random value of a stationary solution and the driving Gaussian process are always dependent. This emphasizes the fact that the concept of invariant distribution is definitely different to the Markovian case.

Keywords: stochastic differential equation; Gaussian process; stationary process; Euler scheme.

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1 Introduction

The study of steady state of dynamical systems is very important for many experimental sciences like Physics, Chemistry, or Biology, since very often measure can only be obtained in that regime. In the Markovian setting the study of long time behavior and stationary solutions of dynamical systems is a classical domain of both Mathematic and Probability. Nevertheless in many situations the driving noise of the dynamical system has long range dependence properties and the solution is not Markovian. In ([5, 6]) a precise definition of a stationary solution to a stochastic differential equation is given when the driving noise is a fractional Brownian motion.

In this paper, we deal with an $\mathbb{R}^d$-valued process $(X_t)_{t\geq 0}$ solution to the following SDE of the following form:

$$dX_t = b(X_t)dt + \sigma dZ_t \tag{1}$$

where $(Z_t)_{t\geq 0}$ is a continuous centered Gaussian process with ergodic stationary increments. For this class of SDEs, our principal aim is to build some stationary solutions under some mean-reverting assumptions on b and weak assumptions on $(Z_t)_{t\geq 0}$ including ergodicity of the discrete increments that will be made precise in the next section. At this

*Laboratoire de Statistiques et Probabilités, Université de Toulouse E-mail: serge.cohen@math.univ-toulouse.fr

†Laboratoire de Statistiques et Probabilités, Université de Toulouse&INSA Toulouse, 135, Avenue de Rangueil, 31077 Toulouse Cedex 4, E-mail: fabien.panloup@.math.univ-toulouse.fr

stage, we already remark the main restriction : the matrix σ is constant. It allows us on one hand to avoid technicalities related to stochastic integration and on the other hand to generalize some results of [5], when the driving noise is not a fractional Brownian motion. Please note that, when $b(x) = -x$, the solution of (1) is an Orstein-Uhlenbeck type process, where the driving process may be more general that a fractional Brownian motion (see [2] for a study of fractional Ornstein-Uhlenbeck processes). We obtain bounds for a discrete version of this generalized Ornstein-Uhlenbeck process, which are an important tool in our proofs and which may have interest of their own (see Lemma 2).
In this work, our approach is quite different to that of [5]. Actually, we choose to first build stationary solutions of an ergodic discrete model associated with (1). Then, stationary solutions of the SDE are exhibited as limits of these stationary solutions. More precisely, in a first step, we study a sequence of functional empirical occupation measures of an Euler scheme $(\bar{X}_{n\gamma})$ with step $\gamma > 0$ associated with (1) and show under some mean-reverting assumptions on b, that, when $n \to +\infty$, this sequence has almost surely ($a.s.$ later on) some weak convergence properties to the distribution of a stationary Euler scheme with step γ of the SDE. This first result shows in particular existence of stationary solutions to the discrete model. Denoting these stationary solutions by $Y^{(\infty,\gamma)}$, we show in a second step, that $(Y^{(\infty,\gamma)})_\gamma$ is tight for the uniform convergence on compact sets and that its weak limits (when $\gamma \to 0$) are stationary solutions to (1).
For a Markovian SDE, this type of approach is used as a way of numerical approximation of the invariant distribution and more generally of the distribution of the Markov process when stationary (see [17], [9], [10], [12], [16], [15]). Here, even if the discrete model can be simulated, we essentially use it as a natural way of construction of stationary solutions of the continuous model and the computation problems are out of the scope of this paper.
In Section 2, we make the mathematical framework precise and we state our main results of convergence to the stationary regime of SDE (1). Then, the sequel of the paper is devoted to the proof of the main results. In Sections 3 and 4, we study the long time behavior of the sequence $(\bar{X}_{n\gamma})_{n\geq 1}$ (when γ is fixed) and the convergence properties (when $n \to +\infty$) of the sequence of functional empirical occupation measures of the continuous-time Euler scheme. We show that this sequence is $a.s.$ tight for the topology of uniform convergence on compact sets and that its weak limits are stationary solutions to the "discretized" SDE. Then, in Section 5, we focus on the behavior of these weak limits when $\gamma \to 0$. Section 6 is an Appendix where we obtain some control of the moment of the supremum of a Gaussian process and where we prove that the initial random value and the driving process of the stationary solution are dependent as soon as the Gaussian process has dependent increments.

2 Framework and main results

Before outlining the sequel of the paper, we list some notations. Throughout this paper, $\mathbb{R}_+ = [0, \infty)$. We denote by $\mathcal{C}(\mathbb{R}_+, \mathbb{R}^d)$ (resp. $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$) the space of continuous (resp. càdlàg functions) endowed with the uniform convergence on compact sets (resp. Skorokhod (see $e.g.$ [1])) topology, and by $\mathcal{C}^k(\mathbb{R}_+, \mathbb{R}^d)$, the set of kth differentiable functions. The Euclidean norm is denoted by $|\,.\,|$. For a measure μ and a μ-measurable function f, we set $\mu(f) = \int f d\mu$. Finally, we will denote by C every non explicit positive constant. In particular, it can change from line to line.
Let us first consider assumptions for the driving noise $(Z_t)_{t\geq 0} = (Z_t^1, \ldots, Z_t^\ell)_{t\geq 0}$: we assume that $(Z_t)_{t\geq 0}$ is a centered Gaussian process with independent coordinates satisfying $Z_0 = 0$ and, for every $i \in \{1, \ldots, \ell\}$, we denote by $c_i : \mathbb{R} \to \mathbb{R}_+$, the following function of

$(Z_t^i)_{t\geq 0}$: for every positive s, t,

$$\mathbb{E}[(Z_t^i - Z_s^i)^2] = c_i(t-s).$$

Note that $c_i(0) = 0$. For every integer $n \geq 0$, let us denote by $\Delta_n = Z_{n\gamma} - Z_{(n-1)\gamma}$ when $\gamma > 0$ is fixed. Setting $\phi_\gamma^i(n) := \mathbb{E}[\Delta_1^i \Delta_{n+1}^i]$ for $i = 1, \ldots, l$, we have:

$$\phi_\gamma^i(n) = \frac{1}{2}\left[c_i((n+1)\gamma) - 2c_i(n\gamma) + c_i((n-1)\gamma)\right]. \tag{2}$$

We denote by $(\bar{Z}_t)_{t\geq 0}$ the "discretized" Gaussian process defined by $\bar{Z}_{n\gamma} := Z_{n\gamma}$ for every $n \geq 0$ and,

$$\bar{Z}_t = \bar{Z}_{n\gamma} \quad \forall t \in [n\gamma, (n+1)\gamma).$$

We introduce assumption $(\mathbf{H_1})$ on the functions c_i, $i \in \{1, \ldots, \ell\}$. More precisely, we impose some conditions on the second derivative of c_i near 0 and $+\infty$ which correspond respectively to some conditions on the local behavior and on the memory of the process.

$(\mathbf{H_1})$ For every $i \in \{1, \ldots, \ell\}$, c_i is continuous on $\mathbb{R}_+$ and $\mathcal{C}^2$ on $(0, +\infty)$. Moreover, there exist $a_i \in (0, 2)$ and $b_i > 0$ such that:

$$|c_i''(t)| \leq \begin{cases} Ct^{-a_i} & \forall t \in (0,1) \\ Ct^{-b_i} & \forall t \geq 1. \end{cases} \tag{3}$$

Let us recall that for a fractional Brownian motion with Hurst index H, these assumptions are satisfied with $a_i = b_i = 2 - 2H$. One can also check that (3) implies that in a neighborhood of 0,

$$c_i(t) \leq C \begin{cases} t & \text{if } a_i \in (0,1), \\ t \ln t & \text{if } a_i = 1, \\ t^{2-a_i} & \text{if } a_i \in (1,2). \end{cases} \tag{4}$$

In particular, the sample paths of $(Z_t)_{t\geq 0}$ are almost surely continuous. Futhermore, we derive from assumption $(\mathbf{H_1})$ that for every $i \in \{1, \ldots, \ell\}$, $\mathbb{E}[\Delta_1^i \Delta_n^i] \to 0$ as $n \to +\infty$. Then, it follows from [3] that $(\Delta_n^i)_{n\geq 1}$ is an ergodic sequence for every $i \in \{1, \ldots, \ell\}$.

Remark 1. Since $(\Delta_n^i)_{n\geq 1}$ is Gaussian, $(\Delta_n^i)_{n\geq 1}$ is in fact strong mixing (see [14]) for every $i \in \{1, \ldots, \ell\}$. It follows from the independence between $(\Delta_n^1)_{n\geq 1}$, …, $(\Delta_n^\ell)_{n\geq 1}$ that the sequence $(\Delta_n)_{n\geq 1}$ is also ergodic.

Let us now introduce some stability assumptions $(\mathbf{H_2})$ and $(\mathbf{H_3})$ concerning the stochastic differential equation

$$dX_t = b(X_t)dt + \sigma dZ_t, \tag{5}$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$, and σ is a matrix with d rows and ℓ columns.
$(\mathbf{H_2})$:

(i) There exists $C > 0$ such that $|b(x)| \leq C(1 + |x|) \quad \forall x \in \mathbb{R}^d$.

(ii) There exist $\beta \in \mathbb{R}$ and $\alpha > 0$ such that

$$\langle x, b(x)\rangle \leq \beta - \alpha|x|^2.$$

$(\mathbf{H_3})$: b is a Lipschitz continuous function and there exist $\alpha > 0$ and $\beta \geq 0$, such that $\forall x, y \in \mathbb{R}^d$,

$$\langle b(x) - b(y), x - y\rangle \leq \beta - \alpha|x - y|^2.$$

Remark 2. The reader can check that $(\mathbf{H_3})$ implies $(\mathbf{H_2})$, but some results are true under the weaker assumption $(\mathbf{H_2})$, so we consider both. When $(\mathbf{H_3})$ holds for $\beta = 0$, we will denote it by $(\mathbf{H_{3,0}})$.

When b is a Lipschitz continuous function, it is obvious using Picard iteration arguments that for any initial random variable ξ a.s. finite there exists a unique solution $(X_t)_{t\geq 0}$ to (5) such that $X_0 = \xi$ which is adapted to the filtration $\sigma(\xi, Z_s, 0 \leq s \leq t)$. Then,

$$X_t = \xi + \int_0^t b(X_s)ds + \sigma Z_t, \quad \forall t > 0. \tag{6}$$

Please note that the integral in (6) is always defined since the sample paths of $(X_t)_{t\geq 0}$ and $(Z_t)_{t\geq 0}$ are continuous.

Let us now define a stationary solution to (5).

Definition 1. Let $(Z_t)_{t\geq 0}$ be a Gaussian process with continuous paths and $b : \mathbb{R}^d \to \mathbb{R}^d$ be a continuous function. We say that $(X_t)_{t\geq 0}$ is a stationary solution to (5) if, $\mathbb{P} - a.s.$,

$$X_t = X_0 + \int_0^t b(X_s)ds + \sigma Z_t, \quad \forall t \geq 0, \tag{7}$$

and if for every $n \in \mathbb{N}$, for every $0 \leq t_1 < t_2 < \ldots < t_n$,

$$(X_{t+t_1}, \ldots, X_{t+t_n}) \overset{\mathcal{L}}{=} (X_{t_1}, \ldots, X_{t_n}) \quad \forall t \geq 0,$$

where $\overset{\mathcal{L}}{=}$ denotes the equality in distribution.

When $(Z_t)_{t\geq 0}$ is a Markovian process, for instance a Brownian motion, it is classical to have X_0 independent of Z, but in general we cannot have such independence as stated later in Proposition 2.

Definition 2. Let ν denote a probability on $\mathbb{R}^d$. We say that ν is an invariant distribution for (1) if there exists a stationary solution $(X_t)_{t\geq 0}$ to (1) such that $\nu = \mathcal{L}(X_0)$.

Remark 3. The fact that X_0 and $(Z_t)_{t\geq 0}$ may be dependent involves that uniqueness of the invariant distribution does not imply uniqueness of stationary solutions to (7).

Let γ be a positive number. We will now discretize equation (5) as follows:

$$\begin{cases} Y_{(n+1)\gamma} - Y_{n\gamma} = \gamma b(Y_{n\gamma}) + \sigma\Delta_{n+1} \quad \forall n \geq 0. \\ Y_t = Y_{n\gamma} \quad \forall t \in [n\gamma, (n+1)\gamma). \end{cases} \tag{$\mathbf{E}_\gamma$}$$

We will say that $(Y_t)_{t\geq 0}$ is a *discretely* stationary solution to $(\mathbf{E}_\gamma)$ is solution of $(\mathbf{E}_\gamma)$ satisfying:

$$(Y_{t_1+k\gamma}, \ldots, Y_{t_n+k\gamma}) \overset{\mathcal{L}}{=} (Y_{t_1}, \ldots, Y_{t_n}) \quad \forall\, 0 < t_1 < \ldots < t_n, \forall n, k \in \mathbb{N}.$$

We denote $(\bar{X}_{n\gamma})$ the Euler scheme defined by: $\bar{X}_0 = x \in \mathbb{R}^d$ and for every $n \geq 0$

$$\bar{X}_{(n+1)\gamma} = \bar{X}_{n\gamma} + \gamma b(\bar{X}_{n\gamma}) + \sigma\Delta_{n+1}. \tag{8}$$

Then, we denote by $(\bar{X}_t)_{t\geq 0}$ the stepwise constant continuous-time Euler scheme defined by:

$$\bar{X}_t = \bar{X}_{n\gamma} \quad \forall t \in [n\gamma, (n+1)\gamma).$$

The process $(\bar{X}_t)_{t\geq 0}$ is a solution to $(\mathbf{E}_\gamma)$ such that $\bar{X}_0 = x$. For every $k \geq 0$, we define by $(\bar{X}_t^{(\gamma k)})_{t\geq 0}$ the (γk)-shifted process: $\bar{X}_t^{(\gamma k)} = \bar{X}_{\gamma k+t}$.
Then, a sequence of random probability measures $(\mathcal{P}^{(n)}(\omega, d\alpha))_{n\geq 1}$ is defined on the Skorokhod space $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$ by

$$\mathcal{P}^{(n,\gamma)}(\omega, d\alpha) = \frac{1}{n}\sum_{k=1}^{n} \delta_{\bar{X}^{(\gamma(k-1))}(\omega)}, (d\alpha)$$

where δ denotes the Dirac measure. For $t \geq 0$, the sequence $(\mathcal{P}_t^{(n)}(\omega, dy))_{n\geq 1}$ of "marginal" empirical measures at time t on $\mathbb{R}^d$ is defined by

$$\mathcal{P}_t^{(n,\gamma)}(\omega, dy) = \frac{1}{n}\sum_{k=1}^{n} \delta_{\bar{X}_t^{(\gamma(k-1))}(\omega)}(dy) = \frac{1}{n}\sum_{k=1}^{n} \delta_{\bar{X}_{\gamma(k-1)+t}(\omega)}(dy).$$

A weak limit of a set $\mathcal{P} \subset \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$ is a limit of any subsequence of $\mathcal{P}$ in $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$. Let us now state the main results.

Theorem 1. *1. Assume* $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$*. Then, there exists* $\gamma_0 > 0$ *such that for every* $\gamma \in (0, \gamma_0)$*,* $(\mathcal{P}^{(n,\gamma)}(\omega, d\alpha))_{n\geq 1}$ *is a.s. tight on* $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$*. Furthermore, every weak limit is a discretely stationary solution to* $(\mathbf{E}_\gamma)$*.*
2. Assume $(\mathbf{H_1})$ *and* $(\mathbf{H_3})$ *and set*

$$\mathcal{U}^{\infty,\gamma}(\omega) := \{\text{weak limits of } (\mathcal{P}^{(n,\gamma)}(\omega, d\alpha))\}.$$

Then, there exists $\gamma_1 \in (0, \gamma_0)$ *such that* $(\mathcal{U}^{\infty,\gamma}(\omega))_{\gamma\leq\gamma_1}$ *is a.s. relatively compact for the uniform convergence topology on compact sets and any weak limit when* $\gamma \to 0$ *of* $(\mathcal{U}^{\infty,\gamma}(\omega))_{\gamma\leq\gamma_1}$ *is a stationary solution to* (5)*.*

The previous theorem states existence of stationary solutions of (5), but one can wonder about uniqueness of the solutions. We will only consider the special case when $(\mathbf{H_{3,0}})$ is enforced which is called in the Markovian setting asymptotic confluence (By asymptotic confluence, we mean that the distance in probability between two solutions starting from two different points x and y tends to 0 when $t \to +\infty$).

Proposition 1. *Assume* $(\mathbf{H_1})$ *and* $(\mathbf{H_{3,0}})$*. Then, there exists a unique stationary solution to* (5) *and to equation* $(\mathbf{E}_\gamma)$*, when* γ *is small enough.*

The next corollary, whose proof is obvious is nevertheless useful.

Corollary 1. *Assume* $(\mathbf{H_1})$ *and* $(\mathbf{H_{3,0}})$*. Denote by* $\mu \in \mathcal{P}(\mathcal{C}(\mathbb{R}_+, \mathbb{R}^d))$*, the distribution of the unique stationary solution to* (1)*. Then,*

$$d_{\mathbb{D}(\mathbb{R}_+,\mathbb{R}^d)}(\mathcal{P}^{(\infty,\gamma)}(\omega, d\alpha), \mu) \xrightarrow{\gamma\to 0} 0 \quad a.s. \tag{9}$$

where $d_{\mathbb{D}(\mathbb{R}_+,\mathbb{R}^d)}$ *denotes a distance on* $\mathcal{P}(\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d))$ *(endowed with the weak topology), the set of probabilities on* $\mathbb{R}^d$*. In particular,*

$$d_{\mathbb{R}^d}(\mathcal{P}^{(\infty,\gamma)}(\omega, d\alpha), \nu) \xrightarrow{\gamma\to 0} 0 \quad a.s. \tag{10}$$

where ν *is the unique invariant distribution of* (5) *and,* $d_{\mathbb{R}^d}$ *is a distance on* $\mathcal{P}(\mathbb{R}^d)$*.*

We will not study the rate of convergence relative to (10) in this paper.

Remark 4. We chose in this paper to work with the stepwise constant Euler scheme because this continuous-time scheme is in a sense the simplest to manage. The default is that the previous convergence result is stated for the Skorokhod topology. Replacing the stepwise constant Euler scheme by a continuous-time Euler scheme built by interpolations would lead to a convergence result for the topology of uniform convergence on compact sets.

Last, the following Proposition shows that the random initial value of a stationary solution can only be independent of a Markovian noise.

Proposition 2. *Assume that $\ell = d$ and that σ is invertible. Let X_0 and $(Z_t)_{t\geq 0}$ denote the random initial value and the driving process of a stationary solution to* (5)*. Then, if X_0 is independent of $(Z_t)_{t\geq 0}$, then $(Z_t)_{t\geq 0}$ has independent increments. As a consequence, $Z = QW$ where W is a standard d-dimensional Brownian Motion and Q is a deterministic matrix.*

Let us start a brief comparison of our results with those of [5] if $(Z_t)_{t\geq 0}$ is fractional Brownian motion. First, our assumption $(\mathbf{H_3})$ is a stability assumption a little weaker than $(\mathbf{A_1})$ in [5]. Likewise $(\mathbf{H_2})$ (i) and b Lipschitz continuous are similar to $(\mathbf{A_2})$ for $N = 1$ with Hairer's notation. In [5] Stochastic Dynamical System (SDS Definition 2.7) and a Feller semigroup $\mathcal{Q}_t$ ((2.4) in [5]) are defined on $\mathbb{R}^d \times \mathcal{C}(\mathbb{R}_+, \mathbb{R}^d)$. The first marginal of a stationary measure μ on $\mathbb{R}^d \times \mathcal{C}(\mathbb{R}_+, \mathbb{R}^d)$ defined in section 2.3 of [5] is what we call an invariant measure in Definition 2. Hence, we consider our results as a generalization of the existence results of [5] to SDEs driven by Gaussian continuous processes with stationary increments. Moreover $\mathcal{P}^{(n,\gamma)}$ for large n and small γ are natural approximations of the stationary measures of [5].

3 Tightness of $(\mathcal{P}^{(n,\gamma)}(\omega, d\alpha))_{n\geq 1}$

The main result of this section is Proposition 3 where we show the first part of Theorem 1, *i.e.* we obtain that $(\mathcal{P}^{(n,\gamma)}(\omega, d\alpha))_{n\geq 1}$ is *a.s.* tight for the Skorokhod topology on $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$ when γ is sufficiently small. A fundamental step for this proposition is to obtain the *a.s.* tightness for the sequence of initial distributions $(\mathcal{P}_0^{(n,\gamma)}(\omega, d\alpha))_{n\geq 1}$. This property is established in the following lemma.

Lemma 1. *Assume* $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$*. Then, there exists $\gamma_0 > 0$ such that for every $\gamma \leq \gamma_0$,*

$$\sup_{n\geq 1} \frac{1}{n} \sum_{k=1}^{n} |\bar{X}_{\gamma(k-1)}|^2 < +\infty \quad a.s. \tag{11}$$

Proof. We have :

$$\begin{aligned} |\bar{X}_{(n+1)\gamma}|^2 = |\bar{X}_{n\gamma}|^2 &+ 2\gamma\langle \bar{X}_{n\gamma}, b(\bar{X}_{n\gamma})\rangle + 2\langle \bar{X}_{n\gamma}, \sigma\Delta_{n+1}\rangle \\ &+ \left(\gamma^2 |b(\bar{X}_{n\gamma})|^2 + 2\gamma\langle b(\bar{X}_{n\gamma}), \sigma\Delta_{n+1}\rangle + |\sigma\Delta_{n+1}|^2\right). \end{aligned}$$

Let $\varepsilon > 0$. Using assumption $(\mathbf{H_2})(i)$ and the elementary inequality $|\langle u, v\rangle| \leq \frac{1}{2}(|\varepsilon u|^2 + |v/\epsilon|^2)$ (for every $u, v \in \mathbb{R}^d$), we have:

$$\begin{aligned} |\langle \bar{X}_{n\gamma}, \sigma\Delta_{n+1}\rangle| &\leq \frac{1}{2}\left(\varepsilon |\bar{X}_{n\gamma}|^2 + \frac{1}{\varepsilon}|\sigma\Delta_{n+1}|^2\right) \quad \text{and} \\ |\langle b(\bar{X}_{n\gamma}), \sigma\Delta_{n+1}\rangle| &\leq \frac{1}{2}\left(\varepsilon C(1 + |\bar{X}_{n\gamma}|^2) + \frac{1}{\varepsilon}|\sigma\Delta_{n+1}|^2\right). \end{aligned}$$

It follows from assumption $(\mathbf{H_2})(ii)$ that for every $\varepsilon > 0$,

$$|\bar{X}_{(n+1)\gamma}|^2 \leq |\bar{X}_{n\gamma}|^2 + 2\gamma(\beta - \alpha|\bar{X}_{n\gamma}|^2) + p(\gamma,\varepsilon)(1 + |\bar{X}_{n\gamma}|^2) + C(\varepsilon,\gamma)|\Delta_{n+1}|^2$$

where $C(\gamma,\varepsilon)$ is a positive constant depending on γ and ε and $p(\gamma,\varepsilon) \leq C(\varepsilon + \gamma\varepsilon + \gamma^2)$. Then, set $\varepsilon = \gamma^2$ (for instance). For γ sufficiently small, $p(\gamma,\varepsilon) \leq \alpha\gamma/2$. Hence, we obtain that there exist $\tilde{\beta} \in \mathbb{R}$ and $\tilde{\alpha} > 0$ such that $\forall n \geq 0$

$$\begin{aligned} |\bar{X}_{(n+1)\gamma}|^2 &\leq |\bar{X}_{n\gamma}|^2 + \gamma(\tilde{\beta} - \tilde{\alpha}|\bar{X}_{n\gamma}|^2) + C|\Delta_{n+1}|^2 \\ &\leq (1 - \gamma\tilde{\alpha})|\bar{X}_{n\gamma}|^2 + C(\gamma + |\Delta_{n+1}|^2). \end{aligned} \tag{12}$$

Finally, by induction, one obtains for every $n \geq 1$:

$$|\bar{X}_{n\gamma}|^2 \leq (1 - \gamma\tilde{\alpha})^n|x|^2 + C\sum_{k=1}^{n}(1 - \gamma\tilde{\alpha})^{n-k}(\gamma + |\Delta_k|^2).$$

Hence, in order to prove (11), it is enough to show that for γ sufficiently small,

$$\sup_{n\geq 1}\frac{1}{n}\sum_{k=1}^{n-1}\sum_{l=1}^{k}(1 - \tilde{\alpha}\gamma)^{k-l}|\Delta_l|^2 < +\infty \quad a.s. \tag{13}$$

But checking that

$$\sum_{k=1}^{n}\sum_{l=1}^{k}(1 - \tilde{\alpha}\gamma)^{k-l}|\Delta_l|^2 = \sum_{k=1}^{n}|\Delta_k|^2\sum_{u=0}^{n-k}(1 - \tilde{\alpha}\gamma)^u \leq C\sum_{k=1}^{n}|\Delta_k|^2 \leq C\sum_{i=1}^{\ell}\sum_{k=1}^{n}(\Delta_k^i)^2,$$

we obtain that it is in fact enough to show that

$$\sup_{n\geq 1}\frac{1}{n}\sum_{k=1}^{n}(\Delta_k^i)^2 < +\infty \quad a.s. \quad \forall i \in \{1,\ldots,\ell\}. \tag{14}$$

Now, by Remark 1, for every $i \in \{1,\ldots,\ell\}$, the sequence $(\Delta_k^i)_{k\geq 1}$ is ergodic. As a consequence,

$$\frac{1}{n}\sum_{k=1}^{n}(\Delta_k^i)^2 \xrightarrow{n\to+\infty} \mathbb{E}[(\Delta_1^i)^2],$$

and (14) is satisfied. This completes the proof. □

Proposition 3. *Assume assumption* $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$. *Then, there exists* $\gamma_0 > 0$ *such that for every* $\gamma \leq \gamma_0$, $(\mathcal{P}^{(n,\gamma)}(\omega, d\alpha))_{n\geq 1}$ *is a.s. tight on* $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$.

Proof. We have to prove the two following points (see *e.g.* [1], Theorem 15.2):

- 1. $\forall T > 0$, $(\mu_T^{(n)}(\omega, dy))$ defined by

$$\mu_T^{(n)}(\omega, dy) = \sum_{k=1}^{n}\delta_{\{\sup_{t\in[0,T]}|\bar{X}_t^{(k-1)}|\}}(dy),$$

  is an *a.s.* tight sequence.

- 2. For every $\eta > 0$,

$$\limsup_{\delta \to 0} \limsup_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} \delta_{\{\omega'_T(\bar{X}^{(k-1)}, \delta) \geq \eta\}} = 0 \quad a.s.$$

with

$$w'_T(x, \delta) = \inf_{\{t_i\}} \{ \max_{i \leq r} \sup_{s,t \in [t_i, t_{i+1})} |x_t - x_s| \}$$

where the infimum extends over finite sets $\{t_i\}$ satisfying:

$$0 = t_0 < t_1 < \ldots < t_r = T \quad \text{and} \quad \inf_{i \leq r} (t_i - t_{i-1}) \geq \delta.$$

In fact, since the process has only jumps at times $n\gamma$ with $n \in \mathbb{N}$, $\omega'_T(\bar{X}^{(k)}, \delta) = 0$ when $\delta < \gamma$. It follows that the second point is obvious. Then, let us prove the first point. By induction, one gets from (12) that, for every $k \geq n$,

$$|\bar{X}_{k\gamma}|^2 \leq |\bar{X}_{n\gamma}|^2 (1 - \gamma\tilde{\alpha})^{k-n} + C \sum_{l=n+1}^{k} (1 - \gamma\tilde{\alpha})^{k-l} (\gamma + |\Delta_l|^2).$$

This implies that

$$\sup_{t \in [0,T]} |\bar{X}_t^{(k-1)}|^2 = \sup_{k \in \{n, \ldots, n+[T/\gamma]\}} |\bar{X}_{k\gamma}|^2 \leq |\bar{X}_{n\gamma}|^2 + C(1 + \sum_{l=n+1}^{n+[T/\gamma]} |\Delta_l|^2).$$

Thus, if $V(x) = |x|^2$, one can deduce:

$$\begin{aligned} \mu^{(n)}(\omega, V) &\leq \frac{1}{n} \sum_{k=1}^{n} V(\bar{X}_{(k-1)\gamma}) + C\Big(1 + \frac{1}{n} \sum_{k=1}^{n} \sum_{l=k+1}^{k+[T/\gamma]} |\Delta_l|^2\Big) \\ &\leq \sup_{n \geq 1} \mathcal{P}_0^{(n,\gamma)}(\omega, V) + C\Big(1 + \frac{1}{n} \left[\frac{T}{\gamma}\right] \sup_{n \geq 1} \frac{1}{n} \sum_{k=1}^{n+[T/\gamma]} |\Delta_k|^2\Big) < +\infty \quad a.s. \end{aligned}$$

thanks to Lemma 1 and (14). Therefore, $\sup_{n \geq 1} \mu_T^{(n)}(\omega, V) < +\infty$ $a.s$ which implies that $(\mu_T^{(n)}(\omega, dy))$ is $a.s.$ tight on $\mathbb{R}^d$ (see *e.g.*. [4], Proposition 2.1.6). □

4 Identification of the weak limits of $(\mathcal{P}^{(n,\gamma)}(\omega, d\alpha))_{n \geq 1}$

In the following proposition, we show that every weak limit of $(\mathcal{P}^{(n,\gamma)}(\omega, d\alpha))_{n \geq 1}$ is $a.s$ a stationary Euler scheme with step γ of SDE (5).

Proposition 4. *Assume* $(\mathbf{H_1})$ *and let* $\mathcal{P}^{(\infty,\gamma)}(\omega, d\alpha)$ *denote a weak limit of* $(\mathcal{P}^{(n,\gamma)}(\omega, d\alpha))_{n \geq 1}$. *Then, a.s.,* $\mathcal{P}^{(\infty,\gamma)}(\omega, d\alpha)$ *is the distribution of a càdlàg process denoted by* $Y^{(\infty,\gamma)}$ *such that, a.s. in* ω,

(a) $(Y^{(\infty,\gamma)}_{l\gamma+t})_{t \geq 0} \overset{\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)}{=} (Y^{(\infty,\gamma)}_t)_{t \geq 0}$ *for every* $l \in \mathbb{N}$ *where* $\overset{\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)}{=}$ *denotes the equality in distribution on* $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$.

(b) $N^{(\infty,\gamma)}$ *defined by*

$$N_t^{(\infty,\gamma)} = Y_t^{(\infty,\gamma)} - Y_0^{(\infty,\gamma)} - \int_0^{\underline{t}_\gamma} b(Y_s^{(\infty,\gamma)})ds$$

is equal in law to $\sigma\bar{Z}^\gamma$ *with* $\underline{t}_\gamma = \gamma[t/\gamma]$.

Remark 5. It follows from the previous proposition that $(Y_t^{(\infty,\gamma)})_{t\geq 0}$ is a discretely stationary solution to $(\mathbf{E}_\gamma)$.

Proof. (a) Let $\mathcal{T}$ denote a countable dense subset of $\mathbb{R}_+$ and $\mathcal{S}_r^K$, a countable dense subset of the space of continuous functions $f : \mathbb{R}^r \to \mathbb{R}$ with compact support. It suffices to prove that $a.s.$, $\forall r \geq 0$, for every $f \in \mathcal{S}_r^K$, for every $t_1, \ldots, t_r \in \mathcal{T}$,$\forall l \in \mathbb{N}$,

$$\int f(\alpha_{t_1}, \ldots, \alpha_{t_r})\mathcal{P}^{(\infty,\gamma)}(\omega, d\alpha) = \int f(\alpha_{t_1+l\gamma}, \ldots, \alpha_{t_r+l\gamma})\mathcal{P}^{(\infty,\gamma)}(\omega, d\alpha).$$

Since $\mathcal{T}$ and $\mathcal{S}_r^K$ are countable, we only have to prove that $\forall r \geq 0$, for every $f \in \mathcal{S}_r^K$, for every $t_1, \ldots, t_r \in \mathcal{T}$, $\forall l \in \mathbb{N}$,

$$\int f(\alpha_{t_1}, \ldots, \alpha_{t_r})\mathcal{P}^{(\infty,\gamma)}(\omega, d\alpha) = \int f(\alpha_{t_1+l\gamma}, \ldots, \alpha_{t_r+l\gamma})\mathcal{P}^{(\infty,\gamma)}(\omega, d\alpha) \quad a.s. \tag{15}$$

Let now $f \in \mathcal{S}_r^K$, $l \in \mathbb{N}$ and $t_1, \ldots, t_r \in \mathcal{T}$. On the one hand,

$$\begin{aligned}
&\frac{1}{n}\sum_{k=1}^n \left(f(\bar{X}_{t_1}^{(k-1)}, \ldots, \bar{X}_{t_r}^{(k-1)}) - f(\bar{X}_{t_1+l\gamma}^{(k-1)}, \ldots, \bar{X}_{t_r+l\gamma}^{(k-1)}) \right) \\
&= \frac{1}{n}\sum_{k=1}^n f(\bar{X}_{(k-1)\gamma+t_1}, \ldots, \bar{X}_{(k-1)\gamma+t_r}) - \frac{1}{n}\sum_{k=1}^n f(\bar{X}_{(k-1+l)\gamma+t_1}, \ldots, \bar{X}_{(k-1+l)\gamma+t_r}) \\
&= \frac{1}{n}\left(\sum_{k=1}^{l-1} f(\bar{X}_{(k-1)\gamma+t_1}, \ldots, \bar{X}_{(k-1)\gamma+t_r}) - \sum_{k=n+1}^{n+l} f(\bar{X}_{(k-1)\gamma+t_1}, \ldots, \bar{X}_{(k-1)\gamma+t_r}) \right),
\end{aligned}$$

and this last term converges to 0 when $n \to +\infty$ $a.s.$ since f is bounded. On the other hand, since $\mathcal{P}^{(\infty,\gamma)}(\omega, d\alpha)$ denotes a weak limit of $(\mathcal{P}^{(n,\gamma)}(\omega, d\alpha))_{n\geq 1}$, there exists a subsequence $(n_k(\omega))_{k\geq 1}$ such that $(\mathcal{P}^{(n_k(\omega),\gamma)}(\omega, d\alpha))_{k\geq 1}$ converges weakly to $\mathcal{P}^{(\infty,\gamma)}(\omega, d\alpha)$ (for the Skorokhod topology). This convergence implies in particular the finite-dimensional convergence. Therefore,

$$\frac{1}{n}\sum_{k=1}^n f(\bar{X}_{t_1}^{(k-1)}, \ldots, \bar{X}_{t_r}^{(k-1)}) \xrightarrow{n\to+\infty} \int f(\alpha_{t_1}, \ldots, \alpha_{t_r})\mathcal{P}^{(\infty,\gamma)}(\omega, d\alpha) \quad a.s.$$

$$\textit{and,} \quad \frac{1}{n}\sum_{k=1}^n f(\bar{X}_{t_1+l\gamma}^{(k-1)}, \ldots, \bar{X}_{t_r+l\gamma}^{(k-1)}) \xrightarrow{n\to+\infty} \int f(\alpha_{t_1+l\gamma}, \ldots, \alpha_{t_r+l\gamma})\mathcal{P}^{(\infty,\gamma)}(\omega, d\alpha) \quad a.s.$$

Therefore, (15) follows.

(b) Let $\Phi_\gamma : \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d) \to \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$ be defined by:

$$(\Phi_\gamma(\alpha))_t = \alpha_t - \alpha_0 - \int_0^{\underline{t}_\gamma} b(\alpha_s)ds. \tag{16}$$

Then, $N^{(\infty,\gamma)} = \Phi_\gamma(Y^{(\infty,\gamma)})$. Let $F : \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d) \to \mathbb{R}$ be a bounded continuous functional:

$$\mathbb{E}[F(N^{(\infty,\gamma)})] = \int F(\Phi_\gamma(\alpha))\mathcal{P}^{(\infty,\gamma)}(\omega, d\alpha) = \lim_{k\to+\infty} \int F(\Phi_\gamma(\alpha))\mathcal{P}^{(n_k(\omega),\gamma)}(\omega, d\alpha). \quad (17)$$

For every $t \geq 0$,

$$\Phi_\gamma(\bar{X}^{(k)})_t = \sigma(\bar{Z}^\gamma_{\gamma_k+t} - \bar{Z}^\gamma_{k\gamma}) = \sum_{l=k+1}^{k+[t/\gamma]} \sigma\Delta_l,$$

with the convention $\sum_\emptyset = 0$. Thus, we derive from (17) that

$$\mathbb{E}[F(N^{(\infty,\gamma)})] = \lim_{k\to+\infty} \frac{1}{n_k} \sum_{m=1}^{n_k} F \circ G((\Delta_l)_{l\geq m})$$

where $G : (\mathbb{R}^d)^{\mathbb{N}} \to \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$ is defined by

$$G((u_n)_{n\geq 1})_t = \sum_{l=1}^{[\frac{t}{\gamma}]} \sigma u_l \quad \forall t \geq 0.$$

Now, by Remark 1, $(\Delta_n)_{n\geq 1}$ is an ergodic sequence. As a consequence, *a.s.*,

$$\frac{1}{n_k} \sum_{i=1}^{n_k} F \circ G((\Delta_l)_{l\geq i}) \xrightarrow{k\to+\infty} \mathbb{E}[F \circ G((\Delta_l)_{l\geq 1})] = \mathbb{E}[F(\sigma\bar{Z}^\gamma)].$$

The result follows. □

5 Convergence of $(\mathcal{P}^{(\infty,\gamma)}(\omega, d\alpha))$ when $\gamma \to 0$

The aim of this section is to show that, *a.s.*, $(\mathcal{P}^{(\infty,\gamma)}(\omega, d\alpha))_\gamma$ is *a.s.* tight for the weak topology induced by the topology of uniform convergence on $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$ and that its weak limits when $\gamma \to 0$ are stationary solutions to (5). The main difficulty for this second part of the proof of Theorem 2 is to show that $(\mathcal{P}^{(\infty,\gamma)}(\omega, d\alpha))_\gamma$ is *a.s.* tight on $\mathbb{R}^d$. For this step, we focus in Lemma 2 on the particular case $b(x) = -x$ (when $(X_t)_{t\geq 0}$ is an Ornstein-Uhlenbeck process) where some explicit computations lead to a control of $(\mathcal{P}^{(\infty,\gamma)}(\omega, d\alpha))_\gamma$. Then, in Lemma 3, we show that this control can be extended to SDE's whose drift term satisfies $(\mathbf{H_3})$. Finally, we establish the main result of this section in Proposition 5.

Let $\gamma > 0$. We denote by $(\Sigma_{n\gamma})$ the Euler scheme in the particular case $b(x) = -x$. We have $\Sigma_0 = x$ and:

$$\Sigma_{(n+1)\gamma} = (1-\gamma)\Sigma_{n\gamma} + \sigma\Delta_{n+1} \quad \forall n \geq 0.$$

Lemma 2. *Assume* $(\mathbf{H_1})$ *and let* $\gamma \in (0,1)$. *Then,* $(\mathbb{E}[|\Sigma_{n\gamma}|^2])_{n\geq 0}$ *is a convergent sequence. Denote by* $v(\gamma)$ *its limit. For every* $\gamma_0 \in (0,1)$,

$$\sup_{\gamma\in(0,\gamma_0]} v(\gamma) < +\infty.$$

Proof. First, by induction,

$$\Sigma_{n\gamma} = (1-\gamma)^n x + \sigma \sum_{k=0}^{n-1} (1-\gamma)^k \Delta_{n-k}.$$

Using that the coordinates are independent, it follows that

$$\mathbb{E}[|\Sigma_{n\gamma}|^2] = (1-\gamma)^{2n}|x|^2 + \sum_{i=1}^{\ell} (\sigma^*\sigma)_{i,i} \mathbb{E}\Big[\Big(\sum_{k=0}^{n-1} (1-\gamma)^k \Delta_{n-k}^i\Big)^2\Big].$$

For every $i \in \{1, \dots, \ell\}$,

$$\mathbb{E}\Big[\Big(\sum_{k=0}^{n-1} (1-\gamma)^k \Delta_{n-k}^i\Big)^2\Big] = \sum_{k=0}^{n-1}\sum_{l=0}^{n-1} (1-\gamma)^{k+l} \phi_\gamma^i(l-k),$$

where ϕ_γ^i is defined by (2). Setting $u = k+l$ and $v = l-k$, we deduce that

$$\mathbb{E}\Big[\Big(\sum_{k=0}^{n-1} (1-\gamma)^k \Delta_{n-k}^i\Big)^2\Big] = \sum_{u=0}^{2n-2} (1-\gamma)^u \sum_{v=(u-(2n-2))\vee(-u)}^{(2n-2-u)\wedge u} \phi_\gamma^i(v), \tag{18}$$

with $x \wedge y = \min(x,y)$ and $x \vee y = \max(x,y)$. Then, with the definition of ϕ, one can check that

$$\sum_{v=(u-(2n-2))\vee(-u)}^{(2n-2-u)\wedge u} \phi_\gamma^i(v) = \begin{cases} c_i(\gamma) & \text{if } u = 0 \text{ or } u = 2n-2, \\ f_i^\gamma((2n-2-u)\wedge u) & \text{otherwise,} \end{cases}$$

with $f_i^\gamma(x) = c_i(\gamma(x+1)) - c_i(\gamma x)$. It follows from (18) that,

$$\mathbb{E}\Big[\Big(\sum_{k=0}^{n-1} (1-\gamma)^k \Delta_{n-k}^i\Big)^2\Big] = c_i(\gamma) + \sum_{u=1}^{n-1} (1-\gamma)^u f_i^\gamma(u) + R_n(\gamma),$$

with

$$\begin{aligned} R_n(\gamma) &= \sum_{u=n}^{2n-1} (1-\gamma)^u f_i^\gamma(2n-2-u) + (1-\gamma)^{2n-2} c_i(\gamma), \\ &= \sum_{u=-1}^{n-2} (1-\gamma)^{2n-2-u} f_i^\gamma(u) + (1-\gamma)^{2n-2} c_i(\gamma). \end{aligned}$$

Since c_i is locally bounded and c_i'' is bounded on $[1, +\infty[$, c_i is a subquadratic function, *i.e.*

$$|c_i(u)| \le C(1+|u|^2) \quad \forall u \ge 0.$$

It follows that f_i^γ is also a subquadratic function. Then, using that for every $u \in \{-1, \dots, n-2\}$, $(1-\gamma)^{2n-2-u} \le (1-\gamma)^n$, we obtain that for every $\gamma \in (0,1)$, $R_n(\gamma) \longrightarrow 0$ as $n \to +\infty$.
Using again that f_i^γ is a subquadratic function, we deduce that for every $\gamma \in (0,1)$, for every $i \in \{1, \dots, d\}$,

$$\mathbb{E}\Big[\Big(\sum_{k=0}^{n-1} (1-\gamma)^k \Delta_{n-k}^i\Big)^2\Big] \xrightarrow{n\to+\infty} w_i(\gamma) := c_i(\gamma) + \sum_{u=1}^{+\infty} (1-\gamma)^u f_\gamma^i(u)$$

and that $w_i(\gamma)$ is finite. By a second order Taylor development, we have for every $u \geq 1$:

$$f_i^\gamma(u) = \gamma c_i'(\gamma u) + \gamma^2 r(\gamma, u) \quad \text{with} \quad r(\gamma, u) = c_i''(\gamma(u + \theta_u)), \quad \theta_u \in [0, 1].$$

Hence, using assumption $(\mathbf{H_1})$, it follows that

$$w_i(\gamma) = c_i(\gamma) + \sum_{u=1}^{+\infty} \gamma(1-\gamma)^u \left[c_i'(\gamma u) + \gamma r(\gamma, u)\right] \quad \text{with} \quad |r(\gamma, u)| \leq C g_{i,1}(\gamma u),$$
$$\text{and,} \quad g_{i,1}(t) = t^{-a_i} 1_{\{t \in (0,1)\}} + t^{-b_i} 1_{\{t \geq 1\}}.$$

Let us now control the behavior of $w_i(\gamma)$ when $\gamma \to 0$. First, for every $\gamma \in (0, 1)$, for every $u \geq 1$, $(1-\gamma)^u \leq \exp(-\gamma u)$. Then, since $t \mapsto \exp(-t)$, $t \mapsto g_{i,1}(t)$ are non-increasing on $\mathbb{R}_+^*$, one deduces that for every $u \geq 2$,

$$\gamma(1-\gamma)^u g_{i,1}(\gamma u) \leq \int_{\gamma(u-1)}^{\gamma u} \exp(-t) g_{i,1}(t) dt.$$

Then,

$$|\sum_{u=1}^{+\infty} \gamma^2 (1-\gamma)^u r(\gamma, u)| \leq c_i(\gamma)(1-\gamma) + C\gamma \int_\gamma^{+\infty} \exp(-t) g_{i,1}(t) dt.$$

Using that $a_i < 2$, we easily check that the right-hand side is bounded and tends to 0 when $\gamma \to 0$. We now focus on the first term of $w_i(\gamma)$. First, by assumption $(\mathbf{H_1})$, for every $t > 0$,

$$|c_i'(t)| \leq C(1 + g_{i,2}(t)) \quad \text{where} \quad g_{i,2}(t) = t^{1-a_i-\delta_1} 1_{\{t \in (0,1)\}} + t^{1-b_i+\delta_2} 1_{\{t \geq 1\}}, \tag{19}$$

with $\delta_1 \in (0, 1)$ (resp. $\delta_2 \in (0, 1)$) if $a_i = 1$ (resp. $b_i = 1$) and $\delta_1 = 0$ (resp. $\delta_2 = 0$) otherwise. Second, using that $(1-\gamma)^u \leq C(1-\gamma)^{-1} \exp(-t)$ for every $t \in [\gamma u, \gamma(u+1)]$, one deduces that

$$\gamma(1-\gamma)^u (\gamma u)^\rho \leq C \begin{cases} \int_{\gamma(u-1)}^{\gamma u} \exp(-t) t^\rho dt & \text{if } \rho < 0 \\ \frac{1}{1-\gamma} \int_{\gamma u}^{\gamma(u+1)} \exp(-t) t^\rho dt & \text{if } \rho \geq 0 \end{cases} \tag{20}$$

It follows from (19) and (20) that

$$\limsup_{\gamma \to 0} \left| \sum_{u=1}^{+\infty} \gamma(1-\gamma)^u c_i'(\gamma u) \right| \leq C \int_0^{+\infty} \exp(-t) \left(1 + g_{i,2}(t)\right) dt.$$

The right-hand member is finite. This completes the proof. □

Lemma 3. *Assume* $(\mathbf{H_1})$ *and* $(\mathbf{H_3})$ *and denote by* $\mathcal{P}^{(\infty,\gamma)}(\omega, d\alpha)$ *a weak limit of* $(\mathcal{P}^{(n,\gamma)}(\omega, d\alpha))$. *Then:*
(i) With the notations of Proposition 4, there exists $\gamma_0 > 0$ *such that,*

$$\sup_{0 < \gamma \leq \gamma_0} \mathbb{E}_\omega[|Y_0^{(\infty,\gamma)}|^2] < +\infty \quad a.s. \tag{21}$$

(ii) Assume $(\mathbf{H_1})$ *and* $(\mathbf{H_{3,0}})$. *Then, uniqueness holds for the distribution of stationary solutions to* (7). *Similarly, there exists* $\gamma_0 > 0$ *such that for every* $\gamma \leq \gamma_0$, *uniqueness holds for the distribution of discretely stationary solutions to* $(\mathbf{E}_\gamma)$.

Proof. (i)
Step 1: Let $(\bar{X}_{n\gamma})$ and $(\Sigma_{n\gamma})$ be defined by:

$$\begin{aligned} &\bar{X}_0 = x, \quad \bar{X}_{(n+1)\gamma} = \bar{X}_{n\gamma} + \gamma b(\bar{X}_{n\gamma}) + \sigma\Delta_{n+1} \quad \text{and,} \\ &\Sigma_0 = x, \quad \Sigma_{(n+1)\gamma} = \Sigma_{n\gamma} - \gamma\Sigma_{n\gamma} + \sigma\Delta_{n+1}. \end{aligned} \tag{22}$$

with $\Delta_n = Z_{n\gamma} - Z_{(n-1)\gamma}$. Then,

$$\begin{aligned} |\bar{X}_{(n+1)\gamma} - \Sigma_{(n+1)\gamma}|^2 &= |\bar{X}_{n\gamma} - \Sigma_{n\gamma}|^2 + 2\gamma\langle b(\bar{X}_{n\gamma}) + \Sigma_{n\gamma}, \bar{X}_{n\gamma} - \Sigma_{n\gamma}\rangle + \gamma^2|b(\bar{X}_{n\gamma}) + \Sigma_{n\gamma}|^2 \\ &\le |\bar{X}_{n\gamma} - \Sigma_{n\gamma}|^2 + 2\gamma\langle b(\bar{X}_{n\gamma}) - b(\Sigma_{n\gamma}), \bar{X}_{n\gamma} - \Sigma_{n\gamma}\rangle + 2\gamma^2|b(\bar{X}_{n\gamma} - b(\Sigma_{n\gamma})|^2 \\ &\quad + 2\gamma\langle b(\Sigma_{n\gamma}) + \Sigma_{n\gamma}, \bar{X}_{n\gamma} - \Sigma_{n\gamma}\rangle + 2\gamma^2|b(\Sigma_{n\gamma}) + \Sigma_{n\gamma}|^2. \end{aligned}$$

On the one hand, using that b is Lipschitz continuous and assumption $(\mathbf{H_3})$, one obtains:

$$\gamma\langle b(\bar{X}_{n\gamma}) - b(\Sigma_{n\gamma}), \bar{X}_{n\gamma} - \Sigma_{n\gamma}\rangle + 2\gamma^2|b(\bar{X}_{n\gamma}) - b(\Sigma_{n\gamma})|^2 \le \gamma\left(\beta + |\bar{X}_{n\gamma} - \Sigma_{n\gamma}|^2(-\alpha + C\gamma)\right). \tag{23}$$

On the other hand, using that b is a sublinear function and the elementary inequality $\langle u, v\rangle \le 1/2(\varepsilon^{-1}|u|^2 + \varepsilon|v|^2)$ (with $u = b(\Sigma_{n\gamma}) + \Sigma_{n\gamma}$, $v = \bar{X}_{n\gamma} - \Sigma_{n\gamma}$ and $\varepsilon = \alpha/2$), one also has:

$$\gamma\langle b(\Sigma_{n\gamma}) + \Sigma_{n\gamma}, \bar{X}_{n\gamma} - \Sigma_{n\gamma}\rangle + 2\gamma^2|b(\Sigma_{n\gamma}) + \Sigma_{n\gamma}|^2 \le \gamma\frac{\alpha}{2}|\bar{X}_{n\gamma} - \Sigma_{n\gamma}|^2 + C\gamma(1 + |\Sigma_{n\gamma}|^2). \tag{24}$$

Therefore, the combination of (23) and (24) yields for sufficiently small γ:

$$|\bar{X}_{(n+1)\gamma} - \Sigma_{(n+1)\gamma}|^2 \le (1 - \tilde{\alpha}\gamma)|\bar{X}_{n\gamma} - \Sigma_{n\gamma}|^2 + C\gamma(1 + |\Sigma_{n\gamma}|^2)$$

where $\tilde{\alpha}$ is a positive number. Then, it follows from Lemma 2,

$$\mathbb{E}[|\bar{X}_{(n+1)\gamma} - \Sigma_{(n+1)\gamma}|^2] \le (1 - \tilde{\alpha}\gamma)\mathbb{E}[|\bar{X}_{n\gamma} - \Sigma_{n\gamma}|^2] + \tilde{\beta}\gamma$$

where $\tilde{\beta}$ does not depend on γ. By induction, we obtain:

$$\sup_{n\ge1}\mathbb{E}[|\bar{X}_{n\gamma} - \Sigma_{n\gamma}|^2] \le \tilde{\beta}\gamma\sum_{k=0}^{+\infty}(1 - \tilde{\alpha}\gamma)^k = \frac{\tilde{\beta}}{\tilde{\alpha}} < +\infty.$$

Finally, since

$$\mathbb{E}[|\bar{X}_{n\gamma}|^2] \le 2\left(\mathbb{E}[|\bar{X}_{n\gamma} - \Sigma_{n\gamma}|^2] + \mathbb{E}[|\Sigma_{n\gamma}|^2]\right),$$

it follows from Lemma 2 that there exists $\gamma_0 > 0$ such that

$$\sup_{0<\gamma\le\gamma_0}\sup_{n\ge1}\mathbb{E}[|\bar{X}_{n\gamma}|^2] < +\infty. \tag{25}$$

Step 2: First, since $\sup_{n\ge1}\frac{1}{n}\sum_{k=1}^n|\bar{X}_{(k-1)\gamma}|^2 = \sup_{n\ge1}\mathcal{P}_0^{(n,\gamma)}(\omega, |x|^2) < +\infty$ *a.s.* (by Lemma 1), the fact that $\mathcal{P}^{(\infty,\gamma)}(\omega, d\alpha)$ is *a.s.* a weak limit of $\mathcal{P}^{(n,\gamma)}(\omega, d\alpha)$ implies that

$$\mathbb{E}_\omega[|Y_0^{(\infty,\gamma)}|^2] < +\infty \quad a.s.$$

By Proposition 4(b), there exists *a.s.* a Gaussian process Z^ω with the same distribution as the driving process of the SDE such that

$$Y_{(n+1)\gamma}^{(\infty,\gamma)} = Y_{n\gamma}^{(\infty,\gamma)} + \gamma b(Y_{n\gamma}^{(\infty,\gamma)}) + \sigma\Delta_{n+1}. \tag{26}$$

with $\Delta_n := Z^\omega_{n\gamma} - Z^\omega_{(n-1)\gamma}$. Moreover, by Remark 5, $\mathbb{E}_\omega[|Y^{(\infty,\gamma)}_{n\gamma}|^2]$ does not depend on n since the sequence $(Y^{(\infty,\gamma)}_{n\gamma})$ is stationary. Let now $(\bar{X}^x_{n\gamma})$ be constructed as in (22) with sequence (Δ_n) of (26). By (25), the lemma will be true if we are able to show that for sufficiently small γ,

$$\limsup_{n\to+\infty} \mathbb{E}_\omega[|\bar{X}^x_{n\gamma} - Y^{(\infty,\gamma)}_{n\gamma}|^2] < C \tag{27}$$

where C does not depend on γ. The process of the proof of (28) is quite similar to Step 1. First, using assumption $(\mathbf{H_3})$, one checks that:

$$|\bar{X}_{(n+1)\gamma} - Y^{(\infty,\gamma)}_{(n+1)\gamma}|^2 \le |\bar{X}_{n\gamma} - Y^{(\infty,\gamma)}_{n\gamma}|^2(1-\alpha\gamma+C\gamma^2) + \beta\gamma \quad a.s.$$

For sufficiently small γ, $\alpha\gamma - C\gamma^2 \ge \gamma\alpha/2$. Setting $\tilde{\alpha} = \alpha/2$, one derives from an induction that:

$$\mathbb{E}_\omega[|\bar{X}_{(n+1)\gamma} - Y^{(\infty,\gamma)}_{(n+1)\gamma}|^2] \le (1-\hat{\alpha}\gamma)^n \mathbb{E}_\omega[|\bar{X}^x_0 - Y^{(\infty,\gamma)}_0|^2] + \beta\gamma\sum_{k=0}^{n-1}(1-\hat{\alpha}\gamma)^k \to \frac{\beta}{\hat{\alpha}}. \tag{28}$$

This concludes the proof of (i).

(ii) First, we prove uniqueness for the distribution of a stationary solution to SDE (5): let $(Y_{t,1})_{t\ge0}$ and $(Y_{t,2})_{t\ge0}$ be some stationary solutions to (5) built on $(\Omega_1, \mathcal{F}^1, (\mathcal{F}^1_t), \mathbb{P}_1)$ and $(\Omega_2, \mathcal{F}^2, (\mathcal{F}^2_t), \mathbb{P}_2)$ respectively and denote by Z^1 and Z^2 the corresponding driving processes. We want to show that for every $T > 0$, for every bounded Lipschitz[1] continuous functional $F : \mathcal{C}([0,T], \mathbb{R}^d) \to \mathbb{R}$,

$$\mathbb{E}[F(Y_{t,1}, 0 \le t \le T)] = \mathbb{E}[F(Y_{t,2}), 0 \le t \le T)]. \tag{29}$$

Let $(X^x_{t,1})$ and $(X^x_{t,2})$ be some solutions to (5) starting from x and built with the previous driving processes Z^1 and Z^2 respectively. First, since b is Lipschitz continuous, a classical argument shows that weak uniqueness holds for solutions to (5) starting from any deterministic $x \in \mathbb{R}^d$. As a consequence, $X^x_{\cdot,1}$ and $X^x_{\cdot,2}$ have the same distribution on $\mathcal{C}(\mathbb{R}_+, \mathbb{R}^d)$. Thus, using that $Y_{\cdot,1}$ and $Y_{\cdot,2}$ are stationary, we obtain that for every $s \ge 0$:

$$\begin{aligned}\mathbb{E}[F(Y_{\cdot,1})] - \mathbb{E}[F(Y_{\cdot,2})] &= \mathbb{E}[F(Y_{t+s,1}, 0 \le t \le T)] - \mathbb{E}[F(X^x_{t+s,1}, 0 \le t \le T)] \\ &\quad + \mathbb{E}[F(X^x_{t+s,2}, 0 \le t \le T)] - \mathbb{E}[F(Y_{t+s,2}, 0 \le t \le T)].\end{aligned}$$

Since F is a bounded Lipschitz continuous functional, it follows that for every $s \ge 0$,

$$|\mathbb{E}[F(Y_{\cdot,1})] - \mathbb{E}[F(Y_{\cdot,2})]| \le C\sum_{i=1}^{2}\mathbb{E}[\sup_{t\in[s,s+T]}|Y_{t,i} - X^x_{t,i}| \wedge 1].$$

In order to obtain (29), it is now enough to prove that

$$\sup_{t\ge s}|Y_{t,i} - X^x_{t,i}| \xrightarrow{s\to+\infty} 0 \quad a.s., \quad i = 1,2.$$

Set $V^i_t = |Y_{t,i} - X^x_{t,i}|^2$. We have:

$$dV^i_t = 2\langle b(Y_{t,i}) - b(X^x_{t,i}), Y_{t,i} - X^x_{t,i}\rangle.$$

[1] for the standard distance δ defined for every $\alpha, \beta \in \mathcal{C}([0,T], \mathbb{R}^d)$ by $\delta(\alpha,\beta) = \sup_{t\in[0,T]}|\alpha_t - \beta_t|$.

Thus, it follows from $(\mathbf{H_3})$ with $\beta = 0$ and from the Gronwall lemma that,

$$|Y_{t,i} - X^x_{t,i}|^2 \leq (Y_0 - x)^2 \exp(-2\alpha t).$$

Therefore,

$$\sup_{t\geq s} |Y_{t,i} - X^x_{t,i}|^2 \leq (Y_0 - x)^2 \exp(-2\alpha s) \xrightarrow{s\to+\infty} 0 \quad a.s., \quad i = 1, 2.$$

This concludes the proof of the uniqueness for the distribution of a stationary solution to (5). For Equation $(\mathbf{E}_\gamma)$, the proof is a straightforward adaptation of the previous one. Details are left to the reader. □

Proposition 5. *Assume* $(\mathbf{H_1})$ *and* $(\mathbf{H_3})$. *Then, there exists* $\gamma_0 > 0$ *such that a.s.* $(\mathcal{P}^{(\infty,\gamma)}(\omega, d\alpha))_{\gamma\in(0,\gamma_0)}$ *is relatively compact for the topology of uniform convergence on compact sets. Furthermore, any weak limit of* $(\mathcal{P}^{(\infty,\gamma)}(\omega, d\alpha))_{\gamma\in(0,\gamma_0)}$ *(when* $\gamma \to 0$*) is the distribution of a stationary solution to SDE* (5).

Proof. **Step 1:** A.s. tightness of $(\mathcal{P}^{(\infty,\gamma)}(\omega, d\alpha))$: For $\omega \in \Omega$, we recall that $Y^{(\infty,\gamma)}$ is a càdlàg process with distribution $\mathcal{P}^{(\infty,\gamma)}(\omega, d\alpha)$. According to Theorem VI.3.26 of [7], we have to show the two following points:

- For every $T > 0$, there exists $\gamma_0 > 0$ such that

$$\limsup_{K\to+\infty} \sup_{\gamma\in(0,\gamma_0]} \mathbb{P}(\sup_{t\in[0,T]} |Y^{(\infty,\gamma)}_t| > K) = 0. \tag{30}$$

- For every positive T, ε and η, there exist $\delta > 0$ and $\gamma_0 > 0$ such that for every $\gamma \leq \gamma_0$,

$$\mathbb{P}(\sup_{|t-s|\leq\delta, 0\leq s\leq t\leq T} |Y^{(\infty,\gamma)}_t - Y^{(\infty,\gamma)}_s| \geq \varepsilon) \leq \eta. \tag{31}$$

First, we focus on (30). Let $K > 0$. By Proposition 4, we have:

$$\mathbb{P}(\sup_{t\in[0,T]} |Y^{(\infty,\gamma)}_t| > K) \leq \mathbb{P}\left(|Y^{(\infty,\gamma)}_0| + \int_0^T |b(Y^{(\infty,\gamma)}_s)|ds + \sup_{t\in[0,T]} |\sigma\bar{Z}^\gamma_t| > K\right).$$

Using the Markov inequality, it follows that

$$\mathbb{P}(\sup_{t\in[0,T]} |Y^{(\infty,\gamma)}_t| > K) \leq \frac{1}{K}\left(\mathbb{E}[|Y^{(\infty,\gamma)}_0|] + CT \sup_{n\in\{0,\dots,[T/\gamma]\}} \mathbb{E}[|Y^{(\infty,\gamma)}_{n\gamma}|] + \|\sigma\|\mathbb{E}[\sup_{t\in[0,T]} |Z_t|]\right)$$

where $\|\sigma\| = \sup\{|\sigma x|/|x|, x \in \mathbb{R}^d\}$. Now, since $(Y_{n\gamma})$ is a stationary sequence and $\sup_{t\in[0,T]} |Z_t|$ is integrable (see Proposition 6), one obtains:

$$\mathbb{P}(\sup_{t\in[0,T]} |Y^{(\infty,\gamma)}_t| > K) \leq \frac{C}{K}\left(1 + \mathbb{E}[|Y^{(\infty,\gamma)}_0|]\right),$$

where C does not depend on γ. Finally, the first point follows from Lemma 3.

Let us now prove (31). In fact, using for instance proof of Theorem 8.3 of [1], it is enough to show that for every positive ε, η and T, there exist $\delta > 0$ and $\gamma_0 > 0$ such that for every $\gamma \leq \gamma_0$:

$$\frac{1}{\delta}\mathbb{P}(\sup_{t\leq s\leq t+\delta} |Y^{(\infty,\gamma)}_t - Y^{(\infty,\gamma)}_s| \geq \varepsilon) \leq \eta \qquad \forall\gamma \leq \gamma_0 \quad \text{and} \quad 0 \leq t \leq T. \tag{32}$$

By the Markov inequality, we have for every $p \geq 1$:

$$\mathbb{P}\left(\sup_{t\leq s\leq t+\delta}|Y_t^{(\infty,\gamma)}-Y_s^{(\infty,\gamma)}|\geq\varepsilon\right)\leq\left(\frac{2\|\sigma\|}{\varepsilon}\right)^2\mathbb{E}\left[\left(\int_{\underline{t}_\gamma}^{\underline{t+\delta}_\gamma}|b(Y_s^{(\infty,\gamma)})|ds\right)^2\right] \tag{33}$$
$$+\left(\frac{2\|\sigma\|}{\varepsilon}\right)^p\mathbb{E}\left[\sup_{s\in[t,t+\delta]}|\bar{Z}_s^\gamma-\bar{Z}_t^\gamma|^p\right]$$

On the one hand,

$$\mathbb{E}\left[\left(\int_t^{t+\delta}|b(Y_s^{(\infty,\gamma)})|ds\right)^2\right]\leq\mathbb{E}\left[\left(\sum_{k=[t/\gamma]}^{[(t+\delta)/\gamma]}\sqrt{\gamma}(\sqrt{\gamma}|b(Y_{n\gamma}^{(\infty,\gamma)})|)\right)^2\right]$$
$$\leq\mathbb{E}\left[\left(\sum_{k=[t/\gamma]}^{[(t+\delta)/\gamma]}\gamma\right)\left(\sum_{k=[t/\gamma]}^{[(t+\delta)/\gamma]}\gamma|b(Y_{n\gamma}^{(\infty,\gamma)})|^2\right)\right]$$

thanks to the Cauchy-Schwarz inequality. Now, when γ is sufficiently small

$$\sum_{k=[t/\gamma]}^{[(t+\delta)/\gamma]}\gamma\leq 2\delta.$$

Therefore, using also the fact that b has sublinear growth yields:

$$\mathbb{E}\left[\left(\int_t^{t+\delta}|b(Y_s^{(\infty,\gamma)})|ds\right)^2\right]\leq C\delta^2(1+\sup_{k\in\{[\frac{t}{\gamma}],\ldots,[\frac{(t+\delta)}{\gamma}]\}}\mathbb{E}[|Y_{k\gamma}^{(\infty,\gamma)}|^2])\leq C\delta^2(1+\mathbb{E}[|Y_0^{(\infty,\gamma)}|^2]) \tag{34}$$

thanks to the stationarity of $(Y_{n\gamma}^{(\infty,\gamma)})_{n\geq 0}$.
On the other hand, we deduce from the stationarity of the increments of $(Z_t)_{t\geq 0}$ that

$$\mathbb{E}\left[\sup_{s\in[t,t+\delta]}|\bar{Z}_s^\gamma-\bar{Z}_t^\gamma|^p\right]\leq\mathbb{E}\left[\sup_{s\in[t,t+\delta]}|Z_s-Z_t|^p\right]\leq\mathbb{E}[\sup_{s\in[0,\delta]}|Z_s|^p].$$

Thus, by Proposition 6 (see Appendix), for sufficiently large p,

$$\mathbb{E}\left[\sup_{s\in[t,t+\delta]}|\bar{Z}_s^\gamma-\bar{Z}_t^\gamma|^p\right]\leq C\delta^{1+\rho} \tag{35}$$

where ρ is a positive number.
Then, the combination of (33), (34) and (35) yields for sufficiently small γ:

$$\mathbb{P}\left(\sup_{t\leq s\leq t+\delta}|Y_t^{(\infty,\gamma)}-Y_s^{(\infty,\gamma)}|\geq\varepsilon\right)\leq C\delta^{2\wedge(1+\rho)}$$

and (31) follows from Lemma 3.

Step 2: We want to show that, *a.s*, any weak limit $\mathcal{P}(\omega,d\alpha)$ of $(\mathcal{P}^{(\infty,\gamma)}(\omega,d\alpha))_\gamma$ when $\gamma\to 0$ (for the uniform convergence topology) is the distribution of a stationary process. Let $f:\mathbb{R}^r\to\mathbb{R}$ be a bounded continuous function and let $t>0$ and $t_1,\ldots,t_r$ such that

$0 \le t_1 < \ldots < t_r$. Denoting by $(Y_t)_{t\ge 0}$ a process with distribution $\mathcal{P}(\omega, d\alpha)$, we have to show that:

$$\mathbb{E}[f(Y_{t_1+t}, \ldots, Y_{t_r+t})] = \mathbb{E}[f(Y_{t_1}, \ldots, Y_{t_r})]. \tag{36}$$

First, since $\mathcal{P}(\omega, d\alpha)$ is a weak limit of $(\mathcal{P}^{(\infty,\gamma)}(\omega, d\alpha))_\gamma$, there exist some sequences $(\gamma_n)_{n\ge 0}$ and $(Y^{(\infty,\gamma_n)})_{n\ge 0}$ such that $\mathcal{L}(Y^{(\infty,\gamma_n)}) = \mathcal{P}^{(\infty,\gamma_n)}(\omega, d\alpha)$ and $(Y_t^{(\infty,\gamma_n)})$ converges weakly to (Y_t) for the weak topology induced by the uniform convergence topology on compact sets on $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$. In particular,

$$\mathbb{E}[f(Y_{t_1}^{(\infty,\gamma_n)}, \ldots, Y_{t_r}^{(\infty,\gamma_n)})] \xrightarrow{n\to+\infty} \mathbb{E}[f(Y_{t_1}, \ldots, Y_{t_r})] \quad \text{and,} \tag{37}$$

$$\mathbb{E}[f(Y_{t_1+\underline{t}_{\gamma_n}}^{(\infty,\gamma_n)}, \ldots, Y_{t_r+\underline{t}_{\gamma_n}}^{(\infty,\gamma_n)})] \xrightarrow{n\to+\infty} \mathbb{E}[f(Y_{t_1+t}, \ldots, Y_{t_r+t})]. \tag{38}$$

since $\underline{t}_{\gamma_n} := \gamma_n[t/\gamma_n] \to t$ when $n \to +\infty$. Now, by Proposition 4,

$$\mathbb{E}[f(Y_{t_1}^{(\infty,\gamma_n)}, \ldots, Y_{t_r}^{(\infty,\gamma_n)})] = \mathbb{E}[f(Y_{t_1+\underline{t}_{\gamma_n}}^{(\infty,\gamma_n)}, \ldots, Y_{t_r+\underline{t}_{\gamma_n}}^{(\infty,\gamma_n)})] \quad \forall n \ge 1.$$

(36) follows.

Step 3: Let $\Phi : \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d) \to \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$ be defined by

$$(\Phi(\alpha))_t = \alpha_t - \alpha_0 - \int_0^t b(\alpha_s)ds.$$

With the notations of Step 2, we want to show that $Y := (Y_t)_{t\ge 0}$ is a solution to (5), *i.e.* that $\Phi(Y)$ is equal in law to $\sigma Z := (\sigma Z_t)_{t\ge 0}$. Let (γ_n) and $(Y^{(\infty,\gamma_n)})_{n\ge 0}$ be defined as in Step 2. Then, since Φ is continuous for the uniform convergence topology on compact sets,

$$\Phi(Y^{(\infty,\gamma_n)}) \overset{n\to+\infty}{\Longrightarrow} \Phi(Y), \tag{39}$$

for the weak topology induced by the uniform convergence topology on compact sets. Therefore, we have to prove that

$$\Phi(Y^{(\infty,\gamma_n)}) \overset{n\to+\infty}{\Longrightarrow} \sigma Z, \tag{40}$$

for this topology. With the notations of Proposition 4,

$$\Phi(Y^{(\infty,\gamma_n)}) = N^{(\infty,\gamma_n)} + R^{\gamma_n} \quad \text{where} \quad R_t^{\gamma_n} = -\int_{\underline{t}_{\gamma_n}}^t b(Y_s^{(\infty,\gamma_n)})ds. \tag{41}$$

First, since b is sublinear and $t - \underline{t}_{\gamma_n} \le \gamma_n$, we have for every $T > 0$:

$$|R_t^{\gamma_n}| \le C\gamma_n(1 + \sup_{t\in[0,T]} |Y_t^{(\infty,\gamma_n)}|) \quad \forall t \in [0, T].$$

Now, in Step 1, we showed that $(\sup_{t\in[0,T]} |Y_t^{(\infty,\gamma)}|)_{\gamma\in(0,\gamma_0)}$ is tight on $\mathbb{R}$. It follows easily that

$$\sup_{t\in[0,T]} |R_t^{\gamma_n}| \xrightarrow{n\to+\infty} 0 \quad \text{in probability } \forall T > 0.$$

Therefore, one derives from (39) and (41),

$$N^{(\infty,\gamma_n)} \overset{n\to+\infty}{\Longrightarrow} \Phi(Y).$$

Then, it follows from Proposition 4 that $N^{(\infty,\gamma_n)}$ is a convergent sequence of Gaussian processes such that $N^{(\infty,\gamma_n)} \overset{\mathcal{L}}{=} \sigma\bar{Z}^{\gamma_n}$. This implies the finite-dimensional convergence to $(Z_t)_{t\ge 0}$ and concludes the proof. □

6 Appendix

Proposition 6. *Assume that $(Z_t)_{t\geq 0}$ satisfies $(\mathbf{H_1})$. Then, for every $T>0$, for every $r>0$, $\mathbb{E}[\sup_{t\in[0,T]}|Z_t|^r]<+\infty$. Moreover, there exist $p\geq 1$ and $T_0>0$ such that for every $T\leq T_0$,*

$$\mathbb{E}[\sup_{t\in[0,T]}|Z_t|^p]\leq CT^{1+\rho} \quad \text{with } \rho>0.$$

Proof. First, note that it is enough to prove the result for every coordinate Z^j with $j\in\{1,\dots,\ell\}$. Therefore, it is in fact enough to prove that the results are true for any one-dimensional centered Gaussian process with stationary increments and variance function $(c(t))_{t\geq 0}$ satisfying $(\mathbf{H_1})$. Then, for every $t>0$ and $\varepsilon>0$, $c(t)=\mathbb{E}[Z_t^2]$ and denote by $\mathcal{D}(T,\varepsilon)$ the Dudley integral defined by

$$\mathcal{D}(T,\varepsilon)=\int_0^{\varepsilon}(\log(N(T,u))^{1/2}du,$$

where, for $u>0$,

$$N(T,u)=\inf\{n\geq 1,\exists\, s_1,\dots,s_n \text{ such that } \forall t\in[0,T],\ \exists i\in\{1,\dots,n\} \text{ with } \sqrt{c(t-s_i)}\leq u\}.$$

By the Dudley Theorem (see *e.g.* Theorem 1 of [13] p 179), for every $T>0$,

$$\mathbb{E}[\sup_{t\in[0,T]}|Z_t|]\leq 2\mathbb{E}[\sup_{t\in[0,T]}Z_t]\leq C\mathcal{D}(T,\sqrt{\bar{c}(T)}) \quad \text{with } \bar{c}(T)=\sup_{t\in[0,T]}c(t). \tag{42}$$

Let us control the right-hand member. By assumption $(\mathbf{H_1})$ and (4), for every $\delta\in(0,1)$,

$$\bar{c}(\delta)\leq C\delta^{\mu} \quad \text{where } C \text{ does not depend on } \delta \text{ and}$$

$\mu\in(0,1]$ (depending on the value of a_1). It follows that, for $u>0$,

$$N(T,u)\leq CTu^{2/\mu},$$

where C does not depend on T. For $\varepsilon>0$ small enough,

$$\mathcal{D}(T,\varepsilon)\leq\int_0^{\varepsilon}|\log(CT)+\frac{2}{\mu}\log(u)|^{1/2}du\leq C\varepsilon|\log(\varepsilon)|^{1/2}.$$

It follows from (42) that there exists $T_0>0$ such that for $T\leq T_0$

$$\begin{aligned}\mathbb{E}[\sup_{t\in[0,T]}|Z_t|]&\leq C(\bar{c}(T))^{1/2}|\log(\bar{c}(T))|^{1/2}\\&\leq CT^{\mu/2}|\log(T)|^{1/2}.\end{aligned}$$

Then by Corollary 3.2 in [11] $\mathbb{E}[\sup_{t\in[0,T]}|Z_t|^r]<+\infty$ for every $T>0$ and every $r>0$ and

$$\mathbb{E}[\sup_{t\in[0,T]}|Z_t|^p]\leq CT^{\mu p/2}|\log(T)|^{p/2},$$

for $T\leq T_0$. One can choose p big enough to prove the second inequality in the Proposition. □

Proof of Proposition 2.

Let $X := (X_t)_{t\geq 0}$ be a stationary solution to (5) and assume that X_0 is independent of $Z := (Z_t)_{t\geq 0}$. The matrix σ is assumed to be invertible. We denote by σ^{-1} its inverse. First, note that for every $t \geq 0$, $Z_{t+\cdot} - Z_t = \psi(X_{t+\cdot})$ where $\psi : \mathcal{C}(\mathbb{R}_+, \mathbb{R}^d) \to \mathcal{C}(\mathbb{R}_+, \mathbb{R}^d)$ is defined for every $\alpha \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}^d)$ by

$$\psi(\alpha)_t = \sigma^{-1}(\alpha_t - \alpha_0 - \int_0^t b(\alpha_s)ds) \quad \forall t \geq 0,$$

is continuous. Then, since (X_t) is stationary, it follows that for every bounded continuous functional F (for the topology of uniform convergence on compact sets), for every $s \geq 0$, for every $t \geq 0$, and every bounded continuous function $f : \mathbb{R}^d \to \mathbb{R}$,

$$\begin{aligned}\mathbb{E}[(f(X_0) - \mathbb{E}[f(X_0)])\, F(Z_{s+u} - Z_s, u \geq 0)] &= \mathbb{E}[(f(X_t) - \mathbb{E}[f(X_t)])\, F(\psi(X_{t+s+\cdot}))] \\ &= \mathbb{E}[(f(X_t) - \mathbb{E}[f(X_t)])\, F(Z_{t+s+u} - Z_{t+s}, u \geq 0)].\end{aligned}$$

Since X_0 is independent of Z,

$$\mathbb{E}[(f(X_0) - \mathbb{E}[f(X_0)])\, F(Z_{s+u} - Z_s, u \geq 0)] = 0.$$

This implies that

$$\mathbb{E}[f(X_t)F(Z_{t+s+u} - Z_t, u \geq 0)] = \mathbb{E}[f(X_t)]\mathbb{E}[F(Z_{t+s+u} - Z_{t+s}, u \geq 0)].$$

One deduces that for every $s, t \geq 0$, such that $0 \leq s \leq t$, X_s is independent of $(Z_{t+u} - Z_t)_{u\geq 0}$. As a consequence, for every positive u, t, for every $i, j \in \{1, \ldots, d\}$,

$$\begin{aligned}&\mathbb{E}[Z_t^i(Z_{t+u}^j - Z_t^j)] \\ &\quad= \sum_{m=1}^{\ell} \sigma_{i,m}^{-1} \left(\mathbb{E}[X_t^i(Z_{t+u} - Z_t)] - \int_0^t \mathbb{E}[b^i(X_v)(Z_{t+u}^j - Z_t^j)]dv - \mathbb{E}[X_0^i(Z_{t+u}^j - Z_t^j)] \right), \\ &\quad= \sum_{m=1}^{\ell} \sigma_{i,m}^{-1} \left(\mathbb{E}[X_t^i]\mathbb{E}[Z_{t+u}^j - Z_t^j] - \int_0^t \mathbb{E}[b^i(X_v)]\mathbb{E}[Z_{t+u}^j - Z_t^j]dv - \mathbb{E}[X_0^i]\mathbb{E}[Z_{t+u}^i - Z_t^i] \right), \\ &\quad= 0.\end{aligned}$$

Since Z is a centered Gaussian process, it clearly implies that Z has independent increments.

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