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# State estimation and fault detection for systems described by Takagi-Sugeno nonlinear models

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**Abstract.** This communication addresses the problem of fault detection and isolation (FDI) for nonlinear systems. Despite a lot of works in this area, model-based FDI for nonlinear systems still remains a difficult task. These two last decades have witnessed growing popularity of Takagi-Sugeno models. This attractivity relies essentially on two advantages: (i) that kind of models can be considered as *universal approximator* *i.e.* they offer a good flexibility at the modeling stage; (ii) their structure based on the blending of linear models is able to fully take advantages of the advances of modern control theory. It is then rather natural to trying to extent the well-established model-based FDI methods for linear systems to the systems that can be represented by Takagi-Sugeno models.

## 1 Introduction

The problem of designing observers for nonlinear systems constitutes an important field of research. It is due to a large domain of application such as, estimating of non accessible states, when sensors are not available or too expensive, designing feedback control laws and generating residual signals in order to detect and isolate sensor, actuator or process faults. For systems described by linear time-invariant (LTI) models, the state estimation can be successfully accomplished from the available inputs and outputs thanks to the well known Luenberger observer or Kalman filter. However, it turns out that LTI models fail for accurately describing the dynamic behaviour of relatively complex systems in a large operating range. Indeed, LTI models only provide a good representation of such systems around an operating point (local modeling) and consequently the generated state estimation is not valid far away from this point. Hence, nonlinear models become unavoidable in order to represent the behavior of the considered systems. From the observer design viewpoint, this makes more difficult the modeling task because the accurate and global description of the input-output behaviour of the system must be accomplished with the help of a mathematical model as simple as possible for observer design.

For such a purpose, a multiple model representation of the system under investigation can be used. In this modeling approach, the dynamic behaviour of the system is accurately captured by the judicious interconnection of a set of

linear submodels [15], [27], [31]. It should be noted that various realisations of multiple models can be employed in order to generate the global output of the multiple model [14], [18]. Two essential realisations of multiple models can be distinguished whether homogeneous or heterogeneous submodels are taken into account for modeling the system behaviour. In the first case, an interpolation of the submodel parameters is considered and the submodels share the same state space. This approach leads to the so-called Takagi-Sugeno model. It has been initially proposed, in a fuzzy modeling framework, by Takagi and Sugeno [41] and in a operating regime-based modeling framework by Johansen and Foss [24]. In the second one, only the submodel outputs are interpolated and consequently each submodel has its own state space which can be of different dimension; that kind of models is frequently called decoupled multiple model [33], [44], [47]. In the sequel, we shall be interested in Takagi-Sugeno models only. This communication will be focused on the possibility to use such models in the context of FDI.

## 2 Takagi-Sugeno model

A Takagi-Sugeno (TS) model allows the representation of the behavior of a nonlinear system by the interpolation of a set of linear submodels. Each submodel contributes to the global behavior of the nonlinear system through a weighting function  $\mu_i(\xi(t))$ . The TS structure is given by:

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^r \mu_i(\xi(t))(A_i x(t) + B_i u(t)) \\ y(t) = \sum_{i=1}^r \mu_i(\xi(t))(C_i x(t) + D_i u(t)) \end{cases} \quad (1)$$

where  $x(t) \in \mathbb{R}^n$  is the state vector,  $u(t) \in \mathbb{R}^m$  is the input vector,  $y(t) \in \mathbb{R}^p$  represents the output vector.  $A_i \in \mathbb{R}^{n \times n}$ ,  $B_i \in \mathbb{R}^{n \times m}$ ,  $C_i \in \mathbb{R}^{p \times n}$  and  $D_i \in \mathbb{R}^{p \times m}$  are known matrices. Finally, the functions  $\mu_i(\xi(t))$  are the weighting functions depending on the so-called decision variables  $\xi(t)$  which can be measurable (as the input or the output of the system) or non measurable (as the state of the system). These functions verify the following properties:

$$\begin{cases} \sum_{i=1}^r \mu_i(\xi(t)) = 1 \\ 0 \leq \mu_i(\xi(t)) \leq 1, \quad \forall i \in \{1, 2, \dots, r\} \end{cases} \quad (2)$$

Of course, the construction of a TS model represents an important and basic procedure. Two main approaches are mainly employed :

1. Identification using input-output data. The literature about that problem is very abundant and we refer the interested reader to the following references [1], [2], [5], [17], [31], [41], [44], [47], [50],

2. Derivation from a given nonlinear model. In that case, two different approaches can be used. In the first one, the nonlinear model is linearized around some *a priori* chosen operating points. Linear models are then obtained for each operating zone. Next, optimization techniques are used to identify the parameters of the weighting functions by minimizing a function of the output error [3]. The second approach, probably the most interesting, relies on the well-known transformation by nonlinear sector [42]. Indeed, this transformation allows to obtain an exact TS representation of a nonlinear model with bounded nonlinearities.

Let us recall the main idea of this last approach, by considering the following nonlinear model:

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)) \\ y(t) = h(x(t), u(t)) \end{cases} \quad (3)$$

The system (3) is first rewritten in a quasi-LPV form. Under the hypothesis that  $f(x(t), u(t))$  and  $h(x(t), u(t))$  are continuous and bounded in  $U \subset \mathbb{R}^n$  with  $f(0, \cdot) = 0$  and  $g(0, \cdot) = 0$ , the system (3) can be expressed in a quasi-LPV form:

$$\begin{cases} \dot{x}(t) = A(x(t), u(t))x(t) + B(x(t), u(t))u(t) \\ y(t) = C(x(t), u(t))x(t) + D(x(t), u(t))u(t) \end{cases} \quad (4)$$

Let us denote  $\xi(t) = [x(t) \ u(t)]$ , the so-called decision variable. Every time-varying entry of the matrices  $A(\xi(t))$ ,  $B(\xi(t))$ ,  $C(\xi(t))$  and  $D(\xi(t))$  are considered as a premise variable, denoted  $z_j(\xi(t))$ ,  $j = 1, \dots, p$ .

Notice that the quasi-LPV form (4) for a nonlinear system (3) is not unique. So choosing a quasi-LPV form is equivalent to choose the premise variable set. The choice of the premise variable set is important, because it has an influence on the submodel numbers and on the global model structure. This is a degree of freedom that should be used to ease the controllability, the observability and the stability analysis studies [32].

Let  $z_j(\xi(t)) \in [z_{j,min}, z_{j,max}]$ . A convex polytopic transformation for the  $p$  premise variables can be performed. Let define:

$$F_{j,max}(z_j) = \frac{z_j(\xi(t)) - z_{j,min}}{z_{j,max} - z_{j,min}} \quad (5)$$

$$F_{j,min}(z_j) = \frac{z_{j,max} - z_j(\xi(t))}{z_{j,max} - z_{j,min}} \quad (6)$$

with the property  $F_{j,min}(z_j) + F_{j,max}(z_j) = 1$ , the  $j^{th}$  premise variable can be expressed as:

$$z_j(\xi(t)) = z_{j,min}F_{j,min}(z_j) + z_{j,max}F_{j,max}(z_j) \quad (7)$$

Using this transformation,  $r = 2^p$  submodels characterized by  $p$  premise variables can be generated. The matrices  $A_i$  (resp.  $B_i$ ,  $C_i$  and  $D_i$ ) are evaluated at

the vertices of the polytope defined by the premise variable partitions intervening in  $A(\xi(t))$  (resp.  $B(\xi(t))$ ,  $C(\xi(t))$  and  $D(\xi(t))$ ). To each submodel  $i$  corresponds a  $p$ -uplet  $\sigma_i$  which codes the partitions of the premise variables occurring in the corresponding weighting function  $\mu_i(\xi(t))$ . Denoting  $\sigma_i^k$ , the index at the  $k^{th}$  position in  $\sigma_i$ , the weighting function  $\mu_i(\xi(t))$  can be defined as:

$$\mu_i(\xi(t)) = \prod_{k=1}^p F_{k, \sigma_i^k}(z_k(\xi(t))) \quad (8)$$

For a more detailed description of this approach, the reader is referred to [30], [32] and [42]. Clearly, this modeling approach is very attractive as it corresponds only to a particular re-writing of the original nonlinear model (without any degradation, provided the boundness of the nonlinearities). However, the reader will notice that the state of the system often intervenes in the decision variables. For being able to design an observer for that kind of model, it is therefore necessary to take into account this constraint.

In the following, the considered models will essentially originate from that type of modeling. So, a particular attention will be devoted to the design of observer for TS model with unmeasurable decision variables which is a key point for being able to implement FDI techniques.

### 3 Observer design for TS model with unmeasurable decision variables

In the field of state estimation and diagnosis of nonlinear systems using TS model, most of the published works considered models with measurable decision variables [4], [9], [35]. It is clear that this choice allows an easy generalization of the methods already developed for linear systems. When the decision variables are not measurable, the problem becomes harder. However, this formalism is essential both, as previously mentioned, in the modeling stage as well in diagnosis methods based on observer banks to detect and isolate actuator and sensor faults (see section 5). A few works are devoted to the observer design in that case, nevertheless, we can cite [6], [7] and [34], where the authors proposed to use a fuzzy Thau-Luenberger observer which is an extension of the classical Luenberger observer [43]. The results presented in this section aim to reduce the conservatism of the existing works by reducing the number of LMIs to be solved and relaxing the conditions under which the methods are applicable. Two different approaches are presented. The first one assumes Lipschitz properties on some functions depending on the weighting functions of the TS model. The second one expresses the state equation error as a perturbed system and the observer is designed for guaranteeing that the disturbance attenuation level is smaller than a given threshold [20].

### 3.1 First approach based on Lipschitz hypotheses

#### *Non-perturbed systems*

Consider first the following non-perturbed TS model:

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^r \mu_i(x(t)) (A_i x(t) + B_i u(t)) \\ y(t) = Cx(t) \end{cases} \quad (9)$$

In order to ease the presentation of the method, the output equation of the model is chosen linear with regard to the state, which is frequently the case in practical situation. Let us consider the matrices  $A_0$  and  $\bar{A}_i$  defined in one of the two following ways:

1. The matrix  $A_0$  is the mean of the matrices  $A_i$  then:

$$A_0 = \frac{1}{r} \sum_{i=1}^r A_i \quad (10)$$

2. The matrix  $A_0$  is chosen as the dominant local model of the system. Let the  $j^{th}$  local model being the dominant one, then:

$$A_0 = A_j \quad (11)$$

The matrix  $\bar{A}_i$  is then defined by:

$$\bar{A}_i = A_i - A_0 \quad (12)$$

Substituting  $A_0$  and  $\bar{A}_i$  in the state equation (9) leads to the following equivalent system:

$$\begin{cases} \dot{x}(t) = A_0 x(t) + \sum_{i=1}^r \mu_i(x(t)) (\bar{A}_i x(t) + B_i u(t)) \\ y(t) = Cx(t) \end{cases} \quad (13)$$

The structure of the proposed state observer is based on this re-writing and has the following classical form:

$$\begin{cases} \dot{\hat{x}}(t) = A_0 \hat{x}(t) + \sum_{i=1}^r \mu_i(\hat{x}(t)) (\bar{A}_i \hat{x}(t) + B_i u(t)) + L(y(t) - \hat{y}(t)) \\ \hat{y}(t) = C \hat{x}(t) \end{cases} \quad (14)$$

The state estimation error is given by:

$$e(t) = x(t) - \hat{x}(t) \quad (15)$$

and its dynamic can be expressed as:

$$\dot{e}(t) = (A_0 - LC)e(t) + \Delta(x, \hat{x}, u) \quad (16)$$

where:

$$\Delta(x, \hat{x}, u) = \sum_{i=1}^r (\bar{A}_i(\mu_i(x(t))x(t) - \mu_i(\hat{x}(t))\hat{x}(t)) + B_i(\mu_i(x(t)) - \mu_i(\hat{x}(t))u(t))) \quad (17)$$

This last term is due to the fact that the decision variables of the model ( $x(t)$ ) and those of the observer ( $\hat{x}(t)$ ) are different as the observer cannot be designed on the basis of unknown variables. In order to study the stability of (16), let us consider the following hypotheses:

- **A1.**  $|\mu_i(x(t))x(t) - \mu_i(\hat{x}(t))\hat{x}(t)| < \alpha_i |x(t) - \hat{x}(t)|$
- **A2.**  $|B_i(\mu_i(x(t)) - \mu_i(\hat{x}(t)))| < \beta_i |x(t) - \hat{x}(t)|$
- **A3.**  $|u(t)| < \rho$

where  $\alpha_i > 0$ ,  $\beta_i > 0$  and  $\rho > 0$ .

Using the assumptions **A1**, **A2** and **A3**, the term  $\Delta(x, \hat{x}, u)$  can be bounded as follows:

$$|\Delta(x, \hat{x}, u)| < \gamma |x(t) - \hat{x}(t)| \quad (18)$$

where:

$$\gamma = \sum_{i=1}^r (\bar{\sigma}(\bar{A}_i)\alpha_i + \beta_i\rho) \quad (19)$$

where  $\bar{\sigma}(M)$  represents the maximum singular value of the matrix  $M$ .

**Theorem 1.** *The state estimation error between the TS model (9) and its observer (14) converges asymptotically toward zero, if there exists matrices  $P = P^T > 0$ , a diagonal positive matrix  $Q$  and a gain matrix  $K$  such that the following condition holds:*

$$\begin{bmatrix} A_0^T P + P A_0 - C^T K^T - K C + \gamma^2 Q & P \\ P & -Q \end{bmatrix} < 0 \quad (20)$$

The gain of the observer is computed by  $L = P^{-1}K$ . ■

*Proof.* The convergence condition of the state estimation error is obtained by using a quadratic Lyapunov function:

$$V(t) = e^T(t)P e(t), \quad P = P^T > 0 \quad (21)$$

Its derivative is given by:

$$\dot{V}(t) = \dot{e}^T(t)P e(t) + e^T(t)P \dot{e}(t) \quad (22)$$

By substituting (16) in (22), this derivative can be written as:

$$\dot{V}(t) = e^T(t) (\Phi^T P + P \Phi) e(t) + 2e^T(t)P \Delta(x, \hat{x}, u) \quad (23)$$

where  $\Phi = A_0 - LC$ .

In order to establish the conditions for the negativity of  $\dot{V}(t)$ , let us use the lemma 1 given in appendix with a diagonal positive definite matrix  $Q$ . Therefore (23) can be bounded as follows:

$$e^T(t)(\Phi^T P + P\Phi + PQ^{-1}P)e(t) + \Delta^T(x, \hat{x}, u)Q\Delta(x, \hat{x}, u) < 0 \quad (24)$$

Taking into account (18), the negativity of  $\dot{V}(t)$  is assured if:

$$e^T(t)(\Phi^T P + P\Phi + PQ^{-1}P + \gamma^2 Q)e(t) < 0 \quad (25)$$

Finally, the inequality (25) holds if:

$$(A_0 - LC)^T P + P(A_0 - LC) + PQ^{-1}P + \gamma^2 Q < 0 \quad (26)$$

The conditions (26) are not linear with respect to the variables  $P$ ,  $L$  and  $Q$ . In order to solve them with the classical LMI approaches, the change of variable  $K = PL$ , and the Schur complement [8] (recalled in appendix) are used, leading to the conditions given in theorem 1.

### *Perturbed systems*

Frequently, the considered systems are perturbed by unknown exogenous disturbances  $\omega(t)$  and the corresponding model can be written as:

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^r \mu_i(x(t)) (A_i x(t) + B_i u(t) + E_i \omega(t)) \\ y(t) = Cx(t) \end{cases} \quad (27)$$

Assuming that  $\omega(t) \in \mathcal{L}_2$ , the observer design consists now to compute the gain  $L$  such that the observer error dynamics is asymptotically stable and that the following specified  $\mathcal{L}_2$  norm upper bound is below a given threshold.

$$\frac{\|e(t)\|_2}{\|\omega(t)\|_2} < \xi, \quad \xi > 0 \quad (28)$$

By using the matrices defined in (10), (11) and (12), the dynamical equation of (27) can be written as:

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^r \mu_i(x(t)) (\bar{A}_i x(t) + B_i u(t) + E_i \omega(t)) \quad (29)$$

The proposed observer for this system keeps the structure given in (14) and its existence is defined by the following theorem.



**Theorem 2.** *The robust observer (14) for the system (27) satisfying (28), is determined by minimizing the real positive number  $\bar{\xi}$  under the following LMI constraints in the variables  $P$ ,  $K$ ,  $Q$  and  $\bar{\xi}$ :*

$$\begin{bmatrix} \Theta & P & PE_i \\ P & -Q & 0 \\ E_i^T P & 0 & -\bar{\xi}I \end{bmatrix} < 0, \quad i = 1, \dots, r \quad (30)$$

where:

$$\Theta = A_0^T P + PA_0 - K^T P - PK + \gamma^2 Q + I \quad (31)$$

The gain of the observer is computed by  $L = P^{-1}K$ . The resulting attenuation level is given by  $\xi = \sqrt{\bar{\xi}}$ . ■

*Proof.* The dynamic of the state estimation error between (29) and (14) is given by the following equation:

$$\dot{e}(t) = (A_0 - LC)e(t) + \Delta(x, \hat{x}, u) + \sum_{i=1}^r \mu_i(x(t))E_i\omega(t) \quad (32)$$

where  $\Delta(x, \hat{x}, u)$  is defined in (17). Consider the quadratic Lyapunov function defined in (21), therefore:

$$\dot{V}(t) = e^T(t)(\Phi^T P + P\Phi)e(t) + 2e^T(t)P\Delta(x, \hat{x}, u) + 2\sum_{i=1}^r \mu_i(x(t))e^T(t)PE_i\omega(t) \quad (33)$$

According to the above assumptions and the lemma 1, we have:

$$\dot{V}(t) \leq e^T(t)(\Phi^T P + P\Phi + PQ^{-1}P + \gamma^2 Q)e(t) + 2\sum_{i=1}^r \mu_i(x(t))e^T(t)PE_i\omega(t) \quad (34)$$

The condition which guarantees the boundedness of the  $\mathcal{L}_2$  norm of the transfer from  $\omega(t)$  to  $e(t)$  (which satisfies (28)) is given by:

$$\dot{V}(t) + e^T(t)e(t) - \xi^2\omega^T(t)\omega(t) < 0 \quad (35)$$

Using the derivative of the Lyapunov function (34), we obtain:

$$\begin{aligned} & e^T(t)(\Phi^T P + P\Phi + PQ^{-1}P + \gamma^2 Q + I)e(t) \\ & + 2e^T(t)P\sum_{i=1}^r \mu_i(x(t))E_i\omega(t) - \xi^2\omega^T(t)\omega(t) < 0 \end{aligned} \quad (36)$$

This last inequality can be written in matricial form:

$$\begin{bmatrix} e(t) \\ \omega(t) \end{bmatrix}^T \mathcal{M} \begin{bmatrix} e(t) \\ \omega(t) \end{bmatrix} < 0 \quad (37)$$

where :

$$\mathcal{M} = \sum_{i=1}^r \mu_i(x(t)) \begin{bmatrix} \Theta + PQ^{-1}P & PE_i \\ E_i^T P & -\xi^2 I \end{bmatrix} \quad (38)$$

Using, the change of variables  $K = PL$ ,  $\bar{\xi} = \xi^2$  and the Schur complement, we obtain the following LMIs, which guarantee that (37) holds:

$$\begin{bmatrix} \Theta & P & PE_i \\ P & -Q & 0 \\ E_i^T P & 0 & -\bar{\xi} I \end{bmatrix} < 0, \quad i = 1, \dots, r \quad (39)$$

### ***Enhancement of the observer performances***

For increasing the performances of the observer (in terms of fast and well damped estimation error  $e(t)$  of the observer) it is necessary to assign all the eigenvalues of  $(A_0 - LC)$  in a specific region of the complex-plane. In order to ensure a minimal decay rate  $a$  and a damping ratio, the eigenvalues of the system generating the state estimation error  $e(t)$  are clustered in  $S(a, R, q)$  defined by

$$S(a, R, q) = \{z \in \mathbb{C} / |z + q| < R, \operatorname{Re}(z) < a, q > 0, a > 0\} \quad (40)$$

(see [11], [35]). The observer eigenvalue constraints can be verified by theorem 3.

**Theorem 3.** *The robust observer (14) for the system (27) satisfying (28) and such that the eigenvalues of the matrix  $(A_0 - LC)$  have their eigenvalues in the region  $S(a, R, q)$ , is determined by minimizing the real positive number  $\bar{\xi}$  under the following LMI constraints in the variables  $P$ ,  $K$ ,  $Q$  and  $\bar{\xi}$ :*

$$\begin{bmatrix} \Xi & P & PE_i \\ P & -Q & 0 \\ E_i^T P & 0 & -\bar{\xi} I \end{bmatrix} < 0, \quad i = 1, \dots, r \quad (41)$$

$$\begin{bmatrix} -RP & qI + A_0^T P - C^T K^T \\ qI + PA_0 - KC & -RP \end{bmatrix} < 0 \quad (42)$$

where  $\Xi = A_0^T P + PA_0 - K^T P - PK + \gamma^2 Q + I + 2aP$  ■

*Proof.* The proof is based on adding the constraints on the eigenvalues of the matrix  $(A_0 - LC)$  (see [35]).

### **3.2 Second approach relying on the perturbation attenuation**

As previously mentioned, the main difficulty in designing an observer when considering a TS model with unmeasurable decision variable, more specifically the state of the system  $x(t)$ , is to handle the fact that the weighting functions of the model depend on the actual state  $x(t)$  when that of the observer depend on their estimates  $\hat{x}(t)$ . Therefore, an idea is to express the model using artificially

weighting functions depending on that estimates. Indeed, consider the model described by (9); it can be re-written in the form:

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^r (\mu_i(\hat{x}(t)) (A_i x(t) + B_i u(t)) + \delta_i(t) (A_i x(t) + B_i u(t))) \\ y(t) = Cx(t) \end{cases} \quad (43)$$

where:

$$\delta_i(t) = \mu_i(x(t)) - \mu_i(\hat{x}(t))$$

Let us define:

$$\Delta A(t) = \sum_{i=1}^r \delta_i(t) A_i = \mathcal{A} \Sigma_A(t) E_A \quad (44)$$

$$\Delta B(t) = \sum_{i=1}^r \delta_i(t) B_i = \mathcal{B} \Sigma_B(t) E_B \quad (45)$$

where:

$$\mathcal{A} = [A_1 \dots A_r], \quad E_A = [I_n \dots I_n]^T, \quad \mathcal{B} = [B_1 \dots B_r], \quad E_B = [I_m \dots I_m]^T$$

$$\Sigma_A(t) = \begin{bmatrix} \delta_1(t) I_n & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \delta_r(t) I_n \end{bmatrix}, \quad \Sigma_B(t) = \begin{bmatrix} \delta_1(t) I_m & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \delta_r(t) I_m \end{bmatrix}$$

The convex sum property of the weighting functions implies the inequality:

$$-1 \leq \delta_i(t) \leq 1$$

then:

$$\begin{aligned} \Sigma_A^T(t) \Sigma_A(t) &\leq I \\ \Sigma_B^T(t) \Sigma_B(t) &\leq I \end{aligned} \quad (46)$$

The system (43) becomes:

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^r \mu_i(\hat{x}(t)) ((A_i + \Delta A(t))x(t) + (B_i + \Delta B(t))u(t)) \\ y(t) = Cx(t) \end{cases} \quad (47)$$

Finally, the system (9) with unmeasurable decision variables is transformed into an equivalent ‘‘uncertain’’ TS model with known decision variables (47). Attention, as a matter of fact, the terms are not uncertain, but only unknown (unlike model uncertainties). This writing is used for observer design purpose only.

For the uncertain model (47), the following observer is proposed:

$$\begin{cases} \dot{\hat{x}}(t) = \sum_{i=1}^r \mu_i(\hat{x}(t)) (A_i \hat{x}(t) + B_i u(t) + L_i (y(t) - \hat{y}(t))) \\ \hat{y}(t) = C \hat{x}(t) \end{cases} \quad (48)$$

The gains  $L_i$  must be determined to ensure the asymptotic convergence of the estimated state  $\hat{x}$  to the actual state of the system  $x$ . In the sequel, for the sake of simplicity, the time variable  $t$  will be omitted. The time evolution of the state estimation error  $e = x - \hat{x}$  is described by the following equation:

$$\dot{e} = \sum_{i=1}^r \mu_i(\hat{x}) ((A_i - L_i C)e) + \Delta A x + \Delta B u \quad (49)$$

Notice that it depends on the input  $u(t)$  and on the state  $x(t)$ . Then the problem of designing the observer (48) reduces to finding the gains  $L_i$  in order that the system (49) generating  $e(t)$  is stable and that the influence of  $u(t)$  on  $e(t)$  is minimized.

Let us define the augmented vector  $e_a = [e^T \ x^T]^T$ , from which the following augmented system is obtained:

$$\begin{cases} \dot{e}_a = \sum_{i=1}^r \sum_{j=1}^r \mu_i(\hat{x}) \mu_j(x) (\bar{A}_{ij} e_a + \bar{B}_{ij} u) \\ e = H e_a \end{cases} \quad (50)$$

where

$$\bar{A}_{ij} = \begin{bmatrix} \Phi_i & \Delta A \\ 0 & A_j \end{bmatrix}, \quad \bar{B}_{ij} = \begin{bmatrix} \Delta B \\ B_j \end{bmatrix}, \quad \Phi_i = A_i - L_i C, \quad H = [I \ 0] \quad (51)$$

**Theorem 4.** *The system (50) is stable and the  $\mathcal{L}_2$  gain of the transfer from  $u(t)$  to the state estimation error is bounded by  $\gamma$ , if there exists two positive and symmetric matrices  $P_1$  and  $P_2$ , matrices  $K_i$ , and positive scalars  $\lambda_1$ ,  $\lambda_2$  and  $\bar{\gamma}$  such that the following LMIs hold,  $\forall i, j \in \{1, \dots, r\}$ :*

$$\begin{bmatrix} \Psi_i & 0 & 0 & P_1 A & P_1 B \\ 0 & \Xi_j & P_2 B_j & 0 & 0 \\ 0 & B_j^T P_2 & -\bar{\gamma} I + \lambda_2 E_B^T E_B & 0 & 0 \\ A^T P_1 & 0 & 0 & -\lambda_1 I & 0 \\ B^T P_1 & 0 & 0 & 0 & -\lambda_2 I \end{bmatrix} < 0 \quad (52)$$

where:

$$\Psi_i = A_i^T P_1 + P_1 A_i - K_i C - C^T K_i^T + I \quad (53)$$

$$\Xi_j = A_j^T P_2 + P_2 A_j + \lambda_1 E_A^T E_A \quad (54)$$

*The gains of the observer are computed from  $L_i = P_1^{-1} K_i$  and the resulting  $\mathcal{L}_2$ -gain from  $u(t)$  to  $e(t)$  is defined by  $\gamma = \sqrt{\bar{\gamma}}$  ■*

*Proof.* The proof of the theorem 4 is established by using the following quadratic Lyapunov function candidate:

$$V = e_a^T P e_a, \quad P = P^T > 0 \quad (55)$$

Its derivative with regard to time is given by:

$$\dot{V} = \dot{e}_a^T P e_a + e_a^T P \dot{e}_a \quad (56)$$

By using the dynamics of the augmented estimation error (50), the following is obtained:

$$\dot{V} = \sum_{i=1}^r \sum_{j=1}^r \mu_i(\hat{x}) \mu_j(x) (e_a^T \bar{A}_{ij}^T P e_a + e_a^T P \bar{A}_{ij} e_a + u^T \bar{B}_{ij}^T P e_a + e_a^T P \bar{B}_{ij} u) \quad (57)$$

The system (49) is stable and the gain  $\mathcal{L}_2$  of the transfer from  $u(t)$  to  $e(t)$  is bounded by  $\gamma$  if the following condition holds [8]:

$$\dot{V} + e^T e - \gamma^2 u^T u < 0 \quad (58)$$

Using (50) and substituting (57) in (58), this latter becomes:

$$\begin{aligned} & \sum_{i=1}^r \sum_{j=1}^r \mu_i(\hat{x}) \mu_j(x) (e_a^T \bar{A}_{ij}^T P e_a + e_a^T P \bar{A}_{ij} e_a \\ & + u^T \bar{B}_{ij}^T P e_a + e_a^T P \bar{B}_{ij} u + e_a^T H^T H e_a - \gamma^2 u^T u) < 0 \end{aligned} \quad (59)$$

or equivalently:

$$\sum_{i=1}^r \sum_{j=1}^r \mu_i(\hat{x}) \mu_j(x) \begin{bmatrix} e_a \\ u \end{bmatrix}^T \begin{bmatrix} X_{ij} & P \bar{B}_{ij} \\ \bar{B}_{ij}^T P & -\gamma^2 I \end{bmatrix} \begin{bmatrix} e_a \\ u \end{bmatrix} < 0 \quad (60)$$

where:

$$X_{ij} = \bar{A}_{ij}^T P + P \bar{A}_{ij} + H^T H \quad (61)$$

According to the convex sum property of the weighting functions  $\mu_i$ , the inequality (60) holds if (62) is satisfied:

$$\begin{bmatrix} \bar{A}_{ij}^T P + P \bar{A}_{ij} + H^T H & P \bar{B}_{ij} \\ \bar{B}_{ij}^T P & -\gamma^2 I \end{bmatrix} < 0, \quad \forall i, j \in \{1, \dots, r\} \quad (62)$$

Let us consider the following particular form of the matrix  $P$ :

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \quad (63)$$

Using (51), (63) and the definition of  $H$ , equation (62) can be written as:

$$\begin{bmatrix} \Phi_i^T P_1 + P_1 \Phi_i + I & P_1 \Delta A & P_1 \Delta B \\ \Delta A^T P_1 & A_j^T P_2 + P_2 A_j & P_2 B_j \\ \Delta B^T P_1 & B_j^T P_2 & -\gamma^2 I \end{bmatrix} < 0 \quad (64)$$

Notice that the inequality (64) is time-dependent due to the terms  $\Delta A(t)$  and  $\Delta B(t)$ . However, these latter are bounded.

Firstly let us re-write (64) separating the time-dependent terms.

$$\begin{bmatrix} \Phi_i^T P_1 + P_1 \Phi_i + I & 0 & 0 \\ 0 & A_j^T P_2 + P_2 A_j & P_2 B_j \\ 0 & B_j^T P_2 & -\gamma^2 I \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & P_1 \Delta A & P_1 \Delta B \\ \Delta A^T P_1 & 0 & 0 \\ \Delta B^T P_1 & 0 & 0 \end{bmatrix}}_{\mathcal{W}} < 0 \quad (65)$$

The time-dependent matrix  $\mathcal{W}$  is decomposed according to:

$$\mathcal{W} = \mathcal{Q} + \mathcal{Q}^T \quad (66)$$

where:

$$\mathcal{Q} = \begin{bmatrix} 0 & P_1 \Delta A & P_1 \Delta B \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

According to the definition of  $\Delta A(t)$  and  $\Delta B(t)$  given in (44) and (45), the matrix  $\mathcal{Q}$  is written as follows:

$$\mathcal{Q} = \underbrace{\begin{bmatrix} P_1 \mathcal{A} & P_1 \mathcal{B} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\theta} \underbrace{\begin{bmatrix} 0 & \Sigma_A(t) E_A & 0 \\ 0 & 0 & \Sigma_B(t) E_B \end{bmatrix}}_{\psi} \quad (67)$$

Applying lemma 1 to  $\mathcal{W}$  (66) with the following definition of  $\Omega$ :

$$\Omega = \begin{bmatrix} \lambda_1 I & 0 \\ 0 & \lambda_2 I \end{bmatrix} \quad (68)$$

we obtain:

$$\mathcal{W} \leq \theta \Omega^{-1} \theta^T + \psi^T \Omega \psi \quad (69)$$

After some computations using the properties of  $\Sigma_A(t)$  and  $\Sigma_B(t)$  (46), the matrix  $\mathcal{W}$  is bounded by:

$$\mathcal{W} \leq \begin{bmatrix} Y & 0 & 0 \\ 0 & \lambda_1 E_A^T E_A & 0 \\ 0 & 0 & \lambda_2 E_B^T E_B \end{bmatrix} \quad (70)$$

where:

$$Y = \lambda_1^{-1} P_1 \mathcal{A} \mathcal{A}^T P_1 + \lambda_2^{-1} P_1 \mathcal{B} \mathcal{B}^T P_1 \quad (71)$$

Substituting  $\mathcal{W}$  (70) in (65), the following is obtained:

$$\begin{bmatrix} \Xi & 0 & 0 \\ 0 & \Xi_j & P_2 B_j \\ 0 & B_j^T P_2 & -\gamma^2 I + \lambda_2 E_B^T E_B \end{bmatrix} < 0 \quad (72)$$

with:

$$\Xi = \Phi_i^T P_1 + P_1 \Phi_i + \lambda_1^{-1} P_1 \mathcal{A} \mathcal{A}^T P_1 + \lambda_2^{-1} P_1 \mathcal{B} \mathcal{B}^T P_1 + I \quad (73)$$

The matrix inequality (72) is not linear with regard to the variables  $P_1$ ,  $P_2$ ,  $L_i$ ,  $\lambda_1$ ,  $\lambda_2$  and  $\gamma$ . In order to solve these matrix inequalities, it is necessary to linearize them to obtain LMIs. To this end, using the Schur complement and some variable changes  $K_i = P_1 L_i$  and  $\bar{\gamma} = \gamma^2$  allows to obtain (52).

## 4 Design of unknown input observers

Processes are often subjected to disturbances which have harmful effects on the normal behavior of the process and their estimation can be used to conceive a control strategy able to minimize their effects. The disturbances are called unknown inputs when they affect the input of the process and their presence make difficult the state estimation. In the linear system framework, many works have been achieved concerning the estimation of the state and the output of a system subjected to unknown inputs [12], [13], [48]. Some extensions to nonlinear systems have also been proposed [10], [37], [38], [40]. For fault isolation purpose, some inputs are frequently considered as unknown in order to design a bank of observers. Each of them is fed with a different set of variables. For example, for detecting actuator faults, it is convenient to design observers using all the inputs of a considered system but one. The different estimates provided by these observers give information about the occurrence of faults. This section is then dedicated to the design of unknown input observer for TS model.

### 4.1 Partial decoupling observer

Consider the TS model with unmeasurable decision variables and unknown inputs:

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^r \mu_i(x(t)) (A_i x(t) + B_i u(t) + E_i d(t)) \\ y(t) = Cx(t) + Gd(t) \end{cases} \quad (74)$$

where  $d(t) \in \mathbb{R}^d$  is an unknown input vector and with matrices  $E_i$  and  $G$  of compatible dimensions. In the following, it is assumed that the dimension of the unknown input vector is less than that of the output vector ( $d < p$ ). Following the idea developed in the section 3.2, the considered model can be rewritten as an equivalent perturbed TS model with known decision variable:

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^r \mu_i(\hat{x}(t)) (A_i x(t) + B_i u(t) + E_i d(t) + \omega(t)) \\ y(t) = Cx(t) + Gd(t) \end{cases} \quad (75)$$

with :

$$\omega(t) = \sum_{i=1}^r (\mu_i(x(t)) - \mu_i(\hat{x}(t))) (A_i x(t) + B_i u(t) + E_i d(t)) \quad (76)$$

The proposed observer has the following structure:

$$\begin{cases} \dot{z}(t) = \sum_{i=1}^r \mu_i(\hat{x}(t)) (N_i z(t) + G_i u(t) + L_i y(t)) \\ \hat{x}(t) = z(t) - Hy(t) \end{cases} \quad (77)$$

The state estimation error is given by:

$$\begin{aligned} e(t) &= x(t) - \hat{x}(t) \\ &= Px(t) - z(t) + HGd(t) \end{aligned} \quad (78)$$

where:

$$P = I + HC \quad (79)$$

and its dynamics can be expressed as

$$\begin{aligned} \dot{e}(t) &= P\dot{x}(t) - \dot{z}(t) + HG\dot{d}(t) \\ &= \sum_{i=1}^r \mu_i(\hat{x}(t)) ((PA_i - N_i - K_i C)x(t) + (PB_i - G_i)u(t) \\ &\quad + (PE_i - K_i G)d(t) + P\omega(t) + N_i e(t)) + HG\dot{d}(t) \end{aligned} \quad (80)$$

with  $K_i = N_i H + L_i$ . If the following conditions hold:

$$HG = 0 \quad (81)$$

$$N_i = PA_i - K_i C \quad (82)$$

$$PB_i = G_i \quad (83)$$

$$PE_i = K_i G \quad (84)$$

$$L_i = K_i - N_i H \quad (85)$$

the dynamics of the state equation becomes:

$$\dot{e}(t) = \sum_{i=1}^r \mu_i(\hat{x}(t)) (N_i e(t) + P\omega(t)) \quad (86)$$

This equation describes a dynamic perturbed system that can be analyzed using the methods presented in section 3.

Let us first consider a Lipschitz approach. In that case, the term  $\omega(t)$  defined in (76) is assumed to satisfy:

$$|\omega(t)| \leq \gamma |e(t)| \quad (87)$$

where  $\gamma$  is a positive constant.



**Theorem 5.** *An unknown input observer exists for the system (74) if there exists a symmetric positive definite matrix  $X$ , matrices  $M_i$  and  $S$ , and positive scalar  $\lambda$  such that the following LMIs hold,  $\forall i = 1, \dots, r$  :*

$$\begin{bmatrix} \Psi_i & (X + SC) \\ (X + SC)^T & -\lambda I \end{bmatrix} < 0 \quad (88)$$

$$SG = 0 \quad (89)$$

$$(X + SC)E_i = M_iG \quad (90)$$

where:

$$\Psi_i = A_i^T(X + C^T S^T) + (X + SC)A_i - C^T M_i^T - M_i C + \lambda \gamma^2 I \quad (91)$$

The matrices defining the observer are computed according to:

$$H = X^{-1}S \quad (92)$$

$$K_i = X^{-1}M_i \quad (93)$$

$$N_i = (I + HC)A_i - K_i C \quad (94)$$

$$L_i = K_i - N_i H \quad (95)$$

$$G_i = (I + HC)B_i \quad (96)$$

■

*Proof.* The proof of the theorem 5 relies on the existence of a quadratic Lyapunov function  $V(t) = e^T(t)X e(t)$  with  $X = X^T > 0$ . Evaluating its derivative and using the lemma 1 lead to the following inequality:

$$\dot{V}(t) \leq \sum_{i=1}^r \mu_i(\hat{x}) e^T(t) (N_i^T X + X N_i + \lambda \gamma^2 I + \lambda^{-1} X P P^T X) e(t) \quad (97)$$

with  $\lambda > 0$ . Thanks to the convex sum property, the derivatives of the Lyapunov function is negative if:

$$N_i^T X + X N_i + \lambda \gamma^2 I + \lambda^{-1} X P P^T X < 0, \quad i = 1, \dots, r \quad (98)$$

Using (82), these last inequalities can also be written as:

$$(P A_i - K_i C)^T X + X (P A_i - K_i C) + \lambda \gamma^2 I + \lambda^{-1} X P P^T X < 0, \quad i = 1, \dots, r \quad (99)$$

With the change of variable  $M_i = X K_i$  and using the Schur complement, (99) can also be written as the following LMIs,  $\forall i \in \{1, \dots, r\}$ :

$$\begin{bmatrix} A_i^T P^T X + X P A_i - C^T M_i^T - M_i C + \lambda \gamma^2 I & X P \\ P^T X & -\lambda I \end{bmatrix} < 0 \quad (100)$$

Using (81) and the change of variable  $S = XH$ , we have  $XP = X + SC$ ; therefore (100) corresponds to inequality (88). The LMIs (89) and (90) are obtained from (81) and (84) using the previous changes of variables. The observer gains  $N_i$ ,  $L_i$  and  $G_i$  are then defined by (94), (95) and (96).

## 4.2 Disturbance attenuation observer

When it is not possible to use a Lipschitz assumption, the observer can also be designed based on a  $\mathcal{L}_2$  approach. The main result is summarized by the following theorem:

**Theorem 6.** *The unknown input observer (77) for the system (74) such that the state estimation error converges towards zero while minimizing the  $\mathcal{L}_2$  gain of the transfer from  $\omega(t)$  to  $e(t)$  is obtained by solving the following constrained optimization problem:*

$$\min_{X, M_i, S} \bar{\gamma} \quad s.t.$$

$$\begin{bmatrix} \Psi_i & X + SC \\ (X + SC)^T & -\bar{\gamma}I \end{bmatrix} < 0, \quad i = 1, \dots, r \quad (101)$$

$$SG = 0 \quad (102)$$

$$(X + SC)E_i = M_iG, \quad i = 1, \dots, r \quad (103)$$

where:

$$\Psi_i = A_i^T(X + C^T S^T) + (X + SC)A_i - C^T M_i^T - M_i C + I \quad (104)$$

The observer matrices are defined by (92)-(96) and the attenuation level is given by  $\gamma = \sqrt{\bar{\gamma}}$ .  $\blacksquare$

*Proof.* Theorem 6 is easily proved using the bounded real lemma already used in the proof of theorem 2. The state estimation error given in (86) converges towards zero and the  $\mathcal{L}_2$  gain of the transfer from  $\omega(t)$  to  $e(t)$  is bounded by  $\gamma$  if the following LMI holds:

$$\sum_{i=1}^r \mu_i(\hat{x}(t)) \begin{bmatrix} N_i^T X + X N_i + I & X P \\ P^T X & -\gamma^2 I \end{bmatrix} < 0 \quad (105)$$

The convex sum property allows sufficient conditions to be expressed:

$$\begin{bmatrix} N_i^T X + X N_i + I & X P \\ P^T X & -\gamma^2 I \end{bmatrix} < 0, \quad i = 1, \dots, r \quad (106)$$

Using the expression (82) of  $N_i$  and the change of variable  $M_i = X K_i$  and  $\bar{\gamma} = \gamma^2$ , inequalities (106) becomes:

$$\begin{bmatrix} A_i^T P^T X + X P A_i - C^T M_i^T - M_i C + I & X P \\ P^T X & -\bar{\gamma}I \end{bmatrix} < 0, \quad \forall i = 1, \dots, r \quad (107)$$

The remaining part of the proof is then identical to that of theorem 5 and it is omitted for sake of brevity.

### 4.3 Unknown input estimation

Unknown inputs can be estimated on the basis of the previous state estimation methods. Indeed, the unknown input  $d(t) \in \mathbb{R}^d$  intervenes in (74) through the following distribution matrix:

$$W(t) = \begin{bmatrix} \sum_{i=1}^r \mu_i(\hat{x}(t)) E_i \\ G \end{bmatrix} \quad (108)$$

For being able to estimate the unknown input, the rank of the matrix  $W(t)$  must satisfy the condition:

$$\text{rank}(W(t)) = d, \forall t \quad (109)$$

If this condition holds,  $W(t)$  is full column rank and its left pseudo-inverse  $W^-(t)$  exists and is defined by:

$$W^-(t) = (W^T(t)W(t))^{-1} W^T(t) \quad (110)$$

The unknown input can then be computed according to:

$$\hat{d}(t) = W^-(t) \begin{bmatrix} \dot{\hat{x}}(t) - \sum_{i=1}^r \mu_i(\hat{x}(t)) (A_i \hat{x}(t) + B_i u(t)) \\ y(t) - C \hat{x}(t) \end{bmatrix} \quad (111)$$

In that case, the asymptotic convergence of  $\hat{x}$  towards  $x$  implies the asymptotic convergence of  $\hat{d}$  towards  $d$ .

### 4.4 Design of proportional-integral observer

In the context of robust observer design, one of the most successful technique is the use of proportional-integral (PI) observer, in which the unknown inputs are estimated simultaneously with the states of the system. The PI observer was first proposed by Wojciechowsky in [49] for single input-single output LTI systems. A generalization scheme was performed by Kaczorek [25] to multivariable systems. Thereafter, the PI observer has been used in different studies. In [36] a linear PI observer is designed and applied to a physical system. In [28] a PI observer for linear descriptor systems is proposed. However, that kind of observer can be used only if the unknown inputs are constant over the time, nevertheless in practical cases the approach is effective if the variations of the unknown inputs are slow in respect to the dynamic of the system. When it is not the case, the problem can be solved by using multiple integrals in the observer in order to estimate all of the derivatives of the unknown inputs. A proportional multiple integral (PMI) observer was firstly proposed by Jiang in [23]. In [16], [26] a proportional multiple integral observer is proposed to estimate a large class of signals described in a polynomial form for LTI descriptor systems.

Consider the following perturbed system:

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^r \mu_i(\xi(t)) (A_i x(t) + B_i u(t) + E_i d(t) + R_i \omega(t)) \\ y(t) = Cx(t) + Gd(t) + W\omega(t) \end{cases} \quad (112)$$

where  $d(t)$  is the vector of unknown inputs and  $\omega(t)$  a disturbance vector. In the following, we assume the three hypotheses:

- **A1.** the system (112) is stable
- **A2.** the signals  $u(t)$ ,  $d(t)$  and  $\omega(t)$  are bounded
- **A3.**  $\dot{d}(t) = 0$

The proposed observer has the following structure:

$$\begin{cases} \dot{\hat{x}}(t) = \sum_{i=1}^r \mu_i(\hat{x}(t)) \left( A_i \hat{x}(t) + B_i u(t) + E_i \hat{d}(t) + L_{P_i} (y(t) - \hat{y}(t)) \right) \\ \dot{\hat{d}}(t) = \sum_{i=1}^r \mu_i(\hat{x}(t)) (L_{I_i} (y(t) - \hat{y}(t))) \\ \hat{y}(t) = C\hat{x}(t) + G\hat{d}(t) \end{cases} \quad (113)$$

Notice that the observer must involve estimated decision variable. Using the idea introduced in section 3.2, the dynamic equation of the model can be rewritten with weighting functions depending on estimated decision variables:

$$\dot{x}(t) = \sum_{i=1}^r \mu_i(\hat{x}(t)) (A_i x(t) + B_i u(t) + E_i d(t) + R_i \omega(t) + \nu(t)) \quad (114)$$

where:

$$\nu(t) = \sum_{i=1}^r (\mu_i(x(t)) - \mu_i(\hat{x}(t))) (A_i x(t) + B_i u(t) + E_i d(t) + R_i \omega(t)) \quad (115)$$

This term is seen as a bounded vanishing perturbation to minimize. Indeed, due to the assumptions **A1** and **A2** and the definition of the weighting functions,  $\nu(t)$  is bounded and if  $\hat{x} \rightarrow x$  then  $\nu \rightarrow 0$ .

Using **A3**, the system (112) can be written using an augmented state vector:

$$\begin{cases} \dot{x}_a(t) = \sum_{i=1}^r \mu_i(\hat{x}(t)) (\bar{A}_i x_a(t) + \bar{B}_i u(t) + \bar{F}_i \bar{\omega}(t)) \\ y(t) = \bar{C} x_a(t) + \bar{D} \bar{\omega}(t) \end{cases} \quad (116)$$

with:

$$x_a(t) = \begin{bmatrix} x(t) \\ d(t) \end{bmatrix}, \quad \bar{A}_i = \begin{bmatrix} A_i & E_i \\ 0 & 0 \end{bmatrix}, \quad \bar{B}_i = \begin{bmatrix} B_i \\ 0 \end{bmatrix}, \quad \bar{F}_i = \begin{bmatrix} I & R_i \\ 0 & 0 \end{bmatrix}, \quad \bar{\omega}(t) = \begin{bmatrix} \nu(t) \\ \omega(t) \end{bmatrix}$$

$$\bar{C} = [C \quad G], \quad \bar{D} = [0 \quad W]$$

A similar reasoning makes it possible to transform the proposed PI observer (113) in the following augmented form:

$$\begin{cases} \dot{\hat{x}}_a(t) = \sum_{i=1}^r \mu_i(\hat{x}(t)) (\bar{A}_i \hat{x}_a(t) + \bar{B}_i u(t) + \bar{L}_i (y(t) - \hat{y}(t))) \\ \hat{y}(t) = \bar{C} \hat{x}_a(t) \end{cases} \quad (117)$$

where:

$$\bar{L}_i = \begin{bmatrix} L_{P_i} \\ L_{I_i} \end{bmatrix} \quad (118)$$

The augmented state estimation error  $e_a = x_a - \hat{x}_a$  obeys the dynamic equation:

$$\dot{e}_a(t) = \sum_{i=1}^r \mu_i(\hat{x}(t)) ((\bar{A}_i - \bar{L}_i \bar{C}) e_a(t) + (\bar{\Gamma}_i - \bar{L}_i \bar{D}) \bar{\omega}(t)) \quad (119)$$

The bounded-real lemma then allows the following theorem to be formulated:

**Theorem 7.** *The PI observer (117) for the system (112) is obtained by solving, for  $P = P^T > 0$ , the following constrained optimization problem:*

$$\min_{P, \bar{M}_i} \bar{\gamma} \quad s.t.$$

$$\begin{bmatrix} \bar{A}_i^T P + P \bar{A}_i - \bar{M}_i \bar{C} - \bar{C}^T \bar{M}_i^T + I & P \bar{\Gamma}_i - \bar{M}_i \bar{D} \\ \bar{\Gamma}_i^T P - \bar{D}^T \bar{M}_i^T & -\bar{\gamma} I \end{bmatrix} < 0 \quad (120)$$

The gains of the observer are given by  $\bar{L}_i = P^{-1} \bar{M}_i$  and the attenuation level is  $\gamma = \sqrt{\bar{\gamma}}$ . ■

*Proof.* The proof is immediate using the bounded-real lemma.

## 5 Fault detection and isolation

Fault detection and isolation has been an active field of research over the past decades. Many techniques have been proposed especially for sensor and/or actuator faults with application to a wide range of engineering fields. Among them, the model-based techniques have been successfully used in several applications.

### 5.1 Implementation of an observer bank

A possible way to achieve FDI goals is to use a bank of dynamic observers to generate residuals for each monitored system output signal. A decision making system is then designed and tuned to detect and isolate faults. The LMI procedures described in this communication provide practical tools to design banks of nonlinear observers to be used for the FDI residual generation. The advantages of the design technique are that the system nonlinearities are directly accounted for in the observer structure [29].

When using that kind of approach, the output estimation errors of the different observers are used as residuals. To isolate faults, a structured residual set should be designed. For sensor faults, this design is very straightforward. For example, if the output vector  $y = (y_1, \dots, y_m)^T$  driving a specific observer is replaced by  $y = (y_1, \dots, y_{i-1}, y_{i+1}, y_m)^T$ , the corresponding residual will be insensitive to the fault in the  $i$ th sensor. The design of a structured residual set for isolating actuator faults is not straightforward but can be solved via the use of unknown input observers developed in section 4.

Due to lack of space, the use of observers previously developed for FDI is only illustrated with a basic academic example [22]. Let us consider a system represented by a TS model with one input  $u(t)$  and two outputs  $y_i(t), i = 1, 2$ :

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^r \mu_i(\hat{x}(t)) (A_i x(t) + B_i u(t)) \\ y(t) = Cx(t) + f(t) + w(t) \end{cases} \quad (121)$$

where  $f(t)$  represents the sensor fault vector and  $w(t)$  a zero-mean noise vector. A Generalized Observer Scheme (GOS) can be used in order to detect and isolate sensor faults. This scheme is represented in figure 1.

The residual signals are defined as follows:

$$r_{ij}(t) = y_j(t) - \hat{y}_{ij}(t), \quad \forall i, j \in \{1, 2\} \quad (122)$$

where  $i$  represents the observer number and  $j$  the output number.

Clearly, the residual signal  $r_{11}(t)$  is not sensitive to the fault  $f_2(t)$  and similarly  $r_{22}(t)$  is not sensitive to the fault  $f_1(t)$ . That properties can be used to develop a decision logic aiming to detect and isolate faults.

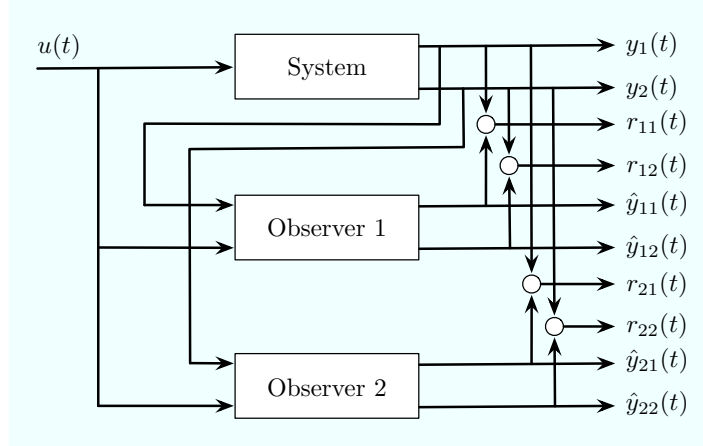
Consider the system (121) with  $r = 2$ , defined by the following matrices:

$$A_1 = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -3 & 0 \\ 2 & 1 & -8 \end{bmatrix}, A_2 = \begin{bmatrix} -3 & 2 & -2 \\ 5 & -3 & 0 \\ 1 & 2 & -4 \end{bmatrix}, B_1 = \begin{bmatrix} 1 \\ 0.5 \\ 0.5 \end{bmatrix}, B_2 = \begin{bmatrix} 0.5 \\ 1 \\ 0.25 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

The weighting functions are defined by:

$$\begin{cases} \mu_1(x) = \frac{1 - \tanh(x_1)}{2} \\ \mu_2(x) = 1 - \mu_1(x) = \frac{1 + \tanh(x_1)}{2} \end{cases}$$

In a first step, an observer has been built on the basis of the knowledge of the input  $u(t)$  and the two noise-free outputs  $y_i(t), i = 1, 2$ , using the procedure described in section 3.2. Minimizing the gain  $\bar{\gamma}$  of the transfer from  $u(t)$  to the state estimation error  $e(t)$  subject to the LMIs given in the theorem 4, gives the



**Fig. 1.** Scheme for sensor fault detection and isolation

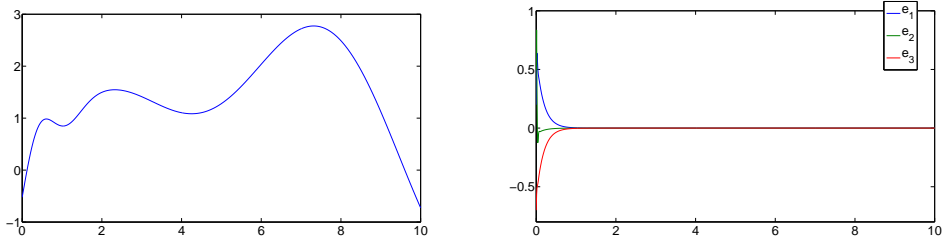
following results:

$$L_1 = \begin{bmatrix} -35.66 & 121.56 \\ 57.61 & 63.59 \\ 15.22 & -9.55 \end{bmatrix}, L_2 = \begin{bmatrix} -37.01 & 121.47 \\ 56.10 & 68.80 \\ -0.25 & -6.09 \end{bmatrix},$$

$$P_1 = \begin{bmatrix} 0.05 & -0.03 & 0.07 \\ -0.03 & 0.40 & -0.06 \\ 0.07 & -0.06 & 0.30 \end{bmatrix}, P_2 = \begin{bmatrix} 3.01 & 1.11 & 0.35 \\ 1.11 & 2.34 & 0.07 \\ 0.35 & 0.07 & 1.09 \end{bmatrix},$$

$$\lambda_1 = 3.47, \lambda_2 = 0.0028 \times 10^{-5}, \gamma = 0.0894$$

The input and the three estimation errors are depicted in figure 2.



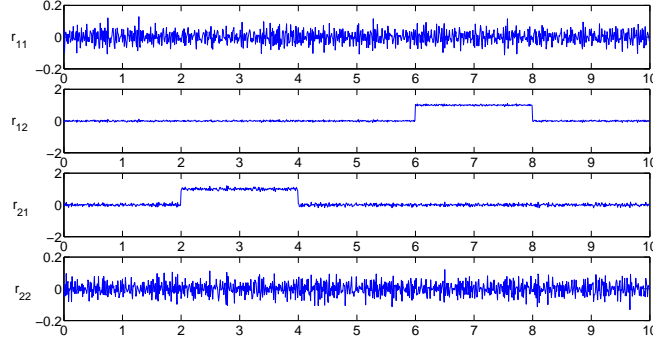
**Fig. 2.** Input of the system (left) and state estimation errors (right)

In a second step, using the same procedure, the two observers described in figure 1 have been designed on the basis of noisy and faulty outputs. The faults

are given by

$$f_1(t) = \begin{cases} 1, & 2 \leq t \leq 4 \\ 0, & \text{elsewhere} \end{cases} \quad f_2(t) = \begin{cases} 1, & 6 \leq t \leq 8 \\ 0, & \text{elsewhere} \end{cases}$$

The time evolution of the four residuals defined in (122) is depicted in figure 3.



**Fig. 3.** Residual signals

As expected, the residual  $r_{11}(t)$  is insensitive to the presence of  $f_2(t)$  (while, at the same time,  $r_{12}(t)$  deviates from zero) and symmetrically the residual  $r_{22}(t)$  is insensitive to the presence of  $f_1(t)$  (while  $r_{21}(t)$  deviates from zero). Clearly, for this low dimension example, the diagnosis scheme allows to detect and isolate the sensor faults.

## 5.2 Fault estimation using PI observer

Actuator faults can often be considered as unknown inputs acting on the system, therefore, the method presented in section 4.4 can be used to estimate these faults. Consider the model (112) (without disturbance  $\omega(t)$ ), defined by the following matrices:

$$A_1 = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -3 & 0 \\ 2 & 1 & -4 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -3 & 2 & -2 \\ 5 & -3 & 0 \\ 0.5 & 0.5 & -4 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 0.3 \\ 0.5 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.5 \\ 1 \\ 0.25 \end{bmatrix},$$

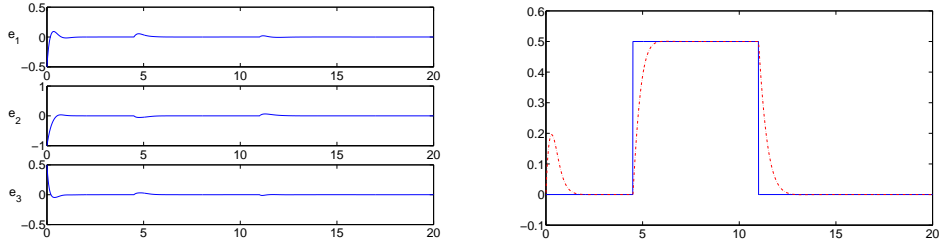
$$C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 0.5 \\ -1 \\ 0.25 \end{bmatrix}, \quad E_2 = \begin{bmatrix} -1 \\ 0.52 \\ 1 \end{bmatrix}, \quad G = \begin{bmatrix} 0.3 \\ 0.9 \end{bmatrix}$$

The considered unknown input is a piecewise constant function:

$$d(t) = \begin{cases} 0.5, & 4.5 \leq t \leq 11 \\ 0, & \text{elsewhere} \end{cases}$$

The PI observer defined by theorem 7 has been designed. The state estimation errors and the estimate of the unknown input are presented in figure 4.





**Fig. 4.** State estimation errors (left); Unknown input and its estimate (right)

Notice that the unknown input  $d(t)$  does not fully satisfy the hypothesis  $\dot{d}(t) = 0$ . There is therefore disturbances on the state estimation errors when the magnitude of the unknown input changes. One can note that the unknown input is correctly estimated. Of course, depending on the application requirements, the convergence rate of the observer can be adjusted by adding some pole placement constraints. If necessary and based on the same idea, a proportional integral multiple observer can also be implemented.

## 6 Conclusion

The results presented in this communication contribute to state estimation and FDI for nonlinear systems represented by Takagi-Sugeno systems. Throughout this work, existence conditions of the different observers have been established using a basic quadratic Lyapunov function. Of course, the conservatism of the obtained solutions could certainly be reduced using more sophisticated non quadratic Lyapunov functions such as piecewise and basis-dependent functions [39]. For discrete TS model, let us also mention the promising approach consisting to analyse the Lyapunov function decrease only every  $k$ -samples [19]. An immediate natural extension of this work concerns the use of fault estimates in an FTC strategy [46], [21]. In the context of FDI, another way of research consists to simultaneously estimate the state and the weighting functions based on input-output data. This approach could be interesting for systems modeled by a set of models representing normal and abnormal modes of operation. Indeed, the knowledge of the time evolution of the weighting functions could provide information about the active model leading to the diagnosis of the system.

## 7 Appendix

The following three lemmas are frequently used for bounding expressions, for transforming inequalities with quadratic terms into linear ones and for minimizing a  $\mathcal{L}_2$  gain of a transfer from a disturbance to an output of a system.

**Lemma 1.** For two matrices  $X$  and  $Y$  with appropriate dimensions, the following property holds:

$$X^T Y + X Y^T < X^T \Omega^{-1} X + Y \Omega Y^T, \quad \Omega > 0$$

**Lemma 2. (Schur complement [8])** Let us consider three matrices  $Q(x) = Q^T(x)$ ,  $R(x) = R^T(x)$  and  $S(x)$  of compatible dimensions depending linearly on the variable  $x$ . The following LMIs are equivalent:

1.  $\begin{pmatrix} Q(x) & S(x) \\ S^T(x) & R(x) \end{pmatrix} > 0$
2.  $R(x) > 0, Q(x) - S(x)R^{-1}(x)S(x) > 0$

**$\mathcal{L}_2$  gain of systems.** Consider the linear system:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases} \quad (123)$$

If the system is stable and  $u(t)$  is bounded then there exists  $\gamma > 0$  such that:

$$\int_0^{+\infty} y(t)^T y(t) dt \leq \gamma^2 \int_0^{+\infty} u(t)^T u(t) dt \quad (124)$$

The constant  $\gamma$  is called  $\mathcal{L}_2$  gain of the system. Particularly, for linear systems such (123),  $\gamma$  is the  $\mathcal{H}_\infty$  norm of the corresponding transfer function.

**Lemma 3. (Bounded-real lemma [8])** The constraint (124) holds for any bounded  $u(t)$  with  $u(t) \neq 0$  if and only if there exists a matrix  $P$  such that:

$$\begin{pmatrix} A^T P + P A + C^T C & P B + C^T D \\ B^T P + D^T C & D^T D - \gamma^2 I \end{pmatrix} < 0 \quad (125)$$

For a given  $\gamma$ , (125) defines an LMI in  $P$ . When it is desirable to minimize the influence of an external input  $u(t)$  on the output  $y(t)$ , the value of  $\gamma$  must be minimized. In order to keep the linearity of the constraint (125), the change of variable  $\bar{\gamma} = \gamma^2$  et the problem is formulated as follows:

$$\min_P \bar{\gamma} \quad s.t. \quad (126)$$

$$P > 0 \quad (127)$$

$$\begin{pmatrix} A^T P + P A + C^T C & P B + C^T D \\ B^T P + D^T C & D^T D - \bar{\gamma} I \end{pmatrix} < 0 \quad (128)$$

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